CONF-8509203 -- 5

Los Alamos National Laboratory is operated by the University of California for the United States Department of Energy under contract W-7405-ENG-36

LA-UR--85-4262

DE86 004731

TITLE: Hausdorff Dimensions for Sets with Broken Scaling Symmetry

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SUBMITTED TO: CNLS Workshop "Dimensions and Entropies in Chaotic Systems"
Proceedings held Sept. 11-16, 1985 at the Pecos River
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Hausdorff Dimensions for Sets with Broken Scaling Symmetry

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Based on Hausdorff's original approach to fractional dimensions, we study systems which are not sufficiently characterized by their "fractal" or scaling dimension. We construct informative examples of such sets and relate them to sets observed in the context of dynamical systems.

1. Introduction

Fractal sets which are not adequately characterized by their Hausdorff-Besicovitch dimensions occur in various physical contexts. One such class of fractals consists of sets which have only either zero or infinite d -dimensional Hausdorff measures.[1] Many random fractals such as those generated by brownian motion in the plane fall into this class [1,2]. In the brownian motion example, it can be shown that the set generated by the motion has a Hausdorff-Besicovitch dimension of 2 and yet is not area filling [2]. Although this set is fundamentally different from an ordinary 2 dimensional set such as a square, the dimension alone gives no indication of this. There is however a more general formalism due to HAUSDORFF [3] which is sensitive to such differences. In this formalism, the Hausdorff measure of a set is defined in terms of functions, called *gauge functions*, that tend to zero as a length scale ϵ tends to zero. The functional form of the particular gauge function which yields a nonzero and finite (referred to throughout simply as finite) Hausdorff measure is used to characterize the set. In the case where a set has the Hausdorff-Besicovitch dimension d_0 and a finite d_0 -dimensional Hausdorff measure, the gauge function of the set is ϵ^{d_0} . If the set has a dimension d_0 and its corresponding d_0 -dimensional Hausdorff measure is either zero or infinite, this gauge function is not a strict power law but is modified by a multiplicative correction which can in many cases be expanded in a product of powers of iterated logarithms in $\frac{1}{\epsilon}$. Thus, sets can be described in terms of

many exponents which can be viewed as higher order *corrections* to the usual Hausdorff-Besicovitch dimension. These corrections can be used to distinguish sets with integer dimensions, but nonfinite Hausdorff measures from *ordinary*, nonfractal sets.

A second class of sets which are not adequately described by their Hausdorff-Besicovitch dimensions are *fat* fractals [4,5,6,7]. These are fractal sets which have integer dimensions and finite volumes (Lebesgue measures). The gauge functions of these sets are pure power laws with integer exponents which are equal to their dimensions. The gauge functions of ordinary sets such as intervals, squares, and 3-tori also have this property. Thus it appears that the Hausdorff scheme is inadequate for describing this class of fractals. This apparent failure has led to other formulations for characterizing these sets [4,5,6]. In these treatments, some coarse-grained Lebesgue measure which approximates the true measure of a set is introduced. This coarse-grained measure depends upon some length scale ϵ . The manner in which it approaches the actual Lebesgue measure of the set as $\epsilon \rightarrow 0$ is used to characterize the set. Since the Lebesgue measure of a fat fractal is just a nonzero multiple of its Hausdorff measure, it is natural to wonder whether or not the Hausdorff formalism can be extended in such a way that fat fractals can be distinguished from ordinary sets.

The purpose of this paper is to discuss the manner in which the two classes of sets just described can be characterized. In particular we use Hausdorff's gauge function approach together with the notion of *capacity* dimension [8] to arrive at a simple scheme for describing these sets. We first present the general formulation where a set's dimension is determined by the scaling properties of its intrinsic gauge function. We then present some simple examples of thin fractals (fractals that are not fat) which illustrate how logarithmic corrections to pure power laws can arise. Next, we examine the characterization of fat fractals in terms of additive corrections to their intrinsic gauge functions. We then define the notion of a *metadimension* which arises from this type of corrections. We will see that this notion is applicable to thin fractals as well.

2. Gauge functions as dimensions

We begin by introducing the Hausdorff-Besicovitch dimension of a set S embedded in a D -dimensional space. Choose an $\epsilon > 0$ and consider every countable covering of S whose elements are D -dimensional cubes of side ϵ or smaller [8,9]. For each of these coverings, form the quantity $\sum_m \epsilon_m^d$, where the sum is over all of the elements of a particular cover and d is some positive real number. Find the infimum of this quantity over all such covers to get the coarse-grained d -dimensional Hausdorff measure of S , $\mathcal{H}^d(\epsilon) = \inf \sum_m \epsilon_m^d$. Then the d -dimensional Hausdorff measure of the set is defined by

$$\nu^{(d)} = \lim_{\epsilon \rightarrow 0} \nu^{(d)}(\epsilon). \quad (1)$$

Now, there exists a critical value of d , say d_0 , such that $\nu^{(d)}$ is infinite for all $d < d_0$ and zero for all $d > d_0$. [3] This critical exponent is called the Hausdorff-Besicovitch dimension of the set.

The concepts just introduced were motivated by the desire to have a generalization of the notion of size which is applicable to sets having nonfinite Lebesgue measures. When d_0 is an integer and $\nu^{(d_0)}$ is finite, the d_0 -dimensional Lebesgue measure is just equal to $\nu^{(d_0)}$ [10]. When d in Eq.(1) is not an integer, it may be considered to be a generalization of the concept of dimension, and $\nu^{(d)}$ may be viewed as a generalized *volume* corresponding to that dimension. The definition of d_0 given above implies nothing about the finiteness of its associated d_0 -dimensional Hausdorff measure. Thus, there can exist sets for which this generalization of size is inadequate. HAUSDORFF [3] recognized this and suggested a more general measure based upon arbitrary *gauge functions*, λ of a non-negative argument ϵ satisfying

$$\lambda(\epsilon) > 0, \quad (2a)$$

$$\lambda(\epsilon_1) < \lambda(\epsilon_2) \quad \text{for } \epsilon_1 < \epsilon_2, \quad (2b)$$

$$\lambda(\epsilon) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0, \quad (2c)$$

$$\lambda(\epsilon) \rightarrow \infty \quad \text{as } \epsilon \rightarrow \infty, \quad (2d)$$

and

$$\begin{vmatrix} \lambda(\epsilon_1) & \epsilon_1^D & 1 \\ \lambda(\epsilon_2) & \epsilon_2^D & 1 \\ \lambda(\epsilon_3) & \epsilon_3^D & 1 \end{vmatrix} > 0 \quad \text{for } \epsilon_1 < \epsilon_2 < \epsilon_3. \quad (2e)$$

Let $\lambda(\epsilon)$ be any function satisfying Eqs.(2). Then, define the coarse-grained Hausdorff measure of a set S with respect to λ by $\nu_\lambda(\epsilon) = \inf \sum_m \lambda(\epsilon_m)$.

This coarse-grained measure is defined in the same way as the d -dimensional measure except that ϵ_m^d has been replaced by its generalization $\lambda(\epsilon_m)$. Then the Hausdorff measure with respect to λ is just

$$\nu_\lambda = \lim_{\epsilon \rightarrow 0} \nu_\lambda(\epsilon). \quad (3)$$

Note that the d -dimensional Hausdorff measure is recovered from this definition by choosing ϵ^d for the gauge function. The particular gauge function λ that gives a finite Hausdorff measure is called the intrinsic gauge function of the set (or simply the gauge function of the set).

The rate at which the intrinsic gauge function of a set vanishes as $\epsilon \rightarrow 0$ is associated with the *dimension* of the set. This is easy to see when the gauge function is a pure power law since the function scales to zero faster for larger values of d_0 . Thus we say that the more quickly a set's intrinsic gauge function vanishes with ϵ , the larger is the dimension the set. In fact, HAUSDORFF [3] refers to a set's intrinsic gauge function itself as the dimension of the set.

An example of a gauge function which corresponds to a dimension that is *between* two power laws is

$$\lambda_1(\epsilon) = \epsilon^{d_0} \left[\log\left(\frac{1}{\epsilon}\right) \right]^{-d_1}. \quad (4)$$

It is easy to see that when $d_1 > 0$, this function goes to zero faster than ϵ^{d_0} and slower than $\epsilon^{(d_0+\delta)}$ for any $\delta > 0$. If d_1 is chosen to be negative, this function vanishes slower than ϵ^{d_0} and faster than $\epsilon^{(d_0-\delta)}$. To carry this a bit further note that for any $d_0, d_1, \delta > 0$, we can find a function that goes to zero faster than the function given in Eq.(4) and slower than the function $\epsilon^{d_0} \left[\log\left(\frac{1}{\epsilon}\right) \right]^{-(d_1+\delta)}$. An example of such a function is

$$\lambda_2(\epsilon) = \lambda_1(\epsilon) \left[\log^2\left(\frac{1}{\epsilon}\right) \right]^{-d_2}$$

where $\log^2(x) = \log \log(x)$ and we have chosen $d_2 > 0$. In his 1919 paper, Hausdorff presents a general expansion of gauge functions that allows for a large number of vanishing rates. It has the form

$$\lambda(\epsilon) = \epsilon^{d_0} \prod_{k=1}^n \left[\log^k\left(\frac{1}{\epsilon}\right) \right]^{-d_k} \quad (5)$$

where $\log^1(x) = \log(x)$, $\log^k(x) = \log \log^{k-1}(x)$, and n is finite. Furthermore, he supplies an algorithm for constructing a linear Cantor set having a gauge function of this form for any finite n and any set of d_k 's provided that the gauge function vanishes slower than ϵ . It should be kept in mind that the form given in Eq.(5) is just one possible expansion. The gauge function of a set is strictly defined as that function $\lambda(\epsilon)$ which satisfies Eqs.(2) and yields a finite Hausdorff measure. However, the expansion is motivated by the study of general scaling properties of functions. It is a special case of what HARDY [11] calls the logarithmico-exponential scale. In cases where the expansion of the form given by Eq.(5) does not exist for a finite n , the expansion can still be used to obtain an approximate gauge function for the set since it is guaranteed that a set's gauge function scales to zero slower at a rate which is *between some two power laws*.

Some comments are in order. First note that the gauge function of a set that is embedded in a D -dimensional space cannot vanish faster than ϵ^D [3]. Otherwise the set would have a dimension that is greater than the space it is

embedded in. Secondly, the intrinsic gauge function of a set is not unique: only an equivalence class of such gauge functions can be defined. We see this as follows: Let $\lambda(\epsilon)$ be a known gauge function of the set S . Choose any number $A > 0$ and any positive function $\xi(\epsilon)$ for which $\lim_{\epsilon \rightarrow 0} \xi(\epsilon) = 1$. Then $\lambda'(\epsilon) = A \lambda(\epsilon) \xi(\epsilon)$ is also an intrinsic gauge function of S . A serves only to change the normalization of the measure. The factor $\xi(\epsilon)$ has no effect on the Hausdorff measure, and, in the sense defined by HARDY [11], it does not affect the scaling rate of the gauge function. This together with the fact that intrinsic gauge functions are determined (in the Hausdorff formalism) only in the $\epsilon \rightarrow 0$ limit makes the definition of a unique intrinsic gauge function impossible.

3. Problems with the Application of the Formalism

The general Hausdorff formalism is difficult to apply in both real and numerical experiments. This is easy to imagine when one considers the definition of the coarse-grained Hausdorff measure. First of all one needs to consider countable covers rather than just finite ones. Secondly, the cover which minimizes the measure given in Eq.(2) must be found before the measure can be estimated. Thus finding intrinsic gauge functions, or even the Hausdorff-Besicovitch dimension, in any real situation, is intractable. When dealing with bounded sets, it is much easier to use the *capacity* notion of dimension [8]. Suppose we modify the definition of the d -dimensional Hausdorff measure of a set in the following manner: Choose an $\epsilon > 0$ as before, but only consider uniform coverings, i.e., coverings whose elements are all D -dimensional cubes of the same length ϵ . Then form the sum $\sum_m \epsilon^d$ over all elements of a given ϵ -cover. Since the set is assumed to be bounded, the sum involves a finite number of equal terms. Then for all such ϵ covers take the infimum of these sums. This infimum is just that particular sum with the smallest number of terms $N(\epsilon)$. Then the capacity version of the d -dimensional measure is

$$\mu^{(d)} = \lim_{\epsilon \rightarrow 0} N(\epsilon) \epsilon^d. \quad (6)$$

The value of d , say d_0 , for which this measure is infinite for all $d < d_0$ and zero for all $d > d_0$ is usually called the capacity dimension of the set. The obvious extension of the definition of the Hausdorff measure with respect to a gauge function λ is

$$\mu_\lambda = \lim_{\epsilon \rightarrow 0} N(\epsilon) \lambda(\epsilon). \quad (7)$$

We call the function λ which yields a finite μ_λ the intrinsic gauge function of the set. To avoid confusion, we call the gauge functions of the Hausdorff formalism *Hausdorff gauge functions*, while referring to those of the capacity

version simply as gauge functions. Since $N(\epsilon)$ diverges as $\epsilon \rightarrow 0$, it is necessarily true that the intrinsic gauge function of any bounded set must vanish with ϵ . Thus, from general scaling considerations, we expect this function to be expandable, in most cases, in the form given by Eq.(5) [11]. Furthermore such a λ will scale to zero no faster than ϵ^D , with D being the dimension of the embedding space.

The foregoing formulation has several practical advantages which are well known [8]. First, the covers to be dealt with are all finite. Second, the members of each cover considered are identical to each other. These properties make it easier to evaluate such quantities as the capacity dimension of a set and, as we shall see, its gauge function. In the rest of this paper, we will be working exclusively with bounded sets and the capacity framework for describing these sets.

4. Examples

We now present some simple examples of Cantor sets which have gauge functions of the form described by Eq.(5) [12]. Before doing so, we introduce some notation that is common to all of the examples we consider, and we outline the method which will be used to determine the gauge functions of these sets. We are interested in simple linear Cantor sets that are constructed in the following manner. Start with a unit interval and in the first stage of construction delete a fraction $h_1 < 1$ from its middle. The resulting stage 1 set is composed of $N_1 = 2^1$ intervals of length $\epsilon_1 = \frac{[1-h_1]}{2}$. In the second stage of construction delete a fraction h_2 from the middle of each of these intervals. The stage 2 set is composed of $N_2 = 2^2$ intervals of length $\epsilon_2 = \frac{[1-h_2]\epsilon_1}{2}$. Next delete a fraction h_3 from the middle of each remaining interval and repeat this process *ad infinitum*. The limit set is a Cantor set and its n^{th} approximant is a set for which

$$N_n = 2^n, \quad (8a)$$

$$\epsilon_n = \frac{1}{2^n} \prod_{k=1}^n [1-h_k], \quad (8b)$$

and

$$L_n = N_n \epsilon_n \quad (8c)$$

where N_n is the number of segments in the set, ϵ_n is the length of each segment, and L_n is the set's Lebesgue measure. We refer to h_k as the *hole function* of a set.

We evaluate the gauge function of these sets as follows: We use the intermediate sets that are generated by the foregoing construction as a

sequence of coverings for the limit set. That is, we examine the behaviour of $N(\epsilon)$ for the sequence of length scales $\{\epsilon_n\}$. We then use the approximation (which becomes exact as $\epsilon \rightarrow 0$)

$$\lambda(\epsilon) \approx \frac{\mu_\lambda}{N(\epsilon)}. \quad (9)$$

This approximation is based on Eq.(7) and the fact that μ_λ is finite. Knowledge of the value of μ_λ is unimportant since we are only interested in the functional form of $\lambda(\epsilon)$. At any rate, we can always normalize the measure to 1 so it is clearly of no consequence. We can equate the expression for the gauge function in Eq.(9) with the expression of Eq.(5) in a convenient manner.

Letting $u = \log(\frac{1}{\epsilon})$, $\theta(u) = -\log \lambda(\epsilon)$, and $\log^0 x = x$, Eq.(5) becomes

$$\theta(u) = \sum_{k=0}^n d_k \log^k u. \quad (10)$$

Taking the negative log of both sides of Eq.(9), using Eq(8a), and restricting ourselves to the coverings that correspond to the ϵ_n we get

$$\theta(u_n) = \log\left(\frac{1}{\mu_\lambda}\right) + n \log 2. \quad (11)$$

where $u_n = \log(\frac{1}{\epsilon_n})$ and n is to be regarded as a function of the u_n . Now taking the negative log of Eq.(8b) results in

$$u_n = n \log 2 - \sum_{k=1}^n \log[1 - h(k)]. \quad (12)$$

So we can find the gauge functions of our examples by solving Eq.(12) for n as a function of u_n , substituting into Eq.(11), and comparing the results with Eq.(10).

Example 1:

In order to show, how this calculation can be done explicitly, we now present an example of a two parameter family of Cantor sets which have the first iterated logarithmic factor in their gauge functions. In the general Cantor set construction outlined in the previous paragraph, choose as the hole function the two parameter family of functions $h_k(b, c) = b + \frac{c}{k}$ where $0 < b$, $|c|$, $|b+c| < 1$. To solve for n as a function of the u_n , use the fact that for $n \gg 1$, $u_n \approx A + n \log[\frac{2}{1-b}] + \frac{c}{(1-b)} \sum_{k=1}^n \frac{1}{k}$ where A is

some constant that depends on b and c . Replace the sum over k by $C + \log n$ (C is the Euler-Mascheroni constant), and solve for n recursively keeping only the terms in u_n which diverge as $n \rightarrow \infty$. The result is

$$n \approx \frac{1}{\log\left[\frac{2}{1-b}\right]} \left[u_n - \frac{c}{1-b} \log u_n \right].$$

Plugging this into Eq.(11) we find that λ has the form of Eq.(5) with

$$d_0 = \frac{\log 2}{\log\left(\frac{2}{1-b}\right)} \quad (13a)$$

and

$$d_1 = -\frac{c \log 2}{(1-b) \left[\log\left(\frac{2}{1-b}\right) \right]}. \quad (13b)$$

and $d_k = 0$ for all $k > 1$.

$b = \frac{1}{3}$. As is well known, the gauge function for this set is a pure power law with exponent $d_0 = \frac{\log 2}{\log 3}$.

get a perturbed classic Cantor set whose gauge function is modified from the unperturbed case by a logarithmic correction. It is important to note that in this example we still have $d_0 = \frac{\log 2}{\log 3}$. The reason for this is that the perturbation from the hole function of the classic Cantor vanishes asymptotically. However this perturbation does not vanish fast enough to make its presence completely unfelt. Thus a logarithmic correction to the power law in the set's gauge function is produced. If we repeat the above analysis on a set generated by $h_k(c) = \frac{1}{3} + c^k$ for some $0 < c < 1$, we would not have observed a logarithmic correction since this perturbation vanishes rapidly with the stage of construction (we will return to the effects of this kind of perturbation in the next section). Returning to the original example 1, we note that, when [218] we set $b = 0$ in the expression for $h_k(b, c)$ and choose $0 < c < 1$, we obtain a dimension 1 object that has zero linear extent or 1-dimensional Lebesgue measure. Fig.(1a) is a picture of this set for $c = \frac{1}{3}$.

Fig.(1b) shows a portion of Fig.(1a) that has been magnified by a factor of 1000. Note that the relative size of the holes appear to be shrinking making the set appear to have positive measure. This should be compared to the corresponding figures for the classic Cantor set and a fat Cantor (Figs.(2) and (3) respectively).

Fig 1A

Fig. 1B

Fig.1. A perturbed classic Cantor set generated by the hole function of Eq.(4) with $b = \frac{1}{3}$ and $c = .1$. Fig. 1a is the a representation of of the entire set and Fig. 1b is a blowup of the region outlined in the box in Fig. 1a. The slowly vanishing perturbation gives rise to a logarithmic correction to the pure power law in the set's intrinsic gauge function.

Fig. 2A

Fig. 2B

Fig. 2. Same as Fig. 1 but for the classic Cantor set obtained from Eq.(4) with $b = \frac{1}{3}$ and $c = 0$. The gauge function for this set is a pure power law.

Fig. 3A

Fig. 3B

Fig.3. A dimension 1 thin fractal obtained by choosing $b = 0$ and $c = \frac{1}{3}$.
 Note from the blowup that the set looks like it may in fact be fat. There is a logarithmic correction, however, that ensures that the Lebesgue measure of this set is zero.

Fig. 4. A plot of $-\log \lambda(\epsilon)$ vs. $\log \epsilon$ where λ is given by Eq.(11) with $d_0 = 1$, $d_1 = 0$, $d_2 = -1$ and $d_k = 0$ for all $k > 2$. The straight line represents the case where $d_0 = 1$ and no logarithmic corrections occur.

From these simple examples we see that a logarithmic gauge function is introduced when a set does not exhibit exact asymptotic self-similarity i.e., when the scale invariance of a set is not exact. We should expect that these corrections occur in nonlinear dynamical systems which have multiple scalings associated with them. For example, in determining the fractal dimension of basin boundaries in asymmetric tent maps, Takasue [13] observed a small oscillation in his $\log N(\epsilon)$ vs. $\log \epsilon$ plot which he attributed to the presence of two scalings associated with the map's asymmetry. Such an oscillation would be seen if the gauge function for the set possessed the first two logarithmic corrections. To see this, we plot $\log \frac{1}{\lambda(\epsilon)}$ vs. $\log \epsilon$ for such a gauge function in Fig.(4). The straight line is what we would see when there is no logarithmic correction. Note that a single oscillation about the line is induced by the logarithmic corrections. This should be compared to Fig.(300) in [13]. It is plausible that the oscillation in this plot is due to these logarithmic corrections. A different kind of oscillations, which are periodic, can arise from the effects of lacunarity as discussed by MANDELBROT [14].

3. Characterization of fat fractals : Metadimensions

We now turn our attention to the characterization of fat fractals. As stated earlier the Hausdorff gauge functions of fat fractals are necessarily pure power laws with integer exponents. Since this is also true of ordinary sets which have finite Lebesgue measure, other methods for characterizing this class of fractals must be employed. In [4] and [5] the discussion is limited to sets which derive their fractal properties from simple *holes* embedded in them. The coarse-grained Lebesgue measure which is chosen is the measure of the complement of all holes having a diameter larger than a size ϵ . Letting $\mu_A(\epsilon)$ be this measure, the exponent $\beta = \lim_{\epsilon \rightarrow 0} \frac{\log[\mu_A(\epsilon) - \mu_A(0)]}{\log \epsilon}$ is used to characterize the set. In [6], a fat fractal is *fattened* by centering balls, each having a radius ϵ , on each point of the set. The coarse-grained measure of interest, which we denote by $\mu_f(\epsilon)$, is the measure of the fattened set, i.e, the measure of the union of all the balls. Then an exponent $\alpha = \lim_{\epsilon \rightarrow 0} \frac{\log[\mu_f(\epsilon) - \mu_f(0)]}{\log \epsilon}$ is used to characterize the set. This second scheme makes use of a method originated by Cantor for estimating the volume of sets.[1] Since the Hausdorff formalism is a generalization of this, it is natural to wonder whether or not fat fractals can be described by generalizing the Hausdorff formalism. It would be nice, for example, if a unique intrinsic Hausdorff gauge function could be defined which can distinguish between fat fractals and ordinary sets. As stated earlier, this appears to be impossible. This is not true of the capacity formalism, however, and this fact together with the numerical results of [4] suggest such a generalization within this framework.

In [4], it is conjectured that bounded chaotic orbits of certain area preserving maps are fat fractals. The numerical procedure used for

estimating β for these orbits is that of box counting. This procedure has been used in many studies to estimate the capacity dimension of strange attractors in dissipative systems [15]. In these studies, a fixed grid of squares ϵ on side is placed on a portion of the phase space accessible to a given orbit. The number of squares needed to cover a subset of the orbit, say $M(\epsilon)$, is computed and it is assumed that $M(\epsilon) \approx N(\epsilon)$. Then the capacity dimension is extracted by fitting a straight line to a plot of $\log M(\epsilon)$ vs. $\log \epsilon$, this being motivated by the definition given in Eq.(6). In [4], the area of the closure of a given chaotic orbit, say $\mu_g(\epsilon)$, is estimated by computing $M(\epsilon)$ for an entire orbit and constructing $\mu_g(\epsilon) = M(\epsilon)\epsilon^2$. It is found that this quantity behaves like $\mu_g(\epsilon) \approx \mu_g(0) + \kappa \epsilon^\gamma$ for ϵ small compared to the diameter of the orbit.

The constants $\mu(0), \kappa$ and, γ are all positive, and it is conjectured that γ is an estimate of β when $\gamma < 1$. The scaling of the measure $\mu_g(\epsilon)$ implies that $M(\epsilon) \approx \epsilon^{-2} [\mu(0) + \kappa \epsilon^\gamma]$. The first term in brackets is just what we would expect to get for the behavior of $M(\epsilon)$ of an ordinary 2-dimensional set. The second term, however, arises from the fractal properties of the sets considered. This term can be viewed as higher order correction to the that part of $M(\epsilon)$ which yields the dimension of the set. This suggests the following description of fat fractals: Let $N(\epsilon)$ again be the minimum number of D dimensional squares of side ϵ needed to cover a d_0 dimensional fat fractal, S . Then define a quantity δ by

$$\delta = \lim_{\epsilon \rightarrow 0} \frac{\log [N(\epsilon)\epsilon^{d_0} - \mu_0]}{\log \epsilon}. \quad (14)$$

where $\mu_0 = \lim_{\epsilon \rightarrow 0} N(\epsilon)\epsilon^{d_0}$. We call δ a *metadimension* of the set S [16]. It describes the rate at which the measure estimated from an optimal uniform ϵ cover converges to the true measure of the set as ϵ tends to zero. By definition, $0 < \delta \leq \infty$. When $\delta \gg 1$, very small changes in ϵ decrease the error in the estimate of a set's true measure dramatically. When $\delta \ll 1$, very large decreases in ϵ are required to make this error small.

Some caution must be exercised in interpreting the exponent γ of [4] as being equivalent to δ . As pointed out in [4], any nonfractal set which is bounded by a smooth curve will have the power law scaling of the measure described above with $\gamma = 1$. This is an artifact of using a fixed grid for a cover. However when $\gamma < 1$, we expect that γ does reflect the fractal nature of the set and equals δ . These issues are discussed in detail elsewhere [17,18].

An equivalent way of viewing the higher order scaling discussed in the foregoing paragraph is that fat fractals can be distinguished from ordinary sets by additive corrections to their gauge functions. This is accomplished by first noting that a unique intrinsic gauge function for a set can be defined through Eq.(7). We simply let the intrinsic gauge function of a set be defined as

$$\lambda(\epsilon) = \frac{\mu_\lambda}{N(\epsilon)}. \quad (15)$$

where μ_λ can be normalized to unity. Thus if we find, for some set of interest, $N(\epsilon) \approx \epsilon^{-d_0} [\mu_0 + \kappa \epsilon^\delta]$ (with $d_0, \mu_0, \kappa, \delta > 0$), we get

$$\lambda(\epsilon) \approx \epsilon^{d_0} \left[1 - \frac{\kappa}{\mu_0} \epsilon^\delta \right].$$

where we have kept only the lowest order terms in ϵ . The first term in the brackets is associated with the dimension. The second term in brackets is a correction that vanishes as $\epsilon \rightarrow 0$. This example has only a power law correction but there is no reason that it cannot scale to zero in some other fashion. This is a point which we return to later.

Example 2:

We now give an example of a fat Cantor set for which δ is easily computed.

Consider the Cantor set construction described in Sec.(2) for a one parameter family of hole functions $h_k(c) = c^k$ for $0 < c < 1$. For a fixed c , it is easy to verify that the one-dimensional Lebesgue measure of the generated set is given by $0 < \mu_0 = L_\infty = \lim_{n \rightarrow \infty} L_n < 1$. To calculate δ for this set, we use

$$\delta = \lim_{n \rightarrow \infty} \frac{\log [L_n - L_\infty]}{\log \epsilon_n} \quad \text{where the } \epsilon_n \text{ correspond to the subsequence of covers defined in Sec.(2). This limit is the same as } \lim_{n \rightarrow \infty} \frac{\log [1 - \prod_{k=n+1}^{\infty} (1 - c^k)]}{n \log \frac{1}{2}}$$

which is easily evaluated to give

$$\delta = \frac{\log \left(\frac{1}{c} \right)}{\log 2}. \quad (16)$$

Thus for $n \gg 1$, we have $L_n \approx L_\infty (1 + \kappa \epsilon^\delta)$ which yields (from Eq.(8c))

$$N(\epsilon) \approx L_\infty \epsilon^{-1} [1 + \kappa \epsilon^\delta]. \quad (17)$$

Therefore, for this fat Cantor set, the gauge function has an additive correction which is a power law in ϵ . Had we done the same calculation for an interval or a collection of intervals, no such additive correction would have been observed. Thus we conjecture that a fat fractal of dimension n is distinguished from a regular set of the same dimension by an additive correction to ϵ^n in its gauge function. We call this additive correction the *metadimension*.

to distinguish it from dimension given in Eq.(5).

Given the above considerations, it is natural to wonder whether or not thin fractals can also exhibit this type of higher order scaling behavior. It turns out that they can. To see this, consider the following Cantor set construction. Choose a two parameter family of hole functions given by $h_k(b, c) = b + c^k$ where $0 < b, c, b + c < 1$. As stated earlier, this set has an intrinsic gauge function which has no multiplicative corrections to a power law. This can be verified by performing the calculation outlined in Sec.(2). The capacity dimension of the set is the same as that for the sets generated by choosing the one parameter family of gauge functions $h_k(b, 0)$, i.e., d_0 is given by Eq.(13a). Now if we modify our definition of δ by replacing μ_0 in Eq.(22) by the set's d_0 dimensional measure (with d_0 the capacity dimension set), we find that δ is the same as that of the fat Cantor set discussed above, i.e., it is given by Eq(16). Thus the notion of a metadimension is applicable to thin fractal sets as well.

The above considerations suggest the following definition: Let S be some bounded set with an intrinsic gauge function $\lambda(\epsilon)$ and measure μ_λ . Furthermore, suppose that $\lambda(\epsilon)$ can be expanded in the form of Eq.(4) for some finite n . Construct the function $\lambda_m(\epsilon) = N(\epsilon) \lambda(\epsilon) - \mu_0$. Then $\lambda_m(\epsilon)$ is a function which vanishes as $\epsilon \rightarrow 0$ and describes the rate at which the measure μ_λ is approached with decreasing length scale. Then define the exponent δ by

$$\delta = \lim_{\epsilon \rightarrow 0} \frac{\log \lambda_m}{\log \epsilon}. \quad (18)$$

There are no geometrical restrictions which require that this function vanish slower than some power in ϵ as there was for $\lambda(\epsilon)$. Thus $\lambda_m(\epsilon)$ can be anything asymptotically. For example, it can have the following logarithmic-exponential[11] form:

$$\lambda_m(\epsilon) = \prod_{j=1}^l [\exp^j(\frac{1}{\epsilon})]^{-q_j} \epsilon^{d_0} \prod_{k=1}^n [\log^k(\frac{1}{\epsilon})]^{-d_k} \quad (19)$$

where $\exp^j(x)$, is the j^{th} iterated exponential, and $q_i > 0$. Many other possibilities for the form of $\lambda_m(\epsilon)$ can be found in HARDY [11].

4. Conclusions

We have discussed the characterization of fractal sets that do not exhibit exact scale invariance. This lack of scale invariance leads to corrections to a pure power law of a set's intrinsic gauge function. We expect these corrections to occur in systems which have many scalings associated with them. Two types of these corrections occur. The first type is multiplicative in nature and can be approximated by an expansion in terms of powers of iterated logarithms. These represent relatively strong perturbations away from scale invariance. These types of corrections occur in many random fractals. A

specific example of this is the two dimensional random walk discussed in LEVY [2]. There, although the set generated by the walk is statistically self-similar, it is not exactly so. Thus a logarithmic correction to a pure power law occurs in the set's gauge function which results in a 2-dimensional set that has zero Lebesgue measure and is thus not area filling. Other cases, in which these corrections seem to appear in a natural way, are associated with fractal basin boundaries as discussed in [13].

A second kind of correction to a pure power law that can be seen in fractals is an additive one. These occur in sets whose deviation from exact scale invariance are relatively weak. Fat fractals are a class of such sets in which these types of corrections occur. It is the deviation from exact scale invariance of these sets which distinguish them from ordinary sets. Examples of such sets occur in many dynamical systems.

Acknowledgements

We are deeply indebted to W. Beyer, who insisted that not everything is well understood about dimensions, and who persuaded us to read Hausdorff's original paper. We also would like to thank J.D. Farmer and R. Eykholt for useful discussions.

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