FIRST ORDER VARIATION OF THE DISPERSION FUNCTION WITH PARTICLE ENERGY DEVIATION*

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1. Summary

The variation of the dispersion function with the particle energy deviation can presently be calculated from second order transfer matrices^(1,2); its periodic solution is determined numerically.

The general differential equations for the dispersion function deduced from the complete equation of motion to second order⁽³⁾ are solved, using Green's function integral leading to an analytical expression⁽⁴⁾ of the periodic solution of the dispersion function D_0 and of the first order perturbation, D_1 , with respect to energy deviation. The same method can be extended to higher order perturbations of the dispersion function.

The determination of the periodic solution as well as the transportation of these two dispersion functions through any element depends only on two particular integrals. These integrals are derived for the general case of a combined function magnet, with up to second order components. The derivation includes the contribution from the edges. Chapter 2 and 3 deal with closed machines, chapter 4 applies these results to beam transport lines.

These analytical expressions are then applied to a typical machine in order to illustrate the most important driving terms; the results do agree with those obtained by optics programs like MAD⁽¹⁾ or DIMAT⁽²⁾ based on second order transfer matrices.

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2. Differential equation for the Dispersion function

The differential equation of particle motion in the horizontal plane in the variables x , y and their derivatives to second order can be expressed as follows⁽²⁾:

$$x'' - h(1 + hx) - x'(hx' + xh') = (1 - \delta + \delta^2) \left[-h + (k - 2h^2)x + (2hk - h^3 + \frac{1}{2}r)x^2 - \frac{1}{2}hx'^2 + \frac{1}{2}(h'' - hk - r)y^2 + 2xy' - \frac{1}{2}hy'^2 \right] \quad (2.1)$$

where the prime indicates the derivative with respect to the azimuth s and

x	is the horizontal coordinate,
y	is the vertical coordinate,
s	is the arc length along the reference orbit,
$h(s) = \frac{1}{\rho}$	is the curvature of the reference orbit,
$k(s) = -\frac{1}{B\rho} \frac{\partial B_y}{\partial x} \Big _{x=0}$	is the normalized quadrupole strength,
$r(s) = -\frac{1}{B\rho} \frac{\partial^2 B_y}{\partial x^2} \Big _{x=0}$	is the normalized sextupole component,
$B\rho$	is the particle rigidity,
$\delta = \frac{\Delta E}{E}$	is the particle energy deviation.

The horizontal motion x of a particle with energy deviation δ can be written in the form

$$x = \bar{x} + D\delta, \quad (2.2)$$

where $\bar{x}$ is the betatron oscillation and $D\delta$ is the closed orbit of this particle. One derives the differential equation for the dispersion function $D(s)$ (also called α_p or η in accelerator theory) by inserting eq. (2.2) with $\bar{x} \equiv 0$ into eq. (2.1). To first order in δ one has

$$D'' + (h^2 - k)D = h + \left[-h + (2h^2 - k + h'D')D + (2hk - h^3 + \frac{1}{2}r)D^2 + \frac{1}{2}hD'^2 \right] \delta. \quad (2.3)$$

The expansion of the dispersion function in the form

$$D = D_0 + D_1\delta \quad (2.4)$$

enables to solve eq. (2.3) successively for each power of δ . The dispersion function $D_0(s)$ is the

periodic solution of the well known differential equation

$$D_0'' + (h^2 - k)D_0 = h. \quad (2.5)$$

The differential equation for the perturbation to the first order in δ , $D_1(s)$, is the periodic solution of

$$D_1'' + (h^2 - k)D_1 = -h + (2h^2 - k + h'D_0')D_0 + (2hk - h^3 + \frac{1}{2}r)D_0^2 + \frac{1}{2}hD_0'^2. \quad (2.6)$$

In a series of reports^(4,5,6) the equation for D_1 has been derived for machines with large bending radius ρ where higher order terms in h as well as the terms coming from combined function magnets (hk) and from the variation of the dispersion function with azimuth (D_0') were disregarded. For small machines these terms cannot be neglected.

3. The periodic solution

The two differential equations (2.5, 2.6) for D_0 and D_1 are of the form

$$D_i'' + (h^2 - k)D_i = f_i \quad \text{with } i = 0, 1 \quad (3.1)$$

where

$$\begin{aligned} f_0 &= h, \\ f_1 &= -h + (2h^2 - k + h'D_0')D_0 + (2hk - h^3 + \frac{1}{2}r)D_0^2 + \frac{1}{2}hD_0'^2. \end{aligned} \quad (3.2)$$

Equation (3.1) is also valid for higher order perturbations of the dispersion function with more complicated driving term functions f_i , $i \geq 2$ in the r.h.s. of this equation.

Assuming that the Twiss parameters (β , α) and the phase advance (μ) along the central trajectory are known, using the usual variable transformations⁽⁷⁾

$$\begin{aligned} \phi(s) &= \int \frac{ds}{Q\beta}, \\ E_i(\phi) &= \frac{D_i}{\sqrt{\beta}} \end{aligned}$$

where Q is the horizontal tune (the number of horizontal betatron oscillations per turn around

the machine) and taking into account the well known relation⁽⁷⁾

$$(h^2 - k)\beta^2 + \frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 = 1$$

the general differential equation (3.1) for the dispersion function can be transformed into the equation of a forced harmonic oscillator:

$$\frac{d^2 E_i}{d\phi^2} + Q^2 E_i = Q^2 \beta^{3/2} f_i(\phi) \quad (3.3)$$

where f_i are the functions defined in eq. (3.2).

3.1 HARMONIC ANALYSIS METHOD

We shall use the Fourier analysis method to solve eq. (3.3). Since β as well as $f_i(\phi)$ are periodic functions with the period 2π , they can be expanded into Fourier series resulting in the periodic solutions:

$$D_i(s) = \sqrt{\beta(s)} \left[a_{i0} + \sum_{p=1}^{\infty} \frac{Q^2}{Q^2 - p^2} (a_{ip} \cos p\phi + b_{ip} \sin p\phi) \right] \quad (3.4)$$

with

$$\begin{aligned} a_{i0} &= \frac{1}{2\pi} \int_0^{2\pi} f_i(\phi) d\phi, \\ a_{ip} &= \frac{1}{\pi} \int_0^{2\pi} f_i(\phi) \cos p\phi d\phi, \\ b_{ip} &= \frac{1}{\pi} \int_0^{2\pi} f_i(\phi) \sin p\phi d\phi. \end{aligned}$$

If the horizontal tune Q approaches an integer, the factor $Q^2/(Q^2 - p^2)$ increases to infinity. As a consequence the Fourier components for the harmonics which are close to the horizontal betatron frequency Q are drastically magnified by this factor. But this does not mean that only these harmonics will contribute to the sum in eq. (3.4). Depending on the element (quadrupoles, bending magnets and sextupoles) distribution in the machine some other harmonics of the function $f_i(\phi)$ can become dominant.

The harmonic analysis method is very useful to determine the driven harmonics⁽⁶⁾ numerically as the summation over p in eq. (3.4) will vary from machine to machine, but it cannot be used to derive closed analytical expressions of the dispersion function.

3.2 GREEN FUNCTION METHOD

The periodic solution of the general differential equation (3.3) can, by means of Green's function, be written in the form⁽⁷⁾

$$E_i(\phi) = \frac{Q}{2 \sin \pi Q} \int_{\phi}^{2\pi+\phi} \beta^{3/2} f_i(\psi) \cos[Q(\pi + \phi - \psi)] d\psi. \quad (3.5)$$

Introducing the original variables

$$\begin{aligned} d\sigma &= Q\beta d\psi, \\ \mu(s) &= Q\phi, \\ D_i(s) &= E_i\sqrt{\beta}, \end{aligned} \quad (3.6)$$

into eq. (3.5) gives

$$D_i(s) = \frac{\sqrt{\beta(s)}}{2 \sin \pi Q} \int_s^{s+L} \sqrt{\beta(\sigma)} f_i(\sigma) \cos[Q\pi + \mu(s) - \mu(\sigma)] d\sigma \quad (3.7)$$

where f_i are the functions defined in eq. (3.2) and where L is the length (circumference) of the reference orbit. Differentiating eq. (3.7) with respect to s leads to the periodic solution for $D'(s)$:

$$D'_i(s) = -D_i(s) \frac{\alpha(s)}{\beta(s)} - \frac{1}{2\sqrt{\beta(s)} \sin \pi Q} \int_s^{s+L} \sqrt{\beta(\sigma)} f_i(\sigma) \sin[Q\pi + \mu(s) - \mu(\sigma)] d\sigma. \quad (3.8)$$

Integration through the various elements

Actually, the integrals in eqs. (3.7), (3.8) can be replaced by sums of integrals over the various magnetic elements which are characterised in the function $f_i(\sigma)$ by the terms containing

$h(hk) \dots$ for (combined function) bending magnets,

- k ... for quadrupoles,
 r ... for sextupoles,
 h' ... for edges of bending magnets.

Assuming the hard edge approximation, where the magnetic field rises abruptly from zero outside the magnet to a constant value inside it, we can solve the integral on the r.h.s. of the eqs. (3.7), (3.8) for these elements analytically.

Defining an element j of length l_j by its transfer matrix

$$R_j(\sigma) = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

and using the values for the functions β_j , α_j and μ_j at the entrance of this element, the variation of the phase μ and the β -function are given by

$$\begin{aligned} \mu(\sigma) &= \arctan \left(\frac{R_{12}(\sigma)}{\beta_j R_{11}(\sigma) - \alpha_j R_{12}(\sigma)} \right), \\ \beta(\sigma) &= \beta_j R_{11}^2(\sigma) - 2\alpha_j R_{11}(\sigma) R_{12}(\sigma) + \frac{1 + \alpha_j^2}{\beta_j} R_{12}^2(\sigma). \end{aligned} \quad (3.9)$$

Taking the origin $s = 0$ as the starting point for the phase $\mu(s)$ and using identities for the inverse trigonometric functions, like

$$\arccos z = \arctan \frac{\sqrt{1-z^2}}{z},$$

one obtains

$$\begin{aligned} \sqrt{\beta(\sigma)} \cos[\pi Q + \mu(s) - \mu_j - \mu(\sigma)] &= \left[\sqrt{\beta_j} R_{11}(\sigma) - \frac{\alpha_j}{\sqrt{\beta_j}} R_{12}(\sigma) \right] \cos \Delta\mu_j + \frac{R_{12}(\sigma)}{\sqrt{\beta_j}} \sin \Delta\mu_j, \\ \sqrt{\beta(\sigma)} \sin[\pi Q + \mu(s) - \mu_j - \mu(\sigma)] &= \left[\sqrt{\beta_j} R_{11}(\sigma) - \frac{\alpha_j}{\sqrt{\beta_j}} R_{12}(\sigma) \right] \sin \Delta\mu_j - \frac{R_{12}(\sigma)}{\sqrt{\beta_j}} \cos \Delta\mu_j, \end{aligned} \quad (3.10)$$

where

$$\begin{aligned} \Delta\mu_j &= \pi Q + \mu(s) - \mu_j & \text{for } \mu(s) < \mu_j, \\ \Delta\mu_j &= -\pi Q + \mu(s) - \mu_j & \text{for } \mu(s) > \mu_j. \end{aligned}$$

Thus eqs. (3.7), (3.8) can be written in the following form:

$$\begin{aligned}
 D_i(s) &= \frac{\sqrt{\beta(s)}}{2 \sin \pi Q} \sum_j \left\{ \sqrt{\beta_j} \cos \Delta \mu_j \int_0^{l_j} f_i(\sigma) R_{11}(\sigma) d\sigma \right. \\
 &\quad \left. - \frac{1}{\sqrt{\beta_j}} (\alpha_j \cos \Delta \mu_j - \sin \Delta \mu_j) \int_0^{l_j} f_i(\sigma) R_{12}(\sigma) d\sigma \right\}, \\
 D_i'(s) &= -D_i(s) \frac{\alpha(s)}{\beta(s)} - \frac{1}{2\sqrt{\beta(s)} \sin \pi Q} \sum_j \left\{ \sqrt{\beta_j} \sin \Delta \mu_j \int_0^{l_j} f_i(\sigma) R_{11}(\sigma) d\sigma \right. \\
 &\quad \left. - \frac{1}{\sqrt{\beta_j}} (\alpha_j \sin \Delta \mu_j + \cos \Delta \mu_j) \int_0^{l_j} f_i(\sigma) R_{12}(\sigma) d\sigma \right\},
 \end{aligned} \tag{3.11}$$

where j runs over the elements of the machine. These expressions were already derived⁽⁴⁾ for the perturbation D_1 . In fact, eqs. (3.11) are valid for any order of the expansion of the dispersion function. Their periodic solutions are completely determined by the following integrals

$$\int_0^{l_j} f_i(\sigma) R_{11}(\sigma) d\sigma \quad \text{and} \quad \int_0^{l_j} f_i(\sigma) R_{12}(\sigma) d\sigma \tag{3.12}$$

over the elements for which the functions f_i are different from zero.

Equations (3.2) shows that the function D_0 is only driven by the bending magnets ($h \neq 0$) while the function D_1 is driven by the combined function bending magnets ($hk \neq 0$), their edges ($h' \neq 0$) and by the quadrupoles ($k \neq 0$) and the sextupoles ($r \neq 0$).

The computation of the above integrals is derived in APPENDIX I for all elements up to second order by replacing the transfer matrix elements by their corresponding values.

4. Transport of the Dispersion function

Although the equations (3.11) define the periodic solution of D_i and D_i' respectively everywhere in the machine, it is important to know the variation of D_i and D_i' due to the different elements in a ring. Moreover, this facilitates the calculation of the perturbation of the dispersion function along transfer lines.

Supposing the values of the Dispersion functions D_i and their derivatives D_i' are known at the point s_0 either by determination of the periodic solution in a ring using eqs. (3.11) or as given initial values for a transfer line, the transported values of the functions D_i and D_i' can then be determined at any point s of the transfer channel using the transfer matrix M defined as

$$M = \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta(s_0)}} [\cos \Delta\mu + \alpha(s_0) \sin \Delta\mu] & \sqrt{\beta(s)\beta(s_0)} \sin \Delta\mu \\ \frac{\alpha(s_0) - \alpha(s)}{\sqrt{\beta(s_0)\beta(s)}} \cos \Delta\mu - \frac{1 + \alpha(s_0)\alpha(s)}{\sqrt{\beta(s_0)\beta(s)}} \sin \Delta\mu & \sqrt{\frac{\beta(s_0)}{\beta(s)}} [\cos \Delta\mu + \alpha(s) \sin \Delta\mu] \end{pmatrix},$$

with $\Delta\mu = \mu(s) - \mu(s_0)$.

In fact, the general solution of eq. (3.3) can be written in the form

$$E_i(\phi) = A_i \cos[Q(\phi - \phi_0)] + B_i \sin[Q(\phi - \phi_0)] + Q \int_{\phi_0}^{\phi} \beta^{3/2} f_i(\psi) \sin[Q(\phi - \phi_0 - \psi)] d\psi$$

where $\phi \equiv \phi(s)$ and $\phi_0 \equiv \phi(s_0)$ and with the coefficients A_i , B_i determined by the initial conditions:

$$A_i = E_i(\phi_0), \quad B_i = \frac{1}{Q} \frac{dE_i(\phi_0)}{d\phi}.$$

Using the original variables as defined in eq. (3.6), the Dispersion functions at any point s are defined by

$$D_i(s) = D_i(s_0)M_{11} + D_i'(s_0)M_{12} + \sqrt{\beta(s)} \int_{s_0}^s \sqrt{\beta(\sigma)} f_i(\sigma) \sin[\mu(s) - \mu(s_0) - \mu(\sigma)] d\sigma. \quad (4.1)$$

Hence, the dispersion functions can be determined successively from the entry (beginning) $D_i(0) \equiv D_{i0}$ to the exit $D_i(l)$ of each element with $M = R(l)$ as the transfer matrix of this

particular element. Introducing eq. (3.10) into eq. (4.1) simplifies the equations for the dispersion functions to

$$\begin{aligned} D_i(l) &= D_{ib}M_{11} + D'_{ib}M_{12} + M_{12} \int_0^l f_i(\sigma)R_{11}(\sigma) d\sigma - M_{11} \int_0^l f_i(\sigma)R_{12}(\sigma) d\sigma, \\ D'_i(l) &= D_{ib}M_{21} + D'_{ib}M_{22} + M_{22} \int_0^l f_i(\sigma)R_{11}(\sigma) d\sigma - M_{21} \int_0^l f_i(\sigma)R_{12}(\sigma) d\sigma. \end{aligned} \quad (4.2)$$

The transport of the Dispersion function D_0 is easy to deduce from the transfer matrix coefficients:

The introduction of

$$\begin{aligned} \int_0^l h [M_{12}R_{11}(\sigma) - M_{11}R_{12}(\sigma)] d\sigma &= M_{13}, \\ \int_0^l h [M_{22}R_{11}(\sigma) - M_{21}R_{12}(\sigma)] d\sigma &= M_{23} \end{aligned}$$

into eq. (4.2) leads to the well known formulae

$$\begin{aligned} D_0(l) &= D_{0b}M_{11} + D'_{0b}M_{12} + M_{13}, \\ D'_0(l) &= D_{0b}M_{21} + D'_{0b}M_{22} + M_{23}. \end{aligned} \quad (4.3)$$

The transport of the Dispersion function D_1 uses the same integrals (3.12) as those needed for the calculation of its periodic solution; they are listed in APPENDIX I.

The complete expressions for the transport of the function D_1 through the different machine elements are summarized in APPENDIX II.

5. Application

The LEP Electron Positron Accumulator (EPA)⁽⁹⁾ provides an ideal application for the calculation of the Dispersion function D_1 . In fact, in this machine all the driving terms of this function are present because of the low bending radius ($\rho = 1.426$ m) and of its combined function bending magnets ($k = 0.5$ m⁻²) with entrance and exit angles.

Moreover, the strong sextupoles ($r = \pm 8.25$ m⁻³) localized in the arcs do perturb the Dispersion function D_1 in the long dispersion free sections ($D_0 = 0$).

The results calculated for one superperiod of EPA using the program COMFORT⁽¹⁰⁾ are displayed in the following table as well as plotted in fig. 1 and fig. 2 for the cases without and with the sextupoles respectively to demonstrate their driving force.

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TABLE

sextupoles OFF						sextupoles ON	
						$r_{sv} = 8.25 \text{ m}^{-3}$	
						$r_{sxh} = -8.25 \text{ m}^{-3}$	
elem	s[m]	DO[m]	DO'	D1[m]	D1'	D1[m]	D1'
BEG	0.00000	0.00000	0.00000	-0.63777	0.00000	2.39719	0.00000
D1	2.92850	0.00000	0.00000	-0.63777	0.00000	2.39719	0.00000
QFWH	3.11850	0.00000	0.00000	-0.62499	0.13407	2.34916	-0.60393
QFWH	3.30850	0.00000	0.00000	-0.58716	0.26277	2.20698	-0.98766
D2	3.77050	0.00000	0.00000	-0.46576	0.26277	1.75068	-0.98766
EDGE	3.77050	0.00000	0.00000	-0.46576	0.19780	1.75068	-0.74346
B	4.33050	0.10998	0.39287	-0.46268	-0.17646	1.22930	-1.10869
EDGE	4.33050	0.10998	0.40821	-0.46262	-0.28906	1.22947	-0.98529
D3	5.98850	0.78679	0.40821	-0.94178	-0.28906	-0.40414	-0.98529
SV	6.28850	0.90925	0.40821	-1.02850	-0.28906	-0.57243	-0.09380
D2	6.75050	1.09785	0.40821	-1.16205	-0.28906	-0.61577	-0.00380
EDGE	6.75050	1.09785	0.58134	-1.17877	-0.27957	-0.63249	-0.00811
B	7.31050	1.52373	0.96001	-1.24345	0.12438	-0.54438	0.39872
EDGE	7.31050	1.52373	1.17255	-1.21124	-1.56338	-0.51217	-1.19154
D2	7.77250	2.06545	1.17255	-1.93352	-1.56338	-1.06267	-1.19154
QFLH	7.96250	2.23468	0.60133	-2.12636	-0.45044	-1.20723	-0.31637
QFLH	8.15250	2.29205	0.00000	-2.10136	0.71503	-1.17998	0.60461
D5	8.51500	2.29205	0.00000	-1.84216	0.71503	-0.96081	0.60461
SXH	8.66500	2.29205	0.00000	-1.73490	0.71503	-1.11687	-2.58538
SXH	8.81500	2.29204	0.00000	-1.62765	0.71503	-1.76642	-5.97537
D5	9.17750	2.29204	0.00000	-1.36846	0.71503	-3.93249	-5.97537
QFLH	9.36750	2.23468	-0.60133	-1.14235	1.65243	-4.00277	-4.19783
QFLH	9.55750	2.06545	-1.17255	-0.74768	2.47713	-5.51624	-2.24014
D2	10.01950	1.52373	-1.17255	0.39676	2.47713	-6.55119	-2.24014
EDGE	10.01950	1.52373	-0.96002	0.36455	1.01366	-6.58339	-4.67276
B	10.57950	1.09784	-0.56135	1.06889	1.42551	-9.07381	-4.29938
EDGE	10.57950	1.09784	-0.40821	1.08561	1.74953	-9.05709	-5.39114
D2	11.04150	0.90924	-0.40821	1.89390	1.74953	-11.54779	-5.39114
SV	11.34150	0.78678	-0.40821	2.41875	1.74953	-13.02499	-4.49968
D3	12.99950	0.10996	-0.40821	5.31947	1.74953	-20.48546	-4.49968
EDGE	12.99950	0.10996	-0.39288	5.31931	2.44345	-20.48563	-7.40522
B	13.55950	-0.00003	-0.00001	6.59377	2.06884	-24.76219	-7.88184
EDGE	13.55950	-0.00003	-0.00001	6.59377	3.01859	-24.76219	-11.33584
D2	14.02150	-0.00004	-0.00001	7.98836	3.01859	-29.99934	-11.33584
QFWH	14.21150	-0.00004	-0.00001	8.39798	1.27882	-31.53763	-4.80236
QFWH	14.40150	-0.00004	0.00000	8.47106	-0.51220	-31.81204	1.92357
D1	17.33000	-0.00003	0.00000	6.97107	-0.51220	-26.17885	1.92357
D6	18.07150	-0.00003	0.00000	6.59127	-0.51220	-24.75252	1.92357
QFWH	18.26650	-0.00003	0.00001	6.42662	-1.17372	-24.13419	4.40780
QFWH	18.46150	-0.00003	0.00001	6.13502	-1.81206	-23.03914	6.80495
D7	20.82400	-0.00001	0.00001	1.85404	-1.81206	-6.96244	6.80495
QDNH	21.01900	-0.00001	0.00001	1.51800	-1.64019	-5.70050	6.16954
QDNH	21.21400	-0.00001	0.00001	1.21224	-1.50103	-4.55224	5.63698
D8	22.02550	0.00000	0.00001	-0.00585	-1.50103	0.02216	5.63698
D8	22.83700	0.00001	0.00001	-1.22394	-1.50103	4.59657	5.63698
QDNH	23.03199	0.00001	0.00001	-1.52982	-1.64139	5.74527	6.16408

QDNH	23.22699	0.00001	0.00001	-1.86620	-1.81448	7.00855	3.81412
D7	25.58949	0.00003	0.00001	-6.15291	-1.81448	23.10691	6.81412
QFNH	25.78449	0.00003	0.00001	-6.44480	-1.17431	24.20309	4.41002
QFNH	25.97949	0.00003	0.00000	-6.60938	-0.51095	24.82116	1.91881
D6	26.72099	0.00003	0.00000	-6.98825	-0.51095	26.24396	1.91881
D6	27.46249	0.00003	0.00000	-7.36712	-0.51095	27.66676	1.91881
QFNH	27.65749	0.00003	0.00000	-7.39366	0.23914	27.76644	-0.89807
QFNH	27.85249	0.00003	0.00000	-7.27416	0.98450	27.31766	-3.69722
D7	30.21499	0.00002	0.00000	-4.94827	0.98450	18.58298	-3.69722
QDNH	30.40999	0.00002	0.00000	-4.80500	0.48739	18.04493	-1.83036
QDNH	30.60499	0.00002	0.00000	-4.75756	0.00000	17.86677	0.00000
D8	31.41649	0.00002	0.00000	-4.75756	0.00000	17.86677	0.00000
D8	32.22799	0.00002	0.00000	-4.75756	0.00000	17.86677	0.00000
QDNH	32.42299	0.00002	0.00000	-4.80500	-0.48739	18.04493	1.83036
QDNH	32.61799	0.00002	0.00000	-4.94827	-0.98450	18.58298	3.69722
D7	34.98049	0.00003	0.00000	-7.27416	-0.98450	27.31766	3.69722
QFNH	35.17549	0.00003	0.00000	-7.39366	-0.23914	27.76644	0.89807
QFNH	35.37049	0.00003	0.00000	-7.36712	0.51095	27.66676	-1.91881
D6	36.11199	0.00003	0.00000	-6.98825	0.51095	26.24396	-1.91881
D6	36.85349	0.00003	0.00000	-6.60938	0.51095	24.82116	-1.91881
QFNH	37.04849	0.00003	-0.00001	-6.44480	1.17431	24.20309	-4.41002
QFNH	37.24349	0.00003	-0.00001	-6.15291	1.81448	23.10691	-6.81412
D7	39.60599	0.00001	-0.00001	-1.86620	1.81448	7.00855	-6.81412
QDNH	39.80099	0.00001	-0.00001	-1.52982	1.64139	5.74527	-6.16408
QDNH	39.99599	0.00001	-0.00001	-1.22394	1.50103	4.59657	-5.63698
C9	40.80749	0.00000	-0.00001	-0.90585	1.50103	0.02216	-5.63698
D8	41.61899	-0.00001	-0.00001	1.21224	1.50103	-4.55224	-5.63698
QDNH	41.81399	-0.00001	-0.00001	1.51800	1.64019	-5.70050	-6.15954
QDNH	42.00899	-0.00001	-0.00001	1.85404	1.81206	-6.96244	-6.80495
D7	44.37149	-0.00003	-0.00001	6.13502	1.81206	-23.03914	-6.80495
QFNH	44.56649	-0.00003	-0.00001	6.42662	1.17372	-24.13419	-4.40780
QFNH	44.76149	-0.00003	0.00000	6.59127	0.51220	-24.75252	-1.92357
D6	45.50299	-0.00003	0.00000	6.97107	0.51220	-26.17885	-1.92357
D1	48.43149	-0.00004	0.00000	8.47106	0.51220	-31.81204	-1.92357
QFNH	48.62149	-0.00004	0.00001	8.39798	-1.27882	-31.53763	4.80236
QFNH	48.81149	-0.00004	0.00001	7.98836	-3.01859	-29.99934	11.33584
D2	49.27349	-0.00003	0.00001	6.59377	-3.01859	-24.76219	11.33584
EDGE	49.27349	-0.00003	0.00001	6.59377	-2.09884	-24.76219	7.88184
B	49.83349	0.10996	0.39288	5.31931	-2.44346	-20.48563	7.40522
EDGE	49.83349	0.10996	0.40821	5.31947	-1.74953	-20.48546	4.49968
D3	51.49149	0.78678	0.40821	2.41875	-1.74953	-13.02499	4.49968
SV	51.79149	0.90924	0.40821	1.89390	-1.74953	-11.54779	5.39114
D2	52.25349	1.09784	0.40821	1.08561	-1.74953	-9.05709	5.39114
EDGE	52.25349	1.09784	0.56135	1.06389	-1.42651	-9.07381	4.29932
B	52.81349	1.52373	0.96002	0.36455	-1.01366	-5.58339	4.67276
EDGE	52.81349	1.52373	1.17256	0.39676	-2.47713	-6.55119	2.24014
D2	53.27549	2.06546	1.17256	-0.74768	-2.47713	-5.51624	2.24014
QFLH	53.46549	2.23468	0.60133	-1.14235	-1.65243	-4.90277	4.19783
QFLH	53.65549	2.29204	0.00000	-1.36845	-0.71503	-3.93249	5.97537
D5	54.01799	2.29204	0.00000	-1.62765	-0.71503	-1.76642	5.97537
SKH	54.16799	2.29205	0.00000	-1.73490	-0.71503	-1.11687	2.68538
SKH	54.31799	2.29205	0.00000	-1.84216	-0.71503	-0.96081	-0.60461
D5	54.68049	2.29205	0.00000	-2.10136	-0.71503	-1.17992	-0.60461
QFLH	54.87049	2.23468	-0.60133	-2.12636	0.45044	-1.20723	0.31637
QFLH	55.06049	2.06545	-1.17255	-1.93352	1.56338	-1.06267	1.19154
D2	55.52249	1.52373	-1.17255	-1.21124	1.56338	-0.51217	1.19154
EDGE	55.52249	1.52373	-0.96001	-1.24345	-0.12438	-0.54438	-0.39872
B	56.08249	1.09785	-0.56134	-1.17877	0.27967	-0.63249	0.00611
EDGE	56.08249	1.09785	-0.40821	-1.16205	0.28906	-0.61577	0.09380
D2	56.54449	0.90925	-0.40821	-1.02850	0.28906	-0.57243	0.09380
SV	56.84449	0.78679	-0.40821	-0.94178	0.28906	-0.40414	0.98529

D3	58.50249	0.10998	-0.40821	-0.46252	0.28906	1.22947	0.98529
EDGE	58.50249	0.10998	-0.39287	-0.46258	0.17646	1.22930	1.10869
B	59.06249	0.00000	0.00000	-0.46570	-0.19780	1.75068	0.74346
EDGE	59.06249	0.00000	0.00000	-0.46570	-0.26277	1.75068	0.98766
D2	59.52449	0.00000	0.00000	-0.58716	-0.26277	2.20698	0.98766
QFWH	59.71449	0.00000	0.00000	-0.62499	-0.13407	2.34916	0.50393
QFWH	59.90449	0.00000	0.00000	-0.63777	0.00000	2.39719	0.00000
D1	62.63299	0.00000	0.00000	-0.63777	0.00000	2.39719	0.00000

Dispersion function D_0 and D_1 for E P A

FIGURE 1

Dispersion function D_0 and D_1 for E P A

sextupoles ON: $r_{sv}=8.25 \text{ m}^{-3}$, $r_{sxh}=-8.35 \text{ m}^{-3}$

FIGURE 2

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APPENDIX I

Contribution of the different elements to the periodic solution of the Dispersion functions

According to the equations (3.11) the periodic solutions of the Dispersion functions are completely determined by the two integrals

$$\int_n^{l_1} f_i(\sigma) R_{11}(\sigma) d\sigma \quad \text{and} \quad \int_0^{l_1} f_i(\sigma) R_{12}(\sigma) d\sigma$$

with f_i defined in eq. (3.2):

$$f_0 = h,$$

$$f_1 = -h + (2h^2 - k + h'D'_0)D_0 + (2hk - h^3 + \frac{1}{2}r)D_0^2 + \frac{1}{2}hD_0'^2.$$

Assuming hard edge approximation, where the magnetic field rises abruptly from zero outside the magnet to a constant value inside it, the two integrals above can be computed analytically for different elements.

COMBINED FUNCTION BENDING MAGNET WITH SEXTUPOLES INCLUDED

In this case the corresponding transfer matrix elements are

$$\begin{aligned} R_{11}(\sigma) &= C(\sigma), & R_{12}(\sigma) &= S(\sigma), & R_{13}(\sigma) &= \frac{h}{K} [1 - C(\sigma)], \\ R_{21}(\sigma) &= -KS(\sigma), & R_{22}(\sigma) &= C(\sigma), & R_{23}(\sigma) &= hS(\sigma), \end{aligned}$$

with the abbreviations

$$\begin{aligned} K = h^2 - k > 0: & \quad C(\sigma) = \cos(\sqrt{K}\sigma), & S(\sigma) &= \frac{\sin(\sqrt{K}\sigma)}{\sqrt{K}} \\ K = h^2 - k < 0: & \quad C(\sigma) = \cosh(\sqrt{|K|}\sigma), & S(\sigma) &= \frac{\sinh(\sqrt{|K|}\sigma)}{\sqrt{|K|}}. \end{aligned}$$

The Dispersion function D_0 is easily deduced from the first order transfer matrix but can

also be determined analytically using eqs. (3.11), since

$$\begin{aligned}\int_0^1 f_0(\sigma) R_{11}(\sigma) d\sigma &= hS \\ \int_0^1 f_0(\sigma) R_{12}(\sigma) d\sigma &= -\frac{h}{K}(C-1)\end{aligned}$$

with the notation $C \equiv C(l)$ and $S \equiv S(i)$.

To calculate the perturbation D_{11} , one replaces, in eq. (3.2), the function $D_0(\sigma)$ by its values at the beginning of that element (index b) and one obtains

$$\begin{aligned}f_1(\sigma) &= -h + (2h^2 - k)D_0(\sigma) + (2hk - h^3 + \frac{1}{2}r)D_0'(\sigma)^2 + \frac{1}{2}hD_0''(\sigma)^2 \\ &= -h + (2h^2 - k)[R_{11}D_{0b} + R_{12}D_{0b}' + R_{13}] \\ &\quad + (2hk - h^3 + \frac{1}{2}r)[R_{11}D_{0b} + R_{12}D_{0b}' + R_{13}]^2 \\ &\quad + \frac{1}{2}h[R_{21}D_{0b} + R_{22}D_{0b}' + R_{23}]^2\end{aligned}$$

Integration leads to

$$\begin{aligned}\int_0^1 f_1(\sigma) R_{11}(\sigma) d\sigma &= -hS + \left[\left(D_{0b} - \frac{h}{K} \right) (l + SC) + S^2 D_{0b}' \right] \left(\frac{k}{2} + \frac{k^2}{K} + \frac{rh}{2K} \right) \\ &\quad + \frac{1}{3}hD_{0b}' \left(D_{0b} - \frac{h}{K} \right) (C^3 - 1) \left[1 - \frac{1}{K} \left(4k + \frac{r}{h} - 2k^2 \right) \right] \\ &\quad + \frac{1}{3}hS \left(D_{0b} - \frac{h}{K} \right)^2 \left[\left(2k + \frac{r}{2h} - h^2 \right) (2 + C^2) + \frac{1}{2}K^2 S^2 \right] \\ &\quad + \frac{1}{2}hS D_{0b}'^2 \left[C^2 + \frac{1}{3}S^2 \left(2k + \frac{r}{h} \right) \right] + S \frac{h}{K} \left(h^2 + \frac{k^2}{K} + \frac{rh}{2K} \right)\end{aligned}$$

and

$$\begin{aligned}\int_0^1 f_1(\sigma) R_{12}(\sigma) d\sigma &= \frac{h}{K}(C-1) + \left[\left(D_{0b} - \frac{h}{K} \right) S^2 + \frac{1}{K} (l - SC) D_{0b}' \right] \left(\frac{k}{2} + \frac{k^2}{K} + \frac{rh}{2K} \right) \\ &\quad + \frac{1}{3}hS^3 D_{0b}' \left(D_{0b} - \frac{h}{K} \right) \left(4k - 2h^2 - K + \frac{r}{h} \right) - \frac{h}{K^2} (C-1) \left(h^2 + \frac{k^2}{K} + \frac{rh}{2K} \right) \\ &\quad + \frac{1}{3}h \left(D_{0b} - \frac{h}{K} \right)^2 \left[\frac{1}{K} (1 - C^3) \left(2k - h^2 + \frac{r}{2h} \right) + 1 + \frac{1}{2}C^3 - \frac{3}{2}C \right] \\ &\quad + \frac{1}{3}D_{0b}'^2 \frac{h}{K} \left[\frac{1}{2} (1 - C^3) + \frac{1}{K} (2 + C^3 - 3C) \left(2k - h^2 + \frac{r}{2h} \right) \right].\end{aligned}$$

In the particular case of $K = h^2 - k \equiv 0$, these equations simplify to

$$\int_0^l f_1(\sigma) d\sigma = l \left\{ -h + h D_{0b} \left[h + \frac{l^2}{3} \left(h^3 + \frac{1}{2} r \right) \right] + h l D'_{0b} \left[h + \frac{l^2}{4} \left(h^3 + \frac{1}{2} r \right) \right] \right. \\ \left. + l D_{0b} D'_{0b} \left(h^3 + \frac{1}{2} r \right) + D_{0b}^2 \left(h^3 + \frac{1}{2} r \right) \right. \\ \left. + D'_{0b}{}^2 \left[\frac{h}{2} + \frac{l^2}{3} \left(h^3 + \frac{1}{2} r \right) \right] + h^2 l^2 \left[\frac{h}{3} + \frac{l^2}{20} \left(h^3 + \frac{1}{2} r \right) \right] \right\},$$

$$\int_0^l f_1(\sigma) \sigma d\sigma = l^2 \left\{ -\frac{h}{2} + \frac{1}{2} h D_{0b} \left[h + \frac{l^2}{2} \left(h^3 + \frac{1}{2} r \right) \right] + h l D'_{0b} \left[\frac{2}{3} h + \frac{l^2}{5} \left(h^3 + \frac{1}{2} r \right) \right] \right. \\ \left. + \frac{2}{3} l D_{0b} D'_{0b} \left(h^3 + \frac{1}{2} r \right) + \frac{1}{2} D_{0b}^2 \left(h^3 + \frac{1}{2} r \right) \right. \\ \left. + \frac{1}{4} D'_{0b}{}^2 \left[h + l^2 \left(h^3 + \frac{1}{2} r \right) \right] + \frac{1}{4} h^2 l^2 \left[h + \frac{l^2}{6} \left(h^3 + \frac{1}{2} r \right) \right] \right\}.$$

The expressions corresponding to elementary elements can then easily be deduced by cancelling the relevant parameters in the general expressions of the integrals and one gets:

Pure drift:

$$\int_0^l f_1(\sigma) R_{11}(\sigma) d\sigma = 0 \\ \int_0^l f_1(\sigma) R_{12}(\sigma) d\sigma = 0$$

Pure sextupole:

$$\int_0^l f_1(\sigma) R_{11}(\sigma) d\sigma = \frac{1}{2} r l \left[D_{0b}^2 + l D_{0b} D'_{0b} + \frac{1}{3} l^2 D'_{0b}{}^2 \right] \\ \int_0^l f_1(\sigma) R_{12}(\sigma) d\sigma = \frac{1}{2} r l^2 \left[\frac{1}{2} D_{0b}^2 + \frac{2}{3} l D_{0b} D'_{0b} + \frac{1}{4} l^2 D'_{0b}{}^2 \right]$$

Pure quadrupole :

$$\int_0^l f_1(\sigma) R_{11}(\sigma) d\sigma = -\frac{1}{2}k [(l+SC)D_{0b} + S^2 D'_{0b}]$$

$$\int_0^l f_1(\sigma) R_{12}(\sigma) d\sigma = -\frac{1}{2} [kS^2 D_{0b} - (l-SC)D'_{0b}]$$

EDGES

With $\theta(\sigma)$ as the angle between the pole face of the element and the particle trajectory the transfer matrix elements are defined as

$$R_{11} = 1, \quad R_{12} = \sigma, \quad R_{13} = 0,$$

$$R_{21} = \int h' \tan \theta d\sigma, \quad R_{22} = 1, \quad R_{23} = 0,$$

since the variation h' of the curvature cannot be neglected. Moreover, for magnetic elements with inclined boundaries whose entrance or exit faces are not normal to the design trajectory, the local quadrupole $k(\sigma)$ and sextupole $r(\sigma)$ components acting on a particle passing through the fringe field of a magnetic element along the design trajectory are given by⁽¹¹⁾

$$k(\sigma) = h' \tan \theta,$$

$$r(\sigma) = -2k'_m \tan \theta \mp h'' \tan^2 \theta + hh' \tan^3 \theta,$$

where

$$\theta = \theta_e - \int h d\sigma.$$

The sign convention for the entrance (exit) angle θ_e is the same as in⁽⁹⁾; the first and second derivatives with respect to the azimuth σ are denoted by ' and '' respectively and k_m is the quadrupole component of the element. The upper sign corresponds to the entrance edge, the lower sign to an exit edge. Introducing these relations for the quadrupole and sextupole fields into the equation for $f_1(\sigma)$ defined in eq. (3.2), integrating over the edges and applying the "hard

edge" approximation ($s \rightarrow 0$) gives

$$\lim_{s \rightarrow 0} \int_0^s f_1(\sigma) R_{11}(\sigma) d\sigma = \lim_{s \rightarrow 0} \int_0^s f_1(\sigma) d\sigma = \pm h \langle D'_0 \rangle_e D_{0e} (1 + \tan^2 \theta_e) - h D_{0e} (1 - h D_{0e}) \tan \theta_e - k D_{0e}^2 \tan \theta_e - \frac{1}{2} h^2 D_{0e}^2 (1 + \frac{1}{2} \tan^2 \theta_e) \tan \theta_e,$$

where the index "m" has been omitted. The index "s" denotes either the entrance of the entrance edge or the exit of the exit edge. After introducing

$$\langle D'_0 \rangle_e = D'_{0e} \pm \frac{1}{2} h D_{0e} \tan \theta_e,$$

the relation for the entrance (upper sign) and for the exit edge (lower sign) is

$$\lim_{s \rightarrow 0} \int_0^s f_1(\sigma) R_{11}(\sigma) d\sigma = \pm h D'_{0e} D_{0e} (1 + \tan^2 \theta_e) - h D_{0e} (1 - h D_{0e}) \tan \theta_e - k D_{0e}^2 \tan \theta_e + \frac{1}{4} h^2 D_{0e}^2 \tan^3 \theta_e.$$

Similarly, one obtains

$$\lim_{s \rightarrow 0} \int_0^s f_1(\sigma) R_{12}(\sigma) d\sigma = \lim_{s \rightarrow 0} \int_0^s \sigma f_1(\sigma) d\sigma = \pm \frac{1}{2} h D_{0e}^2 \tan^2 \theta_e.$$

APPENDIX II

Change of the Dispersion function D_1 due to different elements

Introducing into eqs. (4.2) the expressions for the integrals derived in APPENDIX I and rearranging the different terms, one obtains the relations for the transport of the function D_1 through the different machine elements.

COMBINED FUNCTION BENDING MAGNET WITH SEXTUPOLES INCLUDED

$$\begin{aligned}
 D_1(l) = & D_{1b}C + D'_{1b}S - \frac{h}{K}(1-C) \\
 & + \left[\left(D_{0b} - \frac{h}{K} \right) lS + \frac{1}{K}(S-lC)D'_{0b} \right] \left(\frac{k}{2} + \frac{k^2}{K} + \frac{rh}{2K} \right) \\
 & + \frac{1}{3}SD'_{0b} \left(D_{0b} - \frac{h}{K} \right) (1-C) \left[\frac{1}{K}(4hk+r-2h^3) - h \right] \\
 & + \frac{1}{3} \left(D_{0b} - \frac{h}{K} \right)^2 \left[\frac{1}{K} \left(2hk + \frac{1}{2}r - h^3 \right) (2-C-C^2) + \frac{1}{2}h(1-C)^2 \right] \\
 & + \frac{1}{3}D'_{0b}{}^2 \frac{h}{K} \left\{ C \left(\frac{3}{2}C - \frac{1}{2} - C^3 \right) + \frac{1}{K} [k(1+4C^2-4C-C^4) + h^2C(2+C^3-3C)] \right. \\
 & \quad \left. + \frac{r}{2hK}(1-C)^2 \right\} + \frac{h}{K^2} \left(h^2 + \frac{k^2}{K} + \frac{rh}{2K} \right) (1-C)
 \end{aligned}$$

$$\begin{aligned}
 D'_1(l) = & -D_{1b}KS + D'_{1b}C - hS \\
 & + \left[\left(D_{0b} - \frac{h}{K} \right) (lC+S) + SlD'_{0b} \right] \left(\frac{k}{2} + \frac{k^2}{K} + \frac{rh}{2K} \right) \\
 & + \frac{1}{3}D'_{0b} \left(D_{0b} - \frac{h}{K} \right) (C+1-2C^2) \left[\frac{1}{K}(4hk+r-2h^3) - h \right] \\
 & + \frac{1}{3}hS \left(D_{0b} - \frac{h}{K} \right)^2 \left[\left(2k + \frac{r}{2h} - h^2 \right) (2C+1) + K(1-C) \right] \\
 & + \frac{1}{3}hSD'_{0b}{}^2 \left\{ C^3 + \frac{1}{2} + \frac{1}{K} [k(4+C^3-5C) - h^2(2+C^3-3C)] + \frac{r}{hK}(1-C) \right\} \\
 & + S \frac{h}{K} \left(h^2 + \frac{k^2}{K} + \frac{rh}{2K} \right)
 \end{aligned}$$

In the particular case of $K = h^2 - k \equiv 0$, these equations simplify to

$$D_1(t) = D_{1b} + tD'_{1b} - \frac{1}{2}ht^2 \\ + t^2 \left\{ \frac{1}{2}hD_{0b} \left[h + \frac{t^2}{6} \left(h^3 + \frac{1}{2}r \right) \right] + h t D'_{0b} \left[\frac{h}{3} + \frac{t^2}{20} \left(h^3 + \frac{1}{2}r \right) \right] \right. \\ \left. + \frac{1}{3}tD_{0b}D'_{0b} \left(h^3 + \frac{1}{2}r \right) + \frac{1}{2}D_{0b}^2 \left(h^3 + \frac{1}{2}r \right) \right. \\ \left. + \frac{1}{4}D_{0b}'^2 \left[h + \frac{t^2}{3} \left(h^3 + \frac{1}{2}r \right) \right] + \frac{1}{12}h^2t^2 \left[h + \frac{t^2}{10} \left(h^3 + \frac{1}{2}r \right) \right] \right\}$$

$$D'_1(t) = D'_{1b} - ht \\ + t \left\{ hD_{0b} \left[h + \frac{t^2}{3} \left(h^3 + \frac{1}{2}r \right) \right] + h t D'_{0b} \left[h + \frac{t^2}{4} \left(h^3 + \frac{1}{2}r \right) \right] \right. \\ \left. + tD_{0b}D'_{0b} \left(h^3 + \frac{1}{2}r \right) + D_{0b}^2 \left(h^3 + \frac{1}{2}r \right) \right. \\ \left. + D_{0b}'^2 \left[\frac{h}{2} + \frac{t^2}{3} \left(h^3 + \frac{1}{2}r \right) \right] + h^2t^2 \left[\frac{h}{3} + \frac{t^2}{20} \left(h^3 + \frac{1}{2}r \right) \right] \right\}$$

One can easily deduce the expressions corresponding to elementary elements by cancelling the relevant parameters:

Pure drift :

$$D_1(t) = D_{1b} + D'_{1b}t$$

$$D'_1(t) = D'_{1b}$$

Pure sextupole :

$$D_1(t) = D_{1b} + D'_{1b}t + \frac{1}{2}rt^2 \left(\frac{1}{2}D_{0b}^2 + \frac{1}{3}D_{0b}D'_{0b}t + \frac{1}{12}D_{0b}'^2t^2 \right)$$

$$D'_1(t) = D'_{1b} + \frac{1}{2}rt \left(D_{0b}^2 + D_{0b}D'_{0b}t + \frac{1}{3}D_{0b}'^2t^2 \right)$$

Pure quadrupole :

$$D_1(t) = D_{1b}C + D'_{1b}S - \frac{1}{2}[kD_{0b}tS - (S - tC)D'_{0b}]$$

$$D'_1(t) = D_{1b}kS + D'_{1b}C - \frac{1}{2}k[(S + tC)D_{0b} + D'_{0b}tS]$$

EDGES

Entrance edge : entrance angle θ_1

(D_1 , D'_1 at the end of the entrance edge defined by the values at the beginning, index "b")

$$D_1 = D_{1b} - \frac{1}{2} h D_{0b}^2 \tan^2 \theta_1$$

$$D'_1 = D_{1b} h \tan \theta_1 + D'_{1b}$$

$$+ h D'_{0b} D_{0b} (1 + \tan^2 \theta_1) - h D_{0b} (1 - h D_{0b}) \tan \theta_1 - k D_{0b}^2 \tan \theta_1$$

Exit edge : exit angle θ_2

(D_1 , D'_1 at the end of the exit edge defined by the values at the beginning, index "b")

$$D_1 = D_{1b} + \frac{1}{2} h D_{0b}^2 \tan^2 \theta_2$$

$$D'_1 = D_{1b} h \tan \theta_2 + D'_{1b}$$

$$- h D'_{0b} D_{0b} (1 + \tan^2 \theta_2) - h D_{0b} \tan \theta_2 - k D_{0b}^2 \tan \theta_2 - \frac{1}{2} h^2 D_{0b}^2 \tan^3 \theta_2$$

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