ELEMENTARY PARTICLE THEORY

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TABLE OF CONTENTS

I.	Introductory Overview	
II.	Quantum Electrodynamics	
	A. Local Gauge Invariance	
	B. Photon Mass	
	C. Feynman Rules	
	D. Electron-Electron Scattering	
	E. e^+e^- Annihilation	
	F. Anomalous Magnetic Moments	
	G. Renormalizability and Effective Charge	
III.	Weak Interactions	
IV.	Electroweak Unification	
	A. Weinberg-Salam Model	
	B. Feynman Rules	
	C. Weak Neutral Current	
	D. $W^\pm$ and Z Masses	
	E. $W^\pm$ and Z Decays	
	F. Higgs-Scalar Production and Decay	
V.	Quantum Chromodynamics	
	A. The Quark Model	
	B. Deep-Inelastic Scattering	
	C. Color	
	D. QCD Lagrangian	
	E. Feynman Rules	
	F. Running Coupling and Asymptotic Freedom	
	G. Vacuum Properties of QCD	
	H. Heavy Quarkonia Decays	
	I. QCD Jets	
	J. Quark Confinement	
VI.	Grand Unified Theories	
VII.	Outlook	
	References	
	Appendix A: Dimensional Regularization	

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I. INTRODUCTORY OVERVIEW

During the past decade, significant advances have been made both in the theoretical and experimental aspects of elementary particle physics. A fundamental theory of strong, weak, and electromagnetic interactions has been developed based on the principles of quantum field theory and local gauge invariance. This elegant theory has scored an impressive string of experimental successes, thereby earning its title as the "standard model." That label appropriately describes its acceptance by the physics community as a standard against which experimental findings and alternative theories are to be compared. Although we anticipate that the standard model itself is incomplete and that new physics will emerge as higher energies (shorter distances) are probed, the general feeling is that its basic features are correct and will be maintained by the "ultimate theory."

What is the standard model? It is a local gauge theory based on an $SU(3)_c \times SU(2)_L \times U(1)$ symmetry which describes strong, weak, and electromagnetic interactions among elementary particles.¹ The $SU(3)_c$ part is called quantum chromodynamics (QCD).² It provides the strong interaction dynamics that keeps quarks confined inside hadrons. Appended to QCD is the $SU(2)_L \times U(1)$ Weinberg-Salam model³ of electroweak interactions. Together they explain all observed physical phenomena, excluding gravitational effects. (As yet, an acceptable quantum theory of gravity does not exist.)

What are the elementary particles? As a working definition, I will take an elementary particle to be a point-like entity that has no (as yet observed) constituents. (Examples of non-elementary particles are protons, neutrons, pions, etc.)

The presently known elementary particles and some of their basic properties are illustrated in Table I. The fermions (particles with half-integer spin) are grouped into 3 generations. Within each generation are leptons (non-strongly interacting fermions) and quarks which carry color. The color charge is responsible for strong interactions just as electric charge gives rise to electromagnetic interactions. Why three rather than one generation is unknown.

A comment about masses is perhaps in order. The standard model allows arbitrary fermion masses, i.e. they are parameters determined by experiment. Eventually, we hope to have a theory which will predict all mass ratios. The lepton masses in Table I are directly determinable. As yet only bounds exist for neutrino masses; they may actually be zero. In the case of quarks, the masses given are so-called current quark masses.⁴ These are short-distance values that would be observed if we were able to eliminate long-distance strong interaction effects. For the t quark only a bound is given for its mass. That value comes from lack of observation of top quark produc-

Table I Elementary particles and their properties
in the "standard model"

Particle	Symbol	Spin	Charge	Color	Mass (GeV)	
electron neutrino	ν_e	1/2	0	0	$<6 \times 10^{-8}$	
electron	e	1/2	-1	0	0.51×10^{-3}	1st generation
up quark	u	1/2	2/3	3	5×10^{-3}	
down quark	d	1/2	-1/3	3	9×10^{-3}	
muon neutrino	ν_μ	1/2	0	0	$<0.25 \times 10^{-3}$	
muon	μ	1/2	-1	0	0.106	2nd generation
charm quark	c	1/2	2/3	3	1.25	
strange quark	s	1/2	-1/3	3	0.175	
tau neutrino	ν_τ	1/2	0	0	<0.16	
tau	τ	1/2	-1	0	1.78	3rd generation
top quark	t	1/2	2/3	3	>23	
bottom quark	b	1/2	-1/3	3	4.5	
photon	γ	1	0	0	0	
W boson	$W^\pm$	1	± 1	0	82.2 ± 1.8	gauge bosons
Z boson	Z	1	0	0	93.2 ± 1.5	
gluon	G	1	0	8	0	
Higgs scalar	H	0	0	0	7 GeV-1 TeV	

tion or narrow resonances via $e^+e^- \rightarrow t\bar{t}$ (bar indicates antiparticle) up to $\sqrt{s} \approx 46$ GeV at PETRA.⁵

Quarks and leptons interact by exchanging spin 1 quanta called gauge bosons. In the standard model there are 12 gauge bosons (see Table I). The eight gluons corresponding to generators of $SU(3)_C$ mediate strong interactions while the $W^\pm$, Z , and γ of $SU(2)_L \times U(1)$ are responsible for weak and electromagnetic interactions. (This concept of quanta exchange will be illustrated in subsequent sections by examples.) The strengths of these interactions are parametrized by three independent dimensionless coupling constants g_3 , g_2 , and g_1 , corresponding to $SU(3)_C$, $SU(2)_L$, and $U(1)$ interactions respectively. These couplings are energy dependent; so, for example, the strong coupling $g_3(E)$ is very large at small $E \lesssim 1$ GeV but decreases at high energies (asymptotic freedom). Comparing the couplings at $E = m_H \approx 83$ GeV, one finds

$$g_3(m_H) \approx 1.13,$$

$$g_2(m_H) \approx 0.67,$$

$$g_1(m_H) \approx 0.46. \quad (1.1)$$

Although $g_3(m_W)$ is the largest, they are all not that different. Evolving to yet higher energies, one finds that they become equal at $\approx 10^{14}$ GeV. This meeting is either an incredible accident or more likely indicative of grand unification,⁶ i.e. that the standard model is only part of big simple gauge group theory, and strong, weak, and electromagnetic interactions are merely components of one unified interaction.

The final entry in Table I is the spin 0 Higgs scalar. Its mass is arbitrary (except for theoretical constraint arguments). Whatever its mass, the Higgs scalar is very elusive because its couplings to ordinary matter (e, u, d) is so weak. Much of the theoretical motivation for building a super high energy collider, 20-40 TeV, comes from the Higgs scalar. It is hoped that either the Higgs scalar will be found or new physics will take its place. I consider the latter to be more likely. A detailed discussion of the Higgs scalar will be given in Section IV.

This brief introduction was meant to be a survey of the present state of the art in elementary particle physics. In subsequent sections more details will be provided. I will try to illustrate these ideas by calculational examples. Most of those examples were chosen because I have personally worked on related topics. No claim to completeness or formal rigor is made.

II. QUANTUM ELECTRODYNAMICS

Quantum electrodynamics was the first gauge theory. Its incredible success at describing short-distance electromagnetic interactions gave rise to attempts at describing weak and strong interactions in a similar manner. Ultimately, the standard $SU(3)_C \times SU(2)_L \times U(1)$ gauge theory of strong and electroweak interactions emerged.

In this section I will illustrate some basic features of gauge theories via quantum electrodynamics (QED). Several calculational examples will be given for the purpose of demonstrating concepts.

A. Local Gauge Invariance

Electrodynamics describes a 4-component spinor field $\psi(x)$ interacting with the electromagnetic potential (gauge field) $A_\mu(x)$ which transforms as a vector under Lorentz transformations. Second-quantizing these fields by anticommutation relations for $\psi(x)$ and commutation relations for $A_\mu(x)$ makes them operators on a Fock space rather than merely complex valued quantities. That procedure elevates electrodynamics to quantum electrodynamics (QED) in which $\psi(x)$ and $A_\mu(x)$ describe electron and photon quanta respectively.

At the classical (pre-second-quantized) level this theory is based on the Lagrangian density

$$\mathcal{L}(x) = \mathcal{L}(x)_{\text{free}} + \mathcal{L}(x)_{\text{int}}. \quad (2.1)$$

The free (non-interacting) part is given by⁷

$$\mathcal{L}(x)_{\text{free}} = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \bar{\psi}(x)(i\partial_\mu\gamma^\mu - m_e)\psi(x) \quad (2.2a)$$

where $(\partial_\mu = \partial/\partial x^\mu)$.

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x), \quad (2.2b)$$

$$\bar{\psi}(x) = \psi^\dagger(x) \gamma^0. \quad (2.2c)$$

The principle of least action, $\delta \int d^4 x \mathcal{L}(x)_{\text{free}} = 0$, then implies the free Euler-Lagrange equations of motion (for now we set $\mathcal{L}(x)_{\text{int}} = 0$)

$$\partial_\nu F^{\mu\nu}(x) = 0, \quad (2.3a)$$

$$(i\partial_\mu \gamma^\mu - m_e)\psi(x) = 0. \quad (2.3b)$$

Equation (2.3a) along with

$$\partial_\nu \tilde{F}^{\mu\nu}(x) = 0, \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \quad (2.3c)$$

make up the free Maxwell's equations while Eq. (2.3b) is the free Dirac equation. Interactions between these fields are introduced by the so-called minimal coupling prescription. The partial derivative ∂_μ acting on $\psi(x)$ is replaced by the covariant derivative $\partial_\mu - ieA_\mu(x)$ (e = electric charge of the electron), i.e.

$$\partial_\mu \rightarrow \partial_\mu - ieA_\mu(x). \quad (2.4)$$

This procedure induces an interaction term

$$\mathcal{L}(x)_{\text{int}} = eA_\mu \bar{\psi} \gamma^\mu \psi \quad (2.5)$$

which modifies Eqs. (2.3) to (I will often not exhibit the x dependence)

$$\partial_\nu F^{\mu\nu} = e \bar{\psi} \gamma^\mu \psi, \quad (2.6a)$$

$$(i\partial_\mu \gamma^\mu + eA_\mu \gamma^\mu - m_e)\psi = 0, \quad (2.6b)$$

$$\partial_\nu \tilde{F}^{\mu\nu} = 0. \quad (2.6c)$$

The covariant derivative replacement is perhaps best motivated by the principle of local gauge invariance. The free Lagrangian in Eq. (2.2a) has a global $U(1)$ invariance, i.e. it is invariant under the phase change (also called a global gauge transformation)

$$\psi(x) \rightarrow e^{ie\theta} \psi(x), \quad 0 \leq \theta \leq 2\pi. \quad (2.7)$$

Such phases form a $U(1)$ group. This particular symmetry gives rise to a conserved Noether charge, the electric charge. If we try to perform a local gauge transformation

$$\psi(x) \rightarrow e^{ie\theta(x)} \psi(x) \quad (2.8)$$

on $\mathcal{L}_{\text{free}}$, with θ now dependent on the space-time position x , we find that $\mathcal{L}_{\text{free}}$ is not invariant. If, however, we replace ∂_μ by $\partial_\mu - ieA_\mu$ and require

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \theta(x) \quad (2.9)$$

under local gauge transformations, then the total $\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$ is locally gauge invariant. This symmetry means that we can change the phase of $\psi(x)$ at each point in space-time if we compensate by modifying $A_\mu(x)$ as in Eq. (2.9). The principle of local gauge invariance seems to govern the dynamics of all fundamental interactions.⁸ It is a powerful guiding principle with important implications, as we shall see.

B. Photon Mass

An important consequence of local electromagnetic gauge invariance is that the photon must be massless. To see this, it is easiest to work in the Lorentz gauge s.t. $\partial_\nu A^\nu = 0$. The field equation for A_μ that follows from Eq. (2.6a) is

$$\partial_\nu \partial^\nu A_\mu(x) = -e\bar{\psi}(x)\gamma_\mu\psi(x). \quad (2.10)$$

This is the Klein-Gordon equation for a massless field. We could give the photon a mass by adding a term $-\frac{1}{2}m_\gamma^2 A_\mu A^\mu$ to the Lagrangian density. Such a term would, however, explicitly break the local gauge invariance. At present the experimental bounds on photon mass give⁹

$$m_\gamma < 6 \times 10^{-16} \text{ eV}. \quad (2.11)$$

This is quite good evidence for exact local electromagnetic gauge invariance.

C. Feynman Rules

To go from a classical to a quantum field theory, one either second-quantizes the fields directly or employs Feynman's path integral formulation to define Green's functions. In either case what results are a set of rules for perturbatively calculating quantum amplitudes order by order in the expansion parameter $\alpha = e^2/4\pi \approx 1/137$. I will not discuss quantization or derive the perturbative S-matrix expansion. For those unfamiliar with such techniques I recommend the detailed explanations given in Refs. 7 and 10.

The perturbative rules for calculating transition amplitudes are called Feynman rules. In the case of QED they are quite simple and straightforward to apply. The basic rules and symbols are illustrated in Fig. 1. They are extracted from the Lagrangian and equations of motion. Notice the photon propagator is gauge dependent. I will generally work in the Feynman gauge $\xi = 0$.

With practice, these rules become second nature. Indeed, some elementary particle physicists do much of their thinking (and doodling) in Feynman diagrams. Such diagrams represent amplitudes

Feynman Gauge
Photon Propagator

$$\frac{-i \left(g_{\mu\nu} - \frac{\xi k_\mu k_\nu}{k^2} \right)}{k^2}$$

Electron
Propagator

$$\frac{i}{\not{p} - m_e} = \frac{i(\not{p} + m)}{p^2 - m^2}$$

γ ee Vertex

$$ie \gamma_\mu$$

Fig. 1 Feynman rules for QED; $\xi = 0$ corresponds to the Feynman gauge.

which in turn allow one to compute scattering cross-sections and particle decay rates. I will illustrate their use by several examples.

D. Electron-Electron Scattering

Consider the process $e^- + e^- \rightarrow e^- + e^-$. The amplitude for this reaction (to lowest order) is illustrated in Fig. 2.

Fig. 2 Direct and exchange scattering amplitudes for $e^- + e^- \rightarrow e^- + e^-$.

These diagrams are to be interpreted as follows: Two electrons with asymptotic four momentum p_1 and p_2 propagate in space-time (from the

bottom up). They interact by exchanging a virtual (off-mass-shell) photon. That single photon exchange scatters the electrons into asymptotic outgoing states with momentum p_1 and p_2 . There are two diagrams because the initial and final states contain identical electrons; hence there are two distinct ways this reaction can occur. The Feynman rules tell us that incoming and outgoing solid lines represent an electron spinor, vertices correspond to an $ie\gamma_\mu$ interaction, and the wavy line denotes a photon propagator. In momentum space these diagrams represent the following amplitudes:

$$\mathcal{M}_a = \bar{u}(p'_1)(ie\gamma_\mu)u(p_1) \left(\frac{-ig^{\mu\nu}}{(p_1 - p'_1)^2} \right) \bar{u}(p'_2)(ie\gamma_\nu)u(p_2) , \quad (2.12a)$$

$$\mathcal{M}_b = -\bar{u}(p'_2)(ie\gamma_\mu)u(p_1) \left(\frac{-ig^{\mu\nu}}{(p_1 - p'_2)^2} \right) \bar{u}(p'_1)(ie\gamma_\nu)u(p_2) \quad (2.12b)$$

with $u(p)$ a Dirac spinor in momentum space. The minus sign in Eq. (2.12b) is inserted so the total amplitude $\mathcal{M}_a + \mathcal{M}_b$ changes sign under the interchange of the initial or final state identical fermions.

To get a scattering cross-section, we must square the total amplitude and integrate over all accessible final states. For example, if the initial electrons are unpolarized, one averages over initial spin (that gives a factor of $1/4$) and sums over final spin states. Including phase space and normalization factors (see ref. 7), one finds⁷

$d\sigma =$

$$\frac{d^3 p'_1}{(2\pi)^3} \frac{d^3 p'_2}{(2\pi)^3} \frac{1}{E_1 E_2 E'_1 E'_2} \frac{1}{16s} (2\pi)^4 \delta^4(p'_2 + p'_1 - p_2 - p_1) \frac{1}{4} \sum_{\text{pol.}} |\mathcal{M}_a + \mathcal{M}_b|^2 \quad (2.13)$$

where the δ function explicitly conserves energy-momentum and s is the initial $e^- - e^-$ relative velocity. Squaring the amplitude, summing over electron polarizations, and carrying out all but the angular integration, one finds in the center-of-momentum (c.m.) frame (where $E_1 = E_2 = E'_1 = E'_2 = E$)

$$\frac{d\sigma_{\text{cm}}}{d\Omega} = \frac{\alpha^2}{8E^2} \left[\frac{1 + \cos^4 \theta/2}{\sin^4 \theta/2} + \frac{2}{\sin^2 \theta/2 \cos^2 \theta/2} + \frac{1 + \sin^4 \theta/2}{\cos^4 \theta/2} \right] \quad (2.14)$$

where $\alpha = e^2/4\pi = 1/137$ and θ is the angle between initial and final electron momenta.

The cross-section formula in Eq. (2.14) is fairly typical of QED calculations. Higher-order corrections due to multi-photon exchange or real photon emission are $O(\alpha^3)$; hence, Eq. (2.14) is a very good approximation to the full perturbative expansion. As an exercise in relativistic kinematics, one should calculate the differential cross-section in the rest frame of one of the electrons.

E. e^+e^- Annihilation

The next example I will consider is $e^+ + e^- \rightarrow f + \bar{f}$ where f is some charged fundamental lepton or quark. Such particles are appended to QED by adding a term $\mathcal{L}_f$ to the Lagrangian in Eq. (2.1):

$$\mathcal{L}_f = \bar{\psi}_f(i\partial_\mu\gamma^\mu - eQ_f A_\mu\gamma^\mu - m_f)\psi_f. \quad (2.15)$$

(See Table I for charge assignments and masses.) Their propagators are just like the electron (see Fig. 1) but with $m_e \rightarrow m_f$. Similarly, the photon coupling in Fig. 1 is generalized to $-ieQ_f\gamma_\mu$.

With the above rules in mind, we now compute the cross-section for $e^+ + e^- \rightarrow f + \bar{f}$. The amplitude for this process is illustrated in Fig. 3 (if $f = e$, there is another diagram that must be included).

Fig. 3 Feynman diagram for the scattering process $e^+ + e^- \rightarrow f + \bar{f}$.

The amplitude corresponding to this diagram is

$$\mathcal{M} = -ieQ_f \bar{u}_f(p_1') \gamma^\mu v(p_2') \left[\frac{-ig_{\mu\nu}}{(p_1 + p_2)^2} \right] i\bar{v}(p_2) \gamma^\nu u(p_1). \quad (2.16)$$

To obtain the cross-section we square this amplitude, $|\mathcal{M}|^2$, average over initial and sum over final polarizations, and then integrate over final state phase space. In the center-of-momentum frame ($E_1 = E_2 = E$, $E_1' = E_2' = E'$), one finds (neglecting the electron mass)

$$\frac{d\sigma_{\text{cm}}}{d\Omega} = \frac{\alpha^2 Q_f^2}{4s} 8(2 - \beta^2 \sin^2 \theta), \quad s = 4E^2 \quad (2.17)$$

where $\beta = |\vec{p}'|/E' = (1 - m_f^2/E'^2)^{1/2}$ and θ is the angle between the beam and outgoing fermions (see Fig. 4). Note that there is no angular asymmetry in the cross-section. We shall see in Section IV how weak neutral currents give rise to a C-violating asymmetry.

Fig. 4 Angular definition in $e^+ + e^- \rightarrow f + \bar{f}$.

Integrating Eq. (2.17) over all solid angles finally gives the total cross-section

$$\sigma = \frac{4\pi\alpha^2 Q_f^2}{3s} \beta \left(\frac{3 - \beta^2}{2} \right). \quad (2.18)$$

The dependence on β represents a threshold effect. In the high energy limit $E \gg m_f$, $\beta \rightarrow 1$ and Eq. (2.18) becomes

$$\sigma = \frac{4\pi\alpha^2}{3s} Q_f^2. \quad (2.19)$$

For $f = \mu$ or τ , one just sets $Q_f = -1$. In the case of quarks, $Q_f = +2/3$ or $-1/3$; however they come in three distinct colors (see Table I) so the cross-section must be multiplied by 3.

An important quantity in e^+e^- annihilation physics is the ratio R defined by

$$R \equiv \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}. \quad (2.20)$$

If we neglect strong interaction effects (which is a good approximation at high energies and away from resonances) then from Eq. (2.19)

$$R = 3 \sum_{f=u,d,\dots} Q_f^2 \quad (2.21)$$

where the sum is over all quark flavors whose threshold has been passed. At present PETRA energies,⁵ $E \approx 23$ GeV, the measured value of R is consistent with the 5 observed flavors u, d, c, s, b ; i.e.

$$R = 3[3(-1/3)^2 + 2(2/3)^2] = \frac{11}{3}. \quad (2.22)$$

Passing the top quark threshold should lead to $\Delta R = 4/3$. Such a jump has not been observed, hence the bound $m_t \gtrsim 23$ GeV in Table I.

F. Anomalous Magnetic Moments

QED is an example of a renormalizable quantum field theory. That means that short-distance ultraviolet divergences can be absorbed into the bare parameters of the theory (couplings, masses, etc.). I will say more about this property in the next subsection; however, here I only want to stress its importance. Indeed, renormalizability implies that we can carry out unambiguous calculations in the expansion parameter α , and they can be tested by experiment. In that way one can check the validity of QED and look for new physics via deviations.

There are a number of higher-order calculations in QED that have been verified by experiment. However, perhaps the most impressive are the anomalous magnetic moments of the electron and the muon. In this subsection I wish to present an up-to-date survey of those important parameters.

The magnetic moment of a spin 1/2 charged lepton is given by

$$\mu = \frac{e}{2m_l} g \quad (2.23)$$

where g equals 2, according to the Dirac equation; see Eq. (2.6b). When one includes quantum corrections, g deviates from 2. In examining such deviations, the anomalous magnetic moment a_l is defined

$$a_l \equiv \frac{g - 2}{2} . \quad (2.24)$$

The $O(\alpha)$ contribution to a_l comes from the diagram pictured in Fig. 5; its effect was first calculated by J. Schwinger.¹¹ As (I hope) an instructive exercise, I will sketch a derivation of the Schwinger term

$$a_l = \frac{\alpha}{2\pi} . \quad (2.25)$$

Fig. 5 One-loop contribution to the anomalous magnetic moment.

The diagram in Fig. 5 is called a proper vertex diagram. Its evaluation is actually infinite except in the so-called Landau gauge where the photon propagator is given by $-i(g_{\mu\nu} - k_\mu k_\nu/k^2)/k^2$. In

other gauges, the ultraviolet infinity from this proper vertex diagram is cancelled by the self-energy diagrams in Fig. 6. This cancellation is part of Ward's identity.¹² The magnetic moment part has no infinity, so we can ignore this divergence problem. Our goal is to write the photon- $\ell^+ - \ell^-$ vertex as

$$ie\gamma_\mu F_1(q^2) + \frac{e}{2m_\ell} \sigma_{\mu\nu} q^\nu F_2(q^2) \quad (2.26)$$

Fig. 6 Lepton self-energy corrections to the $\ell\bar{\ell}\gamma$ vertex.

where $q = p_1 - p_2$ and $\sigma_{\mu\nu} = i/2[\gamma_\mu, \gamma_\nu]$. $F_1(q^2)$ and $F_2(q^2)$ are electric and magnetic form factors respectively. $F_1(0) = 1$ by definition of the electric charge e , while the anomalous magnetic moment is given by

$$a_\ell = F_2(0) . \quad (2.27)$$

To separate out $F_2(q^2)$ we will use the fact that this vertex is evaluated between spinors $\bar{u}(p_2)$ and $u(p_1)$ so the Gordon decomposition⁷

$$\bar{u}(p_2) \frac{(p_1 + p_2)_\mu}{2m} u(p_1) = \bar{u}(p_2) \left(\gamma_\mu + \frac{i\sigma_{\mu\nu}(p_1 - p_2)^\nu}{2m} \right) u(p_1) \quad (2.28)$$

can be applied. As we shall see, it is terms proportional to $(p_1 + p_2)_\mu$ that give rise to $F_2(0)$.

The diagram in Fig. 5 is evaluated by integrating over all possible momentum in the loop. Calling this quantity $A_\mu(p_2, p_1)$ one finds, using the Feynman rules in Fig. 1 and momentum routing in Fig. 5,

$$A_\mu(p_2, p_1) = \int \frac{d^4 k}{(2\pi)^4} (ie\gamma^\alpha) \frac{i}{p_2 - k - m_\ell} (ie\gamma_\mu) \frac{i}{p_1 - k - m_\ell} (ie\gamma^\beta) \frac{-ig_{\alpha\beta}}{k^2} \quad (2.29)$$

$$= e^3 \int \frac{d^4 k}{(2\pi)^4} \frac{\gamma^\alpha (p_2 - k + m_\ell) \gamma_\mu (p_1 - k + m_\ell) \gamma_\alpha}{(k^2 - 2p_2 \cdot k)(k^2 - 2p_1 \cdot k)k^2} \quad (2.30)$$

where we have used $p_1^2 = p_2^2 = m_\mu^2$. Using the Dirac algebra identities in Appendix A (with $n = 4$) and dropping terms proportional to γ_μ alone, since they cannot contribute to $F_2(q^2)$, this expression becomes

$$\Lambda_\mu(p_2, p_1) = \gamma_\mu \text{ terms} + e^3 \int \frac{d^4 k}{(2\pi)^4} \frac{\{-2(\not{p}_1 - \not{k})\gamma_\mu(\not{p}_2 - \not{k}) + 4m_\mu(p_1 + p_2 - 2k)_\mu\}}{(k^2 - 2p_2 \cdot k)(k^2 - 2p_1 \cdot k)k^2} \quad (2.31)$$

The denominator factors are combined by using Feynman parameters:

$$\frac{1}{(k^2 - 2p_2 \cdot k)(k^2 - 2p_1 \cdot k)k^2} = \int_0^1 2y dy \int_0^1 dx \frac{1}{(Q^2 - c)^3}, \quad (2.32)$$

$$Q = k - p_2 xy - p_1 y(1 - x),$$

$$c = y^2[m_\mu^2 - q^2 x(1 - x)].$$

Then, making the change in variables $k = Q + p_2 xy + p_1 y(1 - x)$, one finds

$$\Lambda_\mu(p_2, p_1) = \gamma_\mu \text{ terms} + e^3 \int_0^1 2y dy \int_0^1 dx \int \frac{d^4 Q}{(2\pi)^4} \frac{4m_\mu(1 - x - xy)(p_1 + p_2)_\mu}{(Q^2 - c)^3}. \quad (2.33)$$

On carrying out the Q integration, this becomes

$$\Lambda_\mu(p_2, p_1) = \gamma_\mu \text{ terms} - \frac{ie^3}{16\pi^2} \int_0^1 dy \int_0^1 dx \int \frac{4m_\mu(1 - x - xy)(p_1 + p_2)_\mu}{m_\mu^2 - q^2 x(1 - x)}. \quad (2.34)$$

Next, use of the Gordon decomposition in Eq. (2.28) gives

$$F_2(q^2) = \frac{e^2}{2\pi^2} \int_0^1 dy \int_0^1 dx \frac{m_\mu^2(1 - x - xy)}{m_\mu^2 - q^2 x(1 - x)}, \quad (2.35)$$

$$F_2(q^2) = \frac{2\alpha}{\pi} \int_0^1 dx \frac{m_\mu^2(1 - \frac{3}{2}x)}{m_\mu^2 - q^2 x(1 - x)} \quad (2.36)$$

which gives the Schwinger result

$$F_2(0) = \frac{\alpha}{2\pi}. \quad (2.37)$$

(Fill in the missing steps as an exercise.)

To get the $O(\alpha)$ prediction, one employs

$$\alpha^{-1} = 137.035963 \quad (15) \quad (2.38)$$

measured by the Josephson effect¹³ ((15) represent the uncertainty in the last two digits). This gives $a_l = 1.16 \times 10^{-3} + O(\alpha^2)$ for $l = e, \mu, \tau$. Shortly after Schwinger did this calculation, his result was verified by experiment.¹⁴ That confirmation started what was to be one of QED's outstanding success stories. Over the years theoretical calculations and experimental measurements of a_e and a_μ have greatly improved. The experimental values are presently¹⁴

$$\left. \begin{aligned} a_{e-} &= 1\,159\,652\,200\,(40) \times 10^{-12} \\ a_{e+} &= 1\,159\,652\,222\,(50) \times 10^{-12} \\ a_{\mu-} &= 11\,659\,370\,(120) \times 10^{-10} \\ a_{\mu+} &= 11\,659\,110\,(110) \times 10^{-10} \end{aligned} \right\} \text{Experiment.} \quad (2.39)$$

Theoretical calculations have been carried out up to and including terms of order α^4 (4 loops!). The results of those computations are (largely due to the efforts of T. Kinoshita and his collaborators).¹⁵

$$\left. \begin{aligned} a_e &= 1\,159\,652\,460\,(75)\,(127) \times 10^{-12} \\ a_\mu &= 11\,659\,202\,(20) \times 10^{-10} \end{aligned} \right\} \text{Theory.} \quad (2.40)$$

Agreement between theory and experiment is quite spectacular. Notice that theory is behind experiment (its errors are bigger) in the case of a_e while for a_μ the situation is reversed.

The electron's anomalous magnetic moment is a good test of pure QED because hadronic vacuum polarization corrections (which carry uncertainties) to it are very small. On the other hand, the muon's anomalous magnetic moment is more sensitive to those short-distance effects. The hadronic corrections to a_μ are¹⁵

$$a_\mu^{\text{had}} = 702\,(19) \times 10^{-10} \quad (2.41)$$

while the standard model predicts a weak correction¹⁶

$$a_\mu^{\text{weak}} = 19.5 \times 10^{-10} . \quad (2.42)$$

Comparing these contributions to the experimental values in Eq. (2.39), we see that experiments are not yet sensitive to the weak correction (in addition the uncertainty in the hadronic correction is as large as a_μ^{weak}). A precise measurement of a_μ at BNL which would lower its error by a factor of 20 to $\pm 5 \times 10^{-10}$ has been suggested.¹⁷ Such a measurement would be very nice. It would probe a_μ below the level of the standard model's weak corrections. A deviation from theory would point to additional short-distance physics. Indeed, a_e and a_μ already provide the best constraints on lepton substructure.¹⁸

These quantities have an illustrious past and are likely to play an important role in the future of elementary particle physics.

G. Renormalizability and Effective Charge

As stated before, QED is a renormalizable quantum field theory. Short-distance ultraviolet infinities can be absorbed into its bare parameters order by order in perturbation theory.¹⁹ The bare charge, e_0 , and bare mass, m_e^0 , are divided into finite physical parts e and m_e and infinite counterterms δe and δm via

$$\begin{aligned} e_0 &= e - \delta e, \\ m_e^0 &= m_e - \delta m. \end{aligned} \quad (2.43)$$

The counterterms are defined such that m_e is the electron's physical mass and e is the electric charge measured at zero momentum transfer (infinite distance). The fine structure constant $\alpha = e^2/4\pi \approx 1/137$ directly measures this coupling; see Eq. (2.38).

I don't intend to go into much detail about renormalization; however, I do want to demonstrate the basic idea by considering electric charge renormalization. That will allow me to introduce the concept of an effective or running charge.

In QED, we start with a bare charge e_0 which is supposed to be the electron's charge at zero distance. Of course we can never measure e_0 , since we can't get to zero distance (it would require infinite energy). What we can measure is the effective charge at some non-zero distance. If one tries to calculate the difference between these quantities, ultraviolet divergences are encountered. We absorb these infinities into the unmeasurable quantity e_0 . The difference between two experimental measurements is always finite and uniquely predicted in QED. (This procedure works amazingly well as demonstrated by a_e and a_μ in the previous subsection.) The difference between e_0 and e comes about because the vacuum of QED acts like diamagnetic medium. Virtual charged particle-antiparticle fermion pairs screen e_0 . These vacuum polarization effects occur as loop corrections to the photon propagator (see Fig.7.) Note that all charged elementary particles, e , μ , τ , u , d ... $W^\pm$ contribute to the vacuum polarization; however, at large distances the electron's effect dominates because it is the lightest. These contributions change the photon propagator (in the Landau gauge) as follows:

$$-i \frac{\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)}{k^2} \rightarrow -i \frac{\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)}{k^2 \{1 + \pi(k^2)\}}. \quad (2.44)$$

The quantity $\pi(k^2)$ is called the vacuum polarization function. I will compute the one-loop electron contribution to this quantity illustrated in Fig. 7. That Feynman diagram leads to

$$(g_{\mu\nu}k^2 - k_\mu k_\nu)\pi(k^2) = -ie^2\mu^{4-n} \int \frac{d^n q}{(2\pi)^n} \frac{\text{Tr}(\gamma_\mu(q + m_e)\gamma_\nu(k + q + m_e))}{(q^2 - m_e^2)((k + q)^2 - m_e^2)} \quad (2.45)$$

where a minus sign has been included for Fermi statistics.

Fig. 7 Some vacuum polarization contributions to the photon propagator.

This integral diverges in 4 dimensions, so it has been regularized by working in $n < 4$ dimensions²⁰ (see Appendix A). To keep the coupling dimensionless, a mass unit μ^{4-n} has been explicitly included.²¹ Employing the trace rules of dimensional regularization and combining propagator factors by

$$\frac{1}{ab} = \int_0^1 dx \frac{1}{(ax + b(1-x))^2} \quad (2.46)$$

one finds

$$(g_{\mu\nu}k^2 - k_\mu k_\nu)\pi(k^2) = -4ie^2\mu^{n-4} \int_0^1 dx \int \frac{d^n Q}{(2\pi)^n} \frac{g_{\mu\nu} \left(\frac{2-n}{n} Q^2 + m_e^2 + k^2(x-x^2) \right) - 2k_\mu k_\nu(x-x^2)}{(Q^2 - (m_e^2 + k^2(x^2 - x)))^2} \quad (2.47)$$

or

$$\pi(k^2) = \frac{8e^2\mu^{n-4}}{(16\pi^2)^{n/4}} \int_0^1 dx (x-x^2) (m_e^2 + k^2(x^2 - x))^{\frac{n}{2}-2} \Gamma(2 - \frac{n}{2}) \quad (2.48)$$

The ultraviolet divergence manifests itself as a pole at $n = 4$:

$$\Gamma(2 - \frac{n}{2}) = \frac{-2}{n-4} - \gamma + O(n-4) \quad (2.49)$$

where $\gamma = 0.5772\dots$ is the Euler constant. Expanding $\pi(k^2)$ about $n = 4$ one finds

$$\pi(k^2) = \frac{\alpha}{3\pi} \left[\frac{-2}{n-4} - \gamma + 2n(4\pi) - 6 \int_0^1 dx x(1-x) \ln \left(\frac{m_e^2 + k^2(x^2 - x)}{\mu^2} \right) + O(n-4) \right] \quad (2.50)$$

where

$$6 \int_0^1 dx \, x(1-x) \ell_n \left(\frac{m_e^2 + k^2(x^2 - x)}{\mu^2} \right) =$$

$$\ell_n \left(\frac{m_e^2}{\mu^2} \right) + \frac{1}{3} + (3 - B^2) \left[\frac{B}{2} \ell_n \left(\frac{B+1}{B-1} \right) - 1 \right], \quad (2.51a)$$

$$B = \left(1 - \frac{4m_e^2}{k^2} \right)^{1/2}.$$

The pole term is absorbed into the renormalized coupling

$$e^2 \equiv \frac{e_0^2}{1 + \pi(0)}, \quad \alpha = \frac{e^2}{4\pi} = \frac{1}{137.036} \quad (2.52a)$$

or from Eq. (2.50)

$$\frac{1}{e^2} = \frac{1}{e_0^2} - \frac{1}{6\pi^2} \left(\frac{1}{n-4} + \frac{\gamma}{2} - \ell_n \sqrt{4\pi} + \ell_n \left(\frac{m_e}{\mu} \right) \right). \quad (2.52b)$$

Of course all charged particle loops contribute to $\pi(k^2)$ and their effect must be included in (2.52b). In the standard model one finds (including the $W^\pm$ loop)

$$\delta e = \frac{\alpha}{4\pi} \left(\frac{11}{3} \frac{1}{n-4} + \text{finite terms} \right) + O(\alpha^2). \quad (2.53)$$

The definition of the renormalized charge via Eq. (2.52a) is a long-distance definition, i.e. it corresponds to the charge as seen at an infinite distance. As one gets closer to the charge and penetrates the vacuum polarizing cloud of virtual e^+e^- pairs surrounding it, the effective charge $e(k^2)$ grows. For $k^2 \gg m_e^2$, one finds from Eqs. (2.50) and (2.52b)

$$\frac{1}{e^2(k^2)} = \frac{1}{e^2} - \frac{1}{12\pi^2} \ell_n \left(\frac{m_e^2}{k^2} \right). \quad (2.54)$$

As k^2 passes muon and quark thresholds, their vacuum polarization effects are also penetrated and $e^2(k^2)$ grows even faster. It eventually becomes extremely large, but new physics (it is hoped) comes in before that point (called the Landau singularity or pole) is reached.

A more practical way of evolving the effective charge is to employ the $\overline{MS}$ (modified minimal subtraction) definition²² of renormalized charge. That quantity, which we refer to as $e(\mu)$, is obtained by subtracting the $1/(n-4)$ pole, $\gamma/2$, and $\ell_n \sqrt{4\pi}$ terms in Eq. (2.56b):

$$e^2(\mu) = e_0^2 + \frac{e_0^4}{6\pi^2} \left(\frac{1}{n-4} + \frac{\gamma}{2} - \ell_n \sqrt{4\pi} + \dots \right) \quad (2.55)$$

where ... refers to higher orders and the effects of other heavier particles. For $\mu < m_f$, one also subtracts the $\ln(m_f/\mu)$ terms. (My definition may differ somewhat from other people's.) The $\overline{MS}$ coupling $\alpha(\mu) \equiv e^2(\mu)/4\pi$ is related to the usual fine structure constant by (for $\mu \leq m_W \approx 83$ GeV)

$$\frac{1}{\alpha(\mu)} = \frac{1}{\alpha} - \frac{2}{3\pi} \sum_f Q_f^2 \ln\left(\frac{\mu}{m_f}\right) + \frac{1}{6\pi} + \dots \quad (2.56)$$

where the sum is over all flavors with $m_f < \mu$, the $1/6\pi$ comes from $W^\pm$ boson loops and, ... represent higher-order terms.

The evolution of $\alpha(\mu)$ as a function of the energy scale probed is most easily described by its beta function:

$$\mu \frac{\partial}{\partial \mu} \alpha(\mu) = \beta(\alpha) = b_0 \alpha^2 + b_1 \alpha^3 + \dots \quad (2.57)$$

We see from Eq. (2.56) that fermions contribute to b_0 :

$$b_0 = \frac{2}{3\pi} \sum_f Q_f^2 \theta(\mu - m_f) \quad (\text{fermions}) \quad (2.58)$$

where θ is a step function. Because their contribution is positive, they screen the charge. The spin $1/2$ $W^\pm$ particles contribute with the opposite sign:

$$b_0 = \frac{-7}{2\pi} \theta(\mu - m_W) \quad (2.59)$$

i.e. they anti-screen. If we sum over all 3 generations of fermions, we find for $\mu > m_W$

$$b_0 = \frac{11}{6\pi}, \quad \mu > m_W; \quad (2.60)$$

so the fermions win and the coupling $\alpha(\mu)$ continues to grow.

Is there experimental evidence for charge screening by vacuum polarization effects? The answer is yes. Atomic physics effects such as the Lamb shift²³ detect a small increase in the electron's charge at short distance. Also the difference between the anomalous magnetic moments a_μ and a_e (discussed in the previous subsections) to a large extent results from vacuum polarization effects. In the case of the muon one is probing shorter distances; so the effective charge is larger and $a_\mu > a_e$; see Eq. (2.40). Finally, in high energy e^+e^- annihilation experiments at PEP and PETRA one probes the energy range $\sqrt{s} \approx 30$ to 46 GeV. At those energies vacuum polarization effects due to $e, \mu, \tau, u, c, d, s, b$ all enter. One expects

$$\alpha^{-1}(E \approx 35 \text{ GeV}) \approx 130$$

and experiments find (from my own crude analysis) 130 ± 2 .

I have gone into detail about the effective charge and the concept of a running coupling because it plays such an important role in modern-day theory. Other examples of running couplings and the physical implications of that evolution will be given in subsequent sections.

III. WEAK INTERACTIONS

Historically, weak interaction physics had its beginnings in 1896 with Becquerel's discovery of radioactivity.²⁴ That was followed by 38 years of determined experimentation and several clever deductions which finally led to a framework for understanding this feeble weak force.²⁵ Those years were highlighted by Rutherford's discovery²⁶ of the atomic nucleus in 1911, Chadwick's²⁷ observation in 1914 that β -rays (electrons) emitted in weak nuclear decays had a continuum of energies and Pauli's²⁸ speculation in 1931 that a weakly interacting particle (later named the neutrino by Fermi) must be emitted with the electron in order to conserve energy-momentum and angular momentum. Those important breakthroughs combined with Chadwick's discovery of the neutron²⁹ in 1932 provided a description of β -radioactivity as due to the weak decay process $n \rightarrow p + e + \bar{\nu}_e$. Finally in 1934 Fermi developed a theory of weak interactions.³⁰ Inspired by electrodynamics, he wrote down an effective interaction Lagrangian density

$$\mathcal{L}_{\text{int}}(x) = \frac{-G_F}{\sqrt{2}} \bar{\psi}_p(x) \gamma_\alpha \psi_n(x) \bar{\psi}_e(x) \gamma^\alpha \psi_\nu(x) + \text{h.c.} \quad (3.1)$$

($\psi = 4$ component Dirac spinor, h.c. stands for hermitian conjugate) which parametrized the known properties of β -decay. In analogy to Q.E.D., his theory was based on a current-current interaction picture in which the charged hadronic vector current $\bar{\psi}_p \gamma_\alpha \psi_n$ interacted with the leptonic current $\bar{\psi}_e \gamma^\alpha \psi_\nu$. Unlike QED, the interaction took place at a point (so-called local interaction) and its strength was parametrized by a dimensionfull coupling G_F which was determined to be very small.

$$G_F = 1.16638 \pm 0.00002 \times 10^{-5} \text{ GeV}^{-2} \text{ (1984 value)}. \quad (3.2)$$

The β -decay process is illustrated in Fig. 8.

Fig. 8 Local Fermi amplitude for β -decay, $n \rightarrow p + e + \bar{\nu}_e$.

Fermi's effective interaction could be used to compute transition rates via first order perturbation theory. Although it provided an adequate picture of some β -decays, it failed to describe decays in which the nuclear angular momentum changed by one unit $|\Delta J| = 1$ (Gamow-Teller transitions³¹). To accommodate such reactions, additional axial-vector currents of the form $\bar{\psi}_p \gamma_\mu \gamma_5 \psi_n$ had to be appended to Fermi's theory (or alternatively tensor currents). Another major modification was required when Lee and Yang's speculation³² that parity might be violated in weak interactions was experimentally confirmed.³³ Indeed, experiments subsequently found that only left-handed components of fermions (and right-handed components of their antiparticles) participated in charged current weak interactions.³⁴

$$\psi = \psi_L + \psi_R \quad (3.3)$$

$$\psi_L = \frac{1 + \gamma_5}{2} \psi$$

R

These findings resulted in a modified interaction (the V-A theory³⁵)

$$\mathcal{L}_{\text{int}} = \frac{-4G_F}{\sqrt{2}} \bar{\psi}_p \gamma_\alpha \psi_n \bar{\psi}_\nu \gamma^\alpha \psi_e \quad (3.4)$$

The discovery of the muon, μ , and its neutrino partner, ν_μ , as well as strangeness carrying particles Λ^0 , $K^\pm$ etc meant that additional current-current interactions such as

$$\frac{-4G_\mu}{\sqrt{2}} \bar{\psi}_\nu \gamma_\alpha \psi_\mu \bar{\psi}_\nu \gamma^\alpha \psi_e \quad (3.5)$$

had to be added to Eq. (3.4). Rather than dwell on details of the now obsolete Fermi theory, I will illustrate its utility and drawbacks by considering two pure leptonic processes, muon decay and neutrino scattering.

Muon Decay: The muon decay process $\mu \rightarrow e + \bar{\nu}_e + \nu_\mu$ is illustrated in Fig. 9.

Fig. 9 Muon decay amplitude in the four-Fermi theory.

The four-Fermi amplitude induced by Eq. (3.5) for this process is given (to first order in G_F) by the momentum space amplitude

$$\mathcal{M} = -i \frac{4G_F}{\sqrt{2}} \bar{u}_\mu(q_2) \gamma^\alpha \frac{1 - \gamma_5}{2} u_\mu(p) \bar{u}_e(l) \gamma_\alpha \frac{1 - \gamma_5}{2} v_e(q) . \quad (3.6)$$

To obtain the decay rate, one squares this amplitude, sums over final state polarizations, averages over the initial muon polarizations and integrates over all final state momentum configurations consistent with energy-momentum conservation (phase space).³⁶ In the muon's rest frame, this procedure gives for the decay rate $\Gamma(\mu \rightarrow e \bar{\nu}_e \nu_\mu)$

$$\Gamma(\mu \rightarrow e \bar{\nu}_e \nu_\mu) = \frac{(2\pi)^4}{2m_\mu} \int \frac{d^3 \vec{l}}{2l_0 (2\pi)^3} \frac{d^3 \vec{q}_1}{2q_{10} (2\pi)^3} \frac{d^3 \vec{q}_2}{2q_{20} (2\pi)^3} \delta^4(p - l - q_1 - q_2) \frac{1}{2} \sum_{\text{Pol.}} |\mathcal{M}|^2 . \quad (3.7)$$

Using the spinor matrix relationship

$$\sum_{\text{Pol.}} u^j(k) \bar{u}_i(k) = (k + m) \delta_{ij} , \quad i, j = 1, 2, 3, 4 \quad (3.8)$$

one can easily show

$$\frac{1}{2} \sum_{\text{Pol.}} |\mathcal{M}|^2 = 4G_F^2 \text{Tr}(\not{q}_2 \gamma^\alpha \not{p} \gamma^\beta \not{q}_1 \gamma_\beta \frac{1 - \gamma_5}{2}) \quad (3.9a)$$

$$= 64G_F^2 p \cdot q_1 \, l \cdot q_2 . \quad (3.9b)$$

Then employing the integration identity³⁶

$$\int \frac{d^3 \vec{q}_1}{q_{10}} \frac{d^3 \vec{q}_2}{q_{20}} \delta^4(Q - q_1 - q_2) q_{1\alpha} q_{2\beta} = \frac{\pi}{6} (Q^2 g_{\alpha\beta} + 2Q_\alpha Q_\beta) \quad (3.10)$$

the decay rate becomes (with $l_0 = E$)

$$\Gamma(\mu \rightarrow e \bar{\nu}_e \nu_\mu) = \frac{G_F^2}{12\pi^3} \frac{(m_\mu^2 + m_e^2)/2m_\mu}{\int_{m_e}^E dE (E^2 - m_e^2)^{\frac{1}{2}}} \{ 3E(m_\mu^2 + m_e^2) - 4m_\mu E^2 - 2m_\mu m_e^2 \} \quad (3.11a)$$

or

$$\Gamma(\mu \rightarrow e \bar{\nu}_e \nu_\mu) = \frac{G_F^2}{192\pi^3} m_\mu^5 F(m_e^2/m_\mu^2) \quad (3.11b)$$

where

$$F(x) = 1 - 8x + 8x^3 - x^4 - 12x^2 \ln x \xrightarrow{x \rightarrow 0} 1 - 8x. \quad (3.11c)$$

I have left in the m_e dependence because the mass dependence in $F(x)$ may be interesting for other applications. For example, in the case of tau decay $\tau \rightarrow e^- + \bar{\nu}_e + \nu_\tau$, neglecting m_e but leaving in the dependence on m_{ν_τ} one finds

$$\Gamma(\tau \rightarrow e \bar{\nu}_e \nu_\tau) = \frac{G_F^2}{192\pi^3} m_\tau^5 F(m_{\nu_\tau}^2/m_\tau^2). \quad (3.12)$$

Photonic radiative corrections to muon decay in the local four Fermi theory were computed by Kinoshita and Sirlin³⁷ and Berman.³⁸ They are finite and modify the decay rate and lifetime formula to

$$\Gamma = \frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3} \left(1 - \frac{8m_e^2}{m_\mu^2}\right) \left(1 + \frac{\alpha}{2\pi} \left(\frac{25}{4} - \pi^2\right)\right). \quad (3.13)$$

This formula is still used to define the Fermi constant G_F for a given τ_μ . The value of G_F in Eq. (3.2) follows from such a procedure.³⁹

Neutrino Scattering: The scattering process $\nu_\mu + e \rightarrow \nu_e + \mu$ (sometimes called inverse muon decay) also follows from the amplitude in Fig. 9. The cross-section is given by

$$\sigma = \frac{(2\pi)^4}{4p_e \cdot p_{\nu_\mu}} \int \frac{d^3 p_\mu}{(2\pi)^3 2E_\mu} \frac{d^3 p_{\nu_e}}{(2\pi)^3 2E_{\nu_e}} \delta^4(p_e + p_{\nu_\mu} - p_\mu - p_{\nu_e}) \frac{1}{4} \sum_{\text{Pol.}} |\mathcal{M}|^2 \quad (3.14)$$

where $1/4p_e \cdot p_{\nu_\mu}$ is a flux density factor. Neglecting m_e and integrating over phase space one finds

$$\sigma(\nu_\mu + e \rightarrow \nu_e + \mu) = \frac{G_F^2}{\pi} \frac{(s - m_\mu^2)^2}{s}, \quad s = (p_e + p_{\nu_\mu})^2 \geq m_\mu^2. \quad (3.15)$$

In the limit $s \gg m_\mu^2$, this cross-section grows like s

$$\sigma \approx \frac{G_F^2}{\pi} s, \quad \text{large } s \quad (3.16)$$

and thus violates unitarity (conservation of probability). The unitarity limit is exceeded for³⁶

$$\sqrt{s} > (\sqrt{2}\pi/G_F)^{1/2} \approx 600 \text{ GeV}. \quad (3.17)$$

This fundamental shortcoming of the local Fermi theory implies that new physics must manifest itself before $\sqrt{s} = 600$ GeV and restore unitarity. Of course, we now know that new physics in the form of the $W^\pm$ boson does enter at $m_W \approx 83$ GeV.

Intermediate Vector Boson: The idea that weak interactions might be mediated by massive quanta, analogous to the way massless photons mediate electromagnetic interactions, had its origin in the 1935 paper by Yukawa.⁴⁰ Influenced by Fermi's theory, Yukawa speculated that the pion might be the mediator of both strong nuclear binding forces and weak interactions. The pion was found⁴¹ and it is true that it plays an important role in nuclear binding; however, we now know that it is not the mediator of the weak force. The V-A nature of weak interactions (established in 1958) suggests that a spin-1 boson must mediate those forces (the pion has spin 0). Nevertheless, Yukawa deserves enormous credit for originating the idea of massive bosons as mediators of short range forces.

The first models which described charged current weak interactions as due to massive spin-1 $W^\pm$ boson were put forward by Schwinger⁴² and Lee and Yang.⁴³ In their schemes, the lowest order weak force resulted from the exchange of a virtual (off-mass-shell, i.e. $p^2 \neq m_W^2$) $W^\pm$ boson (see Fig. 10).

Fig. 10 Lowest order muon decay amplitude in the intermediate vector boson theory.

The $W^\pm$ propagator was to be of the form

$$g_{\mu\nu} - \frac{p_\mu p_\nu}{m_W^2} - i \frac{p^2 - m_W^2}{p^2 - m_W^2} \quad (3.18a)$$

and its couplings to fermion pairs (such as ν_e -e) given by

$$-i \frac{g}{\sqrt{2}} \gamma_\alpha \frac{1 - \gamma_5}{2} . \quad (3.18b)$$

(Compare with the photon's propagator and couplings in Fig. 1.)

Using these rules, the amplitude for muon decay is given by (neglecting m_μ/m_W terms)

$$\mathcal{M} = \frac{ig^2}{2(p^2 - m_W^2)} \bar{u}_\nu \gamma^\alpha \frac{1 - \gamma_5}{2} u_\mu \bar{e} \gamma_\alpha \frac{1 - \gamma_5}{2} v_\nu. \quad (3.19)$$

Comparing with Eq. (3.6), we see that for $p^2 \ll m_W^2$ contact with Fermi's local theory is accomplished if one makes the identification

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}. \quad (3.20)$$

But how large is m_W ? In this theory g is not fixed; so m_W is not determined by Eq. (3.20). (The effect of m_W on $\Gamma(\mu \rightarrow e \nu_e \nu_\mu)$ is merely to multiply Eq. (3.12) by $1 + 3m_\mu^2/5m_W^2$.⁴³)

For large momentum transfers $> m_W$, the effect of the propagator is significant. In the case of the $\nu_\mu + e \rightarrow \nu_e + \mu$ scattering, the modified amplitude in Eq. (3.19) leads to (for $s \gg m_W^2$)

$$\sigma(\nu_\mu + e \rightarrow \nu_e + \mu) = \frac{G_F^2 m_W^2 s}{\pi(s + m_W^2)} \quad (3.21)$$

We see that for $s \gg m_W^2$, the cross-section becomes constant and unitarity is preserved as long as $m_W \leq 600$ GeV.

The idea of a massive intermediate vector boson as the mediator of the weak force represented an important conceptual advancement. However, by itself, it did not solve most of the problems associated with a weak interaction quantum field theory. Some scattering amplitudes such as $\nu_e + \nu_e \rightarrow W^+ + W^-$ continued to violate unitarity. In addition, higher-order loop corrections were divergent and those divergences could not be removed by renormalization (as is done in QED). What remained to be done was to find a renormalizable theory which incorporated the intermediate vector boson concept. That was accomplished by Weinberg and Salam.⁴⁴ Employing the formalism of Yang and Mills⁴⁵ for self-interacting spin 1 gauge fields, the $SU(2)_L \times U(1)$ symmetry introduced by Glashow⁴⁶ and the Higgs⁴⁷ mechanism for spontaneous symmetry breakdown, they constructed a unified theory of weak and electromagnetic interactions that has been shown to be both unitary and renormalizable.⁴⁸ The Weinberg-Salam model of electroweak interactions is the subject of Section IV.

IV ELECTROWEAK UNIFICATION

In Section II we saw how the principle of local $U(1)$ gauge invariance could be used to motivate the theory of quantum electrodynamics. The extension of that idea to compact non-abelian symmetry groups such as $SU(N)$ has been around since 1954 when Yang and Mills⁴⁵ constructed a field theory with local $SU(2)$ isospin symmetry. The Yang-Mills formalism has since been used to construct a unified elec-

troweak theory, quantum chromodynamics - a theory of strong interactions (see Section V) and even grand unified theories (see Section VI). So, it seems that non-abelian gauge theories provide a realistic description of the fundamental interactions. I will illustrate the concept of non-abelian gauge invariance via the $SU(2)_L \times U(1)$ Weinberg-Salam model⁴⁴ of electroweak interactions.

A. Weinberg-Salam Model

The Weinberg-Salam model⁴⁴ of electroweak interactions is based on the symmetry group $SU(2)_L \times U(1)$ where $SU(2)_L$ is called weak isospin and $U(1)$ is the weak hypercharge symmetry.⁴⁹ $SU(2)_L$ has three non-commuting (non-abelian) generators T^a , $a = 1, 2, 3$ while the $U(1)$ abelian group has a single generator Y . In terms of these quantities, the electric charge operator Q is given by

$$Q = T_3 + Y/2. \quad (4.1)$$

To accommodate the observed maximal parity violation (V-A) of charged current weak interactions (see Section III), left-handed chiral components, $\psi_L = \frac{1}{2}(1 - \gamma_5)\psi$, of fermion spinor fields are assigned to isodoublet representations of $SU(2)_L$ while right-handed components, $\psi_R = \frac{1}{2}(1 + \gamma_5)\psi$, are incorporated as singlets. Because of this left-right asymmetry, i.e. the non-trivial transformation properties of left-handed fermion field components, the weak isospin group is labeled with a subscript L .

The fundamental quarks and leptons of Table I have the following $SU(2)_L$ multiplet structure:

$$\begin{array}{ccc}
 \underbrace{\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L}_{T = \frac{1}{2}, Y = -1} & \underbrace{e_R, \mu_R, \tau_R}_{T = 0, Y = -2} \\
 \underbrace{\begin{pmatrix} u \\ d' \end{pmatrix}_L \quad \begin{pmatrix} c \\ s' \end{pmatrix}_L \quad \begin{pmatrix} t \\ b' \end{pmatrix}_L}_{T = \frac{1}{2}, Y = 1/3} & \underbrace{u_R, c_R, t_R}_{T = 0, Y = +4/3} \\
 & \underbrace{d_R, s_R, b_R}_{T = 0, Y = -2/3}
 \end{array} \quad (4.2)$$

where $T = \frac{1}{2}$ corresponds to $SU(2)_L$ doublets and $T = 0$ denotes singlets. Each of the quark fields actually comes in 3 distinct colors. (That feature will be discussed in Section V.) The primes on d'_L, s'_L, b'_L are meant to indicate that they are linear combinations of the mass eigenstates d, s, b with the two sets of fields related by the KM matrix⁵⁰

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} c_1 & s_1 c_2 & s_1 s_2 \\ -s_1 c_3 & c_1 c_2 c_3 - s_2 s_3 e^{-i\delta} & c_1 s_2 c_3 + c_2 s_3 e^{-i\delta} \\ -s_1 s_3 & c_1 c_2 s_3 + s_2 c_3 e^{-i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{-i\delta} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

$$\begin{aligned} c_i &\equiv \cos \theta_i \\ s_i &\equiv \sin \theta_i \end{aligned} \quad (4.3)$$

where $\theta_1, \theta_2, \theta_3, \delta$ are mixing parameters determined by experiment.⁵¹ I should note that right-handed neutrinos are optional.⁵² If they are added as $SU(2)_L$ singlets, their quantum numbers are $T = 0, Y = 0$.

The free Lagrangian for fermions is given by

$$\mathcal{L}_{\text{free}} = \overline{\psi}_{e_L} (i\gamma^\lambda \partial_\lambda) \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L + \overline{e}_R i\gamma^\lambda \partial_\lambda e_R + \dots \quad (4.4)$$

where ... denotes the other fermions in (4.2) which are included in a similar manner. This Lagrangian leads to the Dirac equation for massless fermions through its equations of motion. $\mathcal{L}_{\text{free}}$ is invariant under global $SU(2)_L$ transformations

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \rightarrow U(\theta) \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L = \exp(i\theta^a \tau^a / 2) \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \quad (4.5)$$

$$e_R \rightarrow e_R$$

where $\tau^a, a = 1, 2, 3$ are the 2×2 Pauli matrices and θ^a are real numbers that parametrize the $SU(2)_L$ rotation. It also has a global $U(1)$ weak hypercharge symmetry,

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \rightarrow \exp(-i\phi/2) \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$$

$$e_R \rightarrow \exp(-i\phi) e_R \quad (4.6)$$

as well as a global $U(1)_{\text{em}}$ symmetry generated by the charge operator Q ,

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \rightarrow \exp[i\delta(\tau^3/2 - \frac{1}{2}1)] \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$$

$$e_R \rightarrow \exp(-i\delta) e_R. \quad (4.7)$$

Explicit mass terms such as $-m_e \bar{e}e$ would, if added to the Lagrangian, break the $SU(2)_L$ and $U(1)$ symmetries but not $U(1)_{\text{em}}$.

To elevate the global $SU(2)_L \times U(1)$ symmetry to a local gauge invariance in which θ^a and ϕ in (4.5) and (4.6) depend on space-time,

the ordinary derivative in (4.4) is replaced by a covariant derivative, $\partial_\lambda \rightarrow D_\lambda$

$$D_\lambda = \partial_\lambda + igT^a A_\lambda^a(x) + ig' \frac{Y}{2} B_\lambda(x) \quad (4.8)$$

where g and g' are $SU(2)_L$ and $U(1)$ couplings and $A_\lambda^a(x)$, $a = 1, 2, 3$, $B_\lambda(x)$ are Lorentz vector gauge fields analogous to the electromagnetic potential in Section II. The $T^a = \tau^a/2$ for fermion doublets and 0 for singlets. Local gauge invariance requires that the gauge fields must transform as

$$A_\lambda(x) = A_\lambda^a(x) \tau^a/2 + U^{-1}(\theta) A_\lambda(x) U(\theta) + \frac{i}{g} U^{-1}(\theta) \partial_\lambda U(\theta),$$

$$B_\lambda(x) \rightarrow B_\lambda(x) - \frac{1}{g'} \partial_\lambda \phi(x). \quad (4.9)$$

In order that in the limit $g, g' \rightarrow 0$ the gauge fields satisfy a free Klein-Gordon equation (appropriate for spin 1 fields), we must add to the Lagrangian the terms

$$-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \quad (4.10a)$$

where

$$F_{\mu\nu}^a(x) = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g\epsilon^{abc} A_\mu^b A_\nu^c, \quad a = 1, 2, 3 \quad (4.10b)$$

$$B_{\mu\nu}(x) = \partial_\mu B_\nu - \partial_\nu B_\mu \quad (4.10c)$$

($\epsilon^{123} = 1$ represents the antisymmetric $SU(2)$ structure constants.)

The terms in (4.10a) are locally gauge invariant. Adding these pieces to the fermion sector gives

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

$$+ \sum_{\ell=e,\mu,\tau} \left(\bar{\psi}_L \not{\partial} \psi_L \right) \gamma^\lambda \left(i\partial_\lambda - g \frac{\tau^a}{2} A_\lambda^a + \frac{g'}{2} B_\lambda \right) \left(\psi_L \right) + \bar{\psi}_R \gamma^\lambda (i\partial_\lambda + g' B_\lambda) \psi_R$$

$$+ \text{quark terms} . \quad (4.11)$$

This Lagrangian, when quantized, describes massless spin $\frac{1}{2}$ fermions interacting with massless spin 1 gauge bosons. In the real world, of course, quarks, charged leptons, and gauge boson mediators of the weak force must be massive. Therefore, in a realistic theory of electroweak interactions, the $SU(2)_L \times U(1)$ gauge symmetry must be broken down to $U(1)_{em}$, which will continue to support a massless photon. We could explicitly break the symmetry by adding mass terms to the Lagrangian in (4.11); however, they would render the resulting

theory non-renormalizable. Instead, Weinberg and Salam showed how to break the symmetry in the quantum vacuum by the Higgs mechanism.⁴⁷ The resulting electroweak theory is both unitary and renormalizable as explicitly proven by G. 't Hooft.⁴⁸

To induce spontaneous symmetry breaking via the Higgs mechanism,^{44,47} one adds $\mathcal{L}_\phi$ to the Lagrangian in (4.10) where

$$\begin{aligned} \mathcal{L}_\phi = & (\partial_\mu \phi^\dagger - ig \phi^\dagger \frac{\tau^a}{2} A_\mu^a - i \frac{g'}{2} \phi^\dagger B_\mu) (\partial_\mu + ig \frac{\tau^a}{2} A_\mu^a + i \frac{g'}{2} B_\mu) \phi \\ & + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 - \sum_{\ell=e,\mu,\tau} G_\ell \left[\bar{\ell}_R \phi^\dagger \begin{pmatrix} \nu_\ell \\ \ell \end{pmatrix}_L + \underbrace{\bar{\nu}_{\ell L} \bar{\ell}_L}_{\text{}} \phi \ell_R \right] \\ & + \text{quark terms,} \end{aligned} \quad (4.12a)$$

$$\phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}. \quad (4.12b)$$

The scalar field ϕ is an $SU(2)_L$ doublet with $Y = +1$. It couples to gauge fields via the covariant derivative and to fermions through the Yukawa terms (with G_ℓ an arbitrary constant) in (4.12a). The scalar potential

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \quad (4.12c)$$

in (4.12a) has been included as a source of symmetry breaking. The classical minima of this potential occur at

$$|\phi|^2 = v^2/2 = \mu^2/2\lambda \quad (\text{minima}) \quad (4.13)$$

rather than $\phi = 0$. If one quantizes about any of these minima, the corresponding vacuum $|0\rangle$ of the theory is not invariant under local $SU(2)_L \times U(1)$ transformations, even though the Lagrangian is. (It is, however, still invariant under local $U(1)_{em}$ gauge transformations generated by the electric charge Q .) This so-called spontaneous symmetry breakdown from $SU(2)_L \times U(1)$ to $U(1)_{em}$ gives rise to 3 massless Goldstone bosons (3 of the 4 components of ϕ) which become longitudinal components of 3 gauge fields and thereby endow them with mass. The fourth neutral scalar component of ϕ remains as a physical spin-0 scalar Higgs field which we will call H . Through its couplings to fermions in (4.11a), the ϕ field also induces fermion masses. It is quite remarkable that in this manner all known masses (and quark mixings) can be generated with a single scalar doublet.

To obtain the physical spectrum of this theory, we can make a non-linear transformation to the unitary gauge using

$$U(\theta) = \exp[-i\theta^a(x)\tau^a/2v] \quad (4.14)$$

where the $\theta^a(x)$ are to be related to the $\phi_i(x)$ in (4.11b) such that

$$\phi \rightarrow U\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ H + v \end{pmatrix} \quad (4.15)$$

with H the neutral physical Higgs scalar. This transformation when applied to (4.12a) induces bilinear mass terms for gauge bosons and fermions. Introducing the combinations

$$\begin{aligned} W_\mu^\pm &= (A_\mu^1 \mp iA_\mu^2)/\sqrt{2} & (W^\pm \text{ bosons}) \\ Z_\mu &= A_\mu^3 \cos\theta_W - B_\mu \sin\theta_W & (Z \text{ boson}) \\ A_\mu &= B_\mu \cos\theta_W + A_\mu^3 \sin\theta_W & (\text{photon}) \\ \tan\theta_W &= g'/g & (\theta_W = \text{weak mixing angle}) \end{aligned} \quad (4.16)$$

we can read off the vector boson masses

$$m_W = gv/2 \quad (4.17a)$$

$$m_Z = m_W/\cos\theta_W \quad (4.17b)$$

$$m_\gamma = 0. \quad (4.17c)$$

Similarly, the Higgs mass is found to be

$$m_H = \sqrt{2} \mu = 2/\sqrt{2\lambda} m_W/g. \quad (4.17d)$$

The photon remains exactly massless due to the local $U(1)_{em}$ symmetry, while the W and Z acquire masses. Through these ϕ couplings fermions also acquire masses, e.g.

$$m_\ell = G_\ell v/\sqrt{2} \quad (4.18)$$

and the quark mixing in (4.3) can be induced.

The above $SU(2)_L \times U(1)$ theory unifies weak and electromagnetic interactions in an elegant manner. Parity violation is put in from the start as an intrinsic left-right asymmetry, and the intermediate vector bosons W and Z acquire masses while the photon remains massless. From the photon-electron coupling we find

$$e = g \sin\theta_W \quad (4.19)$$

and equating $g^2/8m_W^2$ to $G_F/\sqrt{2}$ via the muon decay rate then implies

$$m_W = \left(\frac{\pi\alpha}{\sqrt{2}G_F \sin^2\theta_W} \right)^{1/2} > 37.3 \text{ GeV}, \quad (4.20)$$

a very large W mass. In addition, this theory correctly predicted the existence of a neutral Z boson and its associated low energy phenomenology.

B. Feynman Rules

The Weinberg-Salam model is unitary and renormalizable. That means that transition amplitudes can be perturbatively computed and

short-distance ultraviolet infinities can be systematically absorbed into the bare parameters (couplings, masses, and fields) of the theory. The Feynman rules for carrying out such calculations can be obtained from the Lagrangian described above. A derivation of those rules for arbitrary gauges goes beyond the scope of these lectures.⁵³ However, for use in the subsequent discussion I give in Table II the $SU(2)_L \times U(1)$ model's Feynman rules in the so-called unitary gauge. (In other gauges there are additional ghost fields and unphysical scalars.)

Before proceeding with some illustrative examples, I should make a few comments: From the YW vertex, one learns that the magnetic moment of the $W^\pm$ boson is e/m_W rather than $e/2m_W$, the latter value being what one obtains with minimal coupling. So the $W^\pm$ is predicted to have an anomalous magnetic moment of 1. The Higgs scalar coupling to light fermions is suppressed by gm_f/m_W . As we shall later see, this small coupling makes the Higgs scalar difficult to produce and detect. The Z coupling to fermions has vector and axial-vector components; hence it can give rise to parity violating effects in neutral current phenomena.

C. Weak Neutral Currents

The neutral Z boson of the Weinberg-Salam model gives rise to a variety of novel predictions. I will discuss its effect on neutrino scattering and interference asymmetries in $e^+e^- \rightarrow \mu^+\mu^-$.

Neutrino Scattering: In addition to charged current interactions (described in Section 2), neutrinos can interact through neutral current interactions induced by Z boson exchange; see Fig. 11.

Fig. 11 Amplitude for neutrino-fermion scattering via Z exchange.

Employing the Feynman rules in Table II, this amplitude is given by

$$M = \frac{+ig^2}{4\cos^2\theta_W} \frac{1}{q^2 - m_Z^2} \bar{u}_\nu(p_2') \gamma_\alpha \frac{1 - \gamma_5}{2} u_\nu(p_2) \bar{u}_f(p_1') \gamma^\alpha (a_f \gamma_+ + b_f \gamma_-) u_f(p_1) \quad (4.21a)$$

where

Table II Feynman Rules for the Weinberg-Salam model in the unitary gauge (f stands for a generic fermion)

Propagators: $\frac{W}{K} \rightarrow \text{---}$ $iD_{\mu\nu} = i \frac{K_\mu K_\nu / m_W^2 - g_{\mu\nu}}{K^2 - m_W^2}$

$\frac{Z}{K} \rightarrow \text{---}$ $iD_{\mu\nu} = i \frac{K_\mu K_\nu / m_Z^2 - g_{\mu\nu}}{K^2 - m_Z^2}$

$\frac{\gamma}{K} \rightarrow \text{~~~~~}$ $iD_{\mu\nu} = \frac{-ig_{\mu\nu}}{K^2}$

$\frac{H}{K} \rightarrow \text{---}$ $iD = \frac{i}{K^2 - m_H^2}$

$\frac{f}{p} \rightarrow \text{---}$ $iS = \frac{i}{\not{p} - m_f}$

Vertices: $\nu_\ell \rightarrow \ell$ $\frac{-ig}{\sqrt{2}} \gamma_\mu \frac{1 - \gamma_5}{2}$, $\ell = e, \mu, \tau$

$\downarrow W^-$

$q^{+2/3} \rightarrow q^{-1/3}$ $\frac{-ig}{\sqrt{2}} \gamma_\mu \frac{1 - \gamma_5}{2} \times (\text{KM matrix element})$

$\downarrow W^-$

$$\frac{-ig\gamma_\mu}{2\cos\theta_W} (T_{3f}^L - 2Q_f\sin^2\theta_W - T_{3f}^L\gamma_5)$$

$$T_{3f}^L = \pm \frac{1}{2}$$

$$-ieQ_f\gamma_\mu$$

$$\frac{-igm_f}{2m_W}$$

$$igm_W g_{\mu\nu}$$

$$\frac{igm_Z}{\cos\theta_W} g_{\mu\nu}$$

$$-ig^2\cos^2\theta_W(2g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\beta\nu} - g_{\mu\beta}g_{\alpha\nu})$$

$$(\sin^2\theta_W)$$

$$-ig^2 \sin\theta_W \cos\theta_W (2g_{\mu\nu} g_{\alpha\beta} - g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha})$$

$$+ig^2 (2g_{\mu\nu} g_{\alpha\beta} - g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha})$$

$$\frac{ig^2}{2(\cos^2\theta_W)} g_{\mu\nu}$$

$$-ig \cos\theta_W g_{\alpha\beta} (p, p') (\sin\theta_W)$$

$$v_{\alpha\beta\mu}(p, p') = (p + p')_\mu g_{\alpha\beta} - g_{\alpha\mu} (2p - p')_\beta - g_{\beta\mu} (2p' - p)_\alpha$$

$$\gamma_\pm = \frac{1 \pm \gamma_5}{2},$$

$$a_f = -2Q_f \sin^2\theta_W, \quad b_f = 2T_3^L - 2Q_f \sin^2\theta_W. \quad (4.21b)$$

I have written the amplitude for a generic fermion f , so that the resulting formulas can be applied to a variety of neutral current scattering processes. Squaring this amplitude, averaging over the initial f polarization (the neutrino is purely left-handed), and summing over final state polarizations, one finds

$$\frac{1}{2} \sum_{\text{Pol}} |M|^2 = \frac{g^4}{2 \cos^4 \theta_W} \frac{1}{(q^2 - m_Z^2)^2} (-m_f^2 a_f b_f p_2 \cdot p_2' + a_f^2 p_1 \cdot p_2' p_1' \cdot p_2 + b_f^2 p_1 \cdot p_2 p_1' \cdot p_2') . \quad (4.22)$$

The differential cross-section is given by

$$d\sigma = \frac{(2\pi)^4}{4 p_1 \cdot p_2} \frac{d^3 p_1'}{(2\pi)^3 2E_1'} \frac{d^3 p_2'}{(2\pi)^3 2E_2'} \delta^4(p_1' + p_2' - p_1 - p_2) \frac{1}{2} \sum_{\text{Pol}} |M|^2 . \quad (4.23)$$

The integration over final states is easiest in the center-of-momentum frame where

$$\vec{p}_1 = -\vec{p}_2, \quad \vec{p}_1' = -\vec{p}_2', \quad E_1'^2 = E_1'^2 = E_2'^2 - m_f^2 = E_2'^2 - m_f^2 .$$

Carrying out the momentum integrations, one finds in terms of $x = \cos \theta$ where θ is the angle between $\vec{p}_1$ and $\vec{p}_1'$ (for $E_2 \gg m_f$)

$$\frac{d\sigma}{dx} = \frac{1}{32\pi} \frac{g^4}{2 \cos^4 \theta_W} \frac{1}{(2E_2^2(1-x) - m_Z^2)^2} \left(\frac{1}{4} a_f^2 E_2^2 (1+x)^2 + b_f^2 E_2^2 \right) . \quad (4.24)$$

For $E_2 \ll m_Z$, one finds (using $G_F/\sqrt{2} = g^2/8m_Z^2 \cos^2 \theta_W$)

$$\sigma(\nu f + \nu f) = \frac{G_F^2}{\pi} E_2^2 \left(\frac{1}{3} a_f^2 + b_f^2 \right) \quad (\text{c.m. frame}) \quad (4.25a)$$

or in terms of the lab frame energy E_2^{lab}

$$\sigma(\nu f + \nu f) = \frac{G_F^2}{2\pi} m_f E_\nu^{\text{lab}} \left(\frac{1}{3} a_f^2 + b_f^2 \right) \quad (\text{lab frame}) . \quad (4.25b)$$

These general formulas illustrate that the L-R cross-section is 1/3 of the L-L cross-section. This feature allows us to immediately write down the cross-section for $\bar{\nu} f$, $\nu \bar{f}$, and $\bar{\nu} \bar{f}$ scattering. In the lab system these are

$$\sigma(\bar{\nu} f + \bar{\nu} f) = \frac{G_F^2}{2\pi} m_f E_\nu^{\text{lab}} \left(a_f^2 + \frac{1}{3} b_f^2 \right) \quad (4.26a)$$

$$\sigma(\nu \bar{f} + \nu \bar{f}) = \frac{G_F^2}{2\pi} m_f E_\nu^{\text{lab}} \left(a_f^2 + \frac{1}{3} b_f^2 \right) \quad (4.26b)$$

$$\sigma(\bar{\nu} \bar{f} + \bar{\nu} \bar{f}) = \sigma(\nu f + \nu f) . \quad (4.26c)$$

Applying these formulas to $\nu_\mu e$ and $\bar{\nu}_\mu e$ scattering, using

$$a_e = 2\sin^2\theta_W, \quad b_e = -1 + 2\sin^2\theta_W,$$

gives

$$\sigma(\nu_\mu e + \nu_\mu e) = \frac{G_F^2 m_e E_\nu}{2\pi} (1 - 4\sin^2\theta_W + \frac{16}{3} \sin^4\theta_W) \quad (4.27a)$$

$$\sigma(\bar{\nu}_\mu e + \bar{\nu}_\mu e) = \frac{G_F^2 m_e E_\nu}{2\pi} (\frac{1}{3} - \frac{4}{3} \sin^2\theta_W + \frac{16}{3} \sin^4\theta_W) . \quad (4.27b)$$

The ratio

$$\frac{\sigma(\bar{\nu}_\mu e + \bar{\nu}_\mu e)}{\sigma(\nu_\mu e + \nu_\mu e)} = \frac{1 - 4\sin^2\theta_W + 16\sin^4\theta_W}{3 - 12\sin^2\theta_W + 16\sin^4\theta_W} \quad (4.28)$$

is presently being measured with high precision at BNL⁵⁴ in order to determine $\sin^2\theta_W$ in a purely leptonic reaction. Such an experiment is very difficult because the cross-sections are so small, for $E_\nu \approx 1$ GeV, $\sigma \approx 10^{-42}$ cm².

At present the best determination of $\sin^2\theta_W$ in neutrino scattering experiments comes from measurements⁵⁵ of R_ν

$$R_\nu \equiv \frac{\sigma(\nu_\mu + N \rightarrow \nu_\mu + X)}{\sigma(\nu_\mu + N \rightarrow \mu + X)} , \quad (4.29)$$

the ratio of deep-inelastic neutrino neutral and charged current cross-sections. For isoscalar targets (number of u quarks = number of d quarks) this ratio is predicted to be

$$R_\nu = \frac{1}{2} - \sin^2\theta_W + \frac{20}{27} \sin^4\theta_W . \quad (4.30)$$

After correcting for hadronic effects and higher-order quantum corrections, one finds by averaging the experimental data⁵⁵

$$\sin^2\theta_W = 0.217 \pm 0.014 \quad (4.31)$$

where the renormalized weak mixing angle is defined by⁵⁶

$$\sin^2\theta_W = 1 - m_W^2/m_Z^2 \quad (4.32)$$

$e^+e^- \rightarrow \mu^+\mu^-$ Asymmetry: Interference effects between electromagnetic and Z exchange amplitudes lead to interesting phenomena such as atomic parity violation,⁵⁷ scattering asymmetries in polarized electron scattering,⁵⁸ and charge asymmetries⁵⁹ in $e^+e^- \rightarrow \mu^+\mu^-$. All three of these effects have been observed at the level predicted by the Weinberg-Salam model with $\sin^2\theta_W \approx 0.22$. I will discuss only

$e^+e^- \rightarrow \mu^+\mu^-$ because it is purely leptonic and thus not complicated by hadronic effects.

In leading order, the reaction $e^+e^- \rightarrow \mu^+\mu^-$ can occur through the amplitudes illustrated in Fig. 12. Squaring the sum of these amplitudes and carrying out the phase space integrations, one finds [for $s = (p_{e^+} + p_{e^-})^2 \gg m_\mu^2$]

Fig. 12 Tree diagram amplitudes for $e^+e^- \rightarrow \mu^+\mu^-$.

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s} [R_{\mu\mu}(1 + \cos^2\theta) + B\cos\theta] \quad (4.33)$$

where θ is the angle between $\vec{p}_{e^-}$ and $\vec{p}_{\mu^-}$ and

$$R_{\mu\mu} = 1 - \frac{1}{2}(1 - 4\sin^2\theta_W)^2\chi + \frac{1}{16}(2 - 8\sin^2\theta_W + 16\sin^4\theta_W)^2\chi^2 \quad (4.34a)$$

$$B = -\chi + \frac{1}{2}(1 - 4\sin^2\theta_W)^2\chi^2 \quad (4.34b)$$

$$\chi = \frac{G_F m_Z^2}{2\sqrt{2}\pi\alpha} \frac{s}{m_Z^2 - s} \quad (4.34c)$$

In the integrated cross-section

$$\sigma = \frac{4}{3} \frac{\pi\alpha^2}{s} R_{\mu\mu} \quad (4.35)$$

B goes away. Then the weak neutral current effect is limited to deviations in $R_{\mu\mu}$ from 1. [Of course, there are higher-order QED corrections to $R_{\mu\mu}$ that I am ignoring; they have been computed. Their main effect is to change α to $\alpha(\sqrt{s})$ in (4.35), see Section II.] Unfortunately, for $\sin^2\theta_W = 0.22$ one finds

$$R_{\mu\mu} = 1 - 0.007\chi + 0.06\chi^2 \quad (4.36)$$

which does not deviate very much from 1 at presently measured energies. For example, at $\sqrt{s} = 34.6$ GeV the MARK J collaboration finds⁵⁹

$$R_{\mu\mu}^{\text{exp}} = 0.98 \pm 0.016 \pm 0.04 \quad (4.37a)$$

whereas (4.36) predicts

$$R_{\mu\mu}^{\text{th}}(34.6 \text{ GeV}) = 1.002 . \quad (4.37b)$$

They are in good agreement.

A more easily detected effect is the charge asymmetry due to B in (4.33). Defining the forward-background charge asymmetry $A_{\mu\mu}$

$$A_{\mu\mu} = \frac{N(\theta < 90^\circ) - N(\theta > 90^\circ)}{N(\theta < 90^\circ) + N(\theta > 90^\circ)} \quad (4.38a)$$

one finds from (4.33)

$$A_{\mu\mu} = \frac{3}{8} \frac{B}{R_{\mu\mu}} \quad (4.38b)$$

$$A_{\mu\mu} = -\frac{3}{8} \chi = \frac{-3G_F m_Z^2}{16\sqrt{2}\pi\alpha} \frac{s}{m_Z^2 - s} . \quad (4.38c)$$

The form of (4.38c) is useful because G_F absorbs potentially large radiative corrections.⁶⁰ At $\sqrt{s} = 34.6$ GeV, MARK J finds⁵⁹

$$A_{\mu\mu}(34.6 \text{ GeV}) = -11.7 \pm 1.7 \pm 1\% \quad (4.39a)$$

while (4.38c) yields

$$A_{\mu\mu}(34.6 \text{ GeV}) = -9.3\% . \quad (4.39b)$$

The agreement is quite good.

D. $W^\pm$ and Z Masses

With the discovery of the $W^\pm$ and Z bosons now completed, it becomes important to further scrutinize the standard $SU(2)_L \times U(1)$ model by precise measurements of m_W and m_Z .⁶¹ Those mass values already provide the best determination of $\sin^2\theta_W$ and will eventually test the $SU(2)_L \times U(1)$ model at the level of its quantum loop corrections, just as $g - 2$ measurements test QED.

At the tree level one finds from (4.20) and (3.2)

$$m_W = \left(\frac{\pi\alpha}{\sqrt{2}G_F \sin^2\theta_W} \right)^{1/2} = \frac{37.2804 \pm 0.0003 \text{ GeV}}{\sin\theta_W} . \quad (4.40)$$

If we merely used the value $\sin^2\theta_W = 0.217 \pm 0.014$ extracted from neutral current experiments, then we would anticipate $m_W = 80.0 \pm 2.5$ GeV.

However, the meaning of (4.40) is ambiguous until we give a definition of $\sin^2\theta_W$. In addition, the radiative corrections to this formula are quite significant.⁶²

Employing the definition

$$\sin^2\theta_W = 1 - m_W^2/m_Z^2 \quad (4.41)$$

and computing the one-loop quantum corrections to the γ , $W^\pm$ and Z self-energies as well as the complete $O(\alpha)$ corrections to muon decay, one finds⁶¹

$$m_W = \left(\frac{\pi\alpha}{\sqrt{2}G_F \sin^2\theta_W (1 - \Delta r)} \right)^{1/2} \quad (4.42a)$$

$$m_Z = m_W / \cos\theta_W \quad (4.42b)$$

$$\Delta r = 0.0696 \pm 0.0020 \quad (4.42c)$$

where Δr represents the entire $O(\alpha)$ quantum corrections. Numerically one therefore expects

$$m_W = \frac{38.65 \pm 0.04 \text{ GeV}}{\sin\theta_W} \quad (4.43a)$$

$$m_Z = \frac{77.30 \pm 0.08 \text{ GeV}}{\sin 2\theta_W} \quad (4.43b)$$

Using these formulas and the average experimental values⁶³⁻⁶⁵

$$m_W = 82.2 \pm 1.8 \text{ GeV} \quad (4.44a)$$

$$m_Z = 93.2 \pm 1.5 \text{ GeV} \quad (4.44b)$$

one finds

$$\sin^2\theta_W = 0.221 \pm 0.007 \quad (\text{from } W^\pm + Z \text{ masses}) \quad (4.45)$$

which is in excellent agreement with the result from deep-inelastic neutrino scattering in (4.31). Together these distinct measurements give⁶⁵

$$\sin^2\theta_W = 0.220 \pm 0.006 \quad (\text{world average}) \quad (4.46)$$

As we shall see in Section VI, it is important to determine $\sin^2\theta_W$ with the highest precision possible. In that regard, I note that m_W and m_Z are expected to be measured to within ± 0.1 GeV at Tevatron and LEP. Such measurements will determine $\sin^2\theta_W$ to within ± 0.0005 .

Precise measurements of both m_W and m_Z will test the $SU(2)_L \times U(1)$ at the level of its quantum corrections. For example, eliminating θ_W between (4.42a) and (4.42b) one finds⁶¹

$$\Delta r = 1 - \frac{\pi\alpha}{\sqrt{2} G_F m_W^2 (1 - \frac{m_W^2}{m_Z^2})} = 1 - \frac{(37.2804 \pm 0.0003 \text{ GeV})^2}{m_W^2 (1 - \frac{m_W^2}{m_Z^2})} \quad (4.47)$$

where Δr represents the $O(\alpha)$ radiative corrections. Hence, measurements of both m_W and m_Z to ± 0.1 GeV will determine Δr to within ± 0.013 . That is to be compared with the $SU(2)_L \times U(1)$ prediction in (4.42c). (Present mass values in (4.44) give $\Delta r = 0.074 \pm 0.16$ which is consistent with, but not yet a real test of, the prediction $\Delta r = 0.0696 \pm 0.0020$.) Deviations from the $SU(2)_L \times U(1)$ prediction would signal additional physics beyond the standard model.

E. $W^\pm$ and Z Decays

$W^\pm$ and Z bosons have been produced in proton-antiproton collisions at CERN via quark-antiquark annihilation. At present the UA1 and UA2 collaborations^{63,64,66} have each observed on the order of 50 $W^\pm$ bosons through the decay $W \rightarrow e\nu$ and four Z boson decays $Z \rightarrow e^+e^-$. In addition, UA1 has detected $W \rightarrow \mu\nu$ and $Z \rightarrow \mu^+\mu^-$ events. The measured production cross-sections and decay widths are consistent with the $SU(2)_L \times U(1)$ model's predictions; but the experimental uncertainties are still large. Eventually, one can expect these quantities to be more precisely determined and thus provide an important test of the electroweak model. I will outline a computation of the predicted decay widths.

Consider the lowest-order decay of a generic spin-1 vector boson denoted by B ($B = W^\pm$ or Z) into a pair of spin- $\frac{1}{2}$ fermions $f_1 + \bar{f}_2$ (see Fig. 13).

Fig. 13 Coupling between a vector boson $B = W^\pm$ or Z and $f_1 \bar{f}_2$ pair.

The amplitude for the decay $B \rightarrow f_1 + \bar{f}_2$ is given by⁶⁷

$$M = -ig \epsilon^\lambda(p) \bar{u}_{f_1}(k_1) \gamma_\lambda \frac{a + b\gamma_5}{2} v_{f_2}(k_2) \quad (4.48)$$

where $\epsilon^\lambda(p)$ is the polarization vector of the B. To obtain the decay rate $\Gamma(B \rightarrow f_1 \bar{f}_2)$, we work in the rest frame of the B, sum over final state polarization, average over B polarizations and integrate over phase space, i.e.

$$\Gamma(B \rightarrow f_1 \bar{f}_2) = \frac{(2\pi)^4}{2m_B} \iint \frac{d^3 k_2}{2\omega_2 (2\pi)^3} \frac{d^3 k_1}{2\omega_1 (2\pi)^3} \delta^4(p - k_1 - k_2) \frac{1}{3} \sum_{\text{pol}} |M|^2$$

(4.49)

where

$$\frac{1}{3} \sum_{\text{pol}} |M|^2$$

$$= \frac{g^2}{12} \left(\frac{p \cdot p}{2m_B^2} - g^{\lambda\alpha} \right) \text{Tr}[(k_1 + m_1) \gamma_\lambda (a + b \gamma_5) (k_2 + m_2) \gamma_\alpha (a + b \gamma_5)] .$$

(4.50)

Carrying out the trace and phase space integrations gives⁶⁷

$$\Gamma(B \rightarrow f_1 \bar{f}_2) = \frac{g^2 m_B^2}{48\pi} \left[\left(1 - \frac{m_1^2}{2m_B^2} - \frac{m_2^2}{2m_B^2}\right)^2 - 4 \frac{m_1^2 m_2^2}{m_B^4} \right]^{\frac{1}{2}} \times$$

$$\left[(a^2 + b^2) \left(1 - \frac{m_1^2}{2m_B^2} - \frac{m_2^2}{2m_B^2} - \frac{1}{2} \left(\frac{m_1^2}{m_B^2} - \frac{m_2^2}{m_B^2}\right)^2\right) + 3(a^2 - b^2) \frac{m_1 m_2}{m_B^2} \right] .$$

(4.51)

In the limit of zero fermion masses this becomes

$$\Gamma(B \rightarrow f_1 \bar{f}_2) = \frac{g^2 m_B^2}{48\pi} (a^2 + b^2) .$$

(4.52)

For the W boson, $a = -b = 1/\sqrt{2}$ for leptons and quarks (up to mixing angle effects); however, in the case of quarks, the above formulas must be multiplied by a color factor of 3. Summing over all three generations of fermions, one finds for $m_t = 36$ GeV

$$\Gamma(W \rightarrow \text{all}) \approx 2.8 \text{ GeV} .$$

(4.53)

For Z boson decays $a = (T_3^f - 2Q \sin^2 \theta_W)/\cos \theta_W$, $b = -T_3^f/\cos \theta_W$. Summing over the 3 generations of fermions gives

$$\Gamma(Z \rightarrow \text{all}) \approx 2.8 \text{ GeV} .$$

(4.54)

It is an interesting coincidence that the total widths are essentially the same.

The predicted observable leptonic branching ratios are

$$\frac{\Gamma(W \rightarrow e\nu)}{\Gamma(W \rightarrow \text{all})} \approx \frac{1}{12}$$

(4.55)

$$\frac{\Gamma(Z \rightarrow e^+ e^-)}{\Gamma(Z \rightarrow \text{all})} \approx 0.03 .$$

(4.56)

Deviations from the above predictions could signal new physics such as 4th generation mixing, larger m_t , additional neutrinos, etc. For example, the number of neutrino species, N_ν , is given by^{67,68}

$$N_\nu = 3 + \frac{\Gamma(Z \rightarrow \text{all}) - 2.8 \text{ GeV}}{0.178 \text{ GeV}}. \quad (4.57)$$

The UA2 collaboration⁶⁴ has made a good determination of $\Gamma(W \rightarrow \text{all})/\Gamma(Z \rightarrow \text{all})$ which gives $N_\nu < 6$.

F. Higgs-Scalar Production and Decay

We have seen that the $SU(2)_L \times U(1)$ model predicts the existence of a neutral spin-0 Higgs scalar denoted by H. It is a necessary remnant of the Higgs mechanism used to provide masses for the $W^\pm$, Z, and fermions. Discovery of the H is necessary for final confirmation of the Weinberg-Salam model.

How will the Higgs scalar be produced and detected? The answer depends on its mass, m_H , which is unfortunately a free parameter. A relatively light $H \lesssim 60 \text{ GeV}$ should be detectable via the decays⁶⁹ $Z \rightarrow H\ell^+\ell^-$ ($\ell = e$ or μ), $Z \rightarrow H\gamma$, or toponium $\rightarrow H\gamma$. Somewhat higher masses (up to $\approx 100 \text{ GeV}$) may be observable at LEP II through the reaction $e^+e^- \rightarrow ZH$.⁷⁰ In the case of a very heavy Higgs $> 2m_W \approx 166 \text{ GeV}$, it can best be produced at high-energy hadron-hadron colliders via gluon-gluon fusion.⁷¹ For large enough m_H , the decays $H \rightarrow W^+W^-$ or ZZ decays dominate, and one can then look for a leptonic decay of one of the vector bosons.

The cross-section for $p\bar{p} \rightarrow H + X$ is given by⁷²

$$\sigma(p\bar{p} \rightarrow H) = \frac{\pi}{8m_H^3} \Gamma(H \rightarrow gg) \int_{\frac{2m_H}{\sqrt{s}}}^{\frac{\sqrt{s}}{2m_H}} dy \tau F_G(\sqrt{y}) F_G(\sqrt{y} e^{-y}) \quad (4.58)$$

where $\tau = \frac{2}{m_H^2 s}$, F_G is the proton's gluon distribution function, and $\Gamma(H \rightarrow gg)$ is the 2-gluon decay width of the H. Unfortunately, $\Gamma(H \rightarrow gg)$ is predicted to be very small

$$\Gamma(H \rightarrow gg) = \frac{g^2 m_H^3}{288\pi m_W^2} \left(\frac{\alpha_3(m_H/2)}{\pi} \right)^2 |I|^2 \quad (4.59)$$

where $\alpha_3(m_H/2)$ is the running QCD coupling (≈ 0.1 for $m_H \approx 2m_W$) and⁷³

$$I = 3 \sum_{i=u,d,\dots} (2\lambda_i + \lambda_i(4\lambda_i - 1)G(\lambda_i)) , \quad \lambda_i = \frac{m_i^2}{m_H^2}$$

$$G(\lambda_i) = -2 \left(\arcsin\left(\frac{1}{2\sqrt{\lambda_i}}\right) \right)^2 , \quad \lambda_i \geq \frac{1}{4}$$

$$G(\lambda_i) = \frac{1}{2} \ln^2 \left(\frac{\eta^+}{\eta^-} \right) - \frac{\pi^2}{2} + i\pi \ln \left(\frac{\eta^+}{\eta^-} \right) , \quad \lambda_i < \frac{1}{4}$$

$$\eta^\pm = 1 \pm (1 - 4\lambda_i)^{\frac{1}{2}}. \quad (4.60)$$

At the CERN $p\bar{p}$ collider, one expects⁶⁵ for $\sqrt{s} = 540$ GeV

$$\sigma(p\bar{p} \rightarrow H) \approx 4 \times 10^{-36} \exp[-m_H/21 \text{ GeV}] \text{ cm}^2, \quad 80 \text{ GeV} \leq m_H < 160 \text{ GeV} \quad (4.61)$$

which is too small for observation, given the present luminosity of that machine. However, $\Gamma(H \rightarrow gg)$ may for some reason be much larger, or perhaps some other heavy spin-0 particle couples strongly to two gluons. Indeed, the two-gluon production mechanism may turn out to be our best way of finding new physics.⁷⁴

V QUANTUM CHROMODYNAMICS

Strong interactions are often perceived as powerful short-range forces that bind protons and neutrons together inside nuclei. In that capacity, nuclear binding is described as due to the exchange of mesons, such as the pion, between nucleons. There are many known particles in addition to the proton, neutron, and pion that can interact by this strong force. We collectively call them hadrons. The particle data tables⁷⁵ are filled with literally hundreds of such states, many of which are extremely unstable resonances. A fundamental theory of the strong interaction must be able to account for that observed hadronic spectrum and provide a dynamical description of its interaction properties.

The current view is that the observed hadrons are not elementary but are bound states of more fundamental constituents called quarks, and that the strong interactions between hadrons are epiphenomena of the more fundamental forces between quarks. Those forces between quarks must be very strong, as high energy experiments have failed to reveal free isolated quarks. We now have a fundamental theory of strong interactions with all the above properties. It is called quantum chromodynamics (QCD).⁷⁶ In this section I will describe the developments which gave rise to QCD and discuss some properties of this modern strong interaction theory.

A. The Quark Model

The spectrum of hadrons is generally divided into two categories, mesons which have integer spin 0, 1, 2, ..., such as the pion, kaon, rho, etc., and baryons which have half-integer spin 1/2, 3/2, ..., such as the proton, neutron, lambda, omega, etc. Grouping these particles into multiplets in which members are similar, it was found that they could be ordered by a classification scheme called the eightfold way.⁷⁷ From the regularity of these multiplets, Gell-Mann and Zweig conjectured that all hadrons could be constructed as bound states of a few spin-1/2 fundamental constituents called quarks.⁷⁸ Mesons are quark-antiquark, $q\bar{q}$, bound states, while baryons are made from three quarks qqq (antibaryons are $\bar{q}\bar{q}\bar{q}$ states). Although the hadrons so constructed have integer electric charge, their constituent quarks carry fractional electric charges $\pm 2/3, \pm 1/3$. From a small number of distinct quarks (u, d, c, s, b) one can build all observed hadrons. (We believe that a sixth t quark also exists.)

Some properties of the known quark flavors were already outlined in Table I. All hadrons discovered before 1974 (the year of the J/ψ discovery⁷⁹) could be constructed out of u, d, and s quarks; they constituted the spectrum of the eightfold way. The c and b flavors are relatively new, having been found (through their hadron bound states) during the last decade. Some examples of hadronic quark content are

$$\begin{array}{ll}
 \text{proton: } p = uud, & \text{omega: } \Omega^- = sss \\
 \text{neutron: } n = udd, & \text{lambda: } \Lambda = uds \\
 \text{pion: } \pi^+ = ud, & \text{kaon: } K^+ = u\bar{s} \\
 \text{charmonium: } c\bar{c} & \text{bottomonium: } b\bar{b}
 \end{array} \quad (5.1)$$

Strong interactions between hadrons are flavor independent and flavor conserving. Therefore, any fundamental strong interactions between quarks should exhibit this property.

Given the success of the quark model, it is natural to wonder: Why aren't free quarks observed? Indeed, the apparent impossibility of liberating quarks from their hadronic domains often makes some people skeptical about their actual existence.

B. Deep-Inelastic Scattering

The reality of quarks was rather convincingly demonstrated by deep-inelastic electroproduction experiments at SLAC⁸⁰ carried out in the late 1960s. High energy electrons (≈ 20 GeV) were scattered off protons giving rise to $e + p \rightarrow e + X$ where X denotes multi-hadron states. A schematic amplitude for this reaction is illustrated in Fig. 14.

Fig. 14 Deep-inelastic electron-proton scattering.

Probing the proton at large q (which corresponds to small wavelengths) allows one to examine its short distance structure.

The differential cross-section for deep-inelastic e-p scattering is given by⁸⁰

$$d\sigma = L^{\mu\nu} W_{\mu\nu} \quad (5.2)$$

where $L^{\mu\nu}$ corresponds to the square of the leptonic part of the amplitude and

$$W_{\mu\nu} = \frac{1}{4\pi} \sum_X (2\pi)^4 \delta^4(p + q - p_X) \langle p | J_\mu^{+em}(0) | X \rangle \langle X | J_\nu^{em}(0) | p \rangle. \quad (5.3)$$

(Only the hadronic part is necessary for this discussion.)

Parametrizing $W_{\mu\nu}$ in terms of form factors one has

$$W_{\mu\nu} = -(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) W_1(\nu, Q^2) + \frac{1}{m_p^2} (p_\mu - \frac{p \cdot q}{q^2} q_\mu) (p_\nu - \frac{p \cdot q}{q^2} q_\nu) W_2(\nu, Q^2) \quad (5.4)$$

where $Q^2 = -q^2$ and $\nu = p \cdot q / m_p = E_e - E'_e$ (change in the electron energy). The differential cross-section is given by

$$\frac{d\sigma}{dQ^2 d\nu} = \frac{4\pi\alpha^2}{m_p^4} \frac{E'}{E} \left\{ 2W_1(\nu, Q^2) \sin^2 \frac{\theta}{2} + W_2(\nu, Q^2) \cos^2 \frac{\theta}{2} \right\}. \quad (5.5)$$

In the quark model, one can calculate the above cross-section assuming that the scattering reaction is an incoherent sum of electron-quark cross-sections. In this approach, the form factors $W_1(\nu, Q^2)$ and $(\nu/m_p)W_2(\nu, Q^2)$ turn out to depend only on $x = Q^2/2m_p\nu$. This property is called Bjorken scaling.⁸¹ It has been experimentally confirmed (for $Q^2, m_p\nu \gg m_p^2$). In the quark model one makes the replacements

$$W_1(\nu, Q^2) \rightarrow F_1(x) \quad (5.6a)$$

$$\frac{\nu}{m_p} W_2(\nu, Q^2) \rightarrow F_2(x) \quad (5.6b)$$

in Eq. (5.5) with the prediction

$$F_2(x) = \frac{4}{9} x \{u(x) + \bar{u}(x)\} + \frac{1}{9} x \{d(x) + \bar{d}(x)\} + \frac{1}{9} x \{s(x) + \bar{s}(x)\} + \dots \quad (5.6c)$$

where $q(x)$ represents the number of quarks of type q with fractional momentum between x and $x + dx$. Furthermore, for spin- $\frac{1}{2}$ quarks

$$F_2(x) = 2xF_1(x) \quad (\text{Callan-Gross relation}). \quad (5.6d)$$

(If quarks had spin=0 you would find $F_1(x) = 0$.)

At high energies the electroproduction experiments confirmed the quark model. They observed Bjorken scaling in a manner that was consistent with spin- $\frac{1}{2}$, fractionally charged, point-like constituents in the proton. The quark structure was thus revealed.

Some surprises emerged from these experiments. It was found that the quarks seem to be essentially free when probed at short distances. Strong interaction effects appeared to diminish as one probed near the quark. In addition it was observed that the three charged quarks in the proton carry only about $\frac{1}{3}$ of the proton's momentum; the other $\frac{2}{3}$

appeared to be missing. These two peculiar effects had to await the advent of QCD before they could be understood.

C. Color

There were several problems with the simple quark model. It first failed at the level of quark statistics when applied to baryons.⁸² If quarks are spin- $\frac{1}{2}$ fermions, then baryon wavefunctions should be antisymmetric under the interchange of two identical quarks. But that seemed not to be the case. Take for example the Ω^- which is a bound state of three strange quarks $\Omega^- = |s\rangle|s\rangle|s\rangle$. It has spin $3/2$, so its spin wavefunction is symmetric under the interchange of any 2 quarks. Since the Ω^- is the ground state, one expects each quark to have zero relative orbital angular momenta, so the spatial wave function should also be symmetric. That suggests that their product is symmetric rather than antisymmetric, a violation of Fermi statistics. This apparent inconsistency was overcome by the introduction of color. Each quark was given a new quantum number, color, i.e. they come in three distinct colors q_i , $i = 1, 2, 3$. In that way the baryon wavefunction could be made antisymmetric in the color variable. For example, the Ω^- wavefunction becomes

$$\begin{aligned}\Omega^- &= \frac{1}{\sqrt{6}} \epsilon_{ijk} |s_i\rangle|s_j\rangle|s_k\rangle \\ &= \frac{1}{\sqrt{6}} (|s_1\rangle|s_2\rangle|s_3\rangle - |s_2\rangle|s_1\rangle|s_3\rangle + |s_2\rangle|s_3\rangle|s_1\rangle \\ &\quad - |s_1\rangle|s_3\rangle|s_2\rangle + |s_3\rangle|s_2\rangle|s_1\rangle - |s_3\rangle|s_1\rangle|s_2\rangle) . \quad (5.7)\end{aligned}$$

The meson wavefunctions were given by

$$\frac{1}{\sqrt{3}} (|q_1\rangle|\bar{q}_1\rangle + |q_2\rangle|\bar{q}_2\rangle + |q_3\rangle|\bar{q}_3\rangle) . \quad (5.8)$$

The introduction of color solved some other outstanding problems. The $\pi^0 \rightarrow \gamma\gamma$ decay rate calculated in the quark model had been too small by a factor of $1/9$. Giving the quarks color as in (5.8) increased the decay amplitude by a factor of 3 and the rate by a factor of 9, just what was needed.⁸³ Color also brought the quark model prediction for $R \equiv \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ into accord with experiment; see Eqs. (2.20) and (2.21). Without color, one expected $R = \sum_q Q_q^2 = (4/9 + 1/9 + \dots) = 2/3$ below charm threshold; but experiments found $R \approx 2$. Given the quarks color meant that the prediction was really $3 \times 2/3 = 2$ found experimentally. The three pieces of input — baryon statistics, $\pi^0 \rightarrow \gamma\gamma$ decay rate, and the value of R — together provide convincing evidence for color.

Why was this new quantum number called color? It could have been given any name; however, the nomenclature color provides a convenient way of describing the fact that observed hadrons do not carry this quantum number, they must be colorless.⁸⁴ It is equivalent to the mathematical statement that each quark flavor transforms as a

triplet representation under an internal $SU(3)_c$ color symmetry; but physical hadrons are all singlets.

The quark model plus $SU(3)_c$ color symmetry provides a good description of the observed hadronic spectrum. However, it still fails to account for the dynamics of strong interactions and leaves unanswered some important questions. Why do quarks behave as though they are free inside the nucleus when we probe them in very high-energy experiments? Why aren't free quarks or colored hadrons found in nature? Where is the missing proton momentum? Quantum chromodynamics (QCD) provides answers to all these questions as we shall see.

D. QCD Lagrangian

Although, as pointed out earlier, non-abelian gauge theories have been around since 1954, their most important quantum properties were not appreciated. In 1973, a most remarkable property of such theories was discovered by Politzer⁸⁵ and Gross and Wilczek.⁸⁶ They found that self-interactions among gauge fields led to asymptotic freedom. That is, the strength of the interaction mediated by non-abelian gauge fields becomes vanishingly small at asymptotically high energies (or equivalently at very short distances). Such theories behave almost like free (non-interacting) field theories at high energies. This was precisely the property observed in deep-inelastic scattering experiments. It turns out that only non-abelian gauge theories exhibit asymptotic freedom. A natural conclusion was that the strong forces between quarks must be mediated by non-abelian gauge bosons, and thus QCD emerged.

QCD is based on local gauging of the $SU(3)_c$ symmetry. Analogous to the way we dealt with the $SU(2)_L$ symmetry in Section IV, we introduce eight gauge fields, called gluons, which transform as the adjoint representation of $SU(3)_c$. The QCD Lagrangian density is thus given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \sum_{q=u,d,\dots} \overline{q_1 q_2 q_3} \left(i \partial_\mu \gamma^\mu + g_3 \frac{\lambda^a}{2} A_\mu^a \gamma^\mu - m_q \right) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad (5.9)$$

where λ^a , $a = 1, 2, \dots, 8$ are the 3×3 Gell-Mann matrices, g_3 is the gauge coupling and

$$F_{\mu\nu}^a(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + g_3 f^{abc} A_\mu^b(x) A_\nu^c(x) \quad (5.10)$$

with f^{abc} the $SU(3)_c$ structure constants. This Lagrangian is invariant under the local $SU(3)$ transformations

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \rightarrow U(x) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad (5.11a)$$

$$A_\mu \equiv A_\mu^a \lambda^a / 2 \rightarrow U(x) A_\mu U^{-1}(x) + \frac{i}{g_3} U(x) \partial_\mu U^{-1}(x) \quad (5.11b)$$

$$U(x) = \exp\left[i \frac{\lambda^a}{2} \theta^a(x)\right]. \quad (5.11c)$$

We see that QCD is quite similar to QED in that it is an exact (unbroken) gauge symmetry and hence gluons are massless spin-1 bosons. It differs significantly in that gluons carry color charge and thereby interact with one another, see Eq. (5.10). It is this self-interaction of gluons that endows QCD with such extraordinary properties. It also turns out that gluons in the proton carry the missing momentum.

E. Feynman Rules

To rigorously deduce the Feynman rules of QCD, one must fix the gauge that induces ghost fields, which must generally be included to preserve unitarity.⁷⁶ I will give only the Feynman rules for quarks and gluons in Fig. 15 as a means of illustrating the similarities and differences with QED.

Propagators:

Gluon		$\frac{-i(g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2}) \delta_{ab}}{k^2}$
Quark		$\frac{i}{\not{p} - m_q}$

Vertices:

Gqq		$ig_3 \gamma_\mu \lambda^a / 2$
GGG		$g_3 f^{abc} [(k_1 - k_2)_\sigma g_{\mu\nu} + (k_2 - k_3)_\mu g_{\nu\sigma} + (k_3 - k_1)_\nu g_{\mu\sigma}]$
GGGG		$\begin{aligned} & -ig_3^2 (f_{abe} f_{cde} (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma}) \\ & + f_{ace} f_{bde} (g_{\mu\nu} g_{\sigma\rho} - g_{\mu\rho} g_{\nu\sigma}) \\ & + f_{ade} f_{bce} (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\nu} g_{\sigma\rho})) \end{aligned}$

Fig. 15 QCD Feynman rules for quarks and gluons.

QCD is a renormalizable theory, so ultraviolet infinities which arise at the loop level can be absorbed into the bare parameters of the theory. Hence, one can compute transition rates perturbatively in powers of $\alpha_3 = g_3^2/4\pi$. Of course, such calculations will make sense only if α_3 is small. We shall now see that that is the case at high energies (short distances).

F. Running Coupling and Asymptotic Freedom

The QCD coupling g_3 is renormalized by vacuum polarization effects due to both quarks and gluons (see Fig. 16).

Fig. 16 Some vacuum polarization loop corrections in QCD.

It turns out that the quarks screen the color charge while gluons antiscreen. After defining a renormalized coupling by subtracting the ultraviolet infinities via modified minimal subtraction (MS) one finds (compare with Eq. (2.52b))

$$\frac{1}{g_3^2(\mu)} = \frac{1}{g_{30}^2} + \frac{1}{8\pi^2} \left(11 - \frac{2N_F}{3}\right) \left(\frac{1}{n-4} + \frac{\gamma}{2} - \ln \sqrt{4\pi}\right) + \dots \quad (5.12)$$

where N_F is the number of quark flavors and μ is the 't Hooft unit of mass⁸⁷ in dimensional regularization.

As μ changes, $\alpha_3(\mu) \equiv g_3^2(\mu)/4\pi$ runs in a manner prescribed by the renormalization group equation⁸⁸

$$\mu \frac{\partial}{\partial \mu} \alpha_3(\mu) = \beta(\alpha_3) = b_0 \alpha_3^2 + b_1 \alpha_3^3 + b_2 \alpha_3^4 + \dots \quad (5.13)$$

where

$$b_0 = -\frac{1}{2\pi} \left(11 - \frac{2N_F}{3}\right) \quad (5.14a)$$

$$b_1 = -\frac{1}{4\pi} \left(51 - \frac{19N_F}{3}\right) \quad (5.14b)$$

$$b_2 = -\frac{1}{64\pi^3} \left[2857 - \frac{5033N_F}{9} + \frac{325N_F^2}{27}\right] \quad (5.14c)$$

Solving this equation leads to

$$\frac{1}{\alpha_3(\mu)} = \frac{1}{\alpha_3(\mu_0)} - b_0 \ln\left(\frac{\mu}{\mu_0}\right) - \frac{b_1}{b_0} \ln\left(\frac{\alpha_3(\mu)}{\alpha_3(\mu_0)}\right) - \frac{1}{b_0^2} (b_2 b_0 - b_1^2) (\alpha_3(\mu) - \alpha_3(\mu_0)) + O(\alpha_3^2) \quad (5.15)$$

with μ_0 a reference mass scale. Given a value for $\alpha_3(\mu_0)$, one can compute $\alpha_3(\mu)$ by iteratively solving (5.15). Of course $\alpha_3(\mu)$ must be small if the perturbative expansion is to be meaningful. Notice that as long as $N_F < 17$ (presently we think $N_F = 6$), $b_0 < 0$ so that $\alpha_3(\mu)$ decreases as we go to large μ . This property

$$\alpha_3(\mu) \xrightarrow{\mu \rightarrow \infty} 0 \quad (5.16)$$

is called asymptotic freedom. It explains Bjorken scaling. At low energies the QCD coupling is large while at high energies it is small. The quark color charge $g_3(\mu)$ is quite small at short distances; however massless gluons antiscreen it so that $g_3(\mu)$ grows as we move further away from it; a most remarkable property.

Although Eq. (5.15) provides a means of interpolating $\alpha_3(\mu)$ from one value of μ to another, it is somewhat cumbersome. It is often more convenient to introduce a reference mass scale Λ which incorporates the interactive procedure and provides a fundamental form for $\alpha_3(\mu)$. Such a μ -independent mass parameter Λ must be of the form⁸⁹

$$\Lambda = \mu \exp\left(\frac{1}{b_0 \alpha_3(\mu)} - \frac{b_1}{b_0^2} \ln\left(\frac{-2}{b_0 \alpha_3(\mu)}\right) + C + \frac{1}{b_0^3} (b_2 b_0 - b_1^2) \alpha_3(\mu) + O(\alpha_3^2)\right) \quad (5.17)$$

where C is an arbitrary constant. (Verify that $(d/d\mu) \Lambda = 0$.) In defining what has become a standard mass scale, $\Lambda_{\overline{MS}}$, one sets $C = 0$ for all N_F . However, since N_F changes discontinuously by 1 unit at $\mu = m_q$, $\Lambda_{\overline{MS}}$ must then jump at each mass threshold, i.e. it is N_F dependent. (One can have an N_F independent $\Lambda_{\overline{MS}}$ at the expense of having N_F dependent C 's in Eq. (5.17)).⁸⁹ Setting $C = 0$ one finds

$$\alpha_3^{-1}(\mu) = -\frac{b_0}{2} \ln(\mu^2 / \Lambda_{\overline{MS}}^{(N_F)})^2 + \frac{b_1}{b_0} \ln \ln(\mu^2 / \Lambda_{\overline{MS}}^{(N_F)})^2 + \dots \quad (5.18)$$

The $\Lambda_{\overline{MS}}^{(N_F)}$ are related to one another by requiring that $\alpha_3(\mu)$ be continuous at each $\mu = m_q$. Given a value for $\alpha_3(\mu)$, $\Lambda_{\overline{MS}}^{(N_F)}$ is determined by (5.17). For example, if we know $\alpha_3(\mu)$ for some $m_c \leq \mu \leq m_b$ (an effective $N_F = 4$ theory), then

$$\Lambda_{\overline{MS}}^{(4)} = \mu \exp\left(\frac{-6}{25} \left(\frac{\pi}{\alpha_3(\mu)} - \frac{77}{50} \ln\left(\frac{12\pi}{25\alpha_3(\mu)}\right)\right)\right), \quad m_c \leq \mu \leq m_b \quad (5.19)$$

Similar expressions exist for other N_F . At present we think $\Lambda_{\overline{MS}}^{(4)} \approx 100$ MeV, which implies (for $m_t = 35$ GeV)^{89,90}

$$\frac{\Lambda(6)}{\overline{MS}} : \frac{\Lambda(5)}{\overline{MS}} : \frac{\Lambda(4)}{\overline{MS}} : \frac{\Lambda(3)}{\overline{MS}} :: 27 : 63 : 100 : 130 . \quad (5.20)$$

A precise determination of $\Lambda_{\overline{MS}}^{(4)}$ is particularly important as a test of grand unification (see Section VI).

G. Vacuum Properties of QCD

The physics of asymptotic freedom is nicely illustrated by the properties of the QCD vacuum.⁹¹ Calling ϵ the color dielectric constant and μ the color magnetic permeability of the vacuum, we know from relativistic invariance

$$\epsilon\mu = 1 . \quad (5.21)$$

(Of course the analogous situation for electrodynamics in a medium need not satisfy (5.21). For example, an electron gas at $T = 0$ has $\epsilon > 1$ and $\mu > 1$.) The static potential between two color charges Q_1 and Q_2 is given by

$$V(r) = \frac{Q_1 Q_2}{\epsilon(r)r} . \quad (5.22)$$

If $\epsilon > 1 + \mu < 1$, the charges are screened, while if $\epsilon < 1 + \mu > 1$ they are antiscreened as in the case for asymptotic freedom. Defining the color magnetic susceptibility χ by

$$\mu = 1 + 4\pi \chi \quad (5.23)$$

we see that $\chi > 0$ corresponds to the QCD vacuum and asymptotic freedom. That is the case for so-called paramagnetic substances. So, asymptotic freedom arises because the QCD vacuum acts like a color paramagnetic medium. The opposite behavior occurs in QED which has a diamagnetic vacuum.

H. Heavy Quarkonia Decays

One of the many examples of QCD perturbation theory is provided by heavy quarkonia (such as $c\bar{c}$ and $b\bar{b}$ bound states) decays.⁹² Consider for example the three-gluon decay of a $1^{--} q\bar{q}$ state (see Fig. 17).

The amplitude is proportional to g_3^3 ; so the rate must go like α_3^3 . Summing over color one finds⁹³

$$\Gamma(1^{--} \rightarrow 3 \text{ gluons}) = \frac{160}{81} (\pi^2 - 9) \frac{|\psi(0)|^2}{M^2} \alpha_3^3 \quad (5.24)$$

where $|\psi(0)|$ is the $q\bar{q}$ wavefunction at the origin and M its mass. This is the main decay mode. Its rate is small because α_3 is quite small and it is further suppressed by $\pi^2 - 9$. To get rid of the uncertainty in the wavefunction, one usually compares this rate with the electromagnetic decay

Fig. 17 Three-gluon decay
of a $1^{--} q\bar{q}$ state.

$$\Gamma(1^{--} \rightarrow \mu^+ \mu^-) = 16\pi Q_q^2 \alpha^2 |\psi(0)|^2 / M^2. \quad (5.25)$$

Taking the ratio one finds

$$\frac{\Gamma(1^{--} \rightarrow 3\text{gluons})}{\Gamma(1^{--} \rightarrow \mu^+ \mu^-)} = \frac{10(\pi^2 - 9)\alpha_3^3(\mu)}{81\pi Q_q^2 \alpha^2}. \quad (5.26)$$

Since $1^{--} \rightarrow 3$ gluons constitutes most of the hadronic decay rate, one can use measurements of $\Gamma(1^{--} \rightarrow \text{hadrons})/\Gamma(1^{--} \rightarrow \mu^+ \mu^-)$ to determine $\alpha_3(\mu)$ and thereby find $\Lambda_{\overline{\text{MS}}}$. Unfortunately, the appropriate value of μ in (5.26) is questionable. Lepage and Mackenzie⁹⁴ have computed the $O(\alpha_s^2)$ corrections to (5.26) to determine the appropriate μ . However, they find that the corrections are too large for what they consider reasonable values of μ and conclude that perturbation theory is breaking down. Brodsky, Lepage, and Mackenzie⁹⁵ suggest that $\mu = 0.157M$ should be used in Eq. (5.26), and for that value the corrections lead to a multiplicative correction factor of $[1 - 14\alpha_3(0.157M)/\pi]$.

A more reliable estimate of α_3 and $\Lambda_{\overline{\text{MS}}}$ can be obtained by comparing the inclusive radiative rate $\Gamma(1^{--} \rightarrow \gamma + 2 \text{ gluons})$ with the 3-gluon rate. Including QCD corrections⁹⁶

$$\frac{\Gamma(1^{--} \rightarrow \gamma g g)}{\Gamma(1^{--} \rightarrow g g g)} = \frac{36Q_q^2}{5} \frac{\alpha (1 + 2.2\alpha_3(0.157M)/\pi)}{\alpha_3(0.157M)}. \quad (5.27)$$

From experimental measurements of this ratio for T decay, Lepage finds⁹⁷

$$\Lambda_{\overline{\text{MS}}}^{(4)} = 100^{+100}_{-50} \text{ MeV}. \quad (5.28)$$

This is in my opinion the most precise experimental determination of $\Lambda_{\overline{\text{MS}}}^{(4)}$. Most experiments are consistent with the range in (5.28).⁹⁸

I. QCD Jets

The fact that QCD's strong interaction coupling, $\alpha_3(\mu)$, decreases as the energy domain probed increases allows one to compute high-energy QCD effects perturbatively. One observed effect that perturbative QCD nicely accounts for is the jet structure of final state hadrons produced in high-energy collisions, i.e. the tendency of hadrons to be produced in streams of narrow jet-like cones with very little transverse momentum. That phenomenon is merely a manifestation of gluon bremsstrahlung. For example, in e^+e^- annihilation into hadrons, one describes the total scattering process as the sum

$$\begin{aligned} &\sigma(e^+e^- \rightarrow q\bar{q}) + \sigma(e^+e^- \rightarrow q\bar{q} + \text{gluon}) \\ &+ \sigma(e^+e^- \rightarrow q\bar{q} + 2 \text{ gluons}) + \sigma(e^+e^- \rightarrow q\bar{q}q\bar{q}) + \dots \quad (5.29) \end{aligned}$$

The first term is of lowest (zeroth) order in $\alpha_3(\mu)$; it corresponds to 2-jet production. The second term gives rise to 2 jets when the gluon is nearly collinear with one of the quarks and 3 jets when the opening angles are large. In a similar manner, 4-, 5-, etc., jet events can be produced. However, it costs an extra power of $\alpha_3(\mu)$ for each additional jet.

Using the Sterman-Weinberg definition⁹⁹ of a 2-jet event as one in which all but at most ϵ ($\epsilon \ll 1$) of the total energy E is concentrated in opposite cones of half-angle $\delta \ll 1$ gives

$$\frac{\sigma(e^+e^- \rightarrow 2 \text{ jets})}{\sigma(e^+e^- \rightarrow \text{hadrons})} = 1 - \frac{4}{3} \frac{\alpha_3(E)}{\pi} \left[(4\ln 2\epsilon + 3)\ln\delta + \frac{\pi^2}{3} - \frac{7}{4} \right]. \quad (5.30)$$

The deviation from 1 is the number of 3-jet events (by definition). So we see that as E increases and $\alpha_3(E)$ decreases, 2-jet events become more and more dominant. This is exactly what one observes experimentally.

J. Quark Confinement

Free quarks, gluons, and colored bound states have never been observed. Color must therefore not be a directly observable quantity. The picture of color confinement that we believe (a proof does not exist) occurs in QCD is one in which the color flux between quarks is funneled into string-like configurations that lead to a linear potential between quarks. If one tries to break a hadron apart into its constituents, the potential energy between quarks grows linearly until enough energy is available to pop quark-antiquark pairs out of the vacuum. In that way rather than a quark being freed, many colorless hadrons emerge.

It is believed that at temperatures of order 300 MeV, quark deconfinement may occur. By colliding ultra-relativistic nuclei against one another one may create such a situation in the laboratory. In those reactions, a new quark-gluon plasma phase of QCD may be created. Observing this phase transition would provide a means of studying the long-distance properties of QCD. There are plans to build a high energy heavy ion collider at BNL¹⁰⁰ which will allow experiments to probe QCD and perhaps uncover other novel phenomena.

VI GRAND UNIFIED THEORIES

The Weinberg-Salam $SU(2)_L \times U(1)$ electroweak model and QCD together constitute the "standard model" which we denote by $SU(3)_c \times SU(2)_L \times U(1)$. This model contains three independent gauge couplings g_3 , g_2 , and g_1 ($g_1 = \sqrt{5/3} g'$) and hence cannot be considered a true unification of strong, weak, and electromagnetic interactions. However, it was observed by Georgi and Glashow¹⁰¹ that these couplings would not be independent if they embedded the "standard model" in a simple compact Lie group such as $SU(5)$. In such theories the bare gauge couplings must be equal $g_3 = g_2 = g_1$, and $\sin^2 \theta_W = 3/8$, while the effective low energy couplings $g_i(\mu)$ are naturally unequal because vacuum polarization effects renormalize them differently. This so-called grand unification idea is very appealing theoretically and gives rise to incredible predictions such as proton decay¹⁰¹ and the existence of superheavy magnetic monopoles.¹⁰²

Rather than giving a detailed discussion of grand unified theories, I will merely illustrate their predictability by presenting some results for the minimal $SU(5)$ model.¹⁰³ In that model one has 12 new gauge bosons $X^{\pm 4/3}$, $Y^{\pm 1/3}$ (each comes in 3 colors). They are degenerate (up to $SU(2)_L$ breaking) with mass m_X and are capable of mediating proton decay processes such as $p \rightarrow e^+ \pi^0$. However, m_X is very heavy; so the proton lifetime τ_p is naturally long ($\tau_p \propto m_X^4$). Using $\Lambda_{\overline{MS}}^{(4)}$ as input, one can predict¹⁰⁴ $\sin^2 \theta_W$, m_W , m_Z , m_X , and τ_p in the minimal $SU(5)$ model (see Table III).

Table III Minimal $SU(5)$ model predictions for a given $\Lambda_{\overline{MS}}^{(4)}$.

$\Lambda_{\overline{MS}}^{(4)}$ (MeV)	$\sin^2 \theta_W$	m_W (GeV)	m_Z (GeV)	m_X (GeV)	τ_p (yr)
25	0.226	81.3	92.3	3×10^{13}	$3 \times 10^{26 \pm 2}$
50	0.222	82.0	92.8	6.2×10^{13}	$3 \times 10^{27 \pm 2}$
100	0.218	82.8	93.6	1.3×10^{14}	$5 \times 10^{28 \pm 2}$
200	0.214	83.5	94.3	2.7×10^{14}	$5 \times 10^{29 \pm 2}$
400	0.210	84.3	94.9	5.5×10^{14}	$1 \times 10^{31 \pm 2}$

For $\Lambda_{\overline{MS}}^{(4)} = 100$ MeV, the prediction for $\sin^2 \theta_W$ is in good agreement with experiment, see (4.46). I have allowed a conservative uncertainty estimate of a factor of 100 in τ_p . The present experimental bound¹⁰⁵

$$\tau_p \gtrsim 2 \times 10^{32} \text{ yr} \quad (6.1)$$

appears to rule out this model. However, before that conclusion can be reached with certainty, we should have better determinations of $\Lambda_{\overline{MS}}^{(4)}$ and $\sin^2 \theta_W$. By appending new particles with masses between m_W and m_X , the prediction for τ_p can be increased.¹⁰⁴

In addition to proton decay, grand unified theories predict the existence of super-heavy magnetic monopoles with mass $O(10^{16} \text{ GeV})$.¹⁰⁶ They carry magnetic charge $\pm 1/2e$ as well as screened color magnetic charge. Recently it was shown by Rubakov¹⁰⁷ and Callan¹⁰³ that these monopoles catalyze proton decay with strong interaction cross-sections. Hence these monopoles should leave a trail of proton decays in a proton decay detector. So far, no such signal has been observed.

VII OUTLOOK

The standard $SU(3)_C \times SU(2)_L \times U(1)$ model of strong and electroweak interactions has been very successful. It accounted for all observed elementary particle phenomena and correctly predicted weak neutral effects as well as the $W^\pm$ and Z masses. One final particle, the Higgs scalar, remains to be found; it promises to be very elusive.

What does the future hold? The CERN $\bar{p}p$ collider has already found mysterious events that suggest new physics at $\approx 160 \text{ GeV}$. Speculations that these events may be harbingers of supersymmetry, technicolor, compositeness, etc., have been given. The physics community anxiously awaits further data. In addition there are plans to build a supercollider with $\sqrt{s} = 40 \text{ TeV}$ in the 1990s. We can look forward to thorough exploration of the 100 GeV $\sim 10 \text{ TeV}$ region and exciting new discoveries.

Grand unified theories predict that the proton decays and magnetic monopoles should exist. Several experiments have been mounted to search for these phenomena. As yet only bounds on τ_p and monopole flux have been set; but those experiments are continuing and being expanded. The detection of a positive signal would be a great discovery.

We now believe that strong and electroweak interactions are described by the standard model and the principle of local gauge invariance. Left out of this picture is gravity. So far, a quantum theory of gravity does not exist. Attempts at incorporating gravity range from those which diminish its role to an induced non-fundamental force to Kaluza-Klein theories that elevate gravity to the most fundamental interaction with the others being induced phenomena. A truly unified theory of all interactions is an outstanding goal.

High energy theory has made tremendous progress. During the next decade I expect new unanticipated phenomena to become manifest and pose novel problems for theorists. Exciting new surprises will continue to guide us in our aim to understand the laws of nature and the universe around us.

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APPENDIX A: DIMENSIONAL REGULARIZATION

Perturbative loop corrections encountered in quantum field theories are generally ultraviolet divergent. They exhibit short-distance (high energy) singularities which are dealt with by renormalization, i.e. the infinities are absorbed into the bare parameters of the theory. At intermediate steps of any calculation, these infinities must be regularized. One of the simplest and most powerful techniques for accomplishing this is dimensional regularization.¹⁰⁹ The basic idea is to continue loop integrations from 4 to n dimensions (1 time and $n - 1$ space dimensions) with n an arbitrary complex variable. The integrations are carried out for $n < 4$ where they are finite and then analytically continued to $n = 4$ where the ultraviolet divergence shows up as a pole at $n = 4$. This method is particularly well suited for preserving symmetries of the theory such as local gauge invariance. In addition, the continuous dimension method can also be used to regularize infrared (long-distance) infinities associated with massless particles such as photons and gluons.¹¹⁰ (It can also be applied to so-called fermion mass singularities,¹¹¹ i.e. divergences which occur in the limit $m_f \rightarrow 0$.)

The technique of dimensional regularization has become an important tool in perturbative quantum field theory. Its utility is particularly evident in defining renormalized couplings. Indeed, the MS (modified minimal subtraction¹¹²) definition associated with dimensional regularization has become a standard for QCD (see Section V) and is extremely useful for analyzing grand unified theories (see Section VI).¹⁰⁴

As an introduction to the relatively modern technique of dimensional regularization, I give in this appendix a variety of useful formulas as well as a simple example, the electron self-energy. Additional discussion is incorporated in the text.

In dimensional regularization, internal and external momentum are taken to have n rather than 4 components. All manipulations must be consistent with this generalization. Propagators and interaction vertex forms are unchanged, except for a $\mu^{2-n/2}$ factor introduced at each vertex to keep couplings dimensionless (μ is called the 't Hooft unit of mass). At the one loop level, Feynman integrals have the generic form

$$\int \frac{d^n k}{(2\pi)^n} F(p, k, n) \quad (A1)$$

where k is the loop integration momentum and p symbolically denotes all external momenta. Manipulations within the integrand are carried out using the following n -dimensional conventions and identities.¹¹¹

Metric $g_{\mu\nu}$: $g_{00} = 1, \quad g_{ij} = -\delta_{ij}, \quad g_{0i} = 0, \quad i, j = 1, 2, \dots, n-1$ (A2)

$$k \cdot p = g_{\mu\nu} k^\mu p^\nu = k_0 p_0 - k_1 p_1 - k_2 p_2 \dots - k_{n-1} p_{n-1} \quad (A3)$$

Dirac algebra: $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} I, \quad \mu, \nu = 0, 1, 2, \dots, n-1$ (A4)

$$\gamma_\mu \gamma^\mu = nI \quad (A5)$$

$$\gamma_\mu \gamma_\alpha \gamma^\mu = (2 - n) \gamma_\alpha \quad (A6)$$

$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma^\mu = 4g_{\alpha\beta} I + (n - 4) \gamma_\alpha \gamma_\beta \quad (A7)$$

$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\lambda \gamma^\mu = -2\gamma_\lambda \gamma_\beta \gamma_\alpha - (n - 4) \gamma_\alpha \gamma_\beta \gamma_\lambda \quad (A8)$$

$$\gamma_5 \equiv (\text{anticommutes with all } \gamma_\mu) \quad (A9)$$

Trace identities: $\text{Tr}(\text{odd number of } \gamma\text{'s}) = 0$ (A10)

$$\text{Tr} I = 4 \quad (A11)$$

$$\text{Tr}(\gamma_\alpha \gamma_\beta) = 4g_{\alpha\beta} \quad (A12)$$

$$\text{Tr}(\gamma_\alpha \gamma_\beta \gamma_\lambda \gamma_\rho) = 4(g_{\alpha\beta} g_{\lambda\rho} - g_{\alpha\lambda} g_{\beta\rho} + g_{\alpha\rho} g_{\beta\lambda}) \quad (A13)$$

$$\text{Tr}(\gamma_5) = 0 \quad (A14)$$

$$\text{Tr}(\gamma_5 \gamma_\alpha \gamma_\beta) = 0 \quad (A15)$$

Feynman parametrization:

$$\frac{1}{a^r b^m} = \frac{\Gamma(r+m)}{\Gamma(r)\Gamma(m)} \int_0^1 dx \frac{x^{r-1} (1-x)^{m-1}}{(ax + b(1-x))^{r+m}} \quad (A16)$$

$$\frac{1}{abc} = \int_0^1 2y dy \int_0^1 dx \frac{1}{(ayx + by(1-x) + c(1-y))^3} \quad (A17)$$

Employing the above formulas, all one-loop momentum integrations can be manipulated into the common form (by translation of variables)

$$\int \frac{d^n Q}{(2\pi)^n} \frac{(Q^2)^r}{(Q^2 - c)^m} \quad (A18)$$

and readily evaluated via¹¹³

$$\int \frac{d^n Q}{(2\pi)^n} \frac{(Q^2)^r}{(Q^2 - c)^m} = \frac{i(-1)^{r-m}}{(16\pi^2)^{n/4}} c^{r-m+n/2} \frac{\Gamma(r+n/2)\Gamma(m-r-n/2)}{\Gamma(n/2)\Gamma(m)} \quad (A19)$$

At this point, ultraviolet divergences will manifest themselves as simple poles at $n = 4$ in the Γ functions. They may be isolated by using the following properties of the Γ function¹¹¹

$$\Gamma(n+1) = n! \quad (\text{for } n \text{ integer}) \quad (A20)$$

$$n\Gamma(n) = \Gamma(n+1) \quad (A21)$$

$$\Gamma(1) = 1 \quad (A22)$$

$$\Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}} \quad (\text{A23})$$

$$\Gamma(n) = \frac{2^{n-1}}{\sqrt{\pi}} \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n+1}{2}\right) \quad (\text{A24})$$

$$\Gamma'(1) = -\gamma \text{ (Euler's constant)} \approx -0.5772 \quad (\text{A25})$$

$$\Gamma''(1) = \gamma^2 + \pi^2/6 \quad (\text{A26})$$

$$\Gamma\left(\frac{n}{2} - 1\right) = 1 - \frac{\gamma}{2}(n-4) + \frac{1}{8}\left(\gamma^2 + \frac{\pi^2}{6}\right)(n-4)^2 + \dots \quad (\text{A27})$$

$$\Gamma\left(2 - \frac{n}{2}\right) = -\frac{2}{n-4} - \gamma + O(n-4) \quad (\text{A28})$$

For higher-order m -loop integrations, the same procedure is followed. In such cases poles of order $1/(n-4)^m$ may in general be encountered.

As an illustration of dimensional regularization, consider the $O(\alpha)$ electron self-energy diagram in Fig. A1.

Fig. A1. One-loop electron self-energy diagram.

Its contribution to the self-energy function $\Sigma(p)$ (denoted by $\Sigma_1(p)$) is given by (in the Feynman gauge)

$$-i\Sigma_1(p) = (ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\mu\nu}}{k^2} \gamma^\mu \frac{i}{\not{p} - \not{k} - m} \gamma^\nu. \quad (\text{A29})$$

This integral diverges and must be regularized. Generalizing it to n dimensions and manipulating the Dirac algebra, it takes the form

$$-i\Sigma_1(p) = -e^2 \mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{(2-n)(\not{p} - \not{k}) + nm}{k^2(k^2 - 2p \cdot k + p^2 - m^2)}. \quad (\text{A30})$$

(Note that p^2 need not equal m^2 , i.e. the electron may be off-mass-shell.) Combining denominator factors by Feynman parametrization and translating integration variables, Eq. (A30) becomes

$$-i\Sigma_1(p) = -e^2 \mu^{4-n} \int_0^1 dx \int \frac{d^n Q}{(2\pi)^n} \frac{(2-n)\not{p}(1-x) + nm}{(Q^2 - C)^2} \quad (\text{A31})$$

$$C = p^2(x^2 - x) + m^2 x.$$

Carrying out the $d^n Q$ integration via Eq. (A19) leads to

$$\begin{aligned}
 & -i\Sigma_1(p) \\
 & = \frac{-ie^2}{(16\pi^2)^{n/4}} \Gamma(2 - \frac{n}{2}) \int_0^1 dx \left[(2-n)x(1-x) + nm \right] \frac{p^2(x^2 - x) + m^2 x}{\mu^2}^{\frac{n}{2}-2}.
 \end{aligned} \quad (A32)$$

Before dealing with the final dx integration, consider how $\Sigma(p)$ affects the electron propagator. The sum of all one-particle reducible contributions to the propagator is represented in Fig. A2.

Fig. A2. Self-energy contributions to the electron propagator.

The blob represents one-particle irreducible contributions to the self-energy (called the proper self-energy). These contributions form a geometric series which formally sums to a "dressed" propagator

$$iS_F(p) = \frac{i}{\not{p} - m - \Sigma(p)}. \quad (A33)$$

Now we note that m in Eq. (A33) is actually the bare mass. Expanding about the physical mass m_e

$$\Sigma(p) = A + B(\not{p} - m_e) + C(p)(\not{p} - m_e)^2 \quad (A34)$$

one finds

$$iS_F(p) = \frac{iZ_2}{(\not{p} - m_e)[1 - Z_2 C(p)(\not{p} - m_e)]} \quad (A35)$$

where

$$Z_2^{-1} = 1 - B \quad (A36a)$$

$$m_e = m + A. \quad (A36b)$$

A and B contain all ultraviolet divergences. The self-mass divergence $\delta m = A$ is absorbed into the bare mass while B goes into a wavefunction renormalization of the electron field via $\psi^0 = Z_2^{1/2} \psi$. I should note that Z_2 is gauge dependent while $\delta m = A$ is not.

Rather than integrating Eq. (A32), it is better to expand $\Sigma(p)$ as in Eq. (A34) and take the $n \rightarrow 4$ limit. In that way one obtains to $O(\alpha)$

$$\begin{aligned}
 \delta m = A & = \frac{e^2 m_e}{(16\pi^2)^{n/4}} \Gamma(2 - \frac{n}{2}) \left(\frac{m_e}{2}\right)^{\frac{n}{2}-2} \int_0^1 dx x^{n-4} [2 + (n-2)x] \\
 & = \frac{3\alpha}{4\pi} m_e \left\{ \frac{-2}{n-4} - \gamma + \ln(4\pi) + \frac{4}{3} - \ln\left(\frac{m_e}{\mu}\right)^2 + O(n-4) \right\}.
 \end{aligned} \quad (A37a)$$

The wave function renormalization factor Z_2 can be similarly obtained. I leave it as an exercise. (Note, it will be infrared divergent.)