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Pattern Selection and Low-Dimensional Chaos in Systems of Coupled Nonlinear Oscillators

2

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1. Abstract

The longtime behavior of a number of one- and two-dimensional driven, dissipative, dispersive, many-degree-of-freedom systems is studied. It is shown numerically that the attractors are characterized by strong mode-locking into a small number of (nonlinear) modes. On the basis of the observed profiles, estimates of chaotic attractor dimensions, and projections into nonlinear mode bases, it is argued that the same few modes may (in these extended systems) give a unified picture of spatial pattern selection, low-dimensional chaos, and coexisting coherence and chaos. Analytic approaches to this class of problem are summarized.

2. Introduction

In recent years considerable attention has been given to the properties of low-dimensional maps as models for complicated dynamics in higher-dimensional dynamical systems [1]. This attention has been merited by the proof of "universal" properties in classes of one-dimensional maps [1]. However, with few exceptions, the low-dimensionality has been introduced explicitly by restricting consideration to models with a very small number of degrees-of-freedom. On the other hand, equally active research has focused on the subject of spatial pattern selection in non-equilibrium nonlinear systems with many degrees-of-freedom (e.g. convection cells [2], reaction-diffusion systems [3]). In these cases mode-locking is very strong and a small number of modes dominate the spatial structure and temporal evolution in a nonlinear partial differential equation (p.d.e.) or large system of coupled ordinary differential equations (o.d.e.'s).

The perspective we will emphasize here is that the phenomena of pattern formation, low-dimensional chaos and coexisting coherence and chaos can be intimately connected in perturbed, dissipative dynamical systems with many degrees-of-freedom. More specifically, chaotic dynamics may develop by chaotic motions of the collective coordinates identifying the dominant (determining) patterns in the quiescent regimes. In this way only a small loss of mode-locking is responsible for the temporal chaos coexisting with spatial coherence -- only a small number of modes become unstable and the many remaining ones retain strong coherence. There are many examples of this scenario [4], which gives the problem of identifying and testing (nonlinear [5]) mode reduction schemes a general mathematical and physical importance [5]. Physically, it is essential in experiments to gain information on both time and space correlations; chaotic diffusion may be dominated by motion of the coherent pattern; possible consequences for fractal structure of the patterns need to be investigated.

In this brief summary we report six examples in the form of driven, damped nonlinear p.d.e.'s (and their discrete analogs) in one and two spatial dimensions. Especially in the 1-d cases, solitons [7] (or near-solitons) are fundamental excitations of the underlying unperturbed Hamiltonian systems, and motivate nonlinear mode reductions. Four of the examples are based on perturbed sine-Gordon (SG) equations

$$\phi_{tt} - \phi_{xx} + \sin \phi = F(x,t) - c\phi_t \quad . \quad (1)$$

Figure 1. Space-time evolutions of $\phi(x,t)$ for the SG system (Case 1) through two driving periods for $\epsilon = 0.2$, $\omega_d = 0.6$, with periodic boundary conditions, and driving strengths (a) $\gamma = 0.8$, which results in periodic time evolution, (b) $\gamma = 1.0$, which results in chaotic kink-antikink motion (nearly repeating every driving period)

Figure 2. Nonlinear spectral decompositions for Case 1, using the "soliton" transform on a periodic line [9]. The time evolution is shown of zeroes of the discriminant in the complex eigenvalue (λ) plane for the associated linear scattering operator [9]. Zeroes occur either: (i) on the imaginary axis, symmetrically about $1/4$. Dots denoting such zeroes above $1/4$ are shown in the top portions of each plot; or (ii) in pairs symmetric about the imaginary axis. Only the zeroes in the positive real quadrant are shown. If these lie on the circle of radius $1/4$ (non-translating states) they are denoted by dots and their angle relative to the real axis is shown in the lower portion of each plot. Otherwise the zeroes are denoted by crosses and only their angle is shown, not their distance from the origin. Figure 2(a) is for Case 1 with $\Gamma = 0.89$ where the attractor is periodic in time and period- $1/2$ in space. Note the periodic evolution of the zeroes (the period of the driving field is $2\pi/0.6 \sim 10.5$). The crosses here correspond to wavelength $k = 2$ and 4 radiation modes. Figure 2(b) shows the "chaotic" evolution at $\Gamma = 1.0$. Note that only a few collective modes dominate even this situation. Complete details and further examples are given in Ref. [9]

Here ϕ is a scalar field and x, t are space and time, respectively. Subscripts denote derivatives. $F(x, t)$ is a forcing field and ϵ a damping constant. Note that neither pattern selection nor chaos occur in (1) without dispersion [6], i.e. in the overdamped limit. In the four examples below we have studied eqn. (1) numerically on a finite line (length L) and with a high density spatial mesh (i.e., approximating the p.d.e.):

Case 1 Here $L = 24$ with 120 grid points and periodic boundary conditions, $\epsilon = 0.2$ and $F(x, t) = \gamma \sin \omega_d t$ with $\omega_d = 0.6$. The initial data is a static "pulse" profile (the actual shape of the pulse is not very important -- the number of co-existing attractors and hysteresis are small [8,9]). As reported elsewhere [8], this system undergoes a spontaneous spatial period doubling for $0.6 < \gamma < 0.9$ but remains simply periodic in time. Typical spatial profiles are shown in Fig. 1a, and should be thought of as "breather-soliton" wavetrains [7-10]. As $\gamma \rightarrow 0.9$, the duration of a "chaotic" initial transient diverges, resulting in temporal chaos for $0.9 < \gamma < 1.4$. Accompanying this chaos are large ($\gg 2\pi$) variations in the spatial average of $\phi(x, t)$ [denoted by $\phi(t)$] and large amplitude ($\sim 2\pi$) spatial variations in ϕ relative to ϕ . Instantaneous spatial correlation functions suggest [8] strong structural disorder. However, following the evolution in detail through a period of the driving field (Fig. 1b) shows transparently that the basic coherent structures in the quiescent pre-chaotic regime are preserved, but that their mode-locking relative to each other has been (chaotically) broken so that the structure fails to repeat by a small amount after each driver period. Consequently, we can decompose the field at each instant of time into either two "breather solitons" or two "kink-solitons" and two "antikink-solitons" (at instants of kink-antikink collision the field ϕ may appear to be flat -- see Fig. 1b). Furthermore, chaotic evolution of $\phi(t)$ through multiples of 2π does not take place via single particle dynamics but rather through the slow diffusion of the kinks (antikinks) -- as with thermally assisted transport in such systems [11]. We have confirmed the identification of a small number of coherent "soliton" modes in the chaotic regime by projecting the field at successive times onto a true soliton basis [9]. An example is shown in Fig. 2. This small number of dominating collective modes suggests that the chaos will be governed by a low-dimensional strange attractor [12]. We have checked the dimension (v) using the algorithm of Grassberger and Procaccia [13]. We estimate [13] that $v(\gamma = 1.0) = 2.5 \pm 0.3$, and in fact the dimension is found to lie within this range throughout the chaotic regime. Thus v is indeed low [14]. [For $\gamma < 0.9$, v is 1.0, as expected. v also approaches 1.0 for large $\gamma \gg 10$ where the nonlinear potential is a small perturbation on the dynamics.]

Case 2 Here conditions are the same as case 1, except that the initial data is a single static kink with periodic boundary conditions mod (2π) . The attractor is found [8] to be a kink plus a "breather" and, for $\gamma \geq 0.9$, these coherent structures move randomly with respect to each other producing temporal chaos. The dimension of the chaos is again correspondingly low: e.g., for $\gamma = 1.0$, we estimate [13] $v = 2.6 \pm 0.4$.

Case 3 Here [15] we adopted boundary conditions appropriate to a finite Josephson junction oscillator in zero magnetic field [16], viz. $\phi(x=0, t) = \phi(x=L, t) = 0$. The number of grid points = 120, $L = 6$, $\epsilon = 0.1$, and $F(x, t) = \gamma_0 + \gamma \sin \omega_d t$ with $\gamma_0 = 0.35$. Even with single kink initial conditions i.e. on the first "zero field step" (ZFS) [16] a great variety of dynamical behaviors (including jumping between ZFS's) are observed [15]. For our present purposes we report an example which illustrates one prevalent source of chaos: for $\omega_d = 1.25$ and $\gamma = 1.2$, the kink initial data is attracted to a nonchaotic "symmetric" state [16] on the third ZFS as shown in Fig. 3a. Increasing γ to 1.5, we find a chaotic long-time evolution in which the strongly mode-locked coherent structures of the third ZFS are still dominant but their relative mode-locking has been (chaotically) broken -- see Fig. 3b. Consistently, we estimate [13] the attractor dimension to be low: $v = 2.5 \pm 0.4$.

Figure 3. Space-time evolutions of $\phi_1(x,t)$ for the SG system (Case 3) through 1.6 driving periods for $\epsilon = 0.1$, $\omega_d = 1.25$, with Neumann boundary conditions, dc driving $\gamma_0 = 0.35$, and ac driving strengths (a) $\gamma = 1.2$, which produces a nonchaotic standing wave state on the third ZFS, (b) $\gamma = 1.5$, which produces a chaotic pair of driven pulses

Figure 4. Space-time evolutions of the in-plane angle $\phi(x,t)$ for the easy-plane ferromagnet (Case 5) through two driving periods for $\varepsilon = 0.1$, $\omega_d = 0.05144$, with periodic boundary conditions, and ac driving strengths (a) $B_0^x = 0.03429$, resulting in a standing wave pattern, (b) $B_0^x = 0.02743$, resulting in intermittency between smooth standing wave structures and chaotic kink-antikink dynamics

Figure 5. Homogeneously driven, damped SG system in two space dimensions (a 10 x 10 square) with periodic boundary conditions -- case 6. Parameters are as for case 1 ($\varepsilon = 0.2$, $\omega_d = 0.06$) and a random pulse profile was used as initial data [19]. Here $\Gamma = 0.9$. The attractor is simply periodic in time but with a spontaneous higher spatial symmetry -- period $\sim 1/2$ on a $\pi/4$ -rotated $\sqrt{2} \times \sqrt{2}$ lattice. [Lines are ϕ -intensity contours: solid and dashed lines signify above and below spatial average ϕ in units of 0.04π]

$= 0$. Chaotic evolution is induced by choosing a spatially inhomogeneous driving field $F(x,t) = \gamma, x \in [15,25], F(x,t) = 0$ elsewhere. Our estimates [13] of ν for $0 \leq \gamma < 5$ show that ν increases from 1.0 at $\gamma = 0$ to 2.8 ± 0.4 for $\gamma = 2$ (with a small increase at the chaotic threshold $\gamma \sim 0.4$) and then rapidly decreases to 1.0 for larger values of γ (as in Case 1). Again the low dimension is entirely consistent with an examination of spatial profiles: once more ϕ evolves by the separation of coherent kink-antikink pairs which are periodically nucleated at the center of the line [17].

Case 5 Here we studied [18] a strongly perturbed magnetic chain of classical spins $S = (S^x, S^y, S^z) = (\cos\theta_n \cos\phi_n, \cos\theta_n \sin\phi_n, \sin\theta_n)$, $n = 1, 2, 3 \dots N$, governed by the Hamiltonian

$$H = - \sum_{n=1}^N \vec{S}_n \cdot \vec{S}_{n+1} + \alpha \sum_{n=1}^N (S_n^z)^2 - \vec{B} \cdot \sum_{n=1}^N \vec{S}_n, \quad (2)$$

where $\alpha > 0$ is an easy-plane anisotropy (ϕ_n and θ_n are the in-plane and out-of-plane angles) and $\vec{B}$ is an applied magnetic field. A Gilbert-Landau dissipation term [18] of strength ε was added to the equations of motion following from (2). Again a great variety of chaotic and non-chaotic evolutions are observed [18], depending on magnetic field configurations and parameter values. However, spontaneous pattern formation, consequent low-dimensional chaos, and coexisting coherence and chaos are once again prevalent phenomena. An example is illustrated in Fig. 4, where we have used periodic boundary conditions, $N = 150$, $\varepsilon = 0.1$, $\alpha = 0.1907$ (so as to relate to the material CuNiF_3 [18]), $\vec{B} = (B^x, B^y, B^z) = (B_0^x \sin(\omega_d t), 0, 0)$, $\omega_d = 0.05144$, and random initial data $\phi_n(t=0), \theta_n(t=0)$. In Fig. 4a, $B_0^x = 0.03429$. The period-4 spatial structure seen in Fig. 4a forms spontaneously as the long time attractor with simply-periodic entrained motion. Decreasing B_0^x to 0.02743 we have entered a chaotic regime characterized (Fig. 4b) by chaotic motions of the coherent nonlinear mode components of the precursor period-4 spatial pattern. The Grassberger-Proccaccia estimate [13] of the attractor dimension is correspondingly low --, e.g., for $B_0^x = 0.02743$, we find $\nu = 1.9 \pm 0.4$ for S^z .

Our final example, Case 6, corresponds to the SG Eq. (1) extended to two spatial dimensions. A 10×10 square was used [19] with 161×161 particles and various boundary conditions and initial data. The general scenario of space-time competi-

Figure 6. Same as Fig. 5 but with $\Gamma = 1.6$. After a "chaotic" initial transient, the time evolution is simply periodic and with a spontaneous higher spatial symmetry. However, there is also pattern conversion from a period-1/2 pattern on a $\sqrt{2} \times \sqrt{2}$ lattice (Fig. 6(a)) to a purely one-dimensional period-1/2 pattern (Fig. 6(b)). Figure 6(c) exhibits this pattern discrimination in terms of the total sine-Gordon energy as a function of time -- note that the one-dimensional attractor has a slightly lower energy

elsewhere, [19] but Fig. 5 shows a typical non-chaotic, higher spatial symmetry precursor attractor (periodic boundary conditions were used) which eventually (for $1.0 \leq \Gamma \leq 1.2$) gives way to low-dimensional ($v \sim 2.4 \pm 0.3$) chaos and coherent lowest-symmetry spatial patterns. The higher spatial dimension also admits new phenomena, including [19] transition from a metastable, simply periodic, higher symmetry two-dimensional pattern to a final attractor which is higher symmetry and simply periodic but with variations in only one spatial direction: an example is illustrated in Fig. 6.

In conclusion, we have illustrated with a diverse collection of driven, dissipative, dispersive p.d.e.'s that a small number of determining modes may be typically responsible for both spontaneous pattern formation in quiescent regimes and for low-dimensional chaos in subsequent chaotic regimes. We emphasize that complex temporal behaviors are certainly observed in general parameter regimes for our systems (subharmonics, quasi-periodicity, etc.), especially if strong symmetries are not imposed through boundary conditions or initial data [8,15,18]. Here, however, we have deliberately concentrated on relatively simple circumstances where the competition between pattern selection and complex time-dependence (which we consider of most fundamental importance) is isolated.

The symbiotic relationship between analysis and numerical simulation has a rich history in studies of strongly nonlinear systems (e.g. the Fermi-Pasta-Ulam problem, the recent history of exact soliton equations). The problems considered here are further fine examples, our results having stimulated new questions and directions of analytic attack on the part of mathematicians and physicists. The kinds of questions being addressed include: (i) Adequately identifying collective coordinates to suggest optimal mode reductions (and low-dimension map constructions) which can be compared with those of the full equation. An important issue here is to clarify the role of the extended modes. Are they or the collective modes the drivers or the slaves? Clearly, in general this is a grey area of self-consistent mutual renormalization with extended modes "dressing" the coherent structures. There is certainly evidence [8,20] for selected extended modes being strongly affected and playing a dominating role. Projections onto unperturbed soliton bases [9,21,35] will allow us to address this issue with more control; (ii) Understanding relationships between the coherent patterns we find and bounds [5] on the number of determining modes, fractal structures, Liapunov exponents [21,22], diffusion coefficients, and chaotic attractor dimensions; (iii) Elucidating criteria for pattern selection in nonlinear dispersive [6,23] systems such as that studied here.

Analytic approaches currently focus on: (a) Existence and (local) [24] stability analyses, which are particularly appropriate if the patterns are low-amplitude (e.g. for $\epsilon \ll 1$, $\gamma \ll 1$ and $w_d \ll 1$, for which a low-amplitude breather may stabilize). Techniques include analyses for a uniform state [25,26], one-breather state [25] (generalizing the approach of Kaup and Newell [27]), and multi-breather wave-trains (using, e.g., algebraic-geometric soliton representations for a periodic SG line [10]). Applications of the Melnikov test for "horseshoe" chaos [28] are also being attempted analytically and numerically; (b) Explicit truncations to a small number of coupled modes, e.g. a breather wave-train [25,33] (plus long-wavelength radiation if "background" or spatial-average motion is important), or small number of most unstable extended modes [20], or modes extracted numerically by projections onto unperturbed solitons [9,21,35]; (c) Adding thermal noise to the equation of motion to yield a Langevin description and allowing construction of a Fokker-Planck equation for the field's probability distribution. This last approach has the synergetic attraction that, after averaging over the noise distribution, non equilibrium statistical mechanics systems can [29] be cast in the form of a finite, anisotropic equilibrium Hamiltonian model in one higher spatial dimension. The Hamiltonians produced in the way [30] have the feature of "competing interactions" which is currently popular [31] in equilibrium statistical mechanics and responsible for (experimentally observed) inhomogeneous mean-field states. These synergetic parallels are evident in the attractors of 1-D studies for $\Gamma \approx 1.5$ [19] and even more strikingly in 2-D [19]

-- as for example in 2-D $\rightarrow$ 1-D pattern transitions (c.f. Fig. 6), which should be compared with, e.g., hexagonal $\rightarrow$ striped discommensuration array phase transitions in 2-D incommensurate systems [31].

Studying chaotic dynamics of p.d.e.'s and other many-degree-of-freedom systems is a most exciting and topical area in dynamical systems theory. Consequently, literature and understanding are rapidly changing, but we would be remiss not to conclude with a representative list of relevant publications. Classifying loosely by the type of equation, these include: SG-like systems with periodic [8,9,21], Neumann [15] or absorbing [17] boundary conditions, including 2-D [19] cases and discrete generalizations [32]; the cubic Schrödinger equation [33], including close analogs in plasma models [22] and bistable optical ring oscillators [20], and complex generalizations [34]; the Toda lattice and generalizations [35]; and classical spin chains [18]. We also recommend the somewhat different but complementary studies of coupled map lattices [36] and of cellular automata [37].

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4. Examples of coexisting coherence and chaos include: clumps and cavities in turbulent plasmas; filamentation in lasing mediums; large scale structures in turbulent fluids (e.g. modon "blocking" patterns controlling atmospheric flow and weather, or gulf stream "rings" in oceanography); and perhaps even the red spot of Jupiter! In all cases the coherent structures are long-lived and with slower dynamics than the single-particle turbulence -- a unifying practical concern is their effect on transport and predictability (in space and time). There are also increasing numbers of controlled laboratory scale observations (e.g. in convection cells, water wave surface solitons) as well as probable applications in biological contexts.
5. In some cases rigorous bounds on the number of determining modes have recently been established (e.g. C. Foias, et al., Phys. Rev. Lett. 50, 1031 (1983)), and it has been possible to bound the attractor (fractal) dimension by the number of determining modes (e.g. O. Manley, et al., preprint (1984), B. Nicolaenko and B. Scheurer, preprint (1984)). In certain cases (e.g. for some reaction-diffusions problems) even a truncated set of linear modes can be accurate. (See J. C. Eilbeck, J. Math. Biol. 16, 233 (1983), B. Nicolaenko et al., Proc. Acad. Sci. Paris 298, 23 (1984)).
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10. N. Ercolani, et al., preprint (1984).
11. See, e.g., M. Büttiker and R. Landauer, in "Physics in One Dimension", eds. J. Bernasconi and T. Schneider (Springer 1981).
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13. P. Grassberger and I. Procaccia, Phys. Rev. Lett. 50, 346 (1983). The calculation of various attractor "dimensions" remains in an early state of development (see Ref. 12). In particular, important questions remain such as sensitivity to the scale of structures and the spatial patterns (similar questions apply to Liapunov exponents -- see refs. 21, 22). We emphasize that our error estimates quoted here are realistically conservative. Typically we used an embedding dimension of 5-10 and 50,000 data points.

14. Assuming only 2 breathers (or 4 kinks) as a truncated modal set, the maximum dimension of the space containing the attractor is 8. Our initial data symmetry reduces this to 4. The presence of dissipation will typically further reduce the "active" dimension. Our estimates of ν are generally in the range 2 - 2.5. This is entirely reasonable in view of our estimates (unpublished) of ν for a chaotic single particle with similar damping and driving strengths: there the maximum dimension is 2 but we generally find $\nu \approx 1.1 - 1.3$.
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