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LONGITUDINAL MOTION IN CIRCULAR ACCELERATORS

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1. INTRODUCTION

Particle accelerators may be classified into high-voltage and cyclic types. High-voltage accelerators, in which particles are accelerated to high energy by a single traversal of a large dc potential difference, are limited by problems of holding voltage (sparking, corona, and so on) to energies of the order of tens of MeV and are of interest for higher energies only as injectors for cyclic accelerators. In addition, particles only pass through a high-voltage accelerator once and problems of particle motion in them are not very challenging or interesting, except perhaps at very high beam intensity.

Cyclic accelerators, in which a lower voltage is used over and over to accelerate particles to high energy, can be further classified into linear and circular accelerators. Particles only pass once through a linear accelerator, as in the high-voltage case, but there are interesting problems of particle motion because linear accelerators can be quite long, because there are complicating effects in the motion that should be understood, and because the requirements on the size and shape of the output beam are frequently stringent because of the uses to which it is to be put. The discussion here will be concerned with circular accelerators, but the methods and results can often be used for treatments of particle motion in linear accelerators.

Cyclic accelerators include induction accelerators, in which the accelerating electric fields are generated by changing magnetic fields, applying Faraday's law of electromagnetic induction. We shall not discuss induction accelerators, because longitudinal motion in them does not have the complications or the interesting features that it has in synchrotrons and storage rings.

A circular accelerator need not be precisely circular; in fact, the ideal path deviates significantly from a circle in most cases. But it is always a closed path and particles traverse paths in the neighborhood of this closed path as they are accelerated. Particles travel considerable distances during acceleration; in the largest accelerators, the distance travelled can be millions of miles or kilometers and the opportunities for loss in this long journey are many. Particle motion is therefore a topic of great importance in circular accelerators. Not every particle is injected with exactly the initial conditions of the ideal particle. The group of particles injected together must be kept together by some containment or focusing system if a useable intensity of particles is to reach the final energy. The design of

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the accelerator must include provision of a volume to contain this particle motion without interference arising from striking material objects. The focusing of particles is so important that many new kinds of accelerators have been invented that make use of well-known methods of acceleration, but differ in having improved means of focusing.

As it is accelerated, the ideal particle moves along a path prescribed by the designer. The motion of other particles about the ideal motion can be split into two separate motions that are largely decoupled (although related), the longitudinal motion along the direction of the prescribed ideal path and the transverse motion in the moving plane perpendicular to the prescribed ideal path. Because of the history of development of understanding of these motions, the transverse motion is frequently called betatron motion or betatron oscillations and the longitudinal motion is frequently called synchrotron motion, synchrotron oscillations, or phase oscillations.

Particles are accelerated as they pass through insulating gaps across which a voltage is applied. This voltage cannot be dc, but must oscillate in time in order that particles can continue to gain energy on every passage. The particles are shielded from the oscillating voltage by conducting walls (drift tubes) as they traverse the rest of the accelerator. Acceleration can take place only when the voltage is in the right direction, so particles are bunched in a fraction of the circumference.

Particles must stay in synchronism with the accelerating voltage, so its frequency must be at least as high as the frequency of revolution, in the kilohertz or megahertz radiofrequency range, depending on the size of the accelerator. The accelerating frequency can also be a harmonic h of the revolution frequency, in which case the particles are accelerated in h bunches equally spaced around the ring.

A bunch of particles necessarily contains particles of slightly different energies, because it is impossible to produce particles of exactly one energy for injection. Particles of different energies have different frequencies of revolution, because they have different speeds and because they do not necessarily travel the same path around the accelerator. Particles of higher energy go faster and therefore tend to have a higher frequency of revolution. On the other hand, particles of higher energy are bent less by the magnetic guide field, so they go on paths that have larger radii of curvature and are therefore longer. They therefore tend to have lower frequencies of revolution. Thus the two effects tend to cancel and a closer look is needed to determine the variation of frequency f with energy E .

Let us consider the effects of this variation. In Fig.1, we show an accelerating wave as a function of time. On succeeding passages, particles come back to positions on this wave. Let us suppose first that $\frac{df}{dE} > 0$, that is, that a particle of higher energy goes around faster. We show two particles, 1 and 2, starting at the same time on one traversal. Particle 2, of higher energy, then arrives at the accelerating gap earlier than particle 1

and is therefore lower on the wave. It then receives less voltage and the energy difference between 1 and 2 is reduced. They are focused in energy. Particle 2 will now go around less rapidly than before and will also be focused in phase relative to particle 1. The bunch of particles can stay together in energy and phase as they are accelerated.

This result is fine for $\frac{df}{dE} > 0$. We must also consider the case when $\frac{df}{dE} < 0$. The same arguments apply and there is focusing if the particles now ride on the back of the voltage wave (the descending half) rather than the front. Thus there is always a position on the wave that gives stable acceleration in any accelerator configuration.

The cancellation of speed-difference and path-length effects was an essential element in Lawrence's original conception of the cyclotron in 1930, because it made it possible for him to use continuous rf and to avoid frequency modulation. But the cancellation begins to fail as effects of the relativistic increase of mass with velocity become noticeable. This limits Lawrence's cyclotron to energies of 20 to 25 MeV. In 1944 and 1945, Veksler and McMillan independently developed (Reference 1) the principle of phase stability, which we have just described. This principle is an essential element in the operation of the synchrocyclotron,* the synchrotron,* the storage ring, and linear accelerators, both travelling wave and standing wave.

In a linear accelerator, all particles travel the same path and the only factor that affects the frequency with which they cross accelerating gaps is the change of speed. Thus $\frac{df}{dE}$ is positive in linear accelerators, but becomes small as the particle speed approaches the speed of light. In a cyclotron, the two effects cancel at all energies and $\frac{df}{dE} = 0$. In a weak-focusing accelerator, orbits are widely spread radially because the focusing is weak and the difference in path length always has a larger effect than the difference in speed, so $\frac{df}{dE} < 0$ at all times. In a strong-focusing (alternating-gradient) synchrotron or storage ring, orbits of different energies are closely packed and the effect of different speeds is greater at low energy. But the difference in speeds for a given energy difference decreases as the particles become relativistic and there is therefore an energy, the transition energy, at which the effects of different speeds and different path lengths cancel. At this energy, the phase of the accelerating radiofrequency voltage must be changed in order to keep particles bunched and focused longitudinally. The transition energy is less than the injection energy in electron rings, so the transition does not need to be considered for electrons. When strong focusing was first invented, this phase shift was a matter of concern, because it was not known whether the shift could be carried out quickly enough to avoid debunching the particles, but it has not been difficult to do in practice. We shall return to this subject more

*McMillan introduced these terms because the motion reminded him of the hunting in phase of a synchronous motor.

quantitatively below.

Although electron rings do not have the complication of transition energy, they do have a complication in that the damping of particle motion resulting from synchrotron radiation is of great importance. The qualitative picture of longitudinal motion is not changed, but damping helps to stabilize the motion.

The general picture of longitudinal motion that we have sketched in this introduction is that of an ideal particle, the synchronous particle, exactly in step with the radiofrequency field and a domain in phase and energy, a bucket, around this particle within which particles are focused around the synchronous particle. The rest of this paper has the purpose of making this general picture more precise and quantitative. In Chapter 2 we discuss the equations of longitudinal motion. In Chapter 3 we discuss their solutions and the resulting motion. In Chapter 4 we discuss applications and amplification of the theory.

2. EQUATIONS OF MOTION

2.1 One-Dimensional Motion of a Particle in a Radiofrequency Field

We treat the longitudinal motion alone, with no coupling to the transverse degrees of freedom.* We therefore neglect the transverse motion and consider the one-dimensional motion of a particle of charge e and rest mass m in an external radiofrequency (rf) field. We call the position of the particle along its path s and its momentum p . The equations of motion of a relativistic particle are

$$\begin{aligned}\dot{s} &= p/m\gamma \\ \dot{p} &= e E_{\parallel},\end{aligned}\tag{2.1}$$

where dots denote derivatives with respect to time, $\gamma = (1 - \dot{s}^2/c^2)^{-1/2}$ is the relativistic mass factor and $E_{\parallel} = E_{\parallel}(s, t)$ is the longitudinal rf electric field. There are also forces from rf magnetic fields, but they do not appear in this one-dimensional model. The first of Eqs. (2.1) defines the momentum and the second is just Newton's second law for a relativistic particle in an electric field.

In a circular accelerator, the ideal orbit is a closed path with a definite length or circumference C . This path is usually not a circle, but we can define an average radius by $R=C/2\pi$. The rf field is periodic in s with period C , as well as periodic in t with period $2\pi/\omega_{rf}$, where ω_{rf} is the

*The interested reader may refer to Ref. 7 for derivation of the synchrotron equations from a relativistic three-dimensional Hamiltonian.

applied circular frequency. It is useful, although not compulsory, to discuss longitudinal motion in a circular accelerator with a scaled angular variable

$$\Theta = s/R. \quad (2.2)$$

Note that if the ideal path in the accelerator is not exactly circular, then Θ is not identical with the azimuthal angle, but both angles increase by 2π per revolution, so they are equal on the average. The momentum corresponding to Θ is an angular momentum

$$P_{\Theta} = R p \quad (2.3)$$

and the equations of motion become

$$\dot{\Theta} = P_{\Theta} / m \gamma R^2 \equiv \omega \quad (2.4)$$

$$\dot{P}_{\Theta} = e R E_{\ell}.$$

In the first equation, the angular frequency of revolution ω is introduced. It should be noted that ω is a function of the angular momentum P_{Θ} or the energy E . In the Introduction, we discussed the effects of the dependence of frequency on energy. Both the particle frequency ω and the accelerating frequency ω_{rf} can vary with time as the particles are accelerated and the accelerating frequency is modulated to track the particles. We write the phase of the accelerating field as

$$\psi(t) = \int \omega_{rf} dt \quad (2.5)$$

to include this frequency modulation.

The accelerating field is produced by a number of rf systems distributed around the circumference. It is by no means necessary that they be distributed uniformly; in a large accelerator there are logistic advantages from concentrating the rf systems in one location on the circumference. The distribution of rf systems affects the coupling between synchrotron and betatron motion, but this is not usually a serious problem. However the systems are distributed, E_{ℓ} is still periodic in Θ and t . Then E_{ℓ} can be expanded in a Fourier series

$$E_{\ell} = \sum_n E_n \cos(n\Theta + \delta_n) \sin \psi(t) \quad (2.6)$$

$$= \frac{1}{2} \sum_n E_n [\sin(\psi + n\Theta + \delta_n) + \sin(\psi - n\Theta - \delta_n)],$$

where n goes through all positive integers. Thus the accelerating field can be expressed as a series of travelling waves. Note that we have implicitly assumed in writing Eq.(2.6) that there is only one accelerating frequency, which is usually the case. Only one component of the travelling wave, that

is, one n , will travel in synchronism with the particles over long periods of time and accelerate particles. To see this, we can write formally from Eq. (2.4),

$$P_{\Theta} \sim \frac{1}{2} eR \sum_n E_n \int_0^t [\sin(\psi + n\Theta + \delta_n) + \sin(\psi - n\Theta - \delta_n)] dt.$$

If $\frac{d}{dt}(\psi - n\Theta)$ differs appreciably from zero, the sine function will oscillate in time and the integral will remain of the order of unity. But if for some value of n , say $n = h$, $\psi - h\Theta$ remains small, the integral will have a leading term proportional to t , reaching large values as t grows large. This is the component of the field that accelerates particles to higher energy. All other terms pale by comparison and we neglect them. The integer h that describes the accelerating component is called the harmonic number. It is the number of rf waves around the circumference and satisfies $h\omega = \omega_{rf}$ for the ideal particle.

We are considering the motion of particles relative to the rf and we therefore introduce a phase variable

$$\phi = \psi(t) - h\Theta + \delta, \quad (2.7)$$

which is the phase of the accelerating rf field relative to that of a particle (ϕ is sometimes defined as the negative of our ϕ). Here δ is a phase constant that will be used to simplify the equations. We choose $\delta = -\delta_h$ and the accelerating field can be written $\frac{1}{2} E_h \sin \phi$. The line integral of the accelerating field $E = \frac{1}{2} E_h$ around the circumference is the voltage per revolution or per turn $V = CE = 2\pi RE$ and we can also write the field as $V \sin \phi / 2\pi$. It should be noted that the choice of the phase constant to write the field in terms of $\sin \phi$ is a convention, not a physical necessity. In fact, it is customary in linear accelerators to choose a convention that uses $\cos \phi$ rather than $\sin \phi$; it was customary in older cyclotron papers to choose to use a phase equal to our $\pi - \phi$. These differences lead to some mystification in the literature.

The equations of motion are now

$$\begin{aligned} \dot{\phi} &= \omega_{rf} - h\omega \\ \dot{P}_{\phi} &= \frac{-\dot{P}_{\Theta}}{h} = \frac{eV}{2\pi h} \sin \phi \end{aligned} \quad (2.8)$$

These compact equations are the basis for discussion of longitudinal motion in an accelerating field. The canonical variable P was introduced by Symon and Sessler (Ref. 2); it was called W in their work.

2.2 Equations of Motion and Hamiltonian Mechanics

The equations we have derived above, beginning with Newton's law in Eq.(2.1), are Hamiltonian in nature. Equations (2.1) are derivable from the Hamiltonian

$$\begin{aligned} H &= E(p) + e\phi(s, t) \\ &= \sqrt{c^2 p^2 + m^2 c^4} - e \int_0^s E_L(s, t) ds. \end{aligned} \quad (2.9)$$

The form of this Hamiltonian is not unique. There are many other Hamiltonian functions from which Eqs. (2.1) can be derived, related to one another by gauge transformations.

The change from s to the angular variable Θ is a canonical transformation derivable from the generating function

$$\begin{aligned} F_2 &= F_2(s, P_\Theta) = sP_\Theta/R \\ \frac{\partial F_2}{\partial s} &= p = P_\Theta/R \\ \frac{\partial F_2}{\partial P_\Theta} &= \Theta = s/R \end{aligned} \quad (2.10)$$

and the new Hamiltonian is

$$\begin{aligned} H &= E(P_\Theta/R) + e\phi(\Theta, t) \\ &= \sqrt{c^2 P_\Theta^2/R^2 + m^2 c^2} - eR \int_0^\Theta E_L(\Theta, t) d\Theta. \end{aligned} \quad (2.11)$$

The change from Θ to ϕ is also a canonical transformation derivable from the generating function

$$\begin{aligned} F_2(\Theta, P_\phi) &= P_\phi(\psi(t) - h\Theta + \delta) \\ \frac{\partial F_2}{\partial \Theta} &= P_\Theta = -hP_\phi \\ \frac{\partial F_2}{\partial P_\phi} &= \phi = \psi(t) - h\Theta + \delta \\ \frac{\partial F_2}{\partial t} &= \omega_{rf} P_\phi \end{aligned} \quad (2.12)$$

and the Hamiltonian from which the equations of motion (2.8) are derived is

$$H = \sqrt{\frac{c^2 h^2}{R^2} P_\phi^2 + m^2 c^4} + P_\phi \omega_{rf} + \frac{eV}{2\pi h} \cos\phi, \quad (2.13)$$

where we have dropped all electric-field components except the traveling wave synchronous with the ideal particle. The most important consequence of the existence of a Hamiltonian is that phase space, the space of the two canonical variables, (θ, P_θ) or (ϕ, P_ϕ) , is preserved during the course of the motion. We shall return to this in the discussion of the motion in Sec. 3.2 below.

2.3 Transit Time

The rf accelerating system units must each include an insulating gap across which the rf voltage appears. But this gap is finite in length and the rf field changes in the time during which a particle is crossing the gap. What effect does this have on the acceleration? To consider this question, let us start from the second of Eqs.(2.8). The change in angular momentum during a gap crossing is

$$\Delta P_\phi = eR \int E_\theta dt$$

We take the gap to be centered at the origin and to extend from $-g/2$ to $g/2$ in length, that is, from $-g/2R$ to $g/2R$ in θ , and we change the integration variable from t to θ in order to make use of these limits. We assume the change in ϕ during the gap crossing is small. Then

$$\begin{aligned} \Delta P_\phi &= \frac{ehRE_0}{\omega_{rf}} \int_{-\frac{g}{2R}}^{\frac{g}{2R}} \sin(h\theta + \phi) d\theta \\ &= \frac{ehRE_0}{\omega_{rf}} \left\{ \sin\phi \int_{-\frac{g}{2R}}^{\frac{g}{2R}} \cos h\theta d\theta + \cos\phi \int_{-\frac{g}{2R}}^{\frac{g}{2R}} \sin h\theta d\theta \right\}. \end{aligned}$$

The second integral vanishes by symmetry. The peak voltage across the gap is $V = E_0 g$ and we can write our result as

$$\Delta P_\phi = F \left[\frac{ehV}{\omega_{rf}} \sin\phi \right] \quad (2.14)$$

where

$$F = \frac{\sin \frac{hg}{2R}}{\frac{hg}{2R}} \quad (2.15)$$

The quantity F is the transit-time factor. It is a function whose maximum value is unity, occurring at $g = 0$. It decreases slowly as g increases, reaching the value 0.84 at $hg/2R = 1$. The major effect of finite gap length on particle motion is that the effective voltage experienced by a particle is decreased by a factor F at the gap crossing.

2.4 Synchrotron Radiation

Charged particles radiate when accelerated. Energy is carried away by the radiation and this energy comes from the kinetic energy of the radiating particle. This energy must be made up if the particle is to be accelerated to higher energy or even stored at fixed energy.

The energy radiated by a particle during the acceleration from gaining energy is extremely small, but the energy radiated by a particle from the acceleration during bending in the magnetic field of a circular accelerator can be large and have profound effects.

The instantaneous power radiated by a particle of mass m , charge e , and total energy E being accelerated with a radius of curvature ρ is

$$P_Y = \frac{dE}{dt} = \frac{cC_Y}{2\pi} \frac{E^4}{\rho^2}, \quad (2.16)$$

where $C_Y = \frac{4\pi}{3} \frac{r_e}{(mc^2)^3}$

and $r_e = \frac{e^2}{mc^2}$

is the classical particle radius. The energy lost per revolution is

$$E_t = RC_Y E^4 / \rho^2.$$

For electrons, $C_Y = 8.85 \times 10^{-5} \text{ m/GeV}^3$. For protons, $C_Y = 7.79 \times 10^{-18} \text{ m/GeV}^3$. Synchrotron radiation by protons has barely been observed and plays no significant role in present proton accelerators; it may be more of a factor in the next generation of proton rings. But synchrotron radiation by electrons rapidly becomes a dominating effect on the design of electron and positron rings as the energy is raised beyond a few hundred MeV, because the power radiated increases so rapidly with energy.

The radiation is largely emitted within a cone of opening angle mc/E , so the act of radiation does not appreciably change the direction of motion of the emitting particle, but reduces its energy. Higher-energy particles lose more energy than lower-energy particles and, as a result, not only is the average energy of a group of particles decreased by radiation, but in addition, the energy spread of the group is also reduced. Thus synchrotron radiation damps the energy spread of a particle group. It also has effects on

betatron oscillations, which we shall not discuss here. One general consequence is that the motion is no longer Hamiltonian and conservation theorems no longer apply. Furthermore, synchrotron radiation is emitted, not continuously, but in discrete quanta and the fluctuation effects of this discreteness determine the ultimate minimum beam size.

The effects of synchrotron radiation are of interest only when it is large, that is, when $E \gg mc^2$ and therefore $P_\phi \cong RE/c$. Then

$$\frac{dP_\phi}{dt} = \frac{R}{c} \frac{dE}{dt} = \frac{KP_\phi^4}{\rho^2},$$

where

$$K = c^4 C_Y / 2\pi R^3.$$

There is thus another term in the time rate of change of P_ϕ . Equations (2.8) become

$$\begin{aligned} \dot{\phi} &= \omega_{rf} - h\omega \\ \dot{P}_\phi &= \frac{eV}{2\pi h} \sin\phi - \frac{KP_\phi^4}{\rho^2} \end{aligned} \quad (2.17)$$

3. DYNAMICS OF LONGITUDINAL MOTION

3.1 Equilibrium motion and the Synchronous Particle

There is an equilibrium motion of particles obeying Eqs.(2.8), but it is not always a static equilibrium. The angular momentum P_ϕ and the energy E can increase monotonically for the equilibrium particle; they may also be constant in the case of motion in a storage ring.

We define equilibrium to be motion at a constant phase angle. The equilibrium particle moves in synchronism with the travelling wave and we call it the synchronous particle. We denote it by the subscript s . Thus we have

$$\begin{aligned} h\omega_s &= \omega_{rf} \\ \dot{P}_{\phi s} &= \frac{eV}{2\pi h} \sin\phi_s \end{aligned} \quad (3.1)$$

and

$$P_{\phi s}(t) = \frac{e}{2\pi h} \sin\phi_s \int_0^t V(t) dt \quad (3.2)$$

In a proton storage ring, the equilibrium particle does not gain or lose energy and the equilibrium phase angle $\phi_s = 0$.

The equilibrium is changed somewhat by synchrotron radiation. Extra rf voltage must be supplied to make up the energy lost in the second term of Eq.

(2.17), but the equilibrium is still motion at a constant phase angle. In an electron storage ring, voltage is needed for this purpose, and if E_t is the energy lost per turn to synchrotron radiation, the synchronous phase angle needed is determined by $E_t = eV \sin \phi_s$.

The synchronous particle in a circular accelerator must also obey the magnetic-field relation $cp_s = e\rho B$ or $cP_{\phi_s} = e\rho RB$ (Gaussian units). We can differentiate this relation and use the second of Eqs.(2.8) to find a relation between the rf voltage and the rate of change of the magnetic guide field in a synchrotron

$$\frac{e}{c} \rho R \dot{B} = \frac{eV}{2\pi} \sin \phi_s \quad (3.3)$$

There is a minimum rf voltage that must be maintained to accelerate the particles in step with the magnetic field in a synchrotron. If the voltage falls below this minimum, particles will not gain enough energy to stay on or close to the ideal orbit and will spiral inward in radius and be lost on the inner-radius wall of the vacuum chamber. On the other hand, there is no maximum rf voltage. If a higher voltage is applied, the equilibrium will move to smaller ϕ_s to satisfy Eq. (3.3).

3.2 Motion About Equilibrium

3.2.1 Equations of motion. We introduce the difference in momentum from the synchronous value

$$\Delta P_{\phi} = P_{\phi} - P_{\phi_s} \quad (3.4)$$

and the equations of motion now become

$$\begin{aligned} \dot{\phi} &= \omega_{rf} - h\omega \\ \frac{d}{dt} (\Delta P_{\phi}) &= \frac{eV}{2\pi h} (\sin \phi - \sin \phi_s) \end{aligned} \quad (3.5)$$

The transformation (3.4) is canonical and can be derived from the generating function

$$F_2 = F_2(\phi, P_{\phi}) = \phi(P_{\phi} - P_{\phi_s}) \quad (3.6)$$

and the new Hamiltonian is

$$H = E\left(\frac{P_{\phi_s}}{R} + \frac{\Delta P_{\phi}}{R}\right) + \omega_{rf} \Delta P_{\phi} + \frac{eV}{2\pi h} (\cos \phi + \phi \sin \phi_s) \quad (3.7)$$

(The term $-\omega_{rf} P_{\phi_s}$ does not contribute to the equations of motion and is omitted.) If we include synchrotron radiation, there is of course no Hamiltonian, but the equations of motion can be written as

$$\dot{\phi} = \omega_{rf} - h\omega$$

(3.8)

$$\frac{d}{dt} (\Delta P_\phi) = \frac{eV}{2\pi h} (\sin\phi - \sin\phi_s) - \frac{4KP_\phi^3}{\rho^2} \Delta P_\phi$$

The momentum difference ΔP_ϕ is small compared with $P_{\phi s}$ in most cases of interest and the equations can be simplified by expanding $E(P)$ in a Taylor's series about $P_{\phi s}$. Then the equations of motion become

$$\begin{aligned} \dot{\phi} &= -h\mu \Delta P_\phi = \frac{h\omega_s \eta}{P_{\phi s}} \Delta P_\phi \\ \frac{d}{dt} (\Delta P_\phi) &= \frac{eV}{2\pi h} (\sin\phi - \sin\phi_s), \end{aligned} \quad (3.9)$$

where

$$\mu = \left(\frac{d\omega}{dP_\phi} \right)_s \quad (3.10 \text{ a})$$

and

$$\eta = \frac{P_s}{T} \left(\frac{dT}{dp} \right)_s = \frac{P_{\phi s}}{T_s} \left(\frac{dT}{dP_\phi} \right)_s = \left(\frac{P_{\phi s}}{\omega_s} \right) \mu \quad (3.10 \text{ b})$$

is the relative change of period T with momentum, which is called the dispersion. The relative change of frequency with energy κ is related to the dispersion η by

$$\kappa = \frac{E}{f} \frac{df}{dE} = -\eta/\beta^2. \quad (3.11)$$

We have already seen from qualitative arguments that the dispersion is important in discussion of synchrotron oscillations. It relates synchrotron and betatron oscillations and its calculation from betatron-oscillation theory will be discussed in the next section.

To first order in the momentum difference, $\Delta P_\phi = \Delta E/\omega$ and we can also write the equations of motion as

$$\dot{\phi} = \frac{h\eta}{P_{\phi s}} (\Delta E) = \frac{h\omega_s \eta}{\beta^2} \left(\frac{\Delta E}{E_s} \right) \quad (3.12)$$

$$\frac{d}{dt} \left(\frac{\Delta E}{\omega} \right) = \frac{eV}{2\pi} (\sin\phi - \sin\phi_s),$$

where we have used

$$P_{\phi s} = \frac{E_s \beta^2}{\omega_s}.$$

Equations (3.12) are the traditional form of the equations of motion, found, for example, in the classic paper of Courant and Snyder (Ref. 3) on the alternating-gradient synchrotron. These equations are not Hamiltonian in form; the importance of Hamiltonian form was not realized until the work of Symon and Sessler (Ref. 2).

3.2.2 Dispersion The dispersion η is the relative change of revolution period with momentum about the synchronous momentum at a given time. It is not in general the same as the change of revolution frequency with synchronous momentum. These two quantities are equal only when the guide field is fixed in time, as in a cyclotron or microtron. Conversely, in an electron synchrotron the particles are extremely relativistic and the revolution frequency does not change with momentum or energy, but η does.

To calculate η , we note that $T = C/v$ and

$$\eta = \frac{p}{T} \frac{dT}{dp} = \frac{p}{C} \frac{dC}{dp} - \frac{p}{v} \frac{dv}{dp}, \quad (3.13)$$

where v is the particle speed and C the orbit length (circumference) of a particle of momentum p . These two terms are just the effects discussed qualitatively in the Introduction. The second term can be calculated directly from relativistic kinematics, since $p = \gamma m v$. Thus

$$\frac{p}{v} \frac{dv}{dp} = \frac{1}{\gamma^2}$$

The second term is a property of the accelerator structure. At any given guide-field level B , there is a closed orbit, the ideal orbit, corresponding to a momentum p_s and other closed orbits corresponding to other momenta $p = p_s + \Delta p$. A particle whose momentum is p executes betatron oscillations about its closed orbit and the total motion is the sum of betatron and synchrotron motions. If the relative field gradients do not change with momentum, the relative configuration of the closed orbits of different momenta is independent of momentum. If this is the case, then there is a parameter of the accelerator structure

$$\alpha = \frac{p}{C} \frac{dC}{dp} \quad (3.14)$$

called the momentum-compaction factor. The name is historical and has one unfortunate aspect. The parameter increases when orbits of different momenta are more widely spread apart and thus describes dilation, not compaction. Sands has suggested calling it the "momentum-dilation factor" for this reason. Whatever it is called, $\alpha = 0$ for a linear accelerator and $\alpha = 1$ for a cyclotron. For synchrotrons and storage rings, α can be calculated from betatron-oscillation theory. If the ideal closed orbit, assumed to lie in a plane, is used as the basis of a coordinate system with radius of curvature $\rho = \rho(s)$, the linearized equations of motion of a particle of momentum p are

$$\frac{d^2x}{ds^2} + \left(\frac{1}{\rho^2} + K(s)\right)x = \frac{1}{\rho} \frac{\Delta p}{p}$$

$$\frac{d^2z}{ds^2} - K(s)z = 0, \quad (3.15)$$

where x is the displacement from the reference orbit in the plane of that orbit, z is the displacement perpendicular to the plane of the reference orbit and

$$K(s) = \frac{1}{B\rho} \frac{\partial B}{\partial x} \quad (3.16)$$

is the magnetic-field gradient. The opposite signs of $K(s)$ in the two equations follow from Maxwell's equations and are a fundamental property of betatron oscillations.

The z motion (traditionally called the vertical motion) does not depend on momentum; all orbits of different momentum lie in the plane of curvature. We do not need to discuss it any further. The x motion (traditionally called the horizontal or radial motion) can be split into two parts

$$x = x_\beta + D \frac{\Delta p}{p} \quad (3.17)$$

which satisfy the differential equations

$$\frac{d^2x_\beta}{ds^2} + \left(\frac{1}{\rho^2} + K(s)\right)x_\beta = 0$$

$$\frac{d^2D}{ds^2} + \left(\frac{1}{\rho^2} + K(s)\right)D = \frac{1}{\rho}. \quad (3.18)$$

The quantity $D(s)$ is the dispersion function. The reader should note that a confusion exists in the literature because $\eta = \eta(s)$ is sometimes used to denote the dispersion function, as well as for the dispersion defined in Eq.(3.10). If η is given as a function of s , it is the dispersion function we call D , not the dispersion.

Let us discuss the solution of the dispersion equation, the second of Eqs.(3.18). Both $K(s)$ and ρ are periodic functions of s and the homogeneous equation obeys Floquet's theorem, so that the form of the solution is known. The solutions of the homogeneous equation can be used in well-known ways to construct a Green's function solution of the inhomogeneous equation. Thus, given $K(s)$ and ρ of a particular accelerator structure, D can be calculated. It can also be computed by any betatron-motion computer program and is most often done in this way. If D is known, the orbit length C can be calculated. The element of length along an orbit $x = x(s)$ is

$$[(1 + x/\rho)^2 + x'^2]^{1/2} ds. \quad (3.19)$$

The angle $x' = dx/ds$ that the orbit makes with the reference orbit is small and appears in the length (3.19) squared, so that it is negligible compared with the first-order term x/ρ . Then

$$C = \int_0^{2\pi R} (1 + \frac{D}{\rho} \frac{\Delta p}{p}) ds \quad (3.20)$$

and

$$\alpha = \frac{p}{C} \frac{\Delta C}{\Delta p} = \frac{1}{2\pi R} \int_0^{2\pi R} \frac{D}{\rho} ds. \quad (3.21)$$

Let us consider the simple case of an azimuthally symmetric ring (for example, a weak-focusing synchrotron without straight sections). Orbits of different momenta are all circles. The solution of the inhomogeneous equation (3.18) is

$$D = \frac{\rho}{1 + \rho^2 K} \quad (3.22)$$

and

$$\alpha = \frac{1}{1 + \rho^2 K} \quad (3.23)$$

where now, of course, both ρ and K are constants. The quantity $\rho^2 K$ is just the negative of the relative field gradient n first introduced by Kerst in 1941 and conventionally used in weak-focusing discussions. The condition $0 < n < 1$ must be satisfied in order to have transverse stability in both directions. Then $1 + \rho^2 K = 1 - n < 1$ and $\alpha > 1$ in this weak-focusing case. The second term in η is then always larger than the first and η is always negative. Thus df/dE is negative and focusing occurs on the back of the voltage wave.

In contrast, $\alpha \ll 1$ in a strong-focusing ring. An accurate calculation of α for a given ring must be done in detail, but we can find a moderately accurate value for α from approximate solutions of Eq.(3.18). We replace the alternating gradient coefficient by a harmonic-oscillator restoring force

$$\frac{v_x^2}{\rho^2},$$

where v_x is the number of horizontal waves of betatron oscillation per revolution. Thus we are smoothing out the alternating-gradient focusing. Then

$$D \approx \frac{\rho}{v_x^2}$$

and

$$\alpha \approx \frac{1}{v_x^2} \quad (3.24)$$

In a strong-focusing accelerator, $v_x \gg 1$ and therefore $\alpha \ll 1$. The first term of η decreases as the energy is raised by acceleration and therefore there is an energy γ_T , the transition energy at which

$$\gamma = \gamma_T = 1/\sqrt{\alpha}. \quad (3.25)$$

Below the transition energy, focusing occurs on the front side of the voltage wave, as in a linear accelerator and $\mu > 0$, $\eta < 0$, $\phi_s < \pi/2$. Above the transition energy, focusing occurs on the back side of the voltage wave, as in a weak-focusing synchrotron and $\mu < 0$, $\eta > 0$, $\phi_s > \pi/2$. At the transition energy, the synchronous phase ϕ_s must be shifted from ϕ_s to $\pi - \phi_s$, which is done by turning off the rf accelerating voltage at the time transition is reached and turning it on again at the new phase. Transition energy will be discussed in more detail in Sec. 3.2.6 below. Injection into electron rings takes place at large γ to satisfy minimum guide-field requirements. Thus $\gamma > \gamma_T$ and transition energy is not crossed in electron rings.

3.2.3 Small-amplitude motion. If the amplitude of the motion is small, we can write $\phi = \phi_s + \Delta\phi$ and neglect powers of $\Delta\phi$ above the first in the equations of motion (corresponding to quadratic terms in the Hamiltonian). Then Eqs. (3.5) can be combined to give harmonic-oscillator equations of motion in either $\Delta\phi$ or ΔP_ϕ :

$$\begin{aligned} \ddot{\Delta\phi} + \omega_\phi^2 \Delta\phi &= 0 \\ \ddot{\Delta P_\phi} + \omega_\phi^2 \Delta P_\phi &= 0 \end{aligned}$$

They oscillate with the same circular frequency

$$\omega_\phi = \sqrt{\frac{h\nu E \cos\phi_s}{2\pi}} = \omega_0 \sqrt{\frac{-h\nu E \cos\phi_s}{2\pi E_s}}, \quad (3.26)$$

where $\omega_0 = \omega_s/\beta$ is the circular frequency of revolution of a particle whose speed $v = c$. The oscillations in momentum and phase are out of phase with one another; each is maximum when the other is zero.

For a given $\sin\phi_s$, there are two possible choices of the synchronous phase angle, ϕ_s and $\pi - \phi_s$. The product $(-\eta \cos\phi_s)$ in the square root of Eq.(3.26) is positive if we choose ϕ_s when η is negative and $\pi - \phi_s$ when η is positive. Then the oscillation frequency ω_ϕ is real and there is stable, focused motion. It is important that this stability is present even when the frequency of revolution and the accelerating frequency are modulated during acceleration. This confirms quantitatively the results on focusing both above and below transition energy that we found qualitatively above. The frequency of synchrotron oscillations is small compared with the revolution frequency in

almost all rings. Because of the factor E_s in the denominator, it continues to decrease throughout acceleration. In the largest proton accelerators, ω_ϕ becomes as low as a few hundred Hertz at high energy. Even in electron rings, where V must be large to compensate for the energy lost to synchrotron radiation, ω_ϕ/ω_s is rarely larger than the order of 0.1. Thus synchrotron oscillations are very low in frequency compared with betatron oscillations; a particle goes through many betatron oscillations as it swings slowly through one synchrotron oscillation. The betatron oscillations therefore average out during a synchrotron oscillation and the two oscillations are rather independent. This independence does not hold when there is a resonance between the two oscillations. Then the averaging does not take place and the effects of betatron oscillations on synchrotron oscillations add up

3.2.4 Larger amplitudes. Solution of the nonlinear equations of motion (3.5) in the sense of finding $\phi = \phi(t)$, $\Delta P_\phi = \Delta P_\phi(t)$ is complicated; it is also unrewarding because we are not primarily interested in finding where individual particles are at arbitrary times. We are much more interested in finding the conditions for stability of the motion and finding the limits in ϕ and ΔP_ϕ of stable motion. We can learn these things about the motion from the Hamiltonian, either the one of Eq. (3.7) or the form with $E(P_\phi)$ expanded that gives the equations of motion (3.9)

$$H = -\frac{1}{2} \mu (\Delta P_\phi)^2 + \frac{eV}{2\pi h} (\cos\phi + \Gamma), \quad (3.27)$$

where $\mu = (d\omega/dP_\phi)_s$ and we have introduced the abbreviation

$$\Gamma = \sin\phi_s. \quad (3.28)$$

For definiteness, let us consider (3.27); our discussion will be equally applicable to (3.7). We discuss the motion in the $(\phi, \Delta P_\phi)$ plane, which is the phase plane because ϕ and ΔP_ϕ are canonically conjugate variables. ΔP_ϕ can vary from $-\infty$ to ∞ , but ϕ is only defined modulo 2π , so that $-\pi \leq \phi \leq \pi$. The phase plane is in effect wrapped around a cylinder of unit radius.

The parameters μ , V , and Γ in Eq. (3.27) can vary in time and therefore H is not exactly constant. But in practice this variation is very slow in the sense that the change in H in one oscillation period is small compared with H ; that is $\omega_\phi \dot{H}/H \ll 1$. Except for this adiabatic variation of parameters, the Hamiltonian is independent of time and is therefore a constant of the motion. (The same constant of the motion could have been derived from the equations of motion (3.9) by elementary methods, but Hamiltonian mechanics gives us insight into preservation of phase-space area, adiabatic variation of parameters, and other items of physical interest.) The motion of a given particle thus takes place along a curve of constant H . The curve is determined by the initial conditions of the particle; once a particle starts on a curve corresponding to a particular value of H , it stays on that curve, even when the curve moves slowly as a result of the adiabatic variations. Note that there is a slow

radial motion $D \Delta p/p$ at the phase-oscillation frequency. It is almost always small compared with the betatron motion.

Fixed points. Let us consider curves of constant H in the phase plane. To start, we can find points that remain constant in time, fixed points, by setting

$$\begin{aligned}\dot{\phi} &= 0 \\ \Delta \dot{P}_{\phi} &= 0.\end{aligned}\tag{3.29}$$

From the Hamiltonian (3.2.7),

$$\begin{aligned}\dot{\phi} &= \frac{\partial H}{\partial \Delta P_{\phi}} = -h\omega \Delta P_{\phi} \\ \Delta \dot{P}_{\phi} &= -\frac{\partial H}{\partial \phi} = \frac{eV}{2\pi h} (\sin\phi - \sin\phi_s).\end{aligned}\tag{3.30}$$

These equations are satisfied by

$$\Delta P_{\phi} = 0, \phi = \phi_s \text{ or } \pi - \phi_s.\tag{3.31}$$

These fixed points all lie on the line $\Delta P_{\phi} = 0$ ($P_{\phi} = P_{\phi_s}$).

The fixed point with $\phi = \phi_s$ is the equilibrium motion. The motion in the neighborhood of this point is the small-amplitude motion discussed in Sec. 3.2.3 above. It is represented by closed ellipses in the phase plane and is therefore stable in the sense that it is bounded. The value of H , which is stationary from Eq. (3.30), is a minimum. The frequency of motion around this stable fixed point is the small-amplitude frequency given by Eq.(3.26).

The second fixed point, separatrix, and buckets. As we increase the value of H , we go to larger amplitudes. The approximation of the $\sin\phi$ by its amplitude is less accurate; the motion is less linear and the phase curves depart from elliptical shapes. The frequency decreases from its small-amplitude value.

Eventually we must reach the value of H for the other fixed point given by Eq.(3.31). This value is

$$H = \frac{eV}{2\pi h} (-\cos\delta_s + \Gamma(\pi - \delta_s)).\tag{3.32}$$

There is a curve of constant H corresponding to this value. It goes through the second fixed point. This fixed point is also a stationary value of H , but in this case it is a maximum. Motion at larger amplitudes is not stable and can go to indefinitely large amplitudes. A particle that starts in the neighborhood of this curve on the unstable side will not remain in the neighborhood. The fixed point is therefore unstable and represents the limit in phase and angular momentum of useful phase oscillations. The curve through

the unstable fixed point separates stable and unstable regions and is called the separatrix. The stable area inside the separatrix is called a bucket. A particle inside the bucket will execute stable phase oscillations and will gain energy as the synchronous particle does. A particle outside the bucket will slip in phase relative to the rf and will not be accelerated. If the guide field increases in time, as in a synchrotron, a particle outside the bucket will eventually spiral in to the inner vacuum-chamber wall.

A particle that starts on the unstable fixed point remains on it and can never reach any other point on the separatrix. The motion can be reversed in time because it is Hamiltonian and thus reversible and thus no other point on the separatrix can reach the fixed point in a finite number of steps. The frequency of the motion on the separatrix is therefore zero, having decreased monotonically from its value at the central stable fixed point. Whereas the phase curves in the vicinity of the stable fixed point are elliptical, the phase curves in the vicinity of the unstable fixed point are hyperbolic. Figure 2 shows sketches of elliptic and hyperbolic fixed points.

Stationary and moving buckets. A system with $\Gamma = 0$ is called a stationary bucket. It does not accelerate the synchronous particle. The stable fixed point is at $\phi_s = 0$, of course, and the unstable fixed point is at $\phi = \pi$. It is also at $\phi = -\pi$, which is the same point, because the phase space is cylindrical. Figure 3 shows a stationary bucket.

A system with $\phi_s \neq 0$ is called a moving bucket. The synchronous particle is accelerated, as are all the particles inside the bucket. The stable fixed point is at $\phi = \phi_s$, of course. The unstable fixed point is at $\pi - \phi_s$ and the separatrix crosses the ΔP_ϕ axis at another value of $\phi = \phi_1$, the solution of

$$\cos \phi_1 + \Gamma \phi_1 = -\cos \phi_s + \Gamma(\pi - \phi_s) \quad (3.33)$$

Figure 4 shows a moving bucket.

The separatrix is given by the curve

$$-\frac{1}{2} h \nu (\Delta P)^2 + \frac{eV}{2\pi h} [\cos \phi + \Gamma] = \frac{eV}{2\pi h} [-\cos \phi_s + \Gamma(\pi - \phi_s)],$$

from which the upper and lower branches of $\Delta P_\phi(\phi)$ are

$$\Delta P_\phi(\phi) = \pm \Lambda [\cos \phi + \cos \phi_s + \Gamma(\phi + \phi_s - \pi)]^{1/2}, \quad (3.34)$$

where we have defined

$$\Lambda = \sqrt{\frac{eV}{\pi h^2 |\mu|}} = \sqrt{\frac{eVE_s}{\pi h |\eta|}} \quad (3.35)$$

The factor Λ appears in a number of bucket formulas. For example, the maximum and minimum values of ΔP_ϕ occur at $\phi = \phi_s$. They are

$$\Delta P_{\phi \max} = \pm \frac{2}{h\omega_0} \Lambda [\cos \phi_s - \Gamma(\frac{\pi}{2} - \phi_s)]^{1/2} \quad (3.36)$$

and

$$\Delta E_{\max} = \pm \frac{2\omega_s \Lambda}{h\omega_0} [\cos \phi_s - \Gamma(\frac{\pi}{2} - \phi_s)]^{1/2} \quad (3.37)$$

The phase-space area of one bucket is

$$A_1 = 2 \int_{\phi_1}^{\pi - \phi_s} \Delta P_{\phi} d\phi$$

The integral can be done exactly for $\Gamma = 0$ and we can write in general

$$A_1 = \frac{2}{h\omega_0} \Lambda \alpha_3(\Gamma), \quad (3.38)$$

where

$$\alpha_3(\Gamma) = \int_{\phi_1}^{\pi - \phi_s} [\cos \phi + \cos \phi_s + \Gamma(\phi + \phi_s - \pi)]^{1/2} d\phi \quad (3.39)$$

is the moving-bucket factor introduced by Symon and Sessler. It is very roughly a straight line of negative slope, going from $\alpha_3(0) = 1$ to $\alpha_3(1) = 0$. Thus bucket area vanishes at $\phi_s = 90^\circ$. It sags from a straight line, so that $\alpha_3(1/2) = 0.333$.

On the h^{th} harmonic, there are h buckets and the total bucket area is

$$A_h = hA_1 = \frac{2}{h\omega_0} \Lambda \alpha_3 \quad (3.40)$$

The bucket factor Λ is proportional to $h^{-1/2}$, so the energy spread and bucket area both decrease as the harmonic number is increased.

Given a ring and an accelerating system of given harmonic number h , the buckets are completely determined by the two parameters V and μ (or η). The synchronous phase angle ϕ_s follows as a consequence of Eq. (3.3) and all the bucket parameters can then be calculated from the results of this section.

If synchrotron radiation can be neglected, particles move on curves of constant H and the area of a bunch is preserved. But, as we have seen, synchrotron radiation introduces damping of the longitudinal motion. This damping is slow enough that there is still a bucket and stable motion takes place inside it, but now the bunch slowly shrinks down to a minimum size within the bucket. Damping helps to overcome the effects of rf noise and errors, as we shall discuss in Sec. 4.2, and makes beam stacking easier, as we shall discuss in Sec. 4.6.

3.2.5 Liouville's theorem. The area of a bucket is not constant in time, but changes as the parameters change. But, within limitations, the area occupied

by a bunch in a bucket is constant by Liouville's theorem, except for synchrotron radiation.

Stated rigorously, Liouville's theorem is that in a system of N particles the density of system points in $6N$ - dimensional phase space (γ space) in the neighborhood of a given point is a constant of the motion. Each system point in γ space represents the state of the complete system of N particles and Liouville's theorem is a statement about an ensemble of such systems. If the particles do not interact with one another, each particle can be considered a separate system and the local density of system points in 6-dimensional phase space (μ space) is also a constant. (In our discussion of one-dimensional motion, phase space is two-dimensional.) If further, the local density of points is constant over the area and if there is a very large number of points, the ensemble of N particles is an incompressible fluid in μ space and the total area is preserved. It can still become filamented by nonlinear forces and occupy a larger effective area, as happens when a system is subjected to non-adiabatic changes. We shall discuss this further in Sec. 4.3 below.

Any accelerating system must satisfy two requirements:

- (i) It must provide enough voltage to accelerate particles, keeping them within the confines of the vacuum chamber. It must provide this energy over and above any energy lost to synchrotron radiation.
- (ii) It must provide enough voltage to give sufficient bucket area to contain the particle-bunch area.

These two requirements are independent; one will require more voltage than the other at any given time. For example, early in the cycle of a sinusoidally excited synchrotron, more voltage is needed for bucket area than for acceleration.

Liouville's theorem does not hold rigorously for a finite number of particles and it is possible to increase the phase-space density by judicious application of external forces. This is the basis of stochastic cooling.

The bunch phase-space area is also called the longitudinal emittance. It and the bucket area are measured in units of electron-volt-seconds (which could be called electron-webers in MKS units). The emittance of a bunch is of the order of 1 eV-sec per bucket in a proton ring and considerably smaller in an electron ring because of synchrotron-radiation damping of phase oscillations.

3.2.6 Transition Energy. Our discussion here is confined to proton or heavy-ion rings, because electron rings always have injection energies above transition. As we saw in Eq. (3.25) and the discussion of it, the transition energy occurs when $\mu = \eta = 0$ at

$$\gamma = \gamma_T = 1/\sqrt{\alpha}$$

and focusing in a synchrotron shifts from the rising side of the rf voltage wave to the falling side. The rf phase must be shifted at that point by turning the accelerating voltage off and then turning it on again at the new phase. If Q is the quality factor of the system, a time of the order of Q cycles is needed for each of the turn-off and turn-on, although this time can be decreased somewhat by driving the system down in power. For a frequency-modulated system in the tens of MHz range, like the Brookhaven AGS or Fermilab Main Ring, the total time required is of the order of milliseconds and the bunches must stay together for this time without help from the rf system.

In practice, the bunches stay together very well. They are helped to do this by the fact that the bucket shape changes as γ_T is approached. We see from Eq. (3.35) that Λ and consequently the angular-momentum and energy spreads become large as η approaches zero. Phase-space area is preserved, so the phase extent of a bunch becomes smaller as ΔP_ϕ max increases. Thus the particles are very tightly bunched. They remain tightly bunched just because they are at transition and revolution-frequency differences are very small. There is enough time to make the needed phase jump. In fact, there is enough time to jump back and forth (the triple jump) to prevent beam blowup at high intensity. Even though there is time, it is difficult in practice to avoid some phase-space enlargement.

In the neighborhood of γ_T , it is no longer possible to truncate the expansion of ω as a function of ΔP_ϕ in the development that led to the equations of motion (3.9) and the Hamiltonian (3.27). Near γ_T , the Hamiltonian becomes

$$H = -\frac{1}{2} h\mu (\Delta P_\phi)^2 - \frac{1}{6} h\mu' (\Delta P_\phi)^3 + \frac{eV}{2\pi h} (\cos\phi + D), \quad (3.41)$$

where

$$\mu' = \left(\frac{d^2\omega}{dP_\phi^2} \right)_s \quad (3.42)$$

The topology of the phase plane is changed radically by the addition of the μ' term. New stable regions grow and then die away as γ_T is traversed. Examples of these are shown in Symon and Sessler (Ref. 2). One such example is shown in Fig. 5.

How does a cyclotron manage exactly at transition energy? In fact it manages just because it is not exactly at transition energy. Lawrence and Livingston found empirically in the first cyclotron that the guide field must decrease slightly with radius in order to have any beam accelerated. They understood qualitatively that this provided vertical focusing ($n > 0$ in the language of weak focusing). With this decrease, orbits of different energy are split farther, so that $df/dE < 0$. There is therefore a bucket, which is

stationary because the accelerating frequency ω_{rf} is constant. Particles are accelerated by moving from the bottom or middle to the top of this very high bucket. They execute only a fraction of a phase oscillation.

The accelerator type in which there really is no phase stability is a highly relativistic electron linear accelerator, where again $\mu = 0$. The electrons stay bunched because they are at transition energy. They are so relativistic that the debunching time is far longer than the transit time through the entire accelerator.

3.2.7 Adiabatic Damping. There is adiabatic damping of synchrotron oscillations, although it is not as marked an effect as in betatron oscillations. In the linear approximation, the Hamiltonian that gives the small-amplitude equations of motion is

$$H = \frac{1}{2} A (\Delta p_\phi)^2 + \frac{1}{2} B (\Delta\phi)^2 \quad (3.43)$$

where

$$A = -h\mu$$

$$B = \frac{eV}{2\pi h} \cos\phi_s.$$

This Hamilton gives ellipses in phase space. The maximum phase is

$$\Delta\phi_m = (H/B)^{1/2}$$

and the maximum momentum deviation is

$$\Delta p_{\phi m} = (H/A)^{1/2} = (B/A)^{1/2} \Delta\phi_m$$

The area A of the ellipse is constant. Then

$$A = \pi \Delta\phi_m \Delta p_{\phi m} = \pi (B/A)^{1/2} (\Delta\phi_m)^2$$

and

$$\Delta\phi_m(t) = \left[\frac{\mu(t)}{\mu(0)} \cdot \frac{V(0)}{V(t)} \right]^{1/4} \Delta\phi_m(0) \quad (3.44)$$

$$\Delta p_{\phi m}(t) = \left[\frac{\mu(0)}{\mu(t)} \frac{V(t)}{V(0)} \right]^{1/4} \Delta p_{\phi m}(0)$$

and we see that phase damping takes place slowly, largely because μ decreases during acceleration. Under the same conditions, the momentum spread Δp increases slowly, although the relative momentum spread $\frac{\Delta p}{p}$ usually decreases, except close to transition energy.

4 APPLICATION OF THE THEORY

4.1 Acceleration.

Acceleration systems in modern synchrotrons and storage rings usually have a number of accelerating stations to provide a total peak voltage of several million volts per turn. The exceptions are proton storage rings, which require no replacement of energy lost to synchrotron radiation, but need only minor amounts of rf voltage for stacking and beam-manipulation purposes. In electron synchrotrons and storage rings, considerable voltage is needed for synchrotron radiation; the loss is some millions of volts per turn in the GeV region. In proton or heavy-ion synchrotrons, it is desirable to accelerate at as high a rate as economically feasible to make the accelerator duty factor (the fraction of time during which beam is provided for experiments) as large as possible.

Acceleration takes place as particles traverse a cavity, a resonant rf structure containing one or more insulating gaps across which the accelerating voltage appears. The gap is also usually a vacuum seal separating the beam tube from the remainder of the cavity. The cavity is excited by a high-power amplifier chain which requires its own power supplies. An excellent discussion of cavity design and powering is given in Ref. 5.

Electrons in modern synchrotrons and storage rings are relativistic enough that no frequency variation is needed, but proton and heavy-ion accelerators often require frequency modulation. For example, the Brookhaven AGS and Fermilab Booster have proton injection energies of 200 MeV and highly relativistic peak energies, so that the final frequency is approximately 1.6 times the initial frequency. The Fermilab Main Ring has an injection energy of 8 GeV, so that approximately 0.5% frequency modulation is required. It would be possible to load cavities with resistance so they could operate untuned over these ranges, but the operating power required would be very large. These systems are tuned by varying the dc bias on ferrite loading of the cavity. A biasing system and power supplies for it are then required.

The presence of ferrite limits the acceleration frequency to 50 MHz or less, because the ferrite cannot be tuned at higher frequencies. The choice of accelerating frequency is also affected by the availability of power tubes for the desired frequency and by the total bucket area available, since it decreases with increasing h or frequency. Higher frequency is desirable because cavities occupy less space around the ring, but there must be enough bucket area.

The designer must also choose the stable phase angle at which the system is to operate. Larger gives a higher acceleration rate, but less bucket area, until at the maximum $\phi_s = \pi/2$, there is no bucket area left. A workable compromise in many cases is acceleration at $\phi_s = \pi/4$. It is not usually desirable, except in special cases, to operate with full buckets, because errors in phase and accelerating voltage can lose particles. Most systems

operate with buckets less than 75% full at maximum. Radiation damping in electron rings shrinks the bunch area rapidly and reduces the fraction of bucket area occupied.

The subjects of filling buckets at injection and stacking particle beams are discussed separately in Sections 4.3 and 4.6 below.

4.2 Low-Level and Feedback Systems.

Operation of the high-power rf system is controlled by the low-level system. The system must track the magnetic guide field B and this is usually done by continuous measurement of B or its time derivative dB/dt and integrating it.

This simple method would work well if the rf system were perfect; after all, phase oscillations are stable. But errors in frequency, voltage and phase can shake particles out of the bucket. There are also resonant effects when the magnet power-supply ripple shakes the beam at the phase-oscillation frequency. Feedback systems using non-destructive measurements of the beam are used to alleviate these practical problems.

Pickup electrodes of various configurations are used to detect the electric or magnetic fields of the particle bunch. The phase and radius of the centroid of the bunch are measured. The phase measurement is fed back to the frequency or phase control of the rf system to damp coherent phase oscillations. This system must have a low-frequency cutoff to allow for slow variation of acceleration parameters and must have sufficient bandwidth to extend well beyond the phase-oscillation frequency in order to provide damping. The phase system can be used to correct a master oscillator that controls the amplifier and cavity frequency and phase or the beam itself can be used as a master oscillator.

The phase system requires a bunched beam in order to have a signal and there must therefore be an independent way to start using an external frequency if the injected beam is unbunched. It is interesting to note that an efficient way to carry out this bunching in practice is to use the unstable fixed point as the center of bunching, because particles oscillate so slowly in its neighborhood.

If there were no simultaneous measurement of radius, the rf system would have no way of knowing what the accelerating frequency should be at any given time. The beam could then be marched radially out of the aperture. Measurement of the radius by a beam detector sensitive to it and feedback of this information to the rf system will keep the beam within the aperture. The radial system must allow the bunch to carry out phase oscillations and its high-frequency cutoff must therefore be much lower than the phase-oscillation frequency.

Feedback systems are used on almost all synchrotrons and storage rings as an efficient method of reducing the effects of rf noise, errors, and power-supply ripple.

4.3 Bunching

If a particle bunch fills a bucket uniformly up to some curve $H = \text{const.}$, it is matched to the bucket and the motion will continue in the bucket until the bucket parameters are changed. But how did it get settled down so snugly? How did it start? What will happen when the voltage is turned off?

What happens depends mostly on how rapidly the rf voltage is turned on and off, that is, on how adiabatic the process is. Let us consider the bunching and capture of an initially unbunched beam, which is a band in phase space that stretches through the full 2π in phase and has a height corresponding to the initial momentum spread $\Delta p_{\phi 0}$. Its longitudinal emittance is therefore $2\pi\Delta p_{\phi 0}$. We shall consider two cases, very slow turn-on and very fast turn-on.

If the turn-on is very slow, the change is adiabatic, that is $\Delta H/H \ll 1$ in one period of oscillation. (This is difficult at the beginning, when the voltage and H are very small and the phase-oscillation period is very long). The motion of each particle takes place along a curve of constant H . The end result at full voltage is a matched bunch of uniform density in an area bounded by the curve of constant H that has area $2\pi\Delta p_{\phi 0}$. We show a sequence of sketches of adiabatic turn-on in Fig. 6. Figures 6, 7, and 8 were generated for us by Dr. J.A. MacLachlan of Fermilab using his computer program ESME. Even though a few particles in the sample end up in the last frame outside the desired area, the slow turn-on is highly successful at preserving phase-space density.

Thus an ideal matched beam is achieved by very slow turn-on. How slow is very slow? Computer simulations show that a time of two phase-oscillation periods at the final voltage is sufficient for adiabaticity. In practice, adiabatic capture does not give minimum phase-space area in most cases even when this criterion is met, perhaps because it is so difficult to achieve very smooth turn-on of rf systems.

In contrast, consider the case of very rapid turn-on. We show a sequence in Fig. 7. Now the bucket appears full-blown superimposed on the initial unbunched beam. Within the 2π spread of the beam, every phase of the bucket is sampled by some particle. The result is that the particle distribution quickly becomes quite non-uniform. Particles initially near the center move on elliptical curves of constant H and the inner part of the distribution rotates around in a rigid band. But particles initially near $\phi_s = \pi$ move very slowly because they are close to the fixed point. They lag behind the rotation of the central part and the whole distribution shears and wraps itself around $\phi_s = 0$, the band becoming narrower as it elongates to preserve

area. Eventually, after some phase oscillations, the part of the distribution that was initially outside the bucket will have slipped in phase enough to be lost. The part of the distribution that was initially inside will have wrapped itself around enough that it will fill the bucket in an approximately uniform distribution. It must fill the entire bucket because there is no change in the system to bring particles from near the separatrix in toward the center. The bucket is now filled with particles at a lower phase-space density than that of the initial unbunched beam. In this way, rapid turn-on provides a simple example of how to dilute phase space, that is, to "stir in some empty space."

4.4 Debunching.

The length and phase spread of a bunch of particles moving through a linear transport system will increase as it goes through the system, because of the differences in speed of particles of different momentum. If synchrotron-radiation effects can be neglected, a bunch of particles circulating in a magnet ring without rf will also debunch; the rate of debunching must clearly be proportional to ΔP_ϕ . In both cases, the momentum spread will not change, because there is no force to change it. The bunch area will also not change, again because there is no external force to change it. The bunch shape in phase space will then be distorted as the upper and lower parts of the bunch drift in opposite direction, as we show in the first two frames in Fig. 8.*

Debunching by drifting can be used to measure the momentum spread of a bunch by measuring the change in phase spread as a function of time. It can be used to increase the phase spread of a bunch, but it does not reduce the momentum spread. If a group of bunches drifts long enough to overlap in phase, the group becomes in effect an unbunched beam occupying the same momentum spread and therefore a larger phase-space area than that of the initial bunches.

The momentum spread can be lowered by debunching by judicious use of an acceleration system. If, for example, a stationary bucket is superimposed on a tight bunch, as in the last two frames of Fig. 8, the bunch will rotate in phase space. After one-fourth of a phase oscillation, the phase spread will have been increased and the momentum spread decreased. This technique is useful when one rf accelerator is used to inject particles into a second rf accelerator of different accelerating frequency, a situation in which the bunching at the end of the first accelerator is not useful in the second. RF

*Which way they drift will depend on the sign conventions chosen and on whether the bunch is above or below transition energy. With the sign convention used here, below the transition energy faster particles will lose phase relative to the synchronous particle (see Eq. (2.7)) and the top of the bunch will therefore move to the left and the bottom to the right.

debunching can reduce the momentum spread injected into the second accelerator and therefore the voltage needed for bucket area.

A second application is to the cooling and capture of antiprotons in a storage ring for accumulation. The antiprotons are produced in a target with large momentum spread, but they can be produced with small phase spread by bunching tightly the particle beam (usually protons) used to produce them. The momentum spread can be converted to phase spread by phase-space rotation in a stationary bucket in the ring. The lower momentum spread takes the place of some of the momentum cooling that must be done on the antiproton beam.

4.5 Phase-Displacement Acceleration.

We have seen that a bunch inside a bucket moves up in momentum with the bucket and is accelerated. But what about particles outside the bucket? In most cases they slip in phase and are eventually lost, although they may stay around indefinitely in a fixed-field ring. There is an interesting special case. Suppose that a moving bucket passes through an initially unbunched circulating beam. As the bucket passes through, particles in the beam are displaced in phase because they cannot move from outside to inside the bucket. If the bucket starts far below and ends far above the beam, it will end as it began, unbunched. But it will not be at the same average momentum, because phase-space area is preserved and area has been carried by the bucket from below to above the beam. The beam is therefore decelerated; if the bucket carries an area A through the beam, the beam is decelerated by $\Delta P_\phi = A/2\pi$.

Conversely, if a bucket is carried down in momentum through the beam, the beam will be displaced upward in phase space, that is, accelerated. This phenomenon, phase-displacement acceleration, has been used, for example, to make up the energy lost to synchrotron radiation in a stored electron beam.

4.6 Beam Stacking.

The idea of using colliding beams to give much higher energies for producing new particles was well known for many years, but the interaction rate is proportional to the density of particles in the target and the density of particles in an accelerated beam used as a target is too small by many orders of magnitude for any events to be detected in the presence of background interactions with residual gas in the vacuum chamber. But, as was first understood by Kerst, it is possible through beam stacking to produce particle densities large enough that colliding-beam experiments are feasible. It is now generally believed that colliding-beam experiments will be the method of choice for reaching much higher energies.

The elementary concept of beam stacking is that groups of particles are accelerated to a desired energy and left to circulate in a fixed magnetic field. Subsequent groups are accelerated and deposited adjacent to the first group and to each other in phase space to form the stack. The circulating

bunches overlap in physical space to increase the particle density. The overlap can be increased by a guide-field lattice that has regions of small D , thereby reducing the radial spread from momentum spread in the beam. Betatron oscillations can also be overlapped in regions of small amplitude function β .

Beam stacking is made much easier for electrons by radiation damping. Damping continually reduces the momentum spread and radial size of an electron beam and its final size is limited only by quantum fluctuations.

For protons or other heavy particles without useful synchrotron radiation, Liouville's theorem holds and area is preserved in longitudinal phase space. If the system starts from an unbunched beam with momentum spread ΔP_ϕ at injection (phase-space area $2\pi\Delta P_\phi$), the minimum momentum spread after stacking n groups is $n\Delta P_\phi$. Each group must be brought to the stack in full buckets in order to avoid mixing in empty space. Even with care, each group brought to the stack will decelerate the stack slightly by phase-displacement acceleration and will scatter the stack edges out slightly. Simulations and experimental practice show that if approximately 10 or more groups are stacked, the density at the center of the stack approaches the theoretical maximum.

There are other forms of stacking that are used in other applications. Bunches can be transferred from buckets in one accelerator to buckets in another and stacked longitudinally around the circumference of a larger accelerator. For example, 13 batches of 3-GeV protons from the Fermilab Booster, each containing 84 bunches, are injected sequentially into the Fermilab Main Ring while the guide field is held constant. This boxcar stacking makes efficient use of the Main Ring size. Care must be taken to match the Booster batches to the Main Ring in frequency, momentum, radius, and phase in order that longitudinal emittance in the Main Ring be kept as small as possible.

The usual form of stacking requires frequency modulation in order that particles can be brought to the stack without continual disturbance of the stacked beam. But it is also possible to stack a beam in a fixed-frequency accelerator by making use of the variation of radius with energy. An accelerating field E that varies with radius is introduced. A radius is chosen at which E vanishes. Particles are continually accelerated to this radius and accumulate there. Eventually each particle slips enough in phase to be decelerated and leaves the stack. In principle, it will later be back in phase and be accelerated back to the stack. A deeper discussion is given in Ref. 6.

The discussion in this chapter has been largely descriptive and qualitative. This flavor has been in part a necessity, because many of the results come from computer simulations. But it has also been purposeful, in order to give the reader insight into the physics of basic synchrotron-motion theory.

The short description in this paper will, it is hoped, show the reader that there is a well-developed theory that explains the phenomena observed in accelerators and storage rings. It should be noted that this paper does not discuss at all coupling between longitudinal and transverse motion or many-particle effects such as coherent longitudinal instabilities, more advanced topics that are covered in other parts of this volume.

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FIGURE CAPTIONS

1. Particles being accelerated on an rf wave.
2. The phase plane and small-amplitude motion in the vicinity of a stable (bottom sketch) and an unstable (top sketch) fixed point.
3. A stationary bucket.
4. A moving bucket.
5. Bucket shapes near transition energy.
6. Adiabatic turn-on of an accelerating bucket.
7. Sudden turn-on of an accelerating bucket.
8. Debunching by drifting and by a bucket.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6a

Fig. 6b

CAPTURE OF DEBUNCHED BEAM WITH ADIABATIC TURN-ON TURN

400

Fig. 6c

CAPTURE OF DEBUNCHED BEAM WITH ADIABATIC TURN-ON TURN

800

Fig. 6d

Fig. 7a

Fig. 7b

CAPTURE OF DEBUNCHED BEAM WITH FAST TURN-ON

TURN

25

Fig. 7c

Fig. 7d

Fig. 8a

Fig. 8b

DEBUNCHING

TURN

9

$S_{\text{bunch}} = 2.000\text{E-}01$ [eVs]

Fig. 8c

Fig. 8d