INELASTIC ANALYSIS
OF TRANSVERSE DEFLECTION OF PLATES
BY THE BOUNDARY ELEMENT METHOD

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ABSTRACT

A numerical scheme for time-dependent inelastic analysis of transverse deflection of plates of arbitrary shape by the boundary element method is presented in this paper. The governing differential equation is the inhomogeneous biharmonic equation for the rate of small transverse deflection. This complicated boundary value problem for an arbitrarily shaped plate is solved by using a novel combination of the boundary element method and finite element methodology. The number of unknowns, however, depend upon the boundary discretization and is therefore less than in a finite element model. A combined creep-plasticity constitutive theory with state variables is used to model material behavior. The computer code developed can solve problems for an arbitrarily shaped plate with clamped or simply supported boundary conditions and an arbitrary loading history. Some illustrative numerical results for clamped and simply supported rectangular and triangular plates, under various loading histories, are presented and discussed.

INTRODUCTION

The boundary element method (BEM-also called the boundary-integral equation method) has been applied to solid mechanics problems for some time. Two recent publications edited by Brebbia [1] and Cruse and Rizzo [2] contain many papers on the subject and references to others. Most of the applications to date address problems in linear elasticity and fracture mechanics while a relatively small number consider nonlinear inelastic problems. Mukherjee and his coworkers recently presented a boundary element method formulation for nonlinear time-dependent inelastic deformation problems [3]. This was followed by numerical implementation of the method for planar problems [4,5] and a comparison of the BEM with the widely used finite element method (FEM) [6]. It was demonstrated in references [5,6] that the BEM is a very powerful method with several potential advantages over the FEM. One of these is that the number of unknowns is the resultant algebraic system obtained by discretization is proportional to the number of boundary nodes in BEM as opposed to the total number of nodes in FEM. Another is the ease of discretization and input data preparation. Even though discretization of the domain must be carried out in these nonlinear problems, this is necessary only for the evaluation of certain integrals with known integrands and the topology of the internal elements is much simpler than for the FEM. Finally, due to the nature of the BEM formulation, the stresses are continuous throughout the domain, are easier to interpret, and are generally more accurate than those obtained by the FEM.

The boundary element method has been applied to problems of elastic transverse deflection of plates [7-10]. Jawson, Maiti et al. [7-8] have solved the associated biharmonic equation by using an integral representation of the solution in terms of certain known kernels and unknown source

densities. Recently, a new integral equation formulation for the elastic problem has been presented independently by Benzine and Gamby [9] and Stern [10].

The problem of time-dependent inelastic transverse deflection of plates is the subject of this paper. The governing biharmonic equation for the rate of transverse deflection of a thin plate in the presence of nonelastic strains is first derived. This differential equation is transformed into a pair of coupled Fredholm type integral equations. The formulation admits any of a number of constitutive models with state variables for material behavior provided that the model has the mathematical structure described in the next section. The governing integral equations are discretized. A computer program capable of solving inelastic deflection problems for plates of arbitrary shape, either clamped or simply supported, subjected to arbitrary loading histories, is developed. Finally, numerical solutions are presented for square and triangular plates. The constitutive model due to Hart [11,12], which has a sound experimental basis [13], is adopted in the numerical calculations. Time integration is carried out by using a recently developed Euler type scheme with automatic time-step control [14].

MATHEMATICAL STRUCTURE OF STATE VARIABLE MODELS

The mathematical structure of many of the state variable models of inelastic deformation can be summarised by the following equations (see [3-6] for references to other state variable models)

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^e + \dot{\epsilon}_{ij}^n + \dot{\epsilon}_{ij}^T, \quad \dot{\epsilon}_{ij}^n = f_{ij}(\sigma_{ij}, q_{ij}^{(k)}, T) \quad (1,2)$$

$$\dot{q}_{ij}^{(k)} = h_{ij}(\sigma_{ij}, q_{ij}^{(k)}, T), \quad \dot{\epsilon}_{kk}^n = 0 \quad (3,4)$$

Here $\dot{\epsilon}_{ij}^e$, $\dot{\epsilon}_{ij}^n$ and $\dot{\epsilon}_{ij}^T$ are the elastic, nonelastic and thermal strain rates respectively, σ_{ij} is the stress tensor, T the temperature and $q_{ij}^{(k)}$ are state variables. The number of state variables vary in the different models and they can be scalars or tensors. These state variables are assumed to completely characterize the present deformation state of the material and the history dependence of the rate of nonelastic strain upto the current time is assumed to be completely taken into account by their current values. It is important to note that the rates of the non-elastic strain and state variables at any time depend only on the current values of the stress, state variables and temperature.

Hart's Model

The specific equations for Hart's constitutive model are presented next. The model in tension is shown in Fig. 1. Material response in dilatation is assumed to be elastic. According to this model, the nonelastic strain is decomposed into two (time-dependent) components

$$\dot{\epsilon}_{ij}^n = \dot{\epsilon}_{ij}^a + \dot{\epsilon}_{ij}^p \quad (5)$$

where ϵ_{ij}^a is the anelastic strain, a stored strain which reflects the magnitude and direction of prior deformation history and ϵ_{ij}^p is the completely irrecoverable and path dependent permanent strain. The two state variables in the model are the anelastic strain and a scalar σ^* , called hardness, which is similar to an isotropic strain hardening parameter. The constitutive equations prescribe flow rules for the anelastic strain tensor and the permanent and nonelastic strain rates. Two auxiliary tensors s_{ij}^a and s_{ij}^f are used, and their sum equals s_{ij} , the deviatoric part of the stress σ_{ij} . Certain scalar invariants of the relevant tensors are defined in the usual way (eg. $\sigma^a = \sqrt{(3/2)s_{ij}^a s_{ij}^a}$) and these are related to each other through the uniaxial equations (see Fig. 1)

$$\sigma^a = M\epsilon^a, \quad \dot{\epsilon}^n = \dot{\epsilon}_0 (\sigma^f / \sigma_0)^M \quad (6,7)$$

$$\dot{\epsilon}^p = \dot{\epsilon}^* (\ln(\sigma^* / \sigma^a))^{-1/\lambda} \quad (8)$$

$$\dot{\epsilon}^* = \dot{\epsilon}_{ST}^* (\sigma^* / \sigma_s^*)^m \exp(Q/RT_B) \exp(-Q/RT) \quad (9)$$

$$\dot{\epsilon}^* = \dot{\epsilon}^p \sigma^* \Gamma(\sigma^*, \sigma^a), \quad \Gamma(\sigma^*, \sigma^a) = (\beta / \sigma^*)^\delta (\sigma^a / \sigma^*)^{\beta / \sigma^*} \quad (10,11)$$

Equation (6) represents a linear anelastic element, (7) a nonlinear dash-pot, (8) and (9) a "plastic" element and, finally, (10) and (11) describe strain hardening. The flow parameters are M , M , m , λ , $\dot{\epsilon}_0$ (at a reference stress level σ_0) and $\dot{\epsilon}_{ST}^*$ (at a reference hardness level σ_s^* and reference temperature T_B); β and δ are strain hardening parameters, R is the gas constant and Q the activation energy for atomic self diffusion. The material parameters, many of which are temperature dependent, are obtained from load relaxation and constant strain rate tension experiments. They are given in Appendix A of the EPRI report by Kumar and others [13].

ANALYTICAL FORMULATION

The analytical formulation of the problem closely follows that for small deflections of thin, laterally loaded elastic plates as discussed in Timoshenko and Woinowsky-Krieger [15]. Let a thin plate of arbitrary shape and thickness h be loaded laterally by a load of intensity q per unit area. The plate deflections are assumed to be small compared to the thickness of the plate and membrane effects are neglected. The effects of shearing forces and the compressive stress produced by the load q on the plate deformation are neglected. The middle plane of the plate is a plane of symmetry and is assumed to undergo no extension or contraction during bending. It is therefore a neutral surface. The inplane strains in the plate are assumed to remain proportional to the distance from the neutral surface even in the presence of inelastic deformation. The formulation is carried out in terms of the time rates of change of the relevant variables.

Let the middle plane of the undeformed plate be the x_1x_2 plane and w be the deflection of the middle plane in the x_3 direction. Since the plate deflections and slopes are small, the curvature rates $\dot{\kappa}_1$ and $\dot{\kappa}_2$ and the rate of twist of the middle surface with respect to the x_1 and x_2 axes, $\dot{\kappa}_{12}$, are related to the deflection rate by the following equations

$$\dot{\kappa}_1 = -\frac{\partial^2 \dot{w}}{\partial x_1^2}, \quad \dot{\kappa}_2 = -\frac{\partial^2 \dot{w}}{\partial x_2^2}, \quad \dot{\kappa}_{12} = \frac{\partial^2 \dot{w}}{\partial x_1 \partial x_2} \quad (12)$$

The strain rates in the absence of thermal strains, are, by assumption

$$\begin{aligned}
 \dot{\epsilon}_{11} &= \dot{\epsilon}_{11}^e + \dot{\epsilon}_{11}^n = x_3 \dot{\kappa}_1 \\
 \dot{\epsilon}_{22} &= \dot{\epsilon}_{22}^e + \dot{\epsilon}_{22}^n = x_3 \dot{\kappa}_2 \\
 \dot{\epsilon}_{12} &= \dot{\epsilon}_{12}^e + \dot{\epsilon}_{12}^n = -x_3 \dot{\kappa}_{12}
 \end{aligned} \tag{13}$$

and the stress rates, from Hooke's Law, become

$$\begin{aligned}
 \dot{\sigma}_{11} &= \frac{E}{1-\nu} (\dot{\epsilon}_{11}^e + \nu \dot{\epsilon}_{22}^e) \\
 &= \frac{E x_3}{1-\nu} (\dot{\kappa}_1 + \nu \dot{\kappa}_2) - \frac{E}{1-\nu} (\dot{\epsilon}_{11}^n + \nu \dot{\epsilon}_{22}^n) \\
 \dot{\sigma}_{22} &= \frac{E x_3}{1-\nu} (\dot{\kappa}_2 + \nu \dot{\kappa}_1) - \frac{E}{1-\nu} (\dot{\epsilon}_{22}^n + \nu \dot{\epsilon}_{11}^n) \\
 \dot{\sigma}_{12} &= 2G \dot{\epsilon}_{12}^e = -2G x_3 \dot{\kappa}_{12} - 2G \dot{\epsilon}_{12}^n
 \end{aligned} \tag{14}$$

In the above E and ν are the Young's modulus and Poisson's ratio, respectively, of the plate material.

The rates of the bending moments $\dot{M}_1$ and $\dot{M}_2$ and the twisting moment $\dot{M}_{12}$ are

$$\begin{aligned}
 \dot{M}_1 &= \int_{-h/2}^{h/2} \dot{\sigma}_{11} x_3 dx_3, \quad \dot{M}_2 = \int_{-h/2}^{h/2} \dot{\sigma}_{22} x_3 dx_3, \\
 \dot{M}_{12} &= - \int_{-h/2}^{h/2} \dot{\sigma}_{12} x_3 dx_3
 \end{aligned} \tag{15}$$

The equilibrium equation of a plate element is

$$\frac{\partial^2 \dot{M}_{11}}{\partial x_1^2} + \frac{\partial^2 \dot{M}_{22}}{\partial x_2^2} - \frac{2\partial^2 \dot{M}_{12}}{\partial x_1 \partial x_2} = -\dot{q} \quad (16)$$

Substituting equations (12) (14) and (15) into (16), the governing differential equation for the deflection rate is obtained as

$$\nabla^4 \dot{w} = \frac{\dot{q}}{D} - \frac{12}{h^3} f^{(n)}(x_1, x_2, t) \equiv g(x_1, x_2, t) \quad (17)$$

where ∇^2 is the Laplacian operator,

$$\begin{aligned} f^{(n)} = & \frac{\partial^2}{\partial x_1^2} \int_{-h/2}^{h/2} (\dot{\epsilon}_{11}^n + \nu \dot{\epsilon}_{22}^n) x_3 dx_3 + \frac{\partial^2}{\partial x_2^2} \int_{-h/2}^{h/2} (\dot{\epsilon}_{22}^n + \nu \dot{\epsilon}_{11}^n) x_3 dx_3 \\ & + 2(1-\nu) \frac{\partial^2}{\partial x_1 \partial x_2} \int_{-h/2}^{h/2} \dot{\epsilon}_{12}^n x_3 dx_3 \end{aligned} \quad (18)$$

and $D = Eh^3/(12(1-\nu^2))$.

Equation (18) can be transformed into a pair of Fredholm type integral equations (see, for example, [16] where the biharmonic equation for the stress function for a plane elastoplastic problem is considered)

$$\alpha \pi \nabla^2 \dot{w} - \int_A (\ell n r) g dA = \int_C \left[\frac{\partial}{\partial n} (\ell n r) \nabla^2 \dot{w} - \ell n r \frac{\partial}{\partial n} (\nabla^2 \dot{w}) \right] dC \quad (19)$$

$$\begin{aligned} 4\alpha \pi \dot{w} - \int_A (r^2 \ell n r) g dA = & \int_C \left[\frac{\partial}{\partial n} [\nabla^2 (r^2 \ell n r)] \dot{w} - \nabla^2 (r^2 \ell n r) \frac{\partial \dot{w}}{\partial n} \right. \\ & \left. + \frac{\partial}{\partial n} (r^2 \ell n r) \nabla^2 \dot{w} - r^2 \ell n r \frac{\partial}{\partial n} (\nabla^2 \dot{w}) \right] dC. \end{aligned} \quad (20)$$

Here A is the area of the plate, C its boundary, r the distance between a source point and a field point and n is the normal to C at the field point. The value of $\alpha = 1$ if the source point is on the boundary C where it is locally smooth, and $\alpha = 2$ if the source point is inside the plate.

The boundary conditions for the problem are

$$\text{clamped plate:} \quad \dot{w} = \frac{\partial \dot{w}}{\partial n} = 0 \quad \text{on } C$$

$$\text{Simply-supported plate:} \quad \dot{w} = \nabla^2 \dot{w} = 0 \quad \text{on } C \quad (21)$$

$$\text{Symmetry line:} \quad \frac{\partial \dot{w}}{\partial n} = \frac{\partial}{\partial n} (\nabla^2 \dot{w}) = 0$$

Thus, two of the four quantities $\dot{w}$, $\frac{\partial \dot{w}}{\partial n}$, $\nabla^2 \dot{w}$, $\frac{\partial}{\partial n} (\nabla^2 \dot{w})$ are prescribed on the boundary of the plate, and equations (19) and (20), (for a boundary point with $\alpha = 1$) are integral equations for the remaining two quantities on the boundary. Once these boundary-integral equations are solved, all four of these quantities are known on the boundary, and the values of $\dot{w}$ and $\nabla^2 \dot{w}$ can be obtained at any inside point by evaluation of known integrals. The curvature rates throughout the plate are then obtained from equation (12), the stress rates from (14) and the moment rates from (15). The curvature rates are obtained by analytical differentiation of equation (20) under the integral sign.

NUMERICAL IMPLEMENTATION AND SOLUTION STRATEGY

Discretized Equations

A discretized version of equation (19) for a point on the boundary is obtained by dividing the boundary C of the plate into N_s straight boundary elements using N_b ($N_b = N_s$) boundary nodes and the interior A of the plate into n_i triangular internal elements

$$\pi \nabla^2 \dot{w} - \sum_{n_i} \int_{\Delta A_i} (\ell n r) g dA = \sum_{N_s} \int_{\Delta C_i} \left\{ \frac{\partial}{\partial n} (\ell n r) \nabla^2 \dot{w} - (\ell n r) \frac{\partial}{\partial n} (\nabla^2 \dot{w}) \right\} dC \quad (22)$$

A similar equation results from equation (20).

The quantities $\dot{w}$, $\frac{\partial \dot{w}}{\partial n}$, $\nabla^2 \dot{w}$ and $\frac{\partial}{\partial n} (\nabla^2 \dot{w})$ are assumed to vary linearly over each boundary element with their values assigned at the nodes which lie at the intersection of the elements. Discontinuities in the normal derivatives across corners, if any, are taken care of by placing a 'zero length' element between two boundary nodes and assigning different values of $\frac{\partial \dot{w}}{\partial n}$ and $\frac{\partial}{\partial n} (\nabla^2 \dot{w})$ at each of the nodes. This approach is similar to that used in earlier work to accommodate traction discontinuities in planar problems [5,6].

Integrals through the thickness of the type

$$I(x_1, x_2, t) = \int_{-h/2}^{h/2} (\dot{\epsilon}_{11}^n + \nu \dot{\epsilon}_{22}^n) x_3 dx_3$$

are evaluated by Gaussian quadrature. Six gauss points are used in the numerical calculations presented later. As mentioned earlier, the integrands are known at any time through the constitutive equations (2) and (4). A quadratic interpolation of such integrals is used on each triangular

internal element in the x_1x_2 plane. This FEM type methodology is more accurate than finite difference techniques for the determination of second partial derivatives of such integrals that occur in equation (18). However, the topology of the internal elements is simpler than for FEM in that every node need not be shared by neighboring elements (see Fig. 1 of [4]).

Substitution of the functional forms of the variables into equation (22) and the discretized version of (20) with $\alpha = 1$ leads to an algebraic system of the type

$$[A]\{\dot{x}\} = [B]\{\dot{y}\} + \{\dot{d}\} \quad (23)$$

The matrices $[A]$ and $[B]$ depend only on the geometry, ie the number of boundary nodes and their distribution. They involve integrals of the type

$$\int_{\Delta C_i} (\ln r) dc, \quad \int_{\Delta C_i} (\ln r) c dc$$

where c is the distance measured along a boundary element. These integrals, some of which become singular when the source point coincides with the field point, are evaluated in closed form. They are available in Appendix 1 and those for the curvature rates are given in Appendix 2. The vector $\{\dot{x}\}$ contains unknown and the vector $\{\dot{y}\}$ known quantities on the boundary. Finally, the vector $\{\dot{d}\}$ contains integrals of the kernels $r^2 \ln r$ and $\ln r$ over the triangular internal elements, and the nonelastic strain rates $\dot{\epsilon}_{11}^n$, $\dot{\epsilon}_{22}^n$ and $\dot{\epsilon}_{12}^n$ through the function g . This vector is known at any time through the constitutive relations (2) and (4).

Solution Strategy

The initial distribution of state variables is prescribed and the initial nonelastic strain is set to zero. The initial

distribution of displacements, stresses and moments are obtained from the solution of the corresponding elastic problem. This is obtained from the same BEM equations in terms of the variables (rather than their rates). The rates of nonelastic strains at $t = 0$ are obtained from equations (2) and (4) and those of the state variables from (3). These nonelastic strains are used in equation (23) which is solved for the unknown components of $\dot{w}$, $\frac{\partial \dot{w}}{\partial n}$, $\nabla^2 \dot{w}$ and $\frac{\partial}{\partial n}(\nabla^2 \dot{w})$ on the boundary. The displacement rate $\dot{w}$ and its Laplacian $\nabla^2 \dot{w}$ are now obtained throughout the plate from discretized versions of equations (19) and (20) for an internal point ($\alpha = 2$). This step only involves evaluation of integrals with known integrands and no new unknowns need to be determined.

The curvature rates are next obtained analytically from the displacement rates. The quantities

$$\left(\frac{\partial^2 \dot{w}}{\partial x_1^2} - \frac{\partial^2 \dot{w}}{\partial x_2^2} \right) \quad \text{and} \quad \frac{\partial^2 \dot{w}}{\partial x_1 \partial x_2}$$

are evaluated at an inside point by analytical differentiation of equation (20) under the integral sign. The curvature rates are obtained by combining these with $\nabla^2 \dot{w}$ obtained from equation (19). The stress rates and moment rates are then obtained from equations (14) and (15).

The values of these quantities at a new time Δt are now obtained by integrating forward in time using a suitable numerical procedure and this process is continued in order to determine the time histories of the variables of interest. Since the state variable equations are generally stiff, choice of a suitable time-integration scheme is crucial. A simple but efficient Euler type strategy with automatic time-step control, described in references [14,5,6], is used to advantage.

Boundary curvature rates

The rates of curvatures on the boundary of the plate could not be obtained sufficiently accurately by using the solution strategy just described. The boundary curvature rates are obtained instead from the formulae given below, which are derived in Appendix 4.

Clamped plate: On boundary $\dot{w} = \frac{\partial \dot{w}}{\partial n} = 0$

$$\begin{aligned} \frac{\partial^2 \dot{w}}{\partial x_1^2} &= (\nabla^2 \dot{w}) \sin^2 \gamma, \quad \frac{\partial^2 \dot{w}}{\partial x_2^2} = (\nabla^2 \dot{w}) \cos^2 \gamma \\ \frac{\partial^2 \dot{w}}{\partial x_1 \partial x_2} &= -(\nabla^2 \dot{w}) \sin \gamma \cos \gamma \end{aligned} \quad (24)$$

Simply supported plate: On boundary $\dot{w} = \nabla^2 \dot{w} = 0$

$$\begin{aligned} \frac{\partial^2 \dot{w}}{\partial x_1^2} &= 2 \frac{\partial}{\partial c} \left(\frac{\partial \dot{w}}{\partial n} \right) \sin \gamma \cos \gamma \\ \frac{\partial^2 \dot{w}}{\partial x_2^2} &= -2 \frac{\partial}{\partial c} \left(\frac{\partial \dot{w}}{\partial n} \right) \sin \gamma \cos \gamma \\ \frac{\partial^2 \dot{w}}{\partial x_1 \partial x_2} &= -\frac{\partial}{\partial c} \left(\frac{\partial \dot{w}}{\partial n} \right) \cos 2\gamma \end{aligned} \quad (25)$$

In the above, n is the outward normal and c the counterclockwise tangential direction along a boundary element and γ is the angle between the x_1 axis and the direction tangential to the boundary element. The Laplacian of $\dot{w}$ on the boundary is obtained directly in the clamped case from equation (23) and the curvature rates are obtained easily from equation (24).

In the simply supported case, $\frac{\partial \dot{w}}{\partial n}$ is obtained directly on the boundary and use of equation (25) involves a first order numerical differentiation along the boundary element. This procedure leads to determination of boundary curvatures with sufficient accuracy. Unfortunately, curvatures on symmetry lines could not be obtained in this simple manner and symmetry conditions could not be taken advantage of in obtaining the numerical results presented in the next section.

NUMERICAL RESULTS AND DISCUSSION

Numerical results for several sample plate bending problems for 304 stainless steel plates at 400°C, using Hart's constitutive model, are presented in this section.

Parameter Values

The values of the various parameters used in the numerical calculations, at 400°C, are given below [13].

$$\lambda = 0.15 \quad M = 7.8 \quad m = 5$$

$$M = 779 \text{ GPa} \quad (.113 \times 10^9 \text{ psi})$$

$$E = 168.24 \text{ GPa} \quad (.244 \times 10^8 \text{ psi}) \quad \nu = 0.298$$

$$\dot{\epsilon}_0 = 1.553 \times 10^4 \text{ sec}^{-1} \quad \text{at} \quad \sigma_0 = 68.95 \text{ MPa} \quad (10 \text{ ksi})$$

$$\dot{\epsilon}_{ST}^* = 1.269 \times 10^{-24} \text{ sec}^{-1} \quad \text{at} \quad \sigma_s^* = 68.95 \text{ MPa}$$

$$T_B = 673^\circ\text{K}$$

$$\beta = 1.234 \text{ GPa} \quad (.179 \times 10^6 \text{ psi}) \quad \delta = 1.33$$

The initial values of the state variables for an annealed sample are

$$\sigma^*(x, 0) = 117.215 \text{ MPa} \quad (17 \text{ ksi})$$

$$\epsilon_{ij}^a(x, 0) = 0$$

Square Plates

Numerical results for square plates are shown in Figs. 2-5. The plates are of side $a = 5.08 \text{ cm}$ (2 in.) and the plate thickness is 2% of 'a' for the clamped plate (Figs. 2-4) and 1% of 'a' for the simply

supported plate in Fig. 5. The loading is an uniform pressure q increasing at a constant rate of 0.5 psi/sec. The boundary and internal mesh are shown as insets in the figures.

The pressure as a function of central displacement is shown in Fig. 2. The short time BEM solution is within 1% of the elastic solution from [15].

In order to better understand the behavior of the inelastic curve in Fig. 2, it is instructive to consider an analogous plate made of a linear viscoelastic material which is modelled as elastic in dilatation and as a three parameter viscoelastic solid in tension. If the free spring has modulus E and the one parallel to the dashpot has a modulus M , the long time Young's modulus and Poisson's ratio for this model are [17]

$$E_{\infty} = \frac{ME}{M+E}, \quad \nu_{\infty} = \frac{1}{2} - \frac{E_{\infty}}{6K}$$

where K is the bulk modulus of the material. In this case, for the plate bending problem, the initial and final slopes, q/w_A , are proportional to the quantities $E/(1-\nu^2)$ and $E_{\infty}/(1-\nu_{\infty}^2)$ respectively.

Returning to Hart's model (Fig. 1), it is observed that increase of stress from zero causes deformation of the anelastic spring but substantial permanent deformation of the "plastic" element only begins when the stress in the upper branch, σ^a , approaches the hardness σ^* . Thus, with the plastic element essentially inactive, the behavior in tension is close to that of a nonlinear three parameter viscoelastic solid. The zero time slope of the inelastic curve in Fig. 2, therefore, should be proportional to $E/(1-\nu^2)$ and this should change over to $E_{\infty}/(1-\nu_{\infty}^2)$ at, say, $w_A/h = 0.08$. Comparison of the numbers shows that the zero time slope is within 1% and the slope at $w_A/h = 0.08$ is within 3% of the above mentioned quantities,

thus confirming the accuracy of the BEM model. Further increase of pressure causes substantial permanent deformation of the plastic element and the curve has the features of an usual elastic-plastic curve. Yielding starts at the midpoints of the outside edges of the plate. The yield front starts moving out along the plate edges and then a second yielding occurs at the center of the plate. The redistribution of stress through the thickness at the mid-point of an edge of the plate (Fig. 3) shows the expected transition from the linear elastic distribution to the nonlinear distribution characteristic of elastic-plastic deformation. Redistribution of the nondimensionalized bending moment M_{11} along a center line of the plate ($x_2 = 0$) is shown in Fig. 4.

The load-deflection curve for a simply supported square plate is shown in Fig. 5. This time the elastic solution is within 1% of the correct one and the slope at $w_A/h = .4$ is within 2% of the slope expected from the analogous viscoelastic case.

Triangular Plates

Deflection of equilateral triangular plates is also analysed by the program. Fig. 6 shows the load-deflection curve for a clamped equilateral triangular plate of side 2.54 cm (1 in.) with $h/a = 2\%$. The elastic solution is within 2% of that obtained by Conway [18] and the inelastic solution has the expected elastic-plastic characteristics. Stress redistribution through the thickness (Fig. 7) exhibits the transition from linear elastic to nonlinear elastic-plastic analogous to Fig. 3. Redistribution of moment along the line $x_2 = 0$ is shown in Fig. 8. The elastic solution for a simply supported equilateral triangular plate under uniform pressure, obtained by the BEM program, is within .04% of the elastic solution [15]. The

nonelastic solution has not been determined in this case.

The number of time-steps and computer times on an IBM 370/168, required for these problems, are given in Table 1.

CONCLUSIONS

A computer program for the solution of transverse time-dependent inelastic deflection of plates of arbitrary shape using the boundary element method has been developed. This class of problems is complicated because the problems are nonlinear and history dependent and involve redistribution of stress through the thickness and across the plate during deformation. Numerical results for several sample problems for square and triangular plates, using a combined creep-plasticity constitutive model with state variables, are presented. These solutions are obtained accurately and within reasonable amounts of computer times. Successful applications of the boundary-element method to inelastic planar (in earlier work) and transverse deflection problems of plates demonstrates the usefulness of this powerful general purpose technique for this important class of problems.

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APPENDIX 1

INTEGRALS IN EQUATIONS (19) AND (20)

The integrals in Equations (19) and (20) are evaluated using a method similar to that described by Riccardella [19]. The notations used for the evaluation of boundary integrals is given in Fig. A-1 and the area integrals in Figs. A-2 and A-3.

The quantities $\dot{w}$, $\frac{\partial \dot{w}}{\partial n}$, $\nabla^2 \dot{w}$ and $\frac{\partial}{\partial n} \nabla^2 \dot{w}$ are interpolated linearly on each boundary segment as follows for $\dot{w}$ (others are similarly interpolated)

$$\dot{w}(P) = \frac{\dot{w}(b) - \dot{w}(a)}{\Delta s} s + \dot{w}(a)$$

where $s = D \tan \theta - z_a$ and $\dot{w}(b)$, $\dot{w}(a)$ are the displacement rates at nodes b and a respectively.

A typical integral of the following type would become

$$\begin{aligned} \int_{\Delta C} \frac{\partial}{\partial n} [\nabla^2 (r^2 \ln r)] \dot{w} dC &= \frac{\dot{w}(b) - \dot{w}(a)}{\Delta s} \int \frac{\partial}{\partial n} [\nabla^2 (r^2 \ln r)] D \tan \theta dC \\ &+ [\dot{w}(a) - \frac{(\dot{w}(b) - \dot{w}(a)) z_a}{\Delta s}] \int \frac{\partial}{\partial n} [\nabla^2 (r^2 \ln r)] dC \end{aligned}$$

$$dC = ds$$

$$r = D / \cos \theta$$

For $D \neq 0$ the Integrals over a boundary segment are as follows:

$$1. \quad A_1 = \int \ln r ds = D [\tan \theta_b (\ln r_b - 1) - \tan \theta_a (\ln r_a - 1) + \theta_b - \theta_a]$$

$$A_2 = \int \ln r ds D \tan \theta = \frac{r_b^2}{4} (2 \ln r_b - 1) - \frac{r_a^2}{4} (2 \ln r_a - 1)$$

$$2. \quad B_1 = \int \frac{\partial}{\partial n} (\ln r) ds = \theta_b - \theta_a$$

$$B_2 = \int \frac{\partial}{\partial n} (\ln r) ds D \tan \theta = D \ln(r_b/r_a)$$

$$3. \quad C_1 = \int r^2 \ln r ds = D/3 \{ r_b^2 \tan \theta_b \ln r_b - r_a^2 \tan \theta_b \ln r_a + D^2 (2 \tan \theta_b \ln r_b - 2 \tan \theta_a \ln r_a) \\ - \frac{D^2}{3} [\tan^3 \theta_b - \tan^3 \theta_a] - 2D^2 (\tan \theta_b - \tan \theta_a) \\ + 2D^2 (\theta_b - \theta_a) \}$$

$$C_2 = \int r^2 \ln r ds D \tan \theta = \frac{r_b^4}{4} \ln r_b - \frac{r_a^4}{4} \ln r_a - \frac{1}{16} (r_b^4 - r_a^4)$$

$$4. \quad D_1 = \int \frac{\partial}{\partial n} (r^2 \ln r) ds = D(2A_1 + \Delta s)$$

$$D_2 = \int \frac{\partial}{\partial n} (r^2 \ln r) ds D \tan \theta = D(2A_2 + 1/2 (r_b^2 - r_a^2))$$

$$5. \quad E_1 = \int \nabla^2 (r^2 \ln r) ds = 4(A_1 + \Delta s)$$

$$E_2 = \int \nabla^2 (r^2 \ln r) ds D \tan \theta = 4(A_2 + \frac{1}{2} (r_b^2 - r_a^2))$$

$$6. \quad F_1 = \int \frac{\partial}{\partial n} [\nabla^2 (r^2 \ln r)] ds = 4(\theta_b - \theta_a)$$

$$F_2 = \int \frac{\partial}{\partial n} [\nabla^2 (r^2 \ln r)] ds D \tan \theta = 4D \ln(r_b/r_a)$$

For $D = 0$ $SGN = \sin\theta$

$$1. \quad A_1 = SGN[r_b(\ln r_b - 1) - r_a(\ln r_a - 1)]$$

$$A_2 = [r_b^2(2\ln r_b - 1) - r_a^2(2\ln r_a - 1)]/4$$

$$2. \quad B_1 = 0$$

$$B_2 = 0$$

$$3. \quad C_1 = SGN[r_b^3(3\ln r_b - 1) - r_a^3(3\ln r_a - 1)]/9$$

$$C_2 = [r_b^4(4\ln r_b - 1) - r_a^4(4\ln r_a - 1)]/16$$

$$4. \quad D_1 = 0$$

$$D_2 = 0$$

$$5. \quad E_1 = 4(A_1 + \Delta s)$$

$$E_2 = 4(A_2 + 1/2(r_b^2 - r_a^2))$$

$$6. \quad F_1 = 0$$

$$F_2 = 0$$

$$\int \frac{\partial}{\partial n} \nabla^2 (r^2 \ln r) \dot{w} ds = \frac{\dot{w}(b)}{\Delta s} [F_2 - z_a F_1] + \dot{w}(a) [F_1 (1 + \frac{z_a}{\Delta s}) - \frac{F_2}{\Delta s}]$$

Area Integrals of the type $\int_{\Delta A_i} \ln r g dA$ are obtained by summing algebraically the contribution of each triangle formed by joining a side of the triangular element to the source point as shown in Fig. A-2. The quantity g is assumed to be constant over the triangular element.

$$7. \quad G = g \int_A \ln r dA = \frac{g}{4} [2DA_1 - D^2 (\tan \theta_b - \tan \theta_a)]$$

$$8. \quad H = g \int r^2 \ln r dA$$

$$= D [4C_1 - D^3 \{ (\tan^3 \theta_b - \tan^3 \theta_a) / 3 + (\tan \theta_b - \tan \theta_a) \}] / 16$$

APPENDIX 2

EVALUATION OF CURVATURES

By analytical differentiation of Eq. (20) the following can be obtained

$$\begin{aligned}
 4\pi \left(\frac{\partial^2 \dot{w}}{\partial x^2} - \frac{\partial^2 \dot{w}}{\partial y^2} \right) - \int_A \frac{\xi^2 - \eta^2}{r^2} g \, dA &= \int_C \left\{ -4\dot{w} \frac{\partial}{\partial n} \left(\frac{\xi^2 - \eta^2}{r^4} \right) + 4 \frac{\partial \dot{w}}{\partial n} \left(\frac{\xi^2 - \eta^2}{r^4} \right) \right. \\
 &\quad \left. + \nabla^2 \dot{w} \frac{\partial}{\partial n} \left(\frac{\xi^2 - \eta^2}{r^2} \right) - \frac{\partial}{\partial n} \nabla^2 \dot{w} \left(\frac{\xi^2 - \eta^2}{r^2} \right) \right\} dC \\
 4\pi \frac{\partial^2 \dot{w}}{\partial x \partial y} - \int_A \frac{\xi \eta}{r^2} g \, dA &= \int_C \left\{ -4\dot{w} \frac{\partial}{\partial n} \frac{\xi \eta}{r^4} + 4 \frac{\partial \dot{w}}{\partial n} \frac{\xi \eta}{r^4} + \nabla^2 \dot{w} \frac{\partial}{\partial n} \frac{\xi \eta}{r^2} - \frac{\partial}{\partial n} (\nabla^2 \dot{w}) \frac{\xi \eta}{r^2} \right\} dC
 \end{aligned}$$

where ξ, η are the x_1 and x_2 distances from the source point to the field point as shown in Fig. A-4.

Integrals

$$\begin{aligned}
 9. \quad I_1 &= \int \frac{\xi^2 - \eta^2}{r^2} \, ds \\
 &= D [2\cos 2\alpha (\theta_b - \theta_a) - \cos 2\alpha (\tan(\theta_b) - \tan(\theta_a)) - 2\sin 2\alpha \ln r_b / r_a] \\
 I_2 &= \int \frac{\xi^2 - \eta^2}{r^2} \, ds \, D \tan \theta \\
 &= D^2 \left[2\cos 2\alpha \ln \frac{r_b}{r_a} - \frac{\cos 2\alpha}{2} (\tan^2 \theta_b - \tan^2 \theta_a) - 2\sin 2\alpha (\tan \theta_b - \tan \theta_a - \theta_b + \theta_a) \right]
 \end{aligned}$$

$$10. \quad J_1 = \int \frac{\partial}{\partial n} \left(\frac{\xi^2 - \eta^2}{r^2} \right) ds$$

$$= 2\cos 2\alpha(\theta_b - \theta_a) - 2\sin 2\alpha \ln(r_b/r_a) - \sin 2(\theta_b + \alpha) + \sin 2(\theta_a + \alpha)$$

$$J_2 = \int \frac{\partial}{\partial n} \left(\frac{\xi^2 - \eta^2}{r^2} \right) ds D \tan \theta = 2D \{ \cos 2\alpha [\cos 2\theta_b - \cos 2\theta_a + 4 \ln(r_b/r_a)] / 2$$

$$+ \sin 2\alpha [2(\theta_b - \theta_a) - (\tan \theta_b - \tan \theta_a)]$$

$$- 1/2(\sin 2\theta_b - \sin 2\theta_a) \}$$

$$11. \quad K_1 = \int \frac{\xi^2 - \eta^2}{r^4} ds$$

$$= \frac{1}{2D} [\sin 2(\theta_b + \alpha) - \sin 2(\theta_a + \alpha)]$$

$$K_2 = \int \left(\frac{\xi^2 - \eta^2}{r^4} \right) ds D \tan \theta$$

$$= \sin 2\alpha (\sin 2\theta_b - \sin 2\theta_a) / 2 - \sin 2\alpha (\theta_b - \theta_a) - \cos 2\alpha (\cos (2\theta_b) - \cos (2\theta_a)) / 2$$

$$- \cos 2\alpha \ln(r_b/r_a)$$

$$12. \quad L_1 = \int \frac{\partial}{\partial n} \left(\frac{\xi^2 - \eta^2}{r^4} \right) ds$$

$$= - \frac{1}{4D^2} [2\sin 2(\theta_b + \alpha) - 2\sin 2(\theta_a + \alpha) + \sin(4\theta_b + 2\alpha) - \sin(4\theta_a + 2\alpha)]$$

$$L_2 = \int \frac{\partial}{\partial n} \left(\frac{\xi^2 - \eta^2}{4r} \right) ds D \tan \theta$$

$$= \frac{1}{4D} [\cos(4\theta_b + 2\alpha) - \cos(4\theta_a + 2\alpha) - 2\cos(2\theta_b + 2\alpha) + 2\cos(2\theta_a + 2\alpha)]$$

$$13. \quad M_1 = \int \frac{\xi \eta}{2r} ds$$

$$= D/2 [2\cos 2\alpha \ln(r_b/r_a) + \sin 2\alpha \{2(\theta_b - \theta_a) - (\tan \theta_b - \tan \theta_a)\}]$$

$$M_2 = \int \frac{\xi \eta}{2r} ds D \tan \theta$$

$$= D^2 \{ \cos 2\alpha (\tan \theta_b - \tan \theta_a - (\theta_b - \theta_a)) + \sin 2\alpha [\ln(r_b/r_a) - 1/4 (\tan^2 \theta_b - \tan^2 \theta_a)] \}$$

$$14. \quad N_1 = \int \frac{\partial}{\partial n} \left(\frac{\xi \eta}{2r} \right) ds$$

$$= \cos 2\alpha \ln(r_b/r_a) + \sin 2\alpha (\theta_b - \theta_a) + 1/2 (\cos(2\theta_b + 2\alpha) - \cos(2\theta_a + 2\alpha))$$

$$N_2 = \int \frac{\partial}{\partial n} \left(\frac{\xi \eta}{2r} \right) ds D \tan \theta$$

$$= D \{ \cos 2\alpha [(\tan \theta_b - \tan \theta_a - 2(\theta_b - \theta_a)) + 1/2 (\sin 2\theta_b - \sin 2\theta_a)]$$

$$+ \sin 2\alpha [2 \ln(r_b/r_a) + 1/2 (\cos 2\theta_b - \cos 2\theta_a)] \}$$

$$15. \quad O_1 = \int \frac{\xi \eta}{4r} ds$$

$$= -\frac{1}{4D} [\cos 2(\theta_b + \alpha) - \cos 2(\theta_a + \alpha)]$$

$$O_2 = \int \frac{\xi \eta}{r^4} ds D \tan \theta$$

$$= 1/2 [\cos 2\alpha \{ (\theta_b - \theta_a) - 1/2 (\sin 2\theta_b - \sin 2\theta_a) \}]$$

$$- \sin 2\alpha \{ 1/2 (\cos 2\theta_b - \cos 2\theta_a) + \ln r_b / r_a \}]$$

$$16. \quad P_1 = \int \frac{\partial}{\partial n} \left(\frac{\xi \eta}{r^4} \right) ds$$

$$= \frac{1}{8D^2} [\cos(4\theta_b + 2\alpha) - \cos(4\theta_a + 2\alpha) + 2(\cos(2\theta_b + 2\alpha) - \cos(2\theta_a + 2\alpha))]$$

$$P_2 = \int \frac{\partial}{\partial n} \left(\frac{\xi \eta}{r^4} \right) ds D \tan \theta = \frac{1}{8D} \{ \sin(4\theta_b + 2\alpha) - \sin(4\theta_a + 2\alpha) \}$$

$$- 2(\sin(2\theta_b + 2\alpha) - \sin(2\theta_a + 2\alpha)) \}$$

APPENDIX 3

PROOF OF $\nabla^2 \dot{w} = 0$ ON SIMPLY SUPPORTED BOUNDARY

Using eqs. (14) and (15), the moment rates can be written as

$$\dot{M}_1 = D[\dot{\kappa}_1 + \nu \dot{\kappa}_2] - \frac{E}{1-\nu} \frac{h/2}{2} \int_{-h/2}^{h/2} (\dot{\epsilon}_{11}^n + \nu \dot{\epsilon}_{22}^n) x_3 dx_3$$

$$\dot{M}_2 = D[\dot{\kappa}_2 + \nu \dot{\kappa}_1] - \frac{E}{1-\nu} \frac{h/2}{2} \int_{-h/2}^{h/2} (\dot{\epsilon}_{22}^n + \nu \dot{\epsilon}_{11}^n) x_3 dx_3$$

with

$$\dot{\kappa}_1 = - \frac{\partial^2 \dot{w}}{\partial x_1^2}, \quad \dot{\kappa}_2 = - \frac{\partial^2 \dot{w}}{\partial x_2^2}$$

For the elastic problem

$$M_1 = D(\kappa_1 + \nu \kappa_2), \quad M_2 = D(\kappa_2 + \nu \kappa_1)$$

$$\sigma_{11} = \frac{12}{h} M_1 x_3, \quad \sigma_{22} = \frac{12}{h} M_2 x_3$$

Consider a simply supported straight edge parallel to the x_2 axis

For the elastic problem, on this edge,

$$M_1 = 0, \quad \frac{\partial^2 w}{\partial x_2^2} = 0 \rightarrow \frac{\partial^2 w}{\partial x_1^2} = 0, \quad M_2 = 0, \quad \sigma_{11} = 0, \quad \sigma_{22} = 0$$

Now for the inelastic problem,

$$\sigma_{11} = 0, \sigma_{22} = 0, \sigma_{33} = 0 \rightarrow \dot{\epsilon}_{11}^n = \dot{\epsilon}_{22}^n = 0, \text{ for all time, provided}$$

$$\epsilon_{ij}^a(\underline{x}, 0) = 0.$$

$$\dot{\epsilon}_{11}^n = 0, \dot{\epsilon}_{22}^n = 0, \dot{M}_1 = 0, \frac{\partial^2 \dot{w}}{\partial x_2^2} = 0 \rightarrow \frac{\partial^2 \dot{w}}{\partial x_1^2} = 0, \quad \nabla^2 \dot{w} = 0$$

It can be shown that $\nabla^2 \dot{w} = 0$ for a simply supported edge of arbitrary orientation.

APPENDIX 4

BOUNDARY CURVATURE RATES

The boundary curvature rates are obtained as follows.

The x_1 and x_2 derivatives in terms of tangential and normal derivatives are

$$\frac{\partial}{\partial x_1} = \sin \gamma \frac{\partial}{\partial n} + \cos \gamma \frac{\partial}{\partial s}$$

$$\frac{\partial}{\partial x_2} = -\cos \gamma \frac{\partial}{\partial n} + \sin \gamma \frac{\partial}{\partial s}$$

where γ is the angle between the x_1 axis and the tangential direction to the boundary element. Hence,

$$\frac{\partial^2 \dot{w}}{\partial x_1^2} = \sin^2 \gamma \frac{\partial^2 \dot{w}}{\partial n^2} + 2 \sin \gamma \cos \gamma \frac{\partial^2 \dot{w}}{\partial n \partial s} + \cos^2 \gamma \frac{\partial^2 \dot{w}}{\partial s^2}$$

$$\frac{\partial^2 \dot{w}}{\partial y_1^2} = \cos^2 \gamma \frac{\partial^2 \dot{w}}{\partial n^2} - 2 \sin \gamma \cos \gamma \frac{\partial^2 \dot{w}}{\partial n \partial s} + \sin^2 \gamma \frac{\partial^2 \dot{w}}{\partial s^2}$$

$$\frac{\partial^2 \dot{w}}{\partial x_1 \partial y_1} = \sin \gamma \cos \gamma \left(\frac{\partial^2 \dot{w}}{\partial s^2} - \frac{\partial^2 \dot{w}}{\partial n^2} \right) + (\sin \gamma - \cos^2 \gamma) \frac{\partial^2 \dot{w}}{\partial n \partial s}$$

Clamped edge

$$\dot{w} = 0 \rightarrow \frac{\partial \dot{w}}{\partial s} = 0 \quad \text{and} \quad \frac{\partial^2 \dot{w}}{\partial s^2} = 0 \rightarrow \frac{\partial^2 \dot{w}}{\partial n^2} = \nabla^2 \dot{w}$$

$$\text{Also,} \quad \frac{\partial \dot{w}}{\partial n} = 0 \rightarrow \frac{\partial^2 \dot{w}}{\partial n \partial s} = 0$$

$$\therefore \frac{\partial^2 \dot{w}}{\partial x_1^2} = \nabla^2 \dot{w} \sin^2 \gamma$$

$$\frac{\partial^2 \dot{w}}{\partial x_2^2} = \nabla^2 \dot{w} \cos^2 \gamma$$

$$\frac{\partial^2 \dot{w}}{\partial x_1 \partial x_2} = -(\nabla^2 \dot{w}) \sin \gamma \cos \gamma$$

Simply supported edge

$$\dot{w} = 0 \rightarrow \frac{\partial^2 \dot{w}}{\partial s^2} = 0$$

$$\nabla^2 \dot{w} = 0 \rightarrow \frac{\partial^2 \dot{w}}{\partial n^2} = 0$$

$$\therefore \frac{\partial^2 \dot{w}}{\partial x_1^2} = 2 \frac{\partial}{\partial c} \left(\frac{\partial \dot{w}}{\partial n} \right) \sin \gamma \cos \gamma$$

$$\frac{\partial^2 \dot{w}}{\partial x_2^2} = -2 \frac{\partial}{\partial c} \left(\frac{\partial \dot{w}}{\partial n} \right) \sin \gamma \cos \gamma$$

$$\frac{\partial^2 \dot{w}}{\partial x_1 \partial x_2} = - \frac{\partial}{\partial c} \left(\frac{\partial \dot{w}}{\partial n} \right) \cos 2\gamma$$

where $c = s$.

APPENDIX 5

INPUT FORMAT FOR BEM PLATE BENDING PROGRAM

CARD 1. TITLE - Title of the Problem

Format: (18A4)

CARD 2. NMOD, NELM, NIPT, NGAUSS, SCAL2

Format: (5I5)

NNOD number of boundary nodes

NELM number of internal elements

NIPT number of nodal points for internal elements

NGAUSS number of gauss points used for integration across
the thickness (max 10)

SCAL2 factor for double node separation
(default - value 0.01)

CARD 3. K12, K13, K14

Format: (5x, 3I5)

K12 = 0 first run

 = 1 a restart run (omit Cards 4-17)

K13 maximum iteration number

K14 record number for restart

CARD 4. YOUNG, PR, SCALE, THICK

Format: (4E10.0)

YOUNG Young's modulus

PR Poisson's ratio

SCALE Factor for viscoplastic limit
(default 0.99)

THICK thickness of the plate

CARD 5. CM, BETA, DELTA, SGMSTS, SCM, DASTR, DEPSTS

Format: (8E10.0) (Hart's Parameters)

CM	M
BETA	β
DELTA	δ
SGMSTS	σ_s^*
SCM	M
DASTR	$\dot{a}^*$
DEPSTS	$\dot{\epsilon}_s^*$

CARD 6. I, (XYP(I,J), J = 1,2)

Format: (I5, 2E10.0)

I	boundary node number
XYP(I,1)	x - coordinate
XYP(I,2)	y - coordinate

Note: Intermediate nodes will be automatically generated, if necessary. Nodes on either side of zero length segments must have different numbers but identical coordinates.

CARD 7. (XIP(I,1), XIP(I,2), NPOS(I), I = 1, NIPT)

Format: (2E10.0, I5)

XIP(I,1)	x coordinate of the internal point I
XIP(I,2)	y coordinate of the internal point I
NPOS(I)	if the internal point is on the boundary then NPOS is the boundary segment number, otherwise it must be omitted. An internal point must not lie on a zero segment element. All the internal nodes lying on the boundary must be numbered first.

CARD 8. (NODIP(I,J), J = 1,6), I = 1, NELM)

Format: (6I5)

NODIP	element connectivity (must be counter-clockwise)
-------	---

CARD 9. NBC

Format: (I5)

NBC Number of boundary conditions cards

Note: Card 10 must be omitted if NBC = 0

CARD 10. NOD, ITYPE(NOD)

Format: (2I5)

NOD boundary node number

ITYPE = 1 simply supported node

 = 0 clamped node (default value)

CARD 11. NLC

Format: (I5)

NLC = 1 uniform load applied on all internal elements

 ≠ 1 value of load on each element is given

Note: Include Card 12 only if NLC = 1

CARD 12. ULOAD

Format: (E10.0)

ULOAD value of uniform load

Note: Include Card 13 only if NLC ≠ 1

CARD 13. (PLOAD(I), I = 1, NELM)

Format: (8E10.0)

PLOAD(I) load on element I

CARD 14. NLC

Format: (I5)

NLC = 1 uniform rate of loading applied on all internal elements

NLC $\neq$ 1 value of rate of loading on each element is given.

Note: Include Card 15 only if NLC = 1

CARD 15. TLOAD

Format: (E10.0)

TLOAD value of uniform rate of loading applied on all internal elements

Note: Include Card 16 only if NLC $\neq$ 1CARD 16. (TLOAD(I), I = 1, NELM)

Format: (8E10.0)

TLOAD(I) rate of load on element I

CARD 17. SIGMST

Format: (E10.0)

SIGMST σ^* hardness parameterCARD 18. TLIMIT, DTIME, EMAX, EMIN, DTMAX, NOUT, NINT

Format: (5E10.0, 2I5)

TLIMIT Maximum time for simulation

DTIME Initial time step

EMAX,
EMIN Maximum and minimum error tolerances

DTMAX Maximum time step size

NINT Intermediate storage at every NINT iterations

NOUT Print out at every NOUT iteration

CARD 19. NST2

Format: (I5)

NST2 number of internal nodes at which stress profile is
 desired

CARD 20. (NSTOUT(I), I = 1, NST2)

Format: (16I5)

NSTOUT node numbers for the above

Tapes Required

1, 2, 3 for intermediate results

4 for keeping track of calculations on interactive
 terminals.

Problem	Number of boundary nodes	Number of Internal elements	Number of Time steps	CPU time or IBM 370/168 (sec)
1. Clamped Square plate	36	32	372	208
2. Simply-supported square plate	36	32	350	210
3. Clamped triangular plate	33	25	200	98

Table 1. BEM Program Statistics.

Figure 1. Hart's model in tension.

Figure 2. Load as a function of central displacement for a clamped square plate subjected to a uniform load increasing at a constant rate.

Figure 3. Redistribution of stress at the mid-point of an edge of the clamped square plate of Fig. 2.

Figure 4. Redistribution of moment along the line $x_2 = 0$ of the clamped square plate of Fig. 2.

Figure 5. Load as a function of central displacement for a simply supported square plate subjected to a uniform load increasing at a constant rate.

Figure 6. Load as a function of centroidal displacement for a clamped triangular plate subjected to a uniform load increasing at a constant rate.

Figure 7. Redistribution of stress at the mid-point of an edge of the clamped triangular plate of Fig. 6.

Figure 8. Redistribution of moment along the line $x_2 = 0$ of the clamped triangular plate of Fig. 6.

Figure A-1. Notation used for the evaluation of boundary integrals.

Figure A-2. Area integration scheme for triangular internal elements.

Figure A-3. Notation used for the evaluation of area integrals.

Figure A-4. Notation used for the evaluation of integrals for calculating curvatures.