PULSE PILE-UP EFFECTS

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ABSTRACT

The energy spectrum containing the effects of all orders of pulse pileup is predicted for an idealized X-ray pulse-height-analysis system measuring randomly occurring events. Two simplifying assumptions used are first a fixed pulse resolution time and second that the measured energy of piled-up pulses is the algebraic sum of the energy associated with each pulse.

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I. INTRODUCTION

Diagnostic systems designed to measure the frequency and energy spectrum of randomly occurring events are limited by effects of "pulse pileup."¹⁻³ Pulse pileup occurs when two or more events occur so close together in time that the diagnostic system responds to them as if they constituted a single event with an energy equal to some sort of sum of the energies of the individual events. The measured event rate is thereby depressed below the true event rate and the measured energy spectrum is distorted from the shape of the true energy spectrum.

A commonly used technique to circumvent the effects of pulse pileup is to pass the signal from an event into two channels, a "fast" channel and a "slow" channel.^{2,4} The signal input to the fast channel triggers a fast pulse generator producing a very short pulse. In the slow channel the amplitude of the signal is measured by some technique. This information is stored in a computer memory for later use provided that during the time of amplitude measurement a second fast pulse is not registered in the fast channel. Thus the stored amplitude is not contaminated with the signals of events arriving during the time of amplitude measurement. However, two or more events arriving within the pulse resolution time of the fast channel will result in the generation of only a single fast pulse and the slow channel will measure the mixture of their signals. In this case, a pileup of events will be recorded as a single event. We analyze these pile-up effects with the help of a simple model of the counting process. We imagine the counting device to be exposed to a large number of "events" and determine how many "counts" will be registered. These counts are then subdivided according to how many events are associated with each count.

II. COUNTING

A counting device is characterized by the following processes: first, stimulation by a suitable event; second, producing some sort of a record of that stimulation such as a voltage rise to some level, a switch opened or closed, etc.; and third, recovering or returning to the initial state of readiness. The time taken to accomplish all three steps of the counting process is the "processing time," the time it takes to "make a count." This time is not necessarily a constant and depends on the details of the counting device.

There are two extreme types of counters.^{5,6} In the type I, or paralyzable counter, the counting process requires an event-free period of time, τ . If the counter is stimulated by an event during the counting process, the counting process is extended until an interval of time between events is encountered that exceeds τ . If events occur with intervals less than τ , the counter is "paralyze" and will not register additional counts. Thus the type I counter counts event intervals in excess of some characteristic time τ rather than events.

In the type II, or nonparalyzable counter, the counting process takes a fixed period of time. During this time the counter is insensitive to any additional events that may occur.

We are interested in counting events that occur at random and at some average rate "a." The temporal distribution of such events is in accordance with the Poisson distribution^{7,8} (Appendix A). Hence the probability of exactly n such events occurring in a time τ is

$$P_n = \frac{(a\tau)^n e^{-a\tau}}{n!} . \quad (1)$$

The probability of no events occurring in a time τ is

$$p_0 = e^{-a\tau}. \quad (2)$$

Alternatively, we can think of p_0 as the probability for the occurrence of an event interval of duration greater than τ . The probability of at least one event occurring in a time τ is

$$\sum_{n=1}^{\infty} p_n = 1 - p_0 = 1 - e^{-a\tau}. \quad (3)$$

Consider a long sequence of random events occurring at an average rate "a" such that their total number, N_0 , is quite large. We now analyze how these N_0 events are counted by the type I and type II counters. First we consider the type II, nonparalyzable counter. Let N_c be the number of counts registered. Now $N_c < N_0$ because for each count the counter will be insensitive to events occurring during the counting time τ . On the average $a\tau$ uncounted events will occur for each of the N_c counts. Since N_0 equals the sum of the counted and uncounted events, we expect

$$N_0 = N_c + a\tau N_c$$

or

$$N_c = \frac{N_0}{(1 + a\tau)}. \quad (\text{Type II}) \quad (4)$$

Consider now the N_c time intervals, each of length τ . Each time interval has been initiated by an event, but during the interval there may

occur any number of events. On the average we expect $a\tau$ events to occur. However the probability of exactly n events occurring in the time τ is just p_n as given by Eq. (1). We conclude that for a large enough N_c , we expect a fraction, p_n , of these counts to be characterized by having exactly n events occur during the counting time τ in addition to the initiating event. We write this number of "n-event counts" as N_{cn} . Thus

$$N_{cn} = p_n N_c = \frac{(a\tau)^n}{n!} e^{-a\tau} N_c. \quad (5)$$

We note:

$$\sum_{n=0}^{\infty} N_{cn} = N_c \sum_{n=0}^{\infty} p_n = N_c e^{-a\tau} \sum_{n=0}^{\infty} \frac{(a\tau)^n}{n!} = N_c.$$

Thus all the counts are accounted for. Furthermore since each of the "n-event counts" accounts for $(n + 1)$ events, we also have

$$\begin{aligned} \sum_{n=0}^{\infty} (n + 1) N_{cn} &= N_c \sum_{n=0}^{\infty} (n + 1) p_n = N_c e^{-a\tau} \sum_{n=0}^{\infty} (n + 1) \frac{(a\tau)^n}{n!} \\ &= N_c e^{-a\tau} e^{+a\tau} (a\tau + 1) = N_c. \end{aligned}$$

Thus all the events are accounted for.

Consider now the type I, paralyzable counter. Let N_c be the number of counts registered when exposed to the sequence of N_0 events. Since the type I counter counts event intervals in excess of the characteristic time τ , and since the N_0 events generate $(N_0 - 1) \approx N_0$ event intervals, we expect $N_c = p_0 N_0$ where p_0 is the probability of no events occurring in the time τ . Thus by Eq. (2) we have

$$N_c = e^{-a\tau} N_0. \quad (\text{Type I}) \quad (6)$$

During the time of the counting process n events may occur before an event interval of duration greater than τ is encountered. To illustrate the calculation of the probability for the occurrence of an "n-event count," we consider the case of $n = 2$. A possible pattern of events is shown in Fig. 1. The probability, δp_2 , of this particular pattern occurring is the conditional probability for independent events given by the expression:

$$\delta p_2 = e^{-at_1} a dt_1 e^{-at_2} a dt_2 e^{-a\tau} \quad (7)$$

where e^{-at_1} is the probability that no event will occur in the time t_1 , $a dt_1$ is the probability that an event will occur in the small time interval dt_1 , and so on. All such patterns of events where $t_1 < \tau$ and $t_2 < \tau$ will produce a "2-event count." Hence the total joint probability, p_2 , that at least one of all such possible patterns of events will occur is just the sum of δp_2 over all acceptable values of t_1 and t_2 . Since the events defining t_1 and t_2 are independent, we have:

$$\begin{aligned} p_2 &= e^{-a\tau} \int_{t_1=0}^{\tau} e^{-at_1} a dt_1 \int_{t_2=0}^{\tau} e^{-at_2} a dt_2 \\ &= e^{-a\tau} (1 - e^{-a\tau})^2. \end{aligned}$$

Thus the probability for an "n-event count" will be seen to be:

$$p_n = e^{-a\tau} (1 - e^{-a\tau})^n. \quad (8)$$

We conclude that for a large enough N_c , we expect a fraction, p_n , of these counts to be "n-event counts." We write this number of counts as N_{cn} .

Thus

$$N_{cn} = p_n N_c = e^{-a\tau} (1 - e^{-a\tau})^n N_c. \quad (9)$$

We note:

$$\begin{aligned} \sum_{n=0}^{\infty} N_{cn} &= N_c \sum_{n=0}^{\infty} p_n = N_c e^{-a\tau} \sum_{n=0}^{\infty} (1 - e^{-a\tau})^n \\ &= N_c e^{-a\tau} e^{+a\tau} = N_c. \end{aligned}$$

Thus all the counts are accounted for.

Furthermore, since each of the "n-event counts" accounts for $(n + 1)$ events, we also have:

$$\sum_{n=0}^{\infty} (n + 1) N_{cn} = N_c \sum_{n=0}^{\infty} (n + 1) p_n = N_c e^{+a\tau} = N_o.$$

Thus all the events are accounted for.

We now apply this discussion of the counting process to an analysis of the pulse pile-up effects that occur in a pulse height analysis system.

III. A PULSE HEIGHT ANALYSIS SYSTEM

We analyze a pulse height analysis system (PHA) that is modeled after the system developed by ORTEC and used on the PDX tokamak. The scheme is outlined in the Introduction. For each event a fast pulse is generated by a

"retriggerable one-shot" circuit. The duration of this fast rising pulse is limited by the occurrence of an event interval in excess of τ_r . Thus the fast pulse generating circuit has the characteristics of a paralyzable, type I counter. The amplitude of the signal produced by an event is measured by a "slow" counter. The "slow" counter is also a paralyzable, type I counter with a characteristic time τ_m where $\tau_m \gg \tau_r$. The slow counter is "gated on" by a gate triggered by a fast pulse. The duration of the gate is limited by the occurrence of an interval between fast pulses in excess of τ_m . Logic circuitry allows the storage of information from only those slow counts for which no fast pulse occurs during the "gated on" time. The wave forms associated with the fast pulses and the slow counter are shown in Fig. 2. We will first analyze which slow counts are stored. It is the pulse height information contained in these stored counts that produce the measured energy spectrum. We will then analyze the way in which the measured energy spectrum is assembled from the stored counts. The effects of pulse pileup stem from the occurrence of more than one event during a fast pulse.

We consider a large number, N_0 , of random events occurring at an average rate "a." These events produce N_p fast pulses and N_c slow counts. From the above analysis of paralyzable counters, we have by Eq. (6):

$$N_p = e^{-a\tau_r} N_0. \quad (10)$$

The probability that a fast pulse will be extended containing n events in addition to the initiating event is given by Eq. (8):

$$p_n = e^{-a\tau_r} \left(1 - e^{-a\tau_r} \right)^n. \quad (11)$$

Therefore the number of "n-event" fast pulses, N_{pn} , is

$$N_{pn} = p_n N_p = e^{-2a\tau_r} \left(1 - e^{-a\tau_r}\right)^n N_o. \quad (12)$$

The slow counter counts those intervals between fast pulses which exceed τ_m in duration. We note the fast pulse intervals and the event intervals have similar distributions since each fast pulse interval, t , corresponds to an event interval of $t + \tau_r$. Hence the number, N , of fast pulse intervals of duration greater than t is expected to be

$$N = e^{-a(t + \tau_r)} N_o = e^{-at} N_p. \quad (13)$$

Therefore taking $t = \tau_m$ gives the number of slow counts, N_c , generated by the N_p fast pulses, namely:

$$N_c = e^{-a\tau_m} N_p. \quad (14)$$

The result expressed by Eq. (13) bears some emphasis. It states that the probability for the fast pulse generating circuit to produce an interval between fast pulses in excess of t , namely N/N_p , is equal to the probability that no event will occur in a time t , namely e^{-at} .

During the gated on time for the slow counter n fast pulses may occur after the initiating fast pulse. In Fig. 3 is shown a possible configuration of fast pulses for the case of $n=1$. The probability, δ_p , for this particular configuration is

$$\delta_p = e^{-at_1} a \tau_1 e^{-a\tau_m}. \quad (15)$$

Here e^{-at_1} is the probability of a fast pulse interval in excess of t_1 . The probability that a fast pulse will occur in the interval t_1 to $t_1 + dt_1$ is taken as adt_1 . This expression is equal to the probability that an event will occur in the interval dt_1 . Such an event will trigger a fast pulse. Finally, $e^{-a\tau_m}$ is the probability that there will be a terminating interval between fast pulses greater than τ_m . Different values of t_1 produce different configurations provided t_1 does not exceed τ_m . Thus the total probability, p_1 , of an "n=1 pulse" slow count is

$$p_1 = \int_{t_1=0}^{\tau_m} e^{-at_1} adt_1 e^{-a\tau_m} = e^{-a\tau_m} \left(1 - e^{-a\tau_m} \right). \quad (16)$$

This result is analogous to the result of Eq. (8). For the probability of an "n-pulse" slow count we have

$$p_n = e^{-a\tau_m} \left(1 - e^{-a\tau_m} \right)^n. \quad (17)$$

We write

$$N_{cn} = p_n N_c, \quad (18)$$

where N_{cn} is the number of slow counts such that during the counting gate n fast pulses occur after the initiating fast pulse. We note Eqs. (17) and (18) are equivalent to Eqs. (8) and (9) with a reinterpretation of the symbols.

Now the logic of the pulse height analysis system stores information from only those slow counts for which $n=0$. The number of $n=0$ slow counts is just

$$\begin{aligned}
 N_{CO} &= P_0 N_C \approx e^{-a\tau_m} N_C \\
 &\approx e^{-2a\tau_m} N_P.
 \end{aligned}
 \tag{19}$$

The stored information is collected over some time, T . If we let N_0 be the number of events that occurs in T , we expect

$$N_0 = aT. \tag{20}$$

Hence the expected number of fast pulses generated in T is

$$N_P = aT e^{-a\tau_r}. \tag{21}$$

Equation (21) enables the event rate "a" to be calculated from counting the fast pulses and knowledge of circuit parameters. However there is an ambiguity in this procedure since Eq. (21) determines "a" to be a double valued function of N_P . [The maximum value of N_P is $T/(\epsilon\tau_r)$ and occurs for $(a\tau_r) = 1.$]

The energy measurement for each of the N_{CO} counts is stored and, taken all together, constitutes the measured energy spectrum. The energy measured for a given one of the N_{CO} counts is the sum of the energies of the events that occur during the fast pulse that initiates the count.

IV. THE MEASURED ENERGY SPECTRUM

The measured energy spectrum is created in the following manner. For each of the N_{co} counts, collected over a time T , the energy of the events which have occurred during the fast pulse that initiated the count is measured. Since these events occur within the event resolution time of the fast pulse generator, their energies are effectively summed by the energy measuring circuits of the slow counter. The energy measurement in digital form serves to label an energy interval, or "energy bin," defined in a computer memory. The N_{co} counts are thereby assigned to a contiguous set of energy bins. The resulting distribution of counts constitutes the measured energy spectrum.

We analyze the measured energy spectrum in the following way. We divide the set of N_{co} counts into subsets according to how many events occur during the initiating fast pulse. The energy distribution of each subset of counts is calculated. The sum of these energy distributions constitutes the final measured energy spectrum.

Now the subset of fast pulses that initiate the N_{co} counts is a random sampling of the entire set of fast pulses. Hence, the distribution of n -event fast pulses will be the same in the subset as in the entire set of fast pulses. Thus the expected fraction of the N_{co} counts that are initiated by an n -event fast pulse is equal to the probability that an n -event fast pulse will occur, and is given by Eq. (11). Writing N_{con} for the number of such counts, we have

$$N_{con} = p_n N_{co} = e^{-a\tau_r} \left(1 + e^{-a\tau_r} \right)^n N_{co}. \quad (22)$$

We note

$$\sum_{n=0}^{\infty} p_n = e^{-a\tau x} \sum_{n=0}^{\infty} \left(1 - e^{-a\tau x}\right)^n = 1. \quad (23)$$

Let $N_n(E_1)\Delta E_1$ (note the single subscript "n") be that subset of the N_{con} counts such that the measured energy falls between E_1 and $E_1 + \Delta E_1$. Summing over all the "energy bins," ΔE_1 , we must have

$$\sum_i N_n(E_i)\Delta E_i = N_{con}. \quad (24)$$

Alternatively, summing over n produces the measured energy spectrum, namely:

$$M(E_1)\Delta E_1 \equiv \sum_{n=0}^{\infty} N_n(E_1)\Delta E_1. \quad (25)$$

Now for large enough values of N_{con} we can write

$$\frac{N_n(E_1)\Delta E_1}{N_{con}} = p_n(E_1) \Delta E_1 \quad (26)$$

where $p_n(E_1)\Delta E_1$ is the probability that the sum of the energies of the $(n + 1)$ events associated with one of the N_{con} counts lies in the range E_1 to $E_1 + \Delta E_1$. We calculate this probability as follows. We assume the subset of events that occur during the fast pulse triggers of the N_{co} counts is a random sampling of the set of N_o events. Hence the two sets of events are expected to have the same energy distribution. Let the energy distribution of the N_o events entering the detector be $f(E)$ where

$$\int_0^{\infty} f(E)dE = 1 \quad (27)$$

and $f(E) = 0$ for $E < 0$.

We interpret the expression $f(E)dE$ to be the probability that a particular event will have an energy falling in the range E to $E + dE$. The conditional probability for the $(n + 1)$ events in a particular configuration, say the j^{th} configuration, to have energies around the values $E_0, E_1, E_2, \dots, E_n$ is given by the product

$$\delta p_j \Delta E_1 = f(E_0)dE_0 f(E_1)dE_1 \dots f(E_n) dE_n. \quad (28)$$

We require, in addition, that

$$E_1 < \sum_{k=0}^n E_k \leq (E_1 + \Delta E_1). \quad (29)$$

(This simplifying requirement assumes that the measured energy of the $n+1$ events will be the algebraic sum of the energy associated with each individual event. In reality the peak amplitude, and hence the measured energy, of the sum of two or more of the event signals will depend on the shape of the signals. We neglect this dependence in this analysis.)

The joint probability that at least one such configuration of $(n + 1)$ events will occur is $p_n(E_1)\Delta E_1$. Thus

$$p_n(E_1)\Delta E_1 = \sum_j \delta p_j \Delta E_1 = \Delta E_1 \sum_j \delta p_j \quad (30)$$

where the summation is over all configurations of $(n + 1)$ events that satisfy the constraint given in Eq. (29). We will work out several examples.

Consider the N_{co0} counts. These counts are single event counts. Since there is only one event in each count, the initiating event, the energy distribution of the count is the energy distribution of the event. So

$$p_o(E)dE = f(E)dE \quad (31)$$

and by Eq. (26)

$$N_o(E)dE = N_{co0} f(E)dE. \quad (32)$$

We note

$$\int_{E=0}^{\infty} N_o(E)dE = N_{co0} \int_0^{\infty} f(E)dE = N_{co0}. \quad (33)$$

We have treated the energy bins as small enough to justify the replacement of the summation over bins with integration over energy.

Consider the N_{co1} counts. These are two event counts, a single event following the initiating event within the duration of the fast pulse. Let E_0 and E_1 be the energy of the initiating and the following event, respectively.

$$\text{Then } \delta p_j dE = f(E_0)dE_0 f(E_1)dE_1$$

where $E = E_0 + E_1$. We take $E_1 = E - E_0$ and $dE_1 = dE$ and write

$$\delta p_j dE = f(E_0)dE_0 f(E-E_0)dE.$$

Noting that E_0 is permitted to take on any energy between 0 and E , we then have by Eq. (30):

$$P_1(E)dE = dE \int_{E_0=0}^{\infty} f(E_0) f(E - E_0) dE_0. \quad (35)$$

Hence by Eq. (26)

$$N_1(E)dE = N_{CO1}dE \int_{E_0=0}^{\infty} f(E_0) f(E - E_0) dE_0. \quad (36)$$

We note:

$$\begin{aligned} \int_{E=0}^{\infty} N_1(E)dE &= N_{CO1} \int_{E=0}^{\infty} dE \int_{E_0=0}^{\infty} f(E_0) f(E-E_0)dE_0 \\ &= N_{CO1} \int_{E_0=0}^{\infty} f(E_0)dE_0 \int_{E=E_0}^{\infty} f(E-E_0)dE \\ &= N_{CO1} \int_{E_0=0}^{\infty} f(E_0)dE_0 \times 1 = N_{CO1}. \end{aligned} \quad (37)$$

Consider the N_{CO2} counts. These are three event counts, two events following the initiating event within the duration of the fast pulse. Here

$$\delta p_j dE = f(E_0) dE_0 f(E_1) dE_1 f(E_2) dE_2,$$

where $E = E_0 + E_1 + E_2$. We define $E_s = E_0 + E_1$, taking $dE_1 = dE_s$ and $dE_2 = dE$, and write

$$\delta E_j dE = f(E_0) dE_0 f(E_s - E_0) dE_s f(E - E_s) dE.$$

Summing over all permissible values of E_0 and E_s , we have

$$N_2(E) dE = N_{co2} dE \int_{E_s=0}^E f(E - E_s) \int_{E_0=0}^{E_s} f(E_s - E_0) f(E_0) dE_0 dE_s. \quad (38)$$

As shown in Appendix B we have

$$\int_{E=0}^{\infty} N_2(E) dE = N_{co2} \quad (39)$$

The pattern is established. Substituting for the $N_n(E)$ in Eq. (25) we can write for the measured energy spectrum

$$\begin{aligned} M(E) dE = & \{ N_{co0} f(E, dE) + N_{co1} dE \int_{E_0=0}^E f(E - E_0) f(E_0) dE_0 \\ & + N_{co2} dE \int_{E_s=0}^E f(E - E_s) \int_{E_0=0}^{E_s} f(E_s - E_0) f(E_0) dE_0 dE_s \\ & + \dots \dots \dots \} \quad (40) \end{aligned}$$

The n th term of Eq. (40) represents that fraction of the stored counts of an $(n + 1)$ event pileup which has a measured piled-up energy in the range E to $E + dE$. The first term represents the contribution to the measured energy spectrum of single event (no pileup) counts. The second term represents the contribution of two event pile-up counts and so on.

Substituting for N_{con} from Eq. (22) we have, finally:

$$\begin{aligned}
M(E)dE = dE N_{co} \{ & e^{-a\tau_r} I_0(E) + \\
& e^{-a\tau_r} (1 - e^{-a\tau_r}) I_1(E) + e^{-a\tau_r} (1 - e^{-a\tau_r})^2 I_2(E) + \\
& \dots + e^{-a\tau_r} (1 - e^{-a\tau_r})^n I_n(E) + \dots \}
\end{aligned} \quad (41)$$

where $I_n(E)$ is a convolution integral defined by

$$I_n(E) \equiv \int_{E'=0}^E f(E - E') I_{n-1}(E') dE' \quad (42)$$

and

$$I_0(E) = f(E). \quad (43)$$

Equation (41) is the principal result of this analysis.

Noting, as shown in Appendix B, that

$$\int_{E=0}^{\infty} I_n(E) dE = 1, \quad (44)$$

we have the satisfactory result by integrating Eq. (41):

$$\int_{E=0}^{\infty} M(E) dE = N_{co} e^{-a\tau_r} \sum_{n=0}^{\infty} (1 - e^{-a\tau_r})^n = N_{co}. \quad (45)$$

Treating $(a\tau_r)$ as a small quantity and expanding Eq. (41) we find:

$$\begin{aligned}
M(E)dE = dE N_{CO} & \left[I_0(E) + (a\tau_r) [-I_0(E) + I_1(E)] \right. \\
& + \frac{(a\tau_r)^2}{2!} [I_0(E) - 3I_1(E) + 2I_2(E)] \\
& + \frac{(a\tau_r)^3}{3!} [-I_0(E) + 7I_1(E) - 12I_2(E) + 6I_3(E)] \\
& \left. + \dots \right].
\end{aligned} \tag{46}$$

Consider the first order approximation

$$M(E)\Delta E \approx \Delta E N_{CO} \{ I_0(E) (1 - a\tau_r) + a\tau_r I_1(E) \} . \tag{47}$$

From this expression one could interpret the measured spectrum falling in the chosen interval ΔE to be the true spectrum modified both by the "loss" of events due to "piling up" with events from other parts of the spectrum and by the "gain" of "pseudoevents" due to the "pileup" of events of lower energy. However, from the full expression, Eq. (41), the more exact interpretation of the measured spectrum is that it is a summation of counts from randomly sampled normalized spectra each spectrum being weighted by the probability for its style of counts to occur. The normalized spectrum given by $I_0(E)$ is the "true" normalized spectrum, $f(E)$. $I_n(E)$ is the normalized spectrum for $(n + 1)$ events added together by "pileup." We note that for $a\tau_r \gg 1$ the spectra become more equally weighted, although the total number of "counts" contained in the measured spectrum becomes exponentially small.

To display only the shape of the measured spectrum we normalize $M(E)$ by dividing by N_{CO} . Thus we define

$$m(E)dE \equiv \frac{M(E)dE}{N_{CO}} = dE e^{-aT_x} \{ I_0(E) + (1 - e^{-aT_x}) I_1(E) + \dots + (1 - e^{-aT_x})^n I_n(E) + \dots \} \quad (48)$$

Note: $\int_{E=0}^{\infty} m(E)dE = e^{-aT_x} \{ 1 + (1 - e^{-aT_x}) + \dots + (1 - e^{-aT_x})^n + \dots \}$

$$= 1. \quad (49)$$

V. EXAMPLES OF SPECTRA

To illustrate the above model for the measured energy spectrum we consider a few artificial spectra chosen for their mathematical simplicity. The first two spectra approximate filtered Bremsstrahlung spectra. All the spectra are normalized such that integration over all energy is unity.

Spectrum I:

$$f(E) = \frac{1}{T} e^{-(E - E_m)/T} \quad \text{for } E \geq E_m$$

$$= 0 \quad \text{for } E < E_m.$$

Substituting $f(E)$ into Eq. (42) we find:

$$I_n(E) = e^{-E/T} \left(\frac{e^{E_m/T}}{T} \right)^{n+1} \frac{(E - (n+1)E_m)^n}{n!}, \quad E > (n+1)E_m$$

$$= 0 \quad E < (n+1)E_m.$$

For $E_m = 0$ all the resulting terms in Eq. (48) can be summed in closed form.

We find

$$m(E) = e^{-a\tau_r} \frac{1}{T} e^{-\frac{E}{T}} e^{-a\tau_r} \quad (50)$$

Examples of $m(E)$ for various values of E_m , T , and $a\tau_r$ are shown in Fig. 4. We note for $E_m \neq 0$ the pile-up effects manifest themselves abruptly at integral multiples of E_m . The larger E_m or $a\tau_r$ and the lower the value of T , the larger the pile-up effect.

Spectrum II

$$f(E) = \frac{b^{(p+1)}}{\Gamma(p+1)} E^p e^{-bE} \quad \begin{matrix} b > 0 \\ p > 0 \end{matrix}$$

where $\Gamma(p+1)$ is the gamma function.

Substituting $f(E)$ into Eq. (42) we find:

$$I_{n-1}(E) = c^{-bE} \frac{c^n (p!)^n E^{(np+n-1)}}{(np+n-1)!}$$

where $c = \frac{b^{(p+1)}}{\Gamma(p+1)}$.

Examples of $m(E)$ are shown in Fig. 5 for the choice of $p = 5$, $b = 2$. The contribution of different orders of pile-up effects for this spectrum are shown in Fig. 6. Note that the contribution of a given order of pileup exceeds the contribution of a lower order of pileup for large enough values of E .

Comparing Figs. 4 and 5, it is clear that the sharp onset of pile-up effects for spectrum I is related to the sharp cutoff at $E = E_m$. Spectrum II has a softer cutoff and a softer onset of pile-up effects. It is also clear, particularly in Fig. 4, that the pile-up effects produce little distortion of the shape of the spectrum for energies below approximately twice the spectrum cutoff energy. For spectrum II the cutoff energy, i.e., the energy for which the filtering has significantly reduced the intensity, is not clearly defined as it is for spectrum I. However a "cutoff energy" for spectrum II of $E \approx 2$ does not seem unreasonable.

Spectrum III

$$\begin{aligned} f(E) &= \alpha f_0(E) + \beta f_L(E) & E > 0 \\ &= 0 & E < 0 \end{aligned} \quad (51)$$

$$\text{where } \int_0^{\infty} f(E) dE = \int_0^{\infty} f_0(E) dE = \int_0^{\infty} f_L(E) dE = 1,$$

and hence $\alpha + \beta = 1$.

Substituting $f(E)$ into Eq. (42) and using the commutative, associative, and distributive properties of the convolution integrals (See Appendix C) we find the result:

$$I_{n-1}(E) = \sum_{k=0}^n \frac{n!}{(n-k)!k!} \alpha^{n-k} \beta^k (f_0^{*(n-k)} * f_L^{*k}) \quad (52)$$

where the following notation has been used:

$$f * g \equiv \int_{E_0=0}^E f(E_0) g(E - E_0) dE_0$$

and

$$f * f \equiv f^{*2}, \quad (53)$$

$$f * f * f \equiv f^{*3}, \text{ etc.}$$

In this notation the convolution of two functions is represented by a "product" of the two functions, the symbol "*" replacing the usual multiplication symbol. The usual "rules" of multiplication then apply. The "*" exponent form is introduced, Eq. (53), as a shorthand notation for multiple convolutions of the same function.

With this notation we recognize $I_{n-1}(E)$ to be an n-fold convolution integral:

$$I_{n-1}(E) = f^{*n} = (\alpha f_0 + \beta f_1)^{*n}. \quad (54)$$

Equation (52) is simply the "binomial expansion" of the last form of Eq. (54).

We use the form of Eq. (51) to model line radiation, $f_l(E)$, superimposed in a background radiation, $f_0(E)$. In particular we approximate a single line of radiation with a δ -function centered around the line energy, E_l , namely:

$$f_l(E) = \delta(E - E_l). \quad (55)$$

Then:

$$f_l^* * f_l = \int_{E_0=0}^E \delta(E_0 - E_l) \delta(E - E_0 - E_l) dE_0 = \delta(E - 2E_l)$$

and

$$f_l^* (f_l * f_l) = \int_{E_0=0}^E \delta(E_0 - E_l) \delta(E - E_0 - 2E_l) dE_0 = \delta(E - 3E_l)$$

and, in general

$$f_l^{*n} = \delta(E - nE_l). \quad (56)$$

Also

$$f_l^* * f_0 = \int_{E_0=0}^E \delta(E_0 - E_l) f_0(E - E_0) dE_0 = f_0(E - E_l)$$

and

$$f_l^* (f_l^* * f_0) = \int_{E_0=0}^E \delta(E_0 - E_l) f_0(E - E_0 - E_l) dE_0 = f_0(E - 2E_l), \quad (57)$$

and so on. Using Eq. (55) in Eq. (51) and substituting into Eq. (48) we find, after some algebra,

$$\begin{aligned} m(E) = & e^{-a\tau_x} \left[\alpha f_0(E) + \beta \delta(E - E_l) \right] + (1 - e^{-a\tau_x}) \left[\alpha^2 I_{10}(E) + 2\alpha\beta f_0(E - E_l) + \right. \\ & \left. + \beta^2 \delta(E - 2E_l) \right] + (1 - e^{-a\tau_x})^2 \left[\alpha^3 I_{20}(E) + 3\alpha^2\beta I_{10}(E - E_l) + \right. \\ & \left. + 3\alpha\beta^2 f_0(E - 2E_l) + \beta^3 \delta(E - 3E_l) \right] + (1 - e^{-a\tau_x})^3 \left[\alpha^4 I_{30}(E) \right. \end{aligned}$$

$$\begin{aligned}
& + 4 \alpha^3 \beta I_{20}(E - E_L) + 6 \alpha^2 \beta^2 I_{10}(E - 2E_L) + 4 \alpha \beta^3 f_0(E - 3E_L) + \beta^4 \delta(E - 4E_L)] \\
& + \dots].
\end{aligned}
\tag{58}$$

Here $I_{n0}(E)$ is equivalent to $I_n(E)$ except that $f_0(E)$ is used instead of $f(E)$ in the defining Eqs. (42) and (43).

Consider the structure of Eq. (58) and, as typical, the terms in the third square bracket. These terms describe three-event pileups. $I_{20}(E)$ is compounded from three events all of which occur in the background distribution $f_0(E)$. $I_{20}(E)$ is defined for all $E > 0$. $I_{10}(E - E_L)$ is compounded from two events from $f_0(E)$ plus one event from the line radiation $\delta(E - E_L)$. Hence $I_{10}(E - E_L)$ is nonvanishing only for $E > E_L$. Now $f_0(E - 2E_L)$ arises from one event from $f_0(E)$ and two events from $\delta(E - E_L)$. The two events from the line radiation pile up into a narrow energy range about an energy of $2E_L$. Note $f_0(E - 2E_L)$ is nonvanishing only for $E > 2E_L$. Finally $\delta(E - 3E_L)$ arises from three events all from the line radiation. We note the last term in each of the square brackets represents the pileup of events from only the line radiation. These terms produce a distinct n^{th} order line at an energy of nE_L . The area under each order line is found by integration over E . We note the ratio of the areas under successive orders of the line radiation is simply $\beta(1 - e^{-\alpha \tau_L})$.

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Interesting and enlightening discussions on pulse pile-up effects with S. Sesnic and S. von Goeler are acknowledged with pleasure.

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APPENDIX A: THE POISSON DISTRIBUTION OF RANDOM EVENTS

Consider events occurring at random but at an average rate "a." The temporal distribution of events can be deduced if one takes the probability, p , for an event to occur in a small interval of time, dt , to be given by the expression: $p = a dt$. Then the probability p_n of exactly n such events occurring in a time τ can be identified with the probability, C_n^N , of exactly n "successes" in N trials where N is that integer nearest to the quotient τ/dt . We know

$$C_n^N = p^n \frac{(1-p)^{N-n} N!}{n! (N-n)!} \quad (A.1)$$

We let $dt \rightarrow 0$ and then consider the limit of C_n^N as $p \rightarrow 0$ and $N \rightarrow \infty$ while n is fixed. Thus:

$$\frac{N!}{(N-n)!} = N(N-1)(N-2)\dots(N-n+1) \rightarrow N^n$$

and

$$(1-p)^{N-n} = \left(1 - \frac{a\tau}{N}\right)^{N-n} = \left(1 - \frac{a\tau}{N}\right)^N \rightarrow e^{-a\tau}.$$

Hence:

$$C_n^N \rightarrow \left(\frac{a\tau}{N}\right)^n \frac{e^{-a\tau} N^n}{n!} = \frac{(a\tau)^n}{n!} e^{-a\tau} \quad (A.2)$$

We take the final expression in Eq. (A.2) to be the above posed probability, p_n , of finding exactly n events in a time τ . Identifying $a\tau$ with the average number of events, m , found in a time τ , we have:

$$P_n = \frac{\mu^n e^{-\mu}}{n!}.$$

(A.3)

Equation (A.3) is the Poisson distribution.

APPENDIX B: CONVOLUTION INTEGRALS

We define:

$$I_n(x) = \int_{x'=0}^x f(x-x') I_{n-1}(x') dx'$$

and

$$I_0(x) = f(x).$$

Thus:

$$I_1(x) = \int_{x'=0}^x f(x-x') f(x') dx'$$

$$I_2(x) = \int_{x'=0}^x f(x-x') \left(\int_{x''=0}^{x'} f(x'-x'') f(x'') dx'' \right) dx'$$

etc.

Consider:

$$I \equiv \int_{x=0}^{\infty} I_n(x) dx = \int_{x=0}^{\infty} \int_{x'=0}^x f(x-x') I_{n-1}(x') dx' dx. \quad (B.1)$$

The region of integration for I is the infinite region in the first quadrant above the line $x=x'$. See Fig. B.1. As written in Eq. (B.1) the integration is done first on x' such as along the horizontal strip in Fig. B.1.

Figure B.1

This result is then integrated (summed) over all values of x . The integration can be done on x first, such as along the vertical strip in Fig. B.1. This result is then summed over all values of x' from 0 to ∞ . Thus we can write:

$$\tau = \int_{x'=0}^{\infty} dx' I_{n-1}(x') \int_{x=x'}^{\infty} f(x-x') dx. \quad (\text{B.2})$$

Consider $f(x)$ to be defined so that

$$\int_{x=0}^{\infty} f(x) dx = 1. \quad (\text{B.3})$$

Then Eq. (B.2) becomes

$$I \equiv \int_{x=0}^{\infty} I_n(x) dx = \int_{x'=0}^{\infty} I_{n-1}(x') dx' \quad (\text{B.4})$$

⋮

$$= \int_{x=0}^{\infty} I_0(x) dx = \int_{x=0}^{\infty} f(x) dx = 1.$$

APPENDIX C: SOME PROPERTIES OF CONVOLUTIONS

Define the convolution of $f(t)$ and $g(t)$ on the interval 0 to x as:

$$I(x) \equiv \int_{t=0}^x f(t)g(x-t)dt \equiv f*g. \quad (C.1)$$

$I(x)$ has commutative, associative, and distributive properties.

To show the commutative property let $u = x - t$ and $du = -dt$.

Then

$$I(x) = \int_{u=x}^0 f(x-u) g(u)(-du) = \int_{u=0}^x g(u) f(x-u)du.$$

$$\text{Thus } I(x) = f*g = g*f. \quad (\text{commutative property}) \quad (C.2)$$

Consider:

$$\begin{aligned} I*h &= ((f*g) * h) \\ &= \int_{t=0}^x \left(\int_{u=0}^t f(u) g(t-u)du \right) h(x-t)dt. \end{aligned} \quad (C.3)$$

The domain of integration in Eq. (C.3) is the cross-hatched triangle in Fig. C.1. This domain is covered by integrating first on u , such as along strip aa' , and secondly on t . The domain can be equally well-covered by integrating first in t , such as along strip bb' , and secondly in u .

83X0251

Figure C.1

Thus we have:

$$I^*h = \int_{u=0}^x du f(u) \int_{t=u}^x g(t-u)h(x-t)dt .$$

Write

$$B(x,u) \equiv \int_{t=u}^x g(t-u)h(x-t)dt.$$

Change variables letting $v = t-u$, $dv = dt$.

Then

$$B(x,u) = \int_{v=0}^{x-u} g(v)h((x-u)-v)dv = B(x-u).$$

We see that $B(x-u)$ is the convolution of $g(v)$ and $h(v)$ on the interval 0 to $x-u$.

Hence:

$$\begin{aligned} I * h &= \int_{u=0}^x du f(u) B(x-u) = f * B \\ &= f * (g * h) \end{aligned}$$

Thus we have shown the associative property of convolutions, namely:

$$(f * g) * h = f * (g * h) \quad (C.3)$$

Finally, convolutions clearly have a distributive property since writing $g(x) = g_1(x) + g_2(x)$ and substituting into Eq. (C.1) we have:

$$\begin{aligned} I(x) &= \int_{t=0}^x f(t) [g_1(x-t) + g_2(x-t)] dt \\ &= f * (g_1 + g_2) = f * g_1 + f * g_2. \end{aligned} \quad (C.4)$$

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FIGURE CAPTIONS

- FIG. 1 A possible temporal pattern of events.
- FIG. 2 Waveforms associated with a sequence of events. N_0 are events, N_p are fast pulses, N_c are slow counter gates, N_{co} and N_{cn} are subsets of N_c .
- FIG. 3 Possible configuration of fast pulses with associated slow counter gate.
- FIG. 4 Normalized measured energy spectra, $m(E)$, for true event energy spectrum I.
- FIG. 5 Normalized measured energy spectra, $m(E)$, for true event energy spectrum II.
- FIG. 6 Decomposition of the measured spectra of Fig. 5 into contributions, $m_n(E)$, from different orders of pulse pileup.

82 X 0832

Fig. 1

83X0308

Fig. 2

#82X0833

Fig. 3

#82X0836

Fig. 4

83x0211

Fig. 5

Fig. 6

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