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Heat Conduction From a Sphere to an Infinite External Region

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I. Introduction

The cooling of a sphere $0 < r < a$ (region 1) by heat conduction to an infinite external region $r > a$ (region 2) is considered. Both regions are assumed to have constant physical parameters with the sphere $0 < r < a$ initially at temperature V and the external region $r > a$ initially at zero temperature. Heat conduction takes place across the boundary $r = a$ so that the temperature and heat flux are continuous. Three separate cases involving differences in the diffusivities K_1 , K_2 and conductivities K_1 , K_2 are considered.

In Case 1, $K_1 = K_2$ and $K_1 = K_2$, making the physical distinction of regions 1 and 2 disappear. This case is solved by both the heat source and Fourier integral methods.

In Case 2, the K_1 and K_1 , $i = 1, 2$, are considered different and the temperature distribution for $v_1(r, t)$ and $v_2(r, t)$ is solved as a two region problem. Case 1 is used as an analytical check when $K_1 = K_2$ and $K_1 = K_2$.

In Case 3, the inner sphere is considered to have infinite conductivity, but retains heat capacity so that $v_1(r, t) = v_1(t)$ and $v_1(t)$ does not vary with r . In this case we have a one region problem for $v_2(r, t)$ where the cooling law for $v_1(t)$ is a boundary condition on region 2.

In all three cases, the temperature distributions are derived along with first-term short and long time asymptotic expressions.

II. Solution of the Heat Conduction Problem

Case 1, $K_1 = K_2 = K$, $K_1 = K_2 = K$

A. Source Solution

The solution of this problem is given in [1, p. 303] and [1, p.293] by the formula

$$v(r, t) = \int_0^{\infty} G(r, r', t) \frac{4\pi r'^2}{r} f(r') dr'$$

where the source solution, or Green's function, is

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The solution of this problem is given in [1, p. 303] and [1, p.293] by the formula

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where the source solution, or Green's function, is

$$G(r, r', t) = \frac{e^{-(r-r')^2/(4kt)} - e^{-(r+r')^2/(4kt)}}{8\pi r r' \sqrt{\pi kt}}$$

and

$$f(r) = \begin{cases} v & 0 < r < a \\ 0 & r > a. \end{cases}$$

The results of the integration can be written in terms of the integrals ($b = 1/(4kt)$)

$$\int_0^a r' e^{-(r \pm r')^2 b} dr' = \frac{\sqrt{\pi}}{2\sqrt{b}} \left[\operatorname{erf}(r\sqrt{b}) - \operatorname{erf}((r \pm a)\sqrt{b}) \right] - \frac{1}{2b} \left[e^{-(r \pm a)^2 b} - e^{-r^2 b} \right]$$

which are evaluated by a change of variables $r \pm r' = w/\sqrt{b}$. Then,

$$v(r, t) = v \left\{ \frac{1}{2} \left[\operatorname{erf}((r+a)\sqrt{b}) - \operatorname{erf}((r-a)\sqrt{b}) \right] + \frac{e^{-(r+a)^2 b} - e^{-(r-a)^2 b}}{2r\sqrt{\pi b}} \right\}$$

For $r = 0$, we take the limit and obtain

$$v(0, t) = v \left\{ \operatorname{erf}(a\sqrt{b}) - \frac{2a\sqrt{b}}{\sqrt{\pi}} e^{-a^2 b} \right\}$$

When t is large, the arguments of the erf and exp functions are small in magnitude and the power series for small arguments are appropriate for the large time asymptotic results,

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \dots \right)$$

$$\exp(-x) = 1 - x + \frac{x^2}{2} - \dots$$

$$v(r, t) \sim \frac{v a^3}{6\sqrt{\pi}(kt)^{3/2}} + \dots \text{ for } t \rightarrow \infty.$$

For small times, the arguments of the erf and exp functions are large in magnitude and we use

$$\operatorname{erf}(x) = 1 - \operatorname{erfc}(x), \operatorname{erfc}(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} \text{ for } x \rightarrow \infty$$

to produce

$$v(r, t) \sim v \left\{ 1 - \frac{1}{2\sqrt{\pi b}} \left[\frac{e^{-(r+a)^2 b}}{(a+r)} + \frac{e^{-(a-r)^2 b}}{(a-r)} - \frac{e^{-(r+a)^2 b} - e^{-(a-r)^2 b}}{r} \right] + \dots \right\} \quad r < a$$

$$v(r, t) \sim \frac{v}{2\sqrt{\pi b}} \left[\frac{e^{-(r-a)^2 b}}{r-a} - \frac{e^{-(r+a)^2 b}}{r+a} + \frac{e^{-(r+a)^2 b} - e^{-(r-a)^2 b}}{r} \right] \quad r > a$$

$$v(0, t) \sim v \left\{ 1 - \frac{e^{-a^2 b}}{\sqrt{\pi}} \left[\frac{1}{a\sqrt{b}} - 2a\sqrt{b} \right] + \dots \right\} \quad \text{for } t \rightarrow 0, \quad b = \frac{1}{4\kappa t}$$

For $r = a$

$$v(a, t) \sim \frac{v}{2} \left[1 - \frac{1}{a\sqrt{\pi b}} + \frac{e^{-4a^2 b}}{2a\sqrt{\pi b}} \right], \quad t \rightarrow 0.$$

B. Fourier Integral Solution

The heat conduction problem solved in Case 1-A may be expressed by the equations

$$\frac{\partial v}{\partial t} = \kappa \left(\frac{\partial^2 v}{\partial x^2} + \frac{2}{r} \frac{\partial v}{\partial r} \right) \quad \begin{matrix} 0 < r < \infty \\ t > 0 \end{matrix}$$

$$v(r, 0) = f(r) = \begin{cases} v & r < a \\ 0 & r > a \end{cases}$$

$$\frac{\partial v}{\partial r}(0, t) = 0, \quad \lim_{r \rightarrow \infty} v(r, t) = 0$$

We solve this problem by transforming the equation, under a change of variables

$$u(r, t) = r v(r, t),$$

to produce a one dimensional equation for u ,

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial r^2}.$$

Separation of variables $u(r, t) = R(r) T(t)$ gives solutions of the form

$$v(r, t) = \frac{1}{r} \int_0^{\infty} e^{-\lambda^2 kt} [A(\lambda) \sin \lambda r + B(\lambda) \cos \lambda r] d\lambda$$

where $A(\lambda)$ and $B(\lambda)$ are "arbitrary" functions of λ . Since $\partial v(0, t)/\partial r = 0$, $B = 0$, and we have

$$v(r, t) = \int_0^{\infty} A(\lambda) \frac{\sin \lambda r}{r} e^{-\lambda^2 kt} d\lambda.$$

To finish the problem we apply the initial condition to obtain an equation for $A(\lambda)$

$$rf(r) = \int_0^{\infty} A(\lambda) \sin \lambda r d\lambda.$$

The Fourier inversion formula gives

$$\begin{aligned} A(\lambda) &= \frac{2}{\pi} \int_0^{\infty} rf(r) \sin \lambda r dr \\ &= \frac{2V}{\pi} \int_0^a r \sin \lambda r dr \\ &= \frac{2V}{\pi \lambda^2} [\sin \lambda a - \lambda a \cos \lambda a] \end{aligned}$$

and

$$v(r, t) = \frac{2V}{\pi} \int_0^{\infty} \frac{[\sin \lambda a - \lambda a \cos \lambda a]}{\lambda^2} \frac{\sin \lambda r}{r} e^{-\lambda^2 kt} d\lambda.$$

This can be reduced to the formula of case 1-A by noticing that

$$\frac{\partial v}{\partial t} = \frac{-2Vkt}{\pi} \int_0^{\infty} [\sin \lambda a - \lambda a \cos \lambda a] \frac{\sin \lambda r}{r} e^{-\lambda^2 kt} d\lambda$$

can be integrated from Fourier transform tables in terms of exponentials after applying the identities

$$\sin x \sin y = \frac{1}{2} [\cos (x-y) - \cos (x+y)]$$

$$\sin y \cos y = \frac{1}{2} [\sin (x+y) + \sin (x-y)] .$$

Integration on t then produces the solution in part A.

Case 2, $\kappa_1 \neq \kappa_2$, $K_1 \neq K_2$

In this case, we solve the two region problem represented by the equations

$$\frac{\partial v_1}{\partial t} = \kappa_1 \left(\frac{\partial^2 v_1}{\partial r^2} + \frac{2}{r} \frac{\partial v_1}{\partial r} \right) \quad 0 < r < a, t > 0$$

$$\frac{\partial v_2}{\partial t} = \kappa_2 \left(\frac{\partial^2 v_2}{\partial r^2} + \frac{2}{r} \frac{\partial v_2}{\partial r} \right) \quad r > a, t > 0$$

with the boundary conditions

$$\frac{\partial v_1}{\partial r}(r, t) = 0 \quad \lim_{r \rightarrow \infty} v_2(r, t) = 0$$

$$v_1(r, 0) = V, \quad v_2(r, 0) = 0$$

$$v_1(a, t) = v_2(a, t)$$

$$K_1 \frac{\partial v_1}{\partial r}(a, t) = K_2 \frac{\partial v_2}{\partial r}(a, t).$$

These latter two conditions express continuity of temperature and heat flux across the boundary $r = a$.

To proceed, we apply the Laplace transform to each equation to obtain solutions of the form [1, p. 303]

$$\bar{v}_1(r, s) = \frac{V}{s} + A \frac{\sinh q_1 r}{r}, \quad \bar{v}_2(r, s) = B \frac{e^{-q_2 r}}{r}$$

$$q_1 = \sqrt{\frac{s}{\kappa_1}}$$

$$q_2 = \sqrt{\frac{s}{\kappa_2}}$$

where

$$\frac{\partial v_1}{\partial r}(0, t) = 0, \quad \lim_{r \rightarrow \infty} v(r, t) = 0$$

$$v_1(r, 0) = V, \quad v_2(r, 0) = 0$$

have been used and s is the Laplace transform parameter.

The constants $A(s)$ and $B(s)$ are unknown, but can be determined from the transformed equations at $r = a$,

$$\frac{V}{s} + A \frac{\sinh q_1 a}{a} = B \frac{e^{-q_2 a}}{a}$$

$$\left[\frac{q_1 \cosh q_1 a}{a} - \frac{\sinh q_1 a}{a^2} \right] K_1 A = K_2 B \left[\frac{-q_2 e^{-q_2 a}}{a} - \frac{e^{-q_2 a}}{a^2} \right].$$

This pair of equations gives

$$A = \frac{-Va}{s} \frac{(q_2 a + 1) K_2}{(q_2 a + 1) K_2 \sinh q_1 a + (q_1 a \cosh q_1 a - \sinh q_1 a) K_1}$$

$$B = \frac{Vae^{q_2 a}}{s} \frac{(q_1 a \cosh q_1 a - \sinh q_1 a) K_1}{(q_2 a + 1) K_2 \sinh q_1 a + (q_1 a \cosh q_1 a - \sinh q_1 a) K_1}$$

and

$$\bar{v}_1(r, s) = \frac{V}{s} \left\{ 1 - \frac{a}{r} \frac{(q_2 a + 1) K_2 \sinh q_1 r}{(q_2 a + 1) K_2 \sinh q_1 a + (q_1 a \cosh q_1 a - \sinh q_1 a) K_1} \right\}$$

$$\bar{v}_2(r, s) = \frac{Va}{sr} \left\{ \frac{(q_1 a \cosh q_1 a - \sinh q_1 a) K_1 e^{-q_2(r-a)}}{(q_2 a + 1) K_2 \sinh q_1 a + (q_1 a \cosh q_1 a - \sinh q_1 a) K_1} \right\} \dots$$

The behavior of $v_1(r, t)$ and $v_2(r, t)$ for small time is obtained by inverting the large s expansions of $\bar{v}_1$ and $\bar{v}_2$. If we multiply the numerators and denominators by $\exp(-q_1 a)$ and factor the denominator we obtain

$$\bar{v}_1(r, s) = \frac{v}{s} \left\{ 1 - \frac{2a}{r} \cdot \frac{(q_2 a + 1) K_2}{R} \cdot \frac{\sinh q_1 r e^{-q_1 a}}{1 + \frac{T}{R} e^{-2q_1 a}} \right\}$$

$$\bar{v}_2(r, s) = \frac{VaK_1}{sr} \left\{ \frac{1}{R} \cdot \frac{q_1 a - 1 + (q_1 a + 1) e^{-2q_1 a}}{1 + \frac{T}{R} e^{-2q_1 a}} e^{-q_2(r-a)} \right\}$$

where

$$T = \left(\frac{aK_1}{\sqrt{\kappa_1}} - \frac{aK_2}{\sqrt{\kappa_2}} \right) \sqrt{s} + K_1 - K_2 = \alpha \sqrt{s} + \theta$$

$$R = \left(\frac{aK_1}{\sqrt{\kappa_1}} + \frac{aK_2}{\sqrt{\kappa_2}} \right) \sqrt{s} + K_2 - K_1 = \gamma \sqrt{s} + \rho$$

$$\frac{T}{R} = \frac{\alpha}{\gamma} \frac{\sqrt{s} + \theta/\alpha}{\sqrt{s} + \rho/\gamma} = \frac{\alpha}{\gamma} \left(1 + \frac{C_1}{\sqrt{s} + C_2} \right)$$

$$\frac{q_1 a - 1}{R} = \frac{\frac{a}{\sqrt{\kappa_1}} \sqrt{s} - 1}{\gamma \sqrt{s} + \rho} = \frac{\alpha_1 \sqrt{s} - 1}{\gamma \sqrt{s} + \rho} = \frac{\alpha_1}{\gamma} \left(1 - \frac{C_4}{\sqrt{s} + C_2} \right)$$

$$\frac{q_2 a + 1}{R} = \frac{\frac{a}{\sqrt{\kappa_2}} \sqrt{s} + 1}{\gamma \sqrt{s} + \rho} = \frac{\alpha_2 \sqrt{s} + 1}{\gamma \sqrt{s} + \rho} = \frac{\alpha_2}{\gamma} \left(1 + \frac{C_3}{\sqrt{s} + C_2} \right)$$

$$\alpha = \frac{aK_1}{\sqrt{\kappa_1}} - \frac{aK_2}{\sqrt{\kappa_2}}, \quad \theta = K_1 - K_2, \quad \alpha_1 = \frac{a}{\sqrt{\kappa_1}}$$

$$\gamma = \frac{aK_1}{\sqrt{\kappa_1}} + \frac{aK_2}{\sqrt{\kappa_2}}, \quad \rho = K_2 - K_1, \quad \alpha_2 = \frac{a}{\sqrt{\kappa_2}}$$

$$c_1 = \frac{B}{\alpha} - \frac{\gamma}{\gamma}, c_2 = \frac{B}{\gamma}, c_3 = \frac{1}{\alpha_2} - \frac{B}{\gamma}, c_4 = \frac{1}{\alpha_1} - \frac{B}{\gamma}$$

and we obtain

$$\begin{aligned} \bar{v}_1(r, s) &= \frac{V}{s} \left\{ 1 - \frac{2\alpha K_2 e^{-q_1 a}}{r} \sinh q_1 r \frac{\alpha_2}{\gamma} \left(1 + \frac{C_3}{\sqrt{s+C_2}} \right) \sum_{n=0}^{\infty} (-1)^n \left(\frac{T}{R} \right)^n e^{-2q_1 a n} \right\} \\ \bar{v}_2(r, s) &= \frac{V\alpha K_1}{sr} \left\{ \frac{(q_1 a + 1)}{R} \cdot \frac{(q_1 a - 1)}{(q_1 a + 1)} \right. \\ &\quad \left. + \frac{q_1 a + 1}{R} \sum_{n=1}^{\infty} (-1)^n \cdot \left(\frac{T}{R} \right)^{n-1} \left[\frac{q_1 a - 1}{q_1 a + 1} \cdot \frac{T}{R} - 1 \right] e^{-2q_1 a n} \right\} e^{-q_2(r-a)} \end{aligned}$$

The leading terms ($n = 0$) produce

$$\begin{aligned} \bar{v}_1(r, s) &= \frac{V}{s} \left\{ 1 - \frac{\alpha \alpha_2 K_2}{r\gamma} \left[e^{-q_1(a-r)} - e^{-q_1(a+r)} \right] \left(1 + \frac{C_3}{\sqrt{s+C_2}} \right) + \dots \right\} \\ \bar{v}_2(r, s) &= \frac{V \alpha \alpha_1 K_1}{s \gamma r} \left(1 - \frac{C_4}{\sqrt{s+C_2}} \right) e^{-q_2(r-a)} + \dots \end{aligned}$$

and inversion [2, pp. 245, 246] gives the small time behavior

$$\begin{aligned} v_1(r, t) &\sim V \left\{ 1 - \frac{\alpha \alpha_2 K_2}{r\gamma} \left[H \left(\frac{a-r}{2\sqrt{K_1 t}} \right) - H \left(\frac{a+r}{2\sqrt{K_1 t}} \right) \right] \right. \\ &\quad \left. + \frac{C_3}{C_2} \left[G \left(\frac{a-r}{2\sqrt{K_1 t}}, C_2 \sqrt{t} \right) - G \left(\frac{a+r}{2\sqrt{K_1 t}}, C_2 \sqrt{t} \right) \right] + \dots \right\} \quad \begin{matrix} r \leq a \\ t \rightarrow 0 \end{matrix} \\ v_2(r, t) &\sim \frac{V \alpha \alpha_1 K_1}{\gamma r} \left[H \left(\frac{r-a}{2\sqrt{K_2 t}} \right) - \frac{C_4}{C_2} G \left(\frac{r-a}{2\sqrt{K_2 t}}, C_2 \sqrt{t} \right) \right] \quad \begin{matrix} r > a \\ t \rightarrow 0 \end{matrix} \end{aligned}$$

where

$$H(x) = \operatorname{erfc}(x)$$

$$G(x, w) = \operatorname{erfc}(x) - e^{-x^2} e^{(x+w)^2} \operatorname{erfc}(x+w).$$

To obtain the general solution, we apply the complex inversion integral to the second term of $\bar{v}_1$ and all of $\bar{v}_2$,

$$v_1(x, t) - V = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \left(\bar{v}_1(x, s) - \frac{V}{s} \right) e^{st} ds, \quad v_2(x, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \bar{v}_2(x, s) e^{st} ds,$$

and evaluate by closing the contour according to Figure 1, using Cauchy's theorem and taking $R \rightarrow \infty$.

Figure 1.

Then, with $w_1 = \sqrt{x/\kappa_1}$ and $w_2 = \sqrt{x/\kappa_2}$,

$$v_1(r, t) - V = \frac{Va\kappa_2}{2\pi i r} \left\{ \int_0^\infty \frac{(w_2 a i + 1) i \sin w_1 r e^{-xt}}{(i w_2 a \kappa_2 + \rho) i \sin w_1 a + i w_1 a \kappa_1 \cos w_1 a} \frac{(-dx)}{(-x)} \right. \\ \left. + \int_0^\infty (\text{conjugate of first integrand}) e^{-xt} \frac{(-dx)}{(-x)} \right\} \\ + \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{s = \epsilon e^{i\theta}} \left(\bar{v}_1(r, s) - \frac{V}{s} \right) e^{st} ds$$

$$v_2(r, t) = -\frac{Va\kappa_1}{2\pi i r} \left\{ \int_0^\infty \frac{(i w_1 a \cos w_1 a - i \sin w_1 a) e^{-i w_2(r-a)} e^{-xt}}{(i w_2 a \kappa_2 + \rho) i \sin w_1 a + i w_1 a \kappa_1 \cos w_1 a} \frac{(-dx)}{(-x)} \right. \\ \left. + \int_0^\infty (\text{conjugate of first integrand}) e^{-xt} \frac{(-dx)}{(-x)} \right\} \\ + \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{s = \epsilon e^{i\theta}} \left(\bar{v}_2(r, s) - \frac{V}{s} \right) e^{st} ds$$

Now,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{s = \epsilon e^{i\theta}} \left(\bar{v}_1(r, s) - \frac{V}{s} \right) e^{st} ds = -V, \quad \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{s = \epsilon e^{i\theta}} \bar{v}_2(r, s) ds = 0$$

and

$$v_1(r, t) = \frac{-Va\kappa_2}{2\pi r} \cdot 2\text{Im} \int_0^\infty \frac{(w_2 a i + 1) \sin w_1 r e^{-xt} dx/x}{(i w_2 a \kappa_2 + \rho) \sin w_1 a + w_1 a \kappa_1 \cos w_1 a} \\ v_2(r, t) = \frac{Va\kappa_1}{2\pi r} \cdot 2\text{Im} \int_0^\infty \frac{(w_1 a \cos w_1 a - \sin w_1 a) e^{-i w_2(r-a)} e^{-xt} dx/x}{(i w_2 a \kappa_2 + \rho) \sin w_1 a + w_1 a \kappa_1 \cos w_1 a}$$

or

$$v_1(r, t) = \frac{Va^2 K_1 K_2}{\pi r} \int_0^\infty \frac{w_2 M(x) \sin w_1 r e^{-xt} dx}{\Delta(x)}, \quad w_2 = \sqrt{\frac{x}{K_2}}, \quad w_1 = \sqrt{\frac{x}{K_1}}$$

$$v_2(r, t) = \frac{VaK_1}{\pi r} \int_0^\infty \frac{N(x)}{\Delta(x)} e^{-xt} dx$$

where

$$M(x) = (\sin w_1 a - w_1 a \cos w_1 a)/x$$

$$N(x) = M(x) \left[(\rho \sin w_1 a + w_1 a K_1 \cos w_1 a) \sin w_2(r-a) + w_2 a K_2 \sin w_1 a \cos w_2(r-a) \right]$$

$$\Delta(x) = (\rho \sin w_1 a + w_1 a K_1 \cos w_1 a)^2 + (w_2 a K_2 \sin w_1 a)^2$$

A change of variables, $x = \lambda^2 K_1$, removes the square root singularity at $x = 0$ and

$$v_1(r, t) = \frac{2Va^2 K_1 K_2}{\pi r} \sqrt{\frac{K_1}{K_2}} \int_0^\infty \frac{I_1(\lambda) \sin \lambda r e^{-\lambda^2 K_1 t}}{D(\lambda)} d\lambda$$

$$v_2(r, t) = \frac{2VaK_1}{\pi r} \int_0^\infty \frac{I_2(\lambda, r) e^{-\lambda^2 K_1 t}}{D(\lambda)} d\lambda$$

where

$$I_1(\lambda) = \sin \lambda a - \lambda a \cos \lambda a$$

$$I_2(\lambda, r) = I_1(\lambda) \left[(\rho \sin \lambda a + \lambda a K_1 \sqrt{\frac{K_1}{K_2}} \cos \lambda a) \sin \left(\lambda \sqrt{\frac{K_1}{K_2}} (r-a) \right) + \lambda a K_2 \sqrt{\frac{K_1}{K_2}} \sin \lambda a \cos \left(\lambda \sqrt{\frac{K_1}{K_2}} (r-a) \right) \right] / \lambda$$

$$D(\lambda) = \left(\rho \sin \lambda a + \lambda a K_1 \cos \lambda a \right)^2 + \left(\lambda a K_2 \sqrt{\frac{\kappa_1}{\kappa_2}} \sin \lambda a \right)^2.$$

Notice that if $\kappa_1 = \kappa_2 = \kappa$ and $K_1 = K_2 = K$, then $\rho = K_2 - K_1 = 0$ and we get the formula of case 1-B for both v_1 and v_2 .

These formulae are exact for all t , but the asymptotic expansion can be obtained for $t \rightarrow \infty$ by expanding $I_1(\lambda) \sin \lambda r / D(\lambda)$ and $I_2(\lambda, r) / D(\lambda)$ about $\lambda = 0$ and integrating term by term. The first terms are

$$\frac{I_1(\lambda)}{D(\lambda)} \sim \frac{(\lambda a)^3 / 3}{[(\rho + K_1) \lambda a]^2} = \frac{(\lambda a)}{3 \kappa_2^2}, \quad \rho = K_2 - K_1$$

$$\begin{aligned} \frac{I_2(\lambda, r)}{D(\lambda)} &\sim \frac{(\lambda a)}{3 \kappa_2^2} \left[\left(\rho + K_1 \sqrt{\frac{\kappa_1}{\kappa_2}} \right) \lambda^2 a \sqrt{\frac{\kappa_1}{\kappa_2}} (r-a) + (\lambda a)^2 K_2 \sqrt{\frac{\kappa_1}{\kappa_2}} \right] / \lambda \\ &\sim \frac{\lambda^2 a^3}{3 \kappa_2^2} \sqrt{\frac{\kappa_1}{\kappa_2}} \left[\left(\rho + K_1 \sqrt{\frac{\kappa_1}{\kappa_2}} \right) \frac{(r-a)}{a} + K_2 \right]. \end{aligned}$$

Then,

$$\begin{aligned} v_1(r, t) &\sim \frac{2Va^3}{3\pi} \frac{K_1}{\kappa_2} \sqrt{\frac{\kappa_1}{\kappa_2}} \int_0^\infty \lambda^2 e^{-\lambda^2 \kappa_1 t} d\lambda \\ v_2(r, t) &\sim \frac{2Va^4}{3\pi r} \frac{K_1}{\kappa_2^2} \left[\left(\rho + K_1 \sqrt{\frac{\kappa_1}{\kappa_2}} \right) \left(\frac{r}{a} - 1 \right) + K_2 \right] \sqrt{\frac{\kappa_1}{\kappa_2}} \int_0^\infty \lambda^2 e^{-\lambda^2 \kappa_1 t} d\lambda \end{aligned}$$

or

$$v_1(r, t) \sim \frac{va^3}{6\sqrt{\pi}} \frac{K_1}{K_2} \sqrt{\frac{K_1}{K_2}} \frac{1}{(\kappa_1 t)^{3/2}} \quad t \rightarrow \infty$$

$$v_2(r, t) \sim \frac{va^3}{6\sqrt{\pi}} \frac{K_1 \left[\left(\rho + K_1 \sqrt{\frac{K_1}{K_2}} \right) \left(1 - \frac{a}{r} \right) + \frac{a}{r} K_2 \right]}{K_2^2} \sqrt{\frac{K_1}{K_2}} \frac{1}{(\kappa_1 t)^{3/2}}.$$

Case 3, $K_1 = \infty$

In this case, the temperature distribution inside $r=a$ is flat and the heat flow out is governed by the ability of the media to give up its heat. Thus, $v_1(r, t) = v_1(t)$ and the heat Q flowing across the surface $r=a$ per unit time is

$$Q = \frac{4}{3} \pi a^3 \rho_1 c_1 \frac{dv_1}{dt} = 4\pi a^2 K_2 \frac{\partial v_2}{\partial r}(a, t)$$

where ρ_1 and c_1 are the density and heat capacity, respectively, of region 1. Now the problem to be solved is expressed by the equations

$$\frac{\partial v_2}{\partial t} = K_2 \left(\frac{\partial^2 v_2}{\partial r^2} + \frac{2}{r} \frac{\partial v_2}{\partial r} \right) \quad a < r < \infty$$

$$v_2(r, 0) = 0, \quad \lim_{r \rightarrow \infty} v_2(r, t) = 0$$

$$v_1(t) = v_2(a, t)$$

$$\frac{dv_1}{dt} = D \frac{\partial v_2}{\partial r}(a, t), \quad D = \frac{3K_2}{8\rho_1 c_1}$$

$$v_1(0) = v.$$

We transform the problem as before,

$$\bar{v}_2(r,s) = B e^{-q_2 r}$$

$$\bar{v}_1(s) = \bar{v}_2(a,s)$$

$$s\bar{v}_1(s) - V = D \frac{\partial \bar{v}_2}{\partial r}(a,s) ,$$

and solve this last equation,

$$\frac{s B e^{-q_2 a}}{a} - V = D B \left[-\frac{q_2 e^{-q_2 a}}{a} - \frac{e^{-q_2 a}}{a^2} \right]$$

for B

$$B = \frac{V a e^{q_2 a}}{\left(s + \frac{D}{\sqrt{\kappa_2}} \sqrt{s + \frac{D}{a}} \right)} .$$

Then

$$\bar{v}_2(r,s) = \frac{V a}{r} \frac{e^{-q_2(r-a)}}{(\sqrt{s+R_1})(\sqrt{s+R_2})}$$

where

$$R_1 = \frac{D}{2\sqrt{\kappa_2}} + \frac{1}{2} \sqrt{\left(\frac{D}{\sqrt{\kappa_2}} \right)^2 - \frac{4D}{a}}$$

$$R_2 = \frac{D}{2\sqrt{\kappa_2}} - \frac{1}{2} \sqrt{\left(\frac{D}{\sqrt{\kappa_2}} \right)^2 - \frac{4D}{a}} .$$

We use partial fractions to express $\bar{v}_2$ in the form

$$\bar{v}_2(r,s) = \frac{Va}{r(R_2-R_1)} \left[\frac{e^{-\sqrt{s}(r-a)/\sqrt{\kappa_2}}}{\sqrt{s} + R_1} - \frac{e^{-\sqrt{s}(r-a)/\sqrt{\kappa_2}}}{\sqrt{s} + R_2} \right]$$

and invert for $v_2(r,t)$

$$v_2(r,t) = \frac{Va}{r(R_2-R_1)} \left[H\left(\frac{r-a}{2\sqrt{\kappa_2 t}}, R_2, t\right) - H\left(\frac{r-a}{2\sqrt{\kappa_2 t}}, R_1, t\right) \right]$$

by the results in [2, p. 246] where

$$H(\alpha, \beta, t) = \beta e^{-\alpha^2} e^{(\alpha + \beta \sqrt{t})^2} \operatorname{erfc}(\alpha + \beta \sqrt{t}).$$

If $r = a$ we get

$$v_1(t) = v_2(a,t) = \frac{V}{R_2-R_1} \left[R_2 e^{R_2^2 t} \operatorname{erfc}(R_2 \sqrt{t}) - R_1 e^{R_1^2 t} \operatorname{erfc}(R_1 \sqrt{t}) \right].$$

For large time t , the argument of erfc is large and

$$e^{x^2} \operatorname{erfc}(x) \sim \frac{1}{x\sqrt{\pi}} - \frac{1}{2x^3\sqrt{\pi}}, \quad x \rightarrow \infty$$

yields

$$v_2(r,t) \sim \frac{Vae^{-(r-a)^2/(4\kappa_2 t)}}{r\sqrt{\pi}} \left\{ \frac{r-a}{2\sqrt{\kappa_2 t} P_1 P_2} + \frac{1/2}{R_2 - R_1} \left(\frac{R_1}{P_1^3} - \frac{R_2}{P_2^3} \right) + \dots \right\}$$

$$P_i = \frac{r-a}{2\sqrt{\kappa_2 t}} + R_i \sqrt{t}, \quad i = 1, 2$$

for $t \rightarrow \infty$. If $r = a$ we get

$$v_1(t) = v_2(a, t) \sim \frac{V}{2t^{3/2}\sqrt{\pi}} \left[\frac{R_2 + R_1}{R_1^2 R_2^2} \right] = \frac{V}{2D\sqrt{\pi}\kappa_2} \cdot \frac{a^2}{t^{3/2}}, \quad t \rightarrow 0$$

$$D = 3\kappa_2 / (a\rho_1 c_1).$$

For small time, $t \rightarrow 0$, we use the large s form of $\bar{v}_2(r, s)$

$$\bar{v}_2(r, s) = \frac{Va}{r} \frac{e^{-\alpha_2(r-a)}}{s} \left[1 - \frac{R_1 + R_2}{\sqrt{s}} + \frac{R_1^2 + R_1 R_2 + R_2^2}{s} + \dots \right]$$

and invert term by term to obtain

$$v_2(r, t) \sim \frac{Va}{r} \left[\operatorname{erfc} \left(\frac{r-a}{2\sqrt{\kappa_2 t}} \right) - (R_1 + R_2) E \left(\frac{r-a}{\sqrt{\kappa_2}}, t \right) + \left(R_1^2 + R_1 R_2 + R_2^2 \right) F \left(\frac{r-a}{\sqrt{\kappa_2}}, t \right) + \dots \right] \quad t \rightarrow 0$$

where

$$E(\alpha, t) = 2 \sqrt{\frac{t}{\pi}} e^{-\alpha^2/(4t)} - \alpha \operatorname{erfc} \left(\frac{\alpha}{2\sqrt{t}} \right)$$

$$F(\alpha, t) = \left(t + \frac{\alpha^2}{2} \right) \operatorname{erfc} \left(\frac{\alpha}{2\sqrt{t}} \right) - \alpha \sqrt{\frac{t}{\pi}} e^{-\alpha^2/(4t)}.$$

For $r = a$ and $t \rightarrow 0$ this reduces to

$$v_1(t) = v_2(a, t) \sim V \left[1 - \frac{2}{\sqrt{\pi}} (R_1 + R_2) \sqrt{t} + (R_1^2 + R_1 R_2 + R_2^2) t \right].$$

for $v_1(t)$.

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