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QUASI-STATIC ELECTRIC FIELD BUILD-UP BETWEEN TWO SPHERICAL CONDUCTORS

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TABLE OF CONTENTS

1. Introduction and Summary.
2. Formulation of the Problem
3. An Outline of the Procedures for Obtaining the Solutions
4. The Numerical Algorithm
5. Discussion and Results
6. Appendices
  - I. Quasi-Static Approximation to Maxwell's Equations
  - II. The Method of Images
  - III. Further Detailed Discussions of the Numerical Code.
  - IV. Some Approximate Cases to the Electric Field Buildup by an Electron Beam in a Medium Enclosed by Conductors
  - V. The Flat Plate Approximation with Time-Dependent conductivity
  - VI. Maximum Field when the Conductors are Fully Illuminated with Constant Conductivity and Spherically Symmetric Charge Density Source.
  - VII. The Numerical Code

QUASI-STATIC ELECTRIC FIELD BUILD-UP BETWEEN TWO SPHERICAL CONDUCTORS

K. S. Yee,* J. Kasher**

1. Introduction and Summary

It has been observed that the exposure of dielectrics to electron beams can produce electric fields of sufficient magnitude to cause dielectric break-down (see e.g. Refs. 1 and 2). It is the purpose of this investigation to calculate the electric fields for two of the geometric shapes of common interest: spherical shells and cylinders. This report will summarize the work done to date on spherical dielectric shells with an inner and outer conducting surface which are traversed by an electron beam through the sphere's center. Future work will deal with a spherical shell when the beam is displaced from the sphere's center and with cylinders.

Prior to dielectric breakdown the illuminated region of the medium is rendered conducting by the electron beam. When the time interval of interest is large compared to the time needed for the electromagnetic signal to travel in the spatial dimension of interest, it is plausible that a quasi-static model, in which one neglects the magnetic field, will supply some useful information. We shall describe our problem in terms of the electric field (or electric potential) and the charge density. The most important contribution of this report is that in solving the Poisson's equation for the electric potential

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(and the electric field) we have avoided the brute-force numerical method. In a 3-dimensional geometry this means great savings of effort and money for those who must produce numbers in their work. In this report the problem will first be formulated in a system of partial differential equations. We then describe our method for finding the electric field or the potential. In the appendices we shall include some exact solutions for special cases of our problem. These will include plane or spherical geometry with special assumptions about the source and the conductivity. Special solutions should be valuable in checking against numerical codes.

2. Formulation of the Problem. In a conducting medium, neglecting the magnetic field, Maxwell's equations (with charge deposition) reduce to the following:*

$$-\nabla^2 \phi = \frac{\rho}{\epsilon} \quad (1a)$$

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon} \rho = \frac{\partial \rho_s}{\partial t} + (\nabla \sigma) \cdot \nabla \phi \quad (1b)$$

$$\vec{E} = -\nabla \phi \quad ; \quad \vec{D} = \epsilon \vec{E}$$

where ϕ is the electric potential, ρ is the charge density, ρ_s is the source-charge density ($\frac{\partial \rho_s}{\partial t}$ represents the rate of charge density deposition to the medium), σ is the conductivity which depends on the source, $\vec{E}$ is the electric field and $\vec{D}$ is the electric displacement.

We are interested in solving (1a) and (1b) in a dielectric region between two concentric spherical conductors. The mathematical problem then involves finding ϕ and ρ as functions of space and

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*See Appendix I for the derivation.

time when σ and ρ_s are given. The auxiliary conditions would be the boundary conditions appropriate for perfect conductors. In terms of the electric field $\vec{E}$ and ρ one can write (1a) and (1b) as

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon} \quad (1a)$$

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon} \rho = \frac{\partial \rho_s}{\partial t} - (\nabla \sigma) \cdot \vec{E} \quad (1b)$$

We shall solve (2a) and (2b) instead of (1a) and (1b). Shown in Figure 1 is an electron beam passing through a dielectric between two concentric spherical conductors. The region where the electron beam passes will be conducting. There will be an equivalent ρ_s due to the charge displacement. The electron beam may or may not pass through the center of the spheres; and the cross-section of the electron beam may be nonuniform.

The effect of the electron beam

is to supply ρ_s and σ for our problem. (Or better yet we

Fig. 1. An electron beam through a medium between two concentric conducting spheres.

assume that we can describe the buildup of the electric field due to the electron beam with given ρ_s and σ). We are interested in finding the maximum electric field $|\vec{E}|$ in the medium at any time to ascertain whether the dielectric break-down field is attained. Two cases will be discussed:

Case 1 Both conductors are grounded (electrically connected).

Case 2 The conductors are not electrically connected.

The boundary condition on the conductors is that the electric field has no tangential components. In terms of potential, the conductors are equipotential surfaces. Thus

For case 1: we find $\vec{E}$ (or ϕ) and ρ which are solutions of (2a) and (2b) (or (1a) and (1b)) in the region $R_i \leq r \leq R_o$ subject to the condition $\phi = 0$ at $r = R_i$ and $r = R_o$.

For case 2: Same differential equations as for case 1. But the boundary conditions are

$$\psi = 0 \quad \text{at} \quad r = R_o \quad (3a)$$

$$\psi = \Phi(t) \quad \text{at} \quad r = R_i \quad (3b)$$

where $\Phi(t)$ is an unknown function to be found along with the solution.

3. An Outline of the Procedures for Obtaining the Solutions: In order to present the idea clearly we will now just describe our method; the details will be discussed more fully in the next section. Our method does not require the usual numerical solution to the Poisson's equation (Eq. (1a)). Instead we will solve (1a) (or (2a)) by the method of images well known in solving electrostatic problems. First we describe the solution to Case 1. The solution to Case 2 makes use of the results of Case 1.

Case 1 (Both conductors are grounded)

The boundary conditions can be satisfied if one employs the method of images to solve (1a) (or (1b)). The charge density ρ is then advanced by means of (1b). Specifically one proceeds as follows:

Given a point charge lying between two grounded concentric spherical conductors, one can find the electric potential (actually we need only the electric field) by summing the contribution due to appropriate charges situated at the various inversion points (multiple images) with respect to the two perfectly conducting spheres. For the problem at hand we assume ρ to be known at $t = t_0$. We subdivide the region between the two concentric spheres into zones. We can then assign the charge in each zone to be located at some representative point of that zone. Then the method of images (see Appendix II) will yield the contribution to the electric field due to the charge in this zone. Next we sum up vectorially the contribution to the electric field due to charges in all zones. We do this summation both at points where there is charge and at points where we want the electric field. The field $\vec{E}$ will be needed to advance ρ from $t = t_0$ to $t = t_0 + \Delta t$.

Case 2 One conductor is grounded (say the outer conductor) and the other is not.

This case is a little more complicated; but because we are solving linear differential equations, we will take advantage of the principle of superposition. We set

$$\phi = \phi_o + \phi_p \quad (4)$$

where we require

$$-\nabla^2 \varphi_p = \frac{\rho}{\epsilon} \quad ; \quad \varphi_p \Big|_{r=R_i} = \varphi_p \Big|_{r=R_o} = 0 \quad (4a)$$

$$\nabla^2 \varphi_o = 0 \quad ; \quad \varphi_o \Big|_{r=R_o} = 0 \quad ; \quad \varphi_o \Big|_{r=R_i} = \Phi(t) \quad (4b)$$

It is easily verified that the potential defined in (4), (4a), (4b) satisfies the boundary conditions in (3a) and (3b).

The total surface charge $Q(t)$ on the inner sphere is

$$Q(t) = \epsilon \int_{r=R_i} \hat{n} \cdot \vec{E} \, dS = - \epsilon \int_{r=R_i} \hat{n} \cdot \nabla \varphi \, dS \quad (5)$$

where $\hat{n}$ is a unit vector pointing in the increasing radial direction.

The rate of change of the total surface charge in the inner conductor is

$$\frac{dQ}{dt} = \dot{Q}_s - \int_{r=R_i} \sigma \hat{n} \cdot \vec{E} \, dS = \dot{Q}_s + \int_{r=R_i} \sigma \frac{\partial \varphi}{\partial r} \, dS \quad (6)$$

where $\dot{Q}_s$ is a known function describing the rate of charge deposition into the inner spherical conductor. φ_p (actually $\vec{E}_p = -\nabla \varphi_p$ is needed) can be founded by the method of images as in Case 1. φ_o is given as follows:

$$\varphi_o = \tilde{\varphi}(\vec{r}) \Phi(t) \quad (7a)$$

where

$$\tilde{\varphi}(r) = \left(\frac{1}{r} - \frac{1}{R_0} \right) / \left(\frac{1}{R_i} - \frac{1}{R_0} \right) \quad (7c)$$

$\tilde{\varphi}$ satisfies:

$$\tilde{\varphi}' \Big|_{r=R_0} = 0, \quad \tilde{\varphi}' \Big|_{r=R_i} = 1$$

Substitution of (7a) and (7b) with ϕ_p into (5) yields

$$Q(t) = -e \left\{ \int_{r=R_i} \frac{\partial \phi_p}{\partial r} dS - e \Phi(t) \right\} \frac{\partial \tilde{\varphi}}{\partial r} dS$$

Or,

$$Q(t) = e \int_{r=R_i} \hat{n} \cdot \vec{E}_p dS + 4\pi e \frac{R_i R_0}{R_0 - R_i} \Phi(t) \quad (8)$$

Equation (8) yields $\Phi(t)$ and hence ϕ_0 . Our procedures for this case then consist of the following steps (given conditions at $t = t_0$).

- i) Find $\nabla \phi_p$ by the method of images.
- ii) Find $\Phi(t)$ by (8) and then find ϕ_0 ; whence $\nabla \phi_0$.
- iii) The electric field at $t = t_0$ is then

$$\vec{E} = -\nabla \phi_0 - \nabla \phi_p$$

- iv) Advance p to $t = t_0 + \Delta t$ by (1b).
- Advance Q to $t = t_0 + \Delta t$ by (6).
- v) Repeat the above procedures for $t = t_0 + \Delta t$.

4. The Numerical Algorithm (for a beam passing the center of two concentric spheres)

In this section we describe in outline form the algorithm used in solving the problem. A more detailed explanation can be found in Appendix III.

At the start of the program the following parameters are initialized:

- number of azimuthal angle zones
- number of time loops
- number of z-zones between spheres
- number of r-zones
- inner and outer sphere radii
- beam radii on entrance and exit
- conductivity (mhos/meter)
- source charge density (coulombs/cubic meter)
- duration of pulse
- whether beam is constant or gaussian (r-direction)
- whether both spheres are grounded or inner one floats
- whether dr are equal or gaussian (i.e., become larger in gaussian fashion as r increases)

The beam width is set at the outer sphere radius if it would be larger than that.

The grid is calculated in the next part of the program. Details are in Appendix III; here figure 2 will be enough to at least show how we set it up. R and Z are the coordinates of the observation points, and RA and ZA are the coordinates of the source points (centered in the volume elements DV).

Next the electron beam functions (time derivative of the source charge density, conductivity, partial derivatives of the conductivity with respect to z and r , charge density, charge in the volume element DV) are calculated at each of the source points.

After this the program loops around all the observation points and calculates for each one the electric field (including image charges) from each of the source points. That is, for observation point $I = 1$, $K = 1$ the field from each source point is calculated; for observation point $I = 2$, $K = 1$ the same is done, and so forth. The charge in the volume elements is set equal to one; then, in the following time loops, as the charge is advanced the fields from each source point are multiplied by the proper charge and summed. This leads to the total electric field at each observation point at each time step. The program currently prints out the numerical calculations each quarter of the time loop. In the first time loop the fields are advanced first, and then the charge density is advanced. Then in the second loop the fields are advanced, then the charge density, and so on. If the potential is calculated in the time loop right after the charge density.

The flow chart in figure 3 may help clarify the outline of the program.

Figure 2. The grid used. (See also Appendix III, figures III. 1 - 3.)

Figure 3. Flow chart for the program

5. Discussion of Results

(a) Input Data

In its current form the code needs as input the peak conductivity and peak source charge density. When the beam is gaussian, these functions have the same standard deviation. Other parameters that can be easily changed are the relative permittivity and the duration of the pulse. The peak conductivity should be entered for a pulse 40 nanoseconds long. The code automatically adjusts if the pulse is longer or shorter. The values used for the runs described below were

$$\sigma_0 = \text{peak conductivity} = 2 \times 10^{-4} \text{ mho/meter}$$

$$\hat{\rho}_s = \text{peak source charge density} = 2 \times 10^{-2} \text{ coulombs/meter}^2$$

$$\text{relative permittivity} = 3$$

$$\text{duration of pulse} = 40 \text{ nanoseconds}$$

(b) A Parameter Study

In order to get a feel for the accuracy of the program we ran a parameter check for the case of a gaussian beam with initial standard deviation of two centimeters spreading to a final standard deviation of 5.1 centimeters. We varied the number of radial zones (gaussian dr) from two to ten, and the number of z-zones from two to six. Our results are presented in table I, along with the computer time in minutes needed for each run. As might be expected, the extreme values are found in the coarsest grid (two and three for both r and z zones); but the overall results show surprisingly little variation--the maximum and minimum

values in the whole table differ by 18.2%. Moreover, in the region where the number of r and z zones are each at least four, the maximum and minimum values differ by less than six percent. When the number of zones is at least five they differ by only two percent.

So the code appears to be fairly stable and accurate, at least as far as we have carried our parameter check.

(c) Three Sample Calculations

A non-spreading, gaussian beam with standard deviation of one centimeter led to a maximum field of 6.84×10^6 volts/meter. A non-spreading, constant beam with a one centimeter radius led to 5.56×10^6 volts/meter. When the one-centimeter gaussian beam was allowed to spread to 5.1 centimeters, the maximum field was 6.17×10^6 volts/meter. The number of r -zones was increased from seven to ten for the spreading beam.

radial zones → z-zones ↓	2	3	4	5	7	10
6	10.17 (0.57)	10.96 (1.1)	11.32 (1.8)	11.55 (2.7)	11.69 (5.1)	11.52 (9.9)
5			11.58 (1.3)		11.77 (3.6)	
4	10.43 (0.27)	11.51 (0.54)	11.94 (0.88)	12.02 (1.3)	11.76 (2.4)	11.33 (4.8)
3			12.31 (0.54)		11.55 (1.5)	
2			12.24 (0.27)		11.11 (0.75)	

x 10⁶ volts/meter
(minutes)

$$F_{\text{av}} = 0 \leq F \leq 40 \times 10^{-4} \text{ sec}$$

$$\sigma = \sigma_{\text{group}} \left(-\frac{R^2}{2\sigma_p^2} \right)$$

$$\rho_s = \rho_{\text{group}} \left(-\frac{R^2}{2\sigma_p^2} \right)$$

Table I. A parameter study of the maximum field from a two-centimeter gaussian beam that spreads to 5.1 centimeters.

APPENDIX I

Quasi-Static Approximation to Maxwell's Equations

The Maxwell's equations in MKS units are*:

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad (1a)$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \quad (1b)$$

$$\vec{D} = \epsilon \vec{E}, \quad \vec{B} = \mu \vec{H} \quad (1c)$$

where in (1b) $\vec{J}$ is the total current density including the source current and the conduction current. It can also be expressed as

$$\vec{J} = \vec{J}_s + \sigma \vec{E} \quad (2)$$

where $\vec{J}_s$ is the source current density and $\sigma \vec{E}$ will be the conduction current density. In addition to the above equations we shall need the continuity equation for the source current density and the source charge density, ρ_s .

$$\frac{\partial \rho_s}{\partial t} = - \nabla \cdot \vec{J}_s \quad (3)$$

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*J. A. Stratton, Electromagnetic Theory. McGraw-Hill Book Co. 1941.

The total charge density ρ and the displacement $\vec{D}$ are related by

$$\nabla \cdot \vec{D} = \rho \quad (4)$$

Equation (4) is also a consequence of (1b) (i.e. $\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J}$)

We assume that our field is predominantly electric. Specifically, we assume that $\mu H^2 \ll \epsilon E^2$ (Electric energy is much larger than the magnetic energy). Therefore we set

$$\vec{B} \equiv 0 \quad (5)$$

As a consequence of (5), equation (1) indicates that $\vec{E}$ can be derived from a potential ϕ .

$$\vec{E} = -\nabla \phi \quad (6)$$

Thus (4) will become

$$-\nabla^2 \phi = \frac{\rho}{\epsilon} \quad (7)$$

In equation (7) ρ will be a function of position and time; and ϵ will be assumed to be constant. For another equation we will go back to the equation of continuity for the total charge density.

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\nabla \cdot \vec{J} = -\nabla \cdot (\vec{J}_s + \sigma \vec{E}) = -\nabla \cdot \vec{J}_s - \nabla \cdot (\sigma \vec{E}) \\ &= \frac{\partial \rho_s}{\partial t} + \nabla \cdot (\sigma \nabla \phi) = \frac{\partial \rho_s}{\partial t} + \nabla \sigma \cdot \nabla \phi - \frac{\sigma}{\epsilon} \rho\end{aligned}$$

Or

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon} \rho = \frac{\partial \rho_s}{\partial t} + \nabla \sigma \cdot \nabla \phi \quad (8)$$

Given ρ_s and σ equations (7) and (8) constitute two equations for the two unknowns ρ and ϕ . And the equations are almost decoupled.

APPENDIX II

The Method of Images with Two Concentric
Spherical Conductors

Shown in Fig. 1 is a point charge q located at a point between two concentric spheres of radii R_i and R_o respectively. The two spheres are perfectly conducting and they are grounded. It is desired to find the electric potential and/or the electric field due to this charge in the presence of these two conductors. The position of the point charge is $\vec{y} = y \hat{y}$. The potential satisfies the equation

$$(1) \quad \nabla^2 \phi = -q \delta(\vec{x} - \vec{y})$$

$$\phi = 0 \quad \text{at} \quad r = |\vec{x}| = R_i \quad \text{and} \quad \text{at} \quad r = R_o.$$

and the electric field is given by

$$(2) \quad \vec{E} = -\nabla \phi.$$

Fig. 1 A point charge between two spheres

The above problem can be solved by the method of images.

We shall, however, write down the solution for a simpler problem.

From the solution of the simpler problem we shall construct the solution to the original problem.

Fig. 2 Conducting sphere of radius a with charge q and image charge q' .

Shown in Fig. 2 is a point charge q and its image charge q' in the presence of a conducting sphere of radius a . The point of observation is $\vec{x} = x \hat{x}$. The potential at an observation point $\vec{x}$ with $x > a$ is (in MKSA units)

$$(3) \quad \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q}{|\vec{x} - \vec{y}|} + \frac{q'}{|\vec{x} - \vec{y}'|} \right\}$$

where

$$q = -\frac{a}{y} q', \quad y' = \frac{a^2}{y}, \quad \vec{y}' = \frac{a^2}{y^2} \vec{y}.$$

$$q' = -\frac{a}{y'} q, \quad y = \frac{a^2}{y'},$$

The expression (3) also represents the electric potential inside the sphere due to a point charge q' located at $\vec{y}' = y' \hat{y}$. The points $\vec{y}$ and $\vec{y}'$ are referred to as inversion points with respect to the sphere of radius a . We shall introduce the following notation in the

original problem

The location of the point charge q is denoted by $\vec{y}_e = \vec{y}$

The inversion point of $\vec{y}$ with respect to the inner sphere is $\vec{y}_i$

The inversion point of $\vec{y}$ with respect to the outer sphere is $\vec{y}_o$

The inversion point of $\vec{y}_i$ with respect to the outer sphere is $\vec{y}_{io}$

The inversion point of $\vec{y}_o$ with respect to the inner sphere is $\vec{y}_{oi}$

The sequence of inversion points with respect to both of the spheres
is (just the subscripts will be enumerated)

(4) $e; e, i; e, i, o; e, i, o, i, o; e, i, o, i, o, i; \dots$

The charge located at $\vec{y}_e = \vec{y}$ will be denoted by $q_e = q$.

The charge located at $\vec{y}_i$ will be denoted by q_i

The charge located at $\vec{y}_o$ will be denoted by q_o

The charge located at $\vec{y}_{io}$ will be denoted by q_{io}

The charge located at $\vec{y}_{oi}$ will be denoted by q_{oi}

Denoting $\vec{y}_{io} = y_{io}$ $\hat{y}$ etc., we see that

$$(5) \quad y_{\dots io} = \frac{R_o^2}{y_{\dots i}} ; \quad y_{\dots oi} = \frac{R_i^2}{y_{\dots o}}$$

$$y_i = \frac{R_i^2}{y} ; \quad y_o = \frac{R_o^2}{y}$$

Using the above recursive relations we thus find that

$$(6) \quad y_i = \frac{R_i^2}{y} ; \quad y_o = \frac{R_o^2}{y}$$

$$y_{io} = \frac{R_o^2}{y_i} = \frac{R_o^2}{\frac{R_i^2}{y}} ; \quad y_{oi} = \frac{R_i^2}{y_o} = \frac{R_i^2}{\frac{R_o^2}{y}} ;$$

$$y_{ioi} = \frac{R_i^2}{y_{io}} = \frac{R_i^2}{\frac{R_o^2}{\frac{R_i^2}{y}}} ; \quad y_{oio} = \frac{R_o^2}{y_{oi}} = \frac{R_o^2}{\frac{R_i^2}{\frac{R_o^2}{y}}} ;$$

$$y_{ioio} = \frac{R_o^2}{y_{ioi}} = \frac{R_o^2}{\frac{R_i^4}{R_o^2 y}} ; \quad y_{oioi} = \frac{R_i^2}{y_{oio}} = \frac{R_i^4}{R_o^4 y} ;$$

The following recursive relations between the image charges can also be derived from repeated applications of (3).

$$(7) \quad q_i = -\frac{R_i}{y} q ; \quad q_o = -\frac{R_o}{y} q ;$$

$$q_{\dots io} = -\frac{R_o}{y_{\dots i}} q_{\dots i} ;$$

$$q_{\dots oi} = -\frac{R_i}{y_{\dots o}} q_{\dots o} .$$

Thus one gets

$$(8) \quad q_{io} = -\frac{R_o}{y_i} q_i = -\frac{R_o}{R_i^2} y \cdot \frac{R_i}{y} q = -\left(\frac{R_o}{R_i}\right) q$$

$$q_{oi} = -\frac{R_i}{y_o} q_o = -\frac{R_i}{R_o^2} y \cdot \frac{R_o}{y} q = -\left(\frac{R_i}{R_o}\right) q$$

$$q_{ioi} = -\frac{R_i}{y_{io}} q_{io} = -\frac{R_i}{R_i^2} \frac{R_i^2}{y} \cdot \frac{R_o}{R_i} q = -\left(\frac{R_i}{R_o}\right) \frac{R_i}{y} q$$

$$q_{oio} = -\frac{R_o}{y_{oi}} q_{oi} = -\frac{R_o}{R_o^2} \frac{R_o^2}{y} \cdot \frac{R_i}{R_o} q = -\left(\frac{R_o}{R_i}\right) \frac{R_o}{y} q$$

$$q_{ioio} = -\frac{R_o}{y_{oi}} q_{ioi} = -\frac{R_o}{R_i^2} \frac{R_i^2}{y} \cdot \left(\frac{R_i}{R_o}\right) \frac{R_o}{y} q = -\left(\frac{R_o}{R_i}\right) \frac{R_i^2}{y^2} q$$

$$q_{oioi} = -\frac{R_i}{y_{io}} q_{oio} = -\frac{R_i}{R_o^2} \left(\frac{R_o^2}{y}\right) \cdot \left(\frac{R_o}{R_i}\right) \frac{R_i}{y} q = -\left(\frac{R_i}{R_o}\right) \frac{R_o^2}{y^2} q$$

We denote

$$\bar{\Phi}_e = \frac{1}{4\pi\epsilon} \frac{1}{|\vec{x} - \vec{y}_e|} \quad ; \quad \bar{\Phi}_i = \frac{1}{4\pi\epsilon} \frac{q_i}{|\vec{x} - \vec{y}_i|} \quad ; \quad \bar{\Phi}_o = \frac{1}{4\pi\epsilon} \frac{q_o}{|\vec{x} - \vec{y}_o|}$$

$$\bar{\Phi}_{io} = \frac{1}{4\pi\epsilon} \frac{q_{io}}{|\vec{x} - \vec{y}_{io}|} \quad ; \quad \bar{\Phi}_{oi} = \frac{1}{4\pi\epsilon} \frac{q_{oi}}{|\vec{x} - \vec{y}_{oi}|}$$

and

$$\vec{E}_e = -\nabla \bar{\Phi}_e \quad \text{etc.}$$

Then the electric potential ϕ and the electric field $\vec{E} = -\nabla \phi$ are given by:

$$(9) \quad \phi = \phi_e + \phi_i + \phi_o + \phi_{io} + \phi_{oi} + \phi_{ioi} + \phi_{oi o} + \dots$$

$$(10) \quad \vec{E} = \vec{E}_e + \vec{E}_i + \vec{E}_o + \vec{E}_{io} + \vec{E}_{oi} + \vec{E}_{ioi} + \vec{E}_{oi o} + \dots$$

Equations (9) and (10) are the series for ϕ and for $\vec{E}$. They converge as the geometry series $\alpha + \alpha^2 + \alpha^3 + \dots$; where $\alpha = \frac{R_i}{R_o}$.

Appendix III.

Further Detailed Discussion of the Numerical Code

A) Further Development of the Image Method

From Appendix II one can see that we must sum a point charge and its images such that the potential on both spheres vanishes. We have chosen to group the sum in clusters of four that fall in the same general form. This simplifies the coding considerably. We have chosen to work in cylindrical coordinates, with the z -axis along the beam ($z = 0$ at the center of the spheres), and coinciding with the $\theta = 0$ axis of the spherical coordinate system centered at $z = 0$. When this is done there is symmetry in the azimuthal angle in both spherical and cylindrical coordinates, and the problem becomes two-dimensional. That is, one need only calculate the fields in any $r - z$ plane and the results (as functions of r and z) will hold for all θ in cylindrical coordinates. One must bear in mind, however, that the charges must be summed over the entire volume between the spheres (at least as far as the charge extends); and this will involve summing in the θ - direction. But the resulting field will only have r and z components, since (if we consider r to be the x - component in cartesian coordinates) the y component caused by a charge element at $(x, +y, z)$ will be cancelled by its symmetric counterpart at $(x, -y, z)$. Thus in our considerations we forget about the y - component, and merely calculate the z and r (or x) ones.

Perhaps it is clearer to set up the potential and field in cartesian coordinates with origin at the center of both spheres, and then transfer to cylindrical coordinates. We choose as the plane in which we

will find the electric field the $y = 0$ plane, and denote the position at which we are calculating the field as $(x, 0, z)$. The charge source will be at (x_i, y_i, z_i) , and all its images will be on a line from the origin through this point. This means that the images will have coordinates at $(a_1 x_i, a_1 y_i, a_1 z_i)$, $(a_2 x_i, a_2 y_i, a_2 z_i)$, etc. That is, since the images are some multiple (greater than or less than one) of the distance of the charge source from the origin, each cartesian coordinate is the same multiple (r_{image} and r_{source} are in spherical coordinates):

$$\begin{aligned} r_{\text{image}} &= a r_{\text{source}} = a \sqrt{x_i^2 + y_i^2 + z_i^2} \\ &= \sqrt{(a x_i)^2 + (a y_i)^2 + (a z_i)^2} \end{aligned} \quad (1)$$

The potential from the charge source is

$$\Phi_e = \frac{1}{4\pi\epsilon} \frac{q}{\sqrt{(x-x_i)^2 + y_i^2 + (z-z_i)^2}} \quad (2)$$

and the electric field is

$$\vec{E}_e = \frac{1}{4\pi\epsilon} \cdot \frac{q \left[\hat{i}(x-x_i) + \hat{k}(z-z_i) \right]}{\left[(x-x_i)^2 + y_i^2 + (z-z_i)^2 \right]^{3/2}} \quad (3)$$

We now change to cylindrical coordinates:

$$x = r \cos \theta = r$$

$$x_i = r_i \cos \theta_i$$

$$y_i = r_i \sin \theta_i$$

$$z = z, z_i = z_i$$

and get

$$\vec{E}_e = \frac{1}{4\pi\epsilon} \frac{q \left[\hat{i} (r - r_i \cos \theta_i) + \hat{k} (z - z_i) \right]}{\left[r^2 - 2 r r_i \cos \theta_i + r_i^2 + (z - z_i)^2 \right]^{3/2}} \quad (4)$$

After some manipulation the fields from the image charges inside the smaller sphere can be cast in the form

$$\vec{E}_{in}^n = \frac{1}{4\pi\epsilon} \frac{q \sqrt{a_n} \left[\hat{i} (r - a_n r_i \cos \theta_i) + \hat{k} (z - a_n z_i) \right]}{\left[r^2 - 2 a_n r r_i \cos \theta_i + a_n^2 r_i^2 + (z - a_n z_i)^2 \right]^{3/2}} \quad (5a)$$

(n=1,3,5,...)

These fields are caused by q_{0i} , q_{0i0i} , q_{0i0i0i} , etc. (notation of appendix II)

$$\vec{E}_{in}^n = -\frac{1}{4\pi\epsilon} \frac{q \sqrt{a_n b} \left[\hat{i} (r - a_n b r_i \cos \theta_i) + \hat{k} (z - a_n b z_i) \right]}{\left[r^2 - 2 a_n b r r_i \cos \theta_i + a_n^2 b^2 r_i^2 + (z - a_n b z_i)^2 \right]^{3/2}}$$

(n=2,4,6,...)

caused by q_i , q_{i0i} , q_{i0i0i} , etc.

$$a_n = \left(\frac{r_i}{r_0} \right)^{n+1} \quad ; \quad b = \frac{2;2}{r_i^2 + z_i^2}$$

The image fields outside the larger sphere take the form

$$\vec{E}_{out}^n = \frac{q}{4\pi\epsilon} \frac{\left[\hat{i} \left(\lambda - \frac{a_i \cos \theta_i}{a_n} \right) + \hat{k} \left(z - \frac{z_i}{a_n} \right) \right]}{\left[\lambda^2 - \frac{2\lambda a_i \cos \theta_i}{a_n} + \frac{a_i^2}{a_n^2} + \left(z - \frac{z_i}{a_n} \right)^2 \right]^{3/2}} \quad (5c)$$

($n=1,3,5,\dots$)

caused by $q_{i0}, q_{i0i0}, q_{i0i0i0},$ etc.

$$\vec{E}_{out}^n = \frac{1}{4\pi\epsilon} \frac{q \left[\hat{i} \left(\lambda - \frac{b a_i \cos \theta_i}{a_n} \right) + \hat{k} \left(z - \frac{b z_i}{a_n} \right) \right]}{\left[\lambda^2 - \frac{2 b \lambda a_i \cos \theta_i}{a_n} + \frac{b^2 a_i^2}{a_n^2} + \left(z - \frac{b z_i}{a_n} \right)^2 \right]^{3/2}} \quad (5d)$$

($n=0,2,4,\dots$)

caused by $q_0, q_{0i0}, q_{0i0i0},$ etc.

Notice that (5a) - (5d), as well as (4), all have the same form:

$$\vec{E} = \frac{1}{4\pi\epsilon} \frac{q B_n \left[\hat{i} \left(\lambda - B_n a_i \cos \theta_i \right) + \hat{k} \left(z - B_n z_i \right) \right]}{\left[\lambda^2 - 2 B_n \lambda a_i \cos \theta_i + B_n^2 a_i^2 + \left(z - B_n z_i \right)^2 \right]^{3/2}}$$

In (4), of course, $B_n = 1$.

Thus a subroutine (which we called EFIELD) can be set up which uses $\vec{E}$ in the form of (6). EFIELD is called by another subroutine (IMAGE) which sums over the original charge and its images until the desired accuracy is reached. In IMAGE the image charges are summed over N as groups of four. That is, for $N = 1$ fields (5a) through (5d) are summed; for $N = 2$ all four are summed again, and so on. After each N loop the four fields are added together and compared to the field from the original charge. When this sum is some small

fraction of the original field the loop is terminated (it takes about 7 to 16 iterations, depending on the distance between the charge and observation points), and IMAGE sends the field calculated back to EINIT, the subroutine that originally called it. In order to save resummation after each time loop the electric fields are calculated with $q = 1$. Then in the time loop quantities are multiplied by the true time-advanced charge in order to get the correct components of the electric fields. So subroutine EINIT finds for each observation point the r and z fields (with $q = 1$) from each of the charge elements. This requires two functions (ERR, EZZ) with four arrays (IS, KS to locate the individual charge element, and I, K to locate the particular observation point). Then in the time loop subroutine EFINAL is called. This subroutine multiplies each ERR and EZZ by the appropriate time-advanced charges, and sums them to get at each observation point the total r and z components of the field.

B) Explanation of the Main Program

The main program begins, as might be expected, by defining all the parameters used and setting up various print statements describing them. They are explained with comments in the program itself, and so we need not discuss them here. It would probably be helpful to explain some of the options, though.

If IGAUS = 0 the electron beam functions (time, derivative of the source charge density, conductivity, partial derivatives of the conductivity with respect to r and z , charge density, and charge contained in DV) are constant in the cylindrical r -direction, and also in the z -direction, unless the beam spreads. In the latter case they weaken in proportion to the width of the beam. If IGAUS = 1 the

electron beam functions fall off as a gaussian in the r-direction, with standard deviation given in the program by SIG. The beam width is taken to be three times the standard deviation at exit, and is cut off if its radius becomes larger than the radius of the outer sphere. (if IGAUS = 0 the beam width is SIG.) Provision is also made for the possibility of a spreading beam for IGAUS = 1.

If IPOT = 0, both spheres are grounded. If IPOT = 1, the outer sphere is grounded and the inner sphere is allowed to float.

If IDRGAU = 0 the increments in the r-direction (to be explained below) are equal, with the possible exception of a small triangle left at the very edge of the beam. If IDRGAU = 1 the dr are gaussian. That is, they are small initially, and increase in a gaussian fashion ($\exp(+r^2/2\sigma^2)$) as r increases. These dr are especially appropriate for a gaussian beam, since the volume elements will each contain the same total amount of charge.

In the first major part of the program the grid is defined. R and Z locate the observation points at which the electric field is calculated, and RA and ZA are the coordinates of the centers of the volume elements of charge. The grid is constructed in one of three ways, depending on whether the beam radius is less than or equal to the inner sphere radius, between the radii of the inner and outer spheres, or equal to the outer sphere radius. Figures III. 1, III. 2, and III. 3 illustrate each case (they are drawn with equal dr). Notice that the radial zones follow the inner sphere, and gradually decrease in curvature as Z increases, until they match the outer sphere. This approach makes it much easier to fit zones at the inner and outer spheres, since there is no overlapping. At each z level the number of

Figure III. 1. The beam radius is smaller than the radius of the inner sphere.

Figure III. 2. The beam radius is between the radii of the inner and outer spheres.

Figure III. 3. The beam radius is the same as the radius of the outer sphere.

radial zones is the same. In Figure III.1 we see that for each z level there is an extra triangle left over at the edge of the electron beam. These are added on, with RA and ZA pinpointing the center of gravity (one-third of the way along the base and one-third of the altitude). In figure III. 2 we see that the same procedure is used for z larger than the width of the beam; and the region where z is less than the beam width is divided into equal segments from $\theta = 0$ to $\theta = \pi/2$. Figure III. 3 shows that there is no need to use the triangles when the beam radius equals that of the outer sphere -- the whole region is like the part in figure III. 2 where z is less than the beam width. As far as the bookkeeping is concerned, the RA and ZA are defined with the same I and K as the R and Z at the lower left corner of the elements shown in figures III. 1, III. 2, and III. 3 (that is, the corner with smaller z , and r closest to the central axis).

Once the grid is set, the various beam functions are defined at RA, ZA for each volume element.

The next major portion, and by far the most time-consuming, is a series of loops covering all of the observation points. Subroutine EINIT is called, and the net electric field at each observation point from each source point (with $q = 1$) is calculated and stored (ERR and EZZ, each with four arrays).

Finally, the program runs through the time loop. First, subroutine EFINAL is called for each observation point. In EFINAL the fields calculated in subroutine EINIT are multiplied by the proper charge and summed, leading to the total ER and EZ fields at each observation point. Then the charge density is advanced one time step,

and in the next time loop new electric fields are calculated, and so on. Currently the program prints out (using subroutine PRINT) various quantities throughout the region of interest four times during the time loops, ending, of course with the last values calculated. The algorithm used to advance the charge comes from the following equation:

$$\frac{\partial \phi}{\partial t} + \frac{\sigma}{\epsilon} \phi = \frac{c^2}{2t} - (4\pi k) \cdot E$$

which is differenced as

$$\frac{\phi_{n+1} - \phi_n}{\Delta t} + \frac{\sigma}{\epsilon} \phi_{n+1} = \frac{c^2}{2t} - (4\pi k) \cdot E_{n+1}$$

leading to

$$\phi_{n+1} = \frac{\phi_n - \Delta t \left(\frac{c^2}{2t} - (4\pi k) \cdot E_{n+1} \right)}{1 + \frac{\sigma \Delta t}{\epsilon}}$$

If the potential on the inner sphere is allowed to float, the last part of the time loop advances the potential. The extra fields are then calculated and added on to the "both-spheres-grounded" fields in subroutine EFINAL. The procedure used is described in section 3 of the main part of this report: equations (4) - (8) with $Q_s = 0$, and steps i) - v).

C) Verification of the Program

As a check on our program we ran it for two test cases that have

been solved analytically -- a parallel-plate capacitor and a fully illuminated sphere. In both cases the rate of charge deposition was constant in space and time, and thus the conductivity was also constant.

The parameters used in both cases were the following:

doserate (peak value) = 1.0 calories/gram

conductivity (peak value) = 2×10^{-4} mhos/meter

source charge density (peak value) = 2×10^{-2} coulombs/cubic meter

Both spheres were grounded.

i) The Parallel-Plate Capacitor

When the radii of the two spheres are very large and/or the separation between them is small, and when the beam width is much smaller than the inner radius (and much larger than the separation), our problem approximates a parallel-plate capacitor. We used inner and outer sphere radii of 995 cm and 1000 cm and a beam radius of 60 cm, and got for our maximum electric field 16.3×10^6 volts/meter, which compares with the analytic value of 16.2×10^6 volts/meter (see Appendix V, section A(a)). We used seven radial and four z-zones, 400 time steps, and 24 angle divisions. (See section (ii) for further explanation of these parameters.)

ii) A Fully Illuminated Sphere

Since the beam covers the entire region of interest, and does not fall off in gaussian fashion from the central axis, we would expect larger contributions than normal for large radial distances. Thus this example is a "worst-case" accuracy test, and a true gaussian beam should lead to more accurate results in our code. The peak electric field should be on the inner sphere and should be 19.0×10^6 volts/meter (see Appendix VI).

After some experimentation we found that 400 time steps are accurate to less than one-fifth of one percent. Also, 24 angle divisions (i.e., half of a disc of charge is divided into 24 parts) were also of sufficient accuracy. (the angle divisions would probably have to be increased if the radial divisions go much larger than ten.) Increasing the number of time steps did not seem to increase running time very much, and is probably nothing to worry about. However, doubling the angle divisions will probably nearly double the code running time. Surprisingly, accuracy does not seem to require a fine z-grid.

We used equal radial zones throughout. As a check on the variation of results with z-zoning we ran off (for seven radial zones) results for two through six z-zones. The results are shown in figure III. 4. It seems clear that as the number of z-zones increases the solution settles down. The change between four and five z-zones is about 1.8%, and between five and six is slightly over one percent.

The main problem is in the radial zoning. We found that for equal radial zones the field on the inner sphere was larger on the radial axis, and agreed to within about 2% on the rest of the inner sphere. For this reason we varied the number of radial zones, and display our results in figure III. 5. (there were 400 time steps, 24 angle divisions, and four z-zones for these runs.) Plotted against the number of radial zones are the electric field over most of the inner sphere (excluding the radial axis), and the ratio of the field on the inner sphere at the radial axis to the field over the rest of the inner sphere. One can readily see that as the number of radial zones increases the central "bump" is smoothing out and the electric field is

approaching the correct value of 19.0×10^6 volts/meter. At ten radial zones the field is off by less than 8%, and the ratio of the value on the radial axis to the rest of the sphere is 1.18.

The running times for the radial zone numbers used were roughly:

- 2 -- 0.3 minutes
- 3 -- 0.6
- 4 -- 1.0
- 5 -- 1.5
- 7 -- 3.0
- 10 -- 6.0

We should add that we made a run with gaussian dr with ten radial and four z-zones, and got results very close to the 19.0×10^6 volts/meter. There was some variation in the numbers from $\theta = 0$ (spherical coordinates, along z-axis) to $\theta = \pi/2$, but the first several values were reasonably accurate. The following are the fields on the inner sphere, from $\theta = 0$ to just past $\theta = \pi/4$, with spacing between successive values increasing in gaussian fashion:

19.5×10^6 volts/meter	
19.0	"
18.9	"
18.8	"
18.7	"
18.5	"
18.2	"
17.6	"
16.3	"
13.4	"

18

17

5

1 2

1

1

1

1

Number of 200's

$$\frac{E_{axis}}{E_{other}}$$

2.2
2.0
1.8
1.6
1.4
1.2

17
15
13
11
9
7
5

E_{max}
 $\times 10^6 \text{ V/m}$

2 3 4 5 6 7 8 9 10

Number of Radial Zones

It would seem that the gaussian dr gives a closer answer for the field than the equal dr . This effect should be enhanced even more when a gaussian beam is used.

iii) Check on the Grounding of Both Spheres

As a further check on our program we calculated the potential on both the inner and outer spheres for various cutoff points for summing the image charges. It was found that the potential on both surfaces could be made arbitrarily small, and thus we conclude that the spheres are, in fact, grounded to the accuracy we desire. The beauty of the method of images is that, as long as the conductors are (sufficiently) grounded, the solution to the problem under analysis is correct. The only question is whether the continuous charge distribution can be approximated accurately by replacing the charge spread over a volume element with a point charge at the center of the element. It would seem that a fine enough grid would resolve this problem.

D. Graphics of the Sample Calculations Discussed in Section 5(c)
(Main Report)

The following twelve pages of graphics should help in visualizing the three runs. Perhaps an explanation of the first set (the first four) will clarify what is represented.

Each mark in the picture is an arrow whose length is proportional to the electric field at that point, with the scale being given by the arrow in the very center of the diagram. Thus the series of "dots" that appear to be outlining the inner sphere are really very tiny arrows indicating that the field is very weak around most of the inner sphere. Since there is one arrow drawn for each of the zones used (as well as

for the points on the inner sphere and the eight "checkpoints" between the spheres), and since the zones are rather small within the beam, the arrows within the beam appear lumped together and somewhat difficult to distinguish. It may help to realize that the field on the inner sphere within the beam is directed into the inner sphere, and the field on the outer sphere within the beam is directed out of the outer sphere. What appears to be a large, fat arrow at each of these regions is really a series of tightly packed arrows, one from each zone. It should be noted that the tail of each arrow starts at the point it represents and, of course, extends away from that point.

The set of four graphs show how the field progresses in time, in steps of 0.25 of the pulse length. The beam enters at the top and exits at the bottom. T is the time, given in nanoseconds. $EMAX$ is the maximum electric field at that time, in volts per meter, and always occurs on the inner sphere at the center of the beam (and on the side of the sphere that the beam entered originally, if the beam spreads). $R-INNER$ and $R-OUTER$ are the radii of the inner and outer spheres, respectively, given in meters. $BEAM\ RADIUS\ (STD.\ DEV.)\ ON\ ENTRY$ is the actual radius of the beam (in meters) if it is constant in the r direction, and is the standard deviation if it is gaussian. $EXIT\ BEAM\ RADIUS$ is the same, but at the point of exit of the beam. $N-ANGS$ is the number of slices that half of a disc of charge is divided into, $T-LOOPS$ is the number of time loops used, $Z-ZONES$ is the number of zones in the z -direction between the inner and outer spheres, and $R-ZONES$ is the number of radial zones used for the beam. The bottom line gives information that, we hope, is clear enough.

BEAM RADIUS(STD. DEV.) ON ENTRY =0.010 EXIT BEAM RADIUS =0.010 METERS

N-RINGS= 24 T-LOOPS= 490 Z-ZONES= 6 R-ZONES= 7

GAUSSIAN BEAM IN R DIRECTION, BOTH SPHERE GROUNDED, GAUSSIAN DR

BEAM RADIUS (STD. DEV.) ON ENTRY = 0.010 EXIT BEAM RADIUS = 0.050 METERS
 N-ANGS = 24 T-LOOPS = 400 Z-ZONES = 6 R-ZONES = 10
 GAUSSIAN BEAM IN R DIRECTION, BOTH SPHERE GROUNDED, GAUSSIAN DR

BEAM RADIUS(STD. DEV.) ON ENTRY =0.010 EXIT BEAM RADIUS =0.050 METERS
 N-ANGS= 24 T-LOOPS= 400 Z-ZONES= 6 R-ZONES= 10
 GAUSSIAN BEAM IN R DIRECTION, BOTH SPHERE GROUNDED, GAUSSIAN DR

Appendix IV

Some Approximate Cases to the Electric Field Buildup By
an Electron Beam in A Medium Enclosed by Conductors

We are interested in the electric field buildup in a medium between two concentric spherical conductors. The source of the electric field is an energetic electron beam which renders the medium conducting. In Fig. 1 we show the two concentric conducting spherical conductors with a conducting region induced by the electron beam. The electrically equivalent one-dimensional problem is also shown.

Fig. 1A

Electrically
Equivalent to

Fig. 1B

The crossed region is conducting and the non-crossed region is not conducting. Figure 1B is an approximation to the problem shown in Figure 1A. We shall discuss below a few simple problems whose solutions one can use to approximate more complicated problems.

A. A Simple One-Dimensional Problem

ϕ : Electric Potential ρ : the total charge density ϵ : the constant dielectric constant ρ_s : the source charge density σ : the conductivity ω : the surface charge density at the plate at $x = L$. Q_s : the total surface charge at the plate at $x = L$. Σ_s : Current density into the plate at $x = L$.

the potential satisfies the following equation

$$-\nabla^2 \phi = \frac{\rho}{\epsilon}$$

or

$$-\frac{\partial^2 \phi}{\partial x^2} = \rho(x,t) / \epsilon \quad (1)$$

And the charge density satisfies the differential equation

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon} \rho = \frac{\partial \rho_s}{\partial t} + \nabla \sigma \cdot \nabla \phi$$

or (on the assumption that σ does not depend on x)

$$\frac{\partial \rho}{\partial t} + \frac{\sigma(x)}{\epsilon} \rho = \frac{\partial \rho_s}{\partial t}$$

We assume the condition $\rho(x,0) = \rho_s(x,0) = 0$.

The charge density is then given by

$$\rho(x,t) = \left\{ \int_0^t \dot{\rho}_s(x,\tau) \exp\left(-\int_0^\tau \frac{\sigma(\xi)}{\epsilon} d\xi\right) d\tau \right\} \exp\left[-\int_0^t \frac{\sigma(\xi)}{\epsilon} d\xi\right] \quad (3)$$

This charge density can be used in Poisson's equation.

We recall that

$$\vec{E}(x,t) = -\hat{x} \frac{\partial \phi}{\partial x} ; \quad \vec{D}(x,t) = \hat{x} D(x,t) = \epsilon \vec{E}(x,t)$$

Poisson's equation can be solved by solving the equation involving the electrical displacement. ✓

$$\frac{\partial D}{\partial x} = \rho(x,t) \quad (4)$$

$$D(x,t) = D(0,t) + \int_0^x \rho(\xi,t) d\xi$$

$$D(L,t) = D(0,t) + \int_0^L \rho(\xi,t) d\xi$$

Subtracting

$$D(x,t) - D(L,t) = 0 + \left(\int_0^x - \int_0^L \right) \rho(\xi,t) d\xi = \int_L^x \rho(\xi,t) d\xi$$

or

$$D(x,t) = D(L,t) + \int_L^x \rho(\xi,t) d\xi = D(L,t) - \int_x^L \rho(\xi,t) d\xi \quad (5)$$

$D(L,t)$ is related to the surface charge density

$$\omega(t) = (-\hat{x}) \cdot \vec{D}(L,t) = -D(L,t)$$

Let A be the area of the plate conductors. We then have

$$\begin{aligned} \frac{d}{dt} Q(t) &= \frac{d}{dt} (A \omega(t)) = A \Sigma_s(t) + A \sigma(L,t) E(L,t) \\ &= A \Sigma_s(t) + A \frac{\sigma(L,t)}{\epsilon} D(L,t) = A \Sigma_s(t) - A \frac{\sigma(L,t)}{\epsilon} \omega(t) \end{aligned}$$

or

$$\frac{d\omega(t)}{dt} + \frac{\sigma(L,t)}{\epsilon} \omega(t) = \Sigma_s(t) \quad (6)$$

The solution is also immediate

$$\omega(t) = \left\{ \int_0^t \Sigma_s(\xi) \exp\left(-\int_0^\xi \frac{\sigma(L,\eta)}{\epsilon} d\eta\right) d\xi \right\} \left\{ \exp\left[-\int_0^t \frac{\sigma(L,\eta)}{\epsilon} d\eta\right] \right\} \quad (7)$$

From the above expressions one can derive results for special yet important cases. Thus if

$$\rho_s = \bar{X}(x) T(x)$$

$$\sigma = \sigma(x)$$

we get

$$\begin{aligned} \rho(x,t) &= \left\{ \int_0^t \bar{X}(x) \dot{T}(\xi) \exp \left(\int_0^\xi \frac{\sigma(\xi)}{\epsilon} d\xi \right) d\xi \right\} \left\{ \exp \left[- \int_0^t \frac{\sigma(\xi)}{\epsilon} d\xi \right] \right\} \\ &= \bar{X}(x) G(t) \end{aligned}$$

where

$$G(t) = \left\{ \int_0^t \dot{T}(\xi) \exp \left(\int_0^\xi \frac{\sigma(\xi)}{\epsilon} d\xi \right) d\xi \right\} \left\{ \exp \left[- \int_0^t \frac{\sigma(\xi)}{\epsilon} d\xi \right] \right\} \quad (7a)$$

Let us examine the expression

$$\int_0^t \frac{\sigma(\xi)}{\epsilon} d\xi$$

We are working in a time interval of the order 10^{-8} seconds. We know that ϵ is of the order 10^{-11} . We therefore make the following observations

- i) If $\sigma(t) \cdot 10^{11} \cdot 10^{-8} \gg 1$, we'll say that the conductivity is appreciable.
- ii) If $\sigma(t) \cdot 10^{11} \cdot 10^{-8} \ll 1$, the conductivity is unimportant.

Asymptotic estimates for the expression (7a) when the conductivity is appreciable and when the conductivity is unimportant can also be obtained.

Thus from (7a)

- i) If $\sigma(t) \cdot 10^{11} \cdot 10^{-8} \gg 1$

with

$$G(t) = \int_0^t d\xi \dot{T}(\xi) \exp \left[- \int_{\xi}^t \frac{\sigma(\xi)}{\epsilon} d\xi \right]$$

Write

$$\begin{aligned} \frac{\sigma(\xi)}{\epsilon} &= \frac{1}{\epsilon} \sigma(t) + \frac{1}{\epsilon} \sigma'(t) (\xi - t) + \dots \\ &= \frac{1}{\epsilon} \sigma(t) - \frac{1}{\epsilon} \sigma'(t) (t - \xi) + \dots \end{aligned}$$

We assume of course that

$$\frac{\sigma(t)}{\epsilon} \gg \frac{1}{\epsilon} \sigma'(t) (t - \xi)$$

Then

$$\begin{aligned} G(t) &\approx \dot{T}(t) \int_0^t \exp \left(- \frac{\sigma(t)}{\epsilon} (t - \xi) \right) d\xi \approx \dot{T}(t) \frac{\epsilon}{\sigma(t)} \\ &= \dot{T}(t) \frac{\epsilon}{\sigma(t)} + o \left(\frac{\epsilon}{\sigma(t)} \right) \end{aligned}$$

And we obtain

$$\rho(x,t) \approx \Sigma(x) \dot{T}(t) \cdot \frac{\epsilon}{\sigma(t)} = \frac{\epsilon}{\sigma(t)} \frac{\partial}{\partial x} \rho_s(x,t) \quad (8a)$$

ii) If $\sigma(t) \cdot 10^{11} \cdot 10^{-8} \ll 1$

$$G(t) \approx \int_0^L \dot{T}(y) dy = T(L) - T(0)$$

and we obtain

$$\rho(x,t) \approx \Sigma(x) T(t) = \rho_s(x,t) \quad (8b)$$

If $\rho(x,t)$ is of one sign (which is usually the case), (5) shows that the maximum electric field occurs at $x = 0$ or $x = L$. This should be compared with the case when both of the plates are maintained at the same potential. In this case the electric field changes polarity in the interval from $x = 0$ to $x = L$. Consequently $|\vec{E}|$ will be less.

B. Another approximate one-dimensional problem (slightly more complicated) shown in Fig. 3 is a pair of parallel conducting plates. Part of the area between the plates is conducting.

Fig. 3

Conducting Area : A_s

Total Area : A

The approximation calls for solving Poisson's equation where there is source charge and conductivity as well as solving Laplace's equation in the source-free region. This approximation neglects the fringing field. In the conducting region with source, (3) is true. Equation (5) for the electric displacement is also true. In the source free non-conducting region the electric field will be a constant function of space. The potential of the plate at $x = L$ will depend on the amount of surface charge on it. Let

$w_0(t)$ be the surface charge density at $x = L$ adjacent to the non-conducting region.

$w(t)$ be the surface charge density at $x = L$ adjacent to the conducting region.

$\phi(t)$ be the potential at $x = L$.

$\phi_0(x,t)$ be the potential in the source free region.

$\phi(x,t)$ be the potential in the region with source.

Then:

$$\varphi_0(x,t) = \frac{x}{L} \bar{\Phi}(t)$$

$$\varphi(x,t) = -\frac{1}{\epsilon} \int_0^x D(\xi,t) d\xi$$

where D is given by the equation (5).

Therefore

$$\begin{aligned} \varphi(x,t) &= -\frac{1}{\epsilon} D(L,t) x + \frac{1}{\epsilon} \int_0^x d\xi \int_{\xi}^L P(\xi,t) d\xi \\ &= -\frac{1}{\epsilon} D(L,t) x + \frac{1}{\epsilon} \left\{ x \int_x^L P(\xi,t) d\xi + \int_0^x \xi P(\xi,t) d\xi \right\} \\ &= -\frac{1}{\epsilon} D(L,t) x + \frac{1}{\epsilon} \left\{ x \int_0^L P(\xi,t) d\xi + \int_0^x (x-\xi) P(\xi,t) d\xi \right\} \end{aligned}$$

If we set $x = L$ we get

$$\bar{\Phi}(t) = -\frac{1}{\epsilon} D(L,t) L + \frac{1}{\epsilon} \int_0^L \xi P(\xi,t) d\xi$$

or

$$-\frac{1}{\epsilon} D(L,t) L = \bar{\Phi}(t) - \frac{1}{\epsilon} \int_0^L \xi P(\xi,t) d\xi \quad (9a)$$

$$\omega_0(t) = \epsilon \left. \frac{\partial}{\partial x} \varphi_0(x,t) \right|_{x=L} = \epsilon \frac{\bar{\Phi}(t)}{L} \quad (9b)$$

$$\omega(t) = \epsilon \left. \frac{\partial}{\partial x} \varphi(x,t) \right|_{x=L} = -D(L,t) \quad (9c)$$

$$Q(t) = A_s \omega(t) + (A - A_s) \omega_0(t)$$

Charge conservation at $x = L$ yields

$$\frac{dQ(t)}{dt} = A_s \Sigma_s(t) + A_s \sigma(L,t) E(L,t) = A_s \Sigma_s(t) - A_s \frac{\sigma(L,t)}{\epsilon} \omega(t)$$

Or

$$A_s \frac{d\omega(t)}{dt} + (A - A_s) \frac{d\omega_0(t)}{dt} = A_s \Sigma_s(t) - A_s \frac{\sigma(L,t)}{\epsilon} \omega(t) \quad (9d)$$

But (9a) - (9c) leads to

$$\omega_0(t) = \frac{\epsilon}{L} \Phi(t) = -V(L,t) + \frac{1}{L} \int_0^L x \rho(x,t) dx = \omega(t) + \frac{1}{L} \int_0^L x \rho(x,t) dx$$

Substituting this into (9d) we get

$$A \frac{d\omega(t)}{dt} + A_s \frac{\sigma(L,t)}{\epsilon} \omega(t) = A_s \Sigma_s(t) - (A - A_s) \frac{1}{L} \int_0^L x \frac{\partial \rho(x,t)}{\partial t} dx$$

Or

$$\boxed{\frac{d\omega(t)}{dt} + \frac{A_s}{A} \frac{\sigma(L,t)}{\epsilon} \omega(t) = \frac{A_s}{A} \Sigma_s(t) - \frac{A - A_s}{A} \frac{1}{L} \int_0^L x \frac{\partial \rho(x,t)}{\partial t} dx} \quad (10)$$

C. A More Realistic Approximate One-Dimensional Problem

Show in Fig. 4 is a pair of parallel plates with conducting and non-conducting regions

Total Plate Area: A

$\Sigma_1 u$ and $\Sigma_2 u$
are the current
densities into the
plate at $x = L$

Area: A_1

Area: A_2

Fig. 4

The mathematical manipulations are similar to those given in case B.

Let D_i ($i = 0, 1, 2$) be the displacement in the region $i = 0, 1, 2$

We have (same as (9a))

$$\begin{aligned} \Phi(x) &= -\frac{1}{\epsilon} D_i(L, t) + \frac{1}{\epsilon} \int_0^L \epsilon_i P_i(\xi, t) d\xi & (10a) \\ w_i(t) &= -D_i(L, t) \quad i = 0, 1, 2 & (10b) \end{aligned}$$

Again the approximate solution for this problem neglects the fringing effect.

The electric displacement in the various regions is given by:

$$D_i(x, t) = D_i(L, t) + \int_L^x P_i(\xi, t) d\xi = D_i(L, t) - \int_x^L P_i(\xi, t) d\xi \quad (10c)$$

Conservation of charge at the plate at $x = L$ yields (similar to (9d))

$$\begin{aligned} A_0 \frac{\partial \omega_0(x)}{\partial x} + A_1 \frac{\partial \omega_1(x)}{\partial x} + A_2 \frac{\partial \omega_2(x)}{\partial x} &= A_1 \Sigma_1(x) + A_2 \Sigma_2(x) - A_1 \frac{\sigma_1(L,t)}{\epsilon} \omega_1(x) - A_2 \frac{\sigma_2(L,t)}{\epsilon} \omega_2(x) \\ \frac{\epsilon}{L} (A_0 + A_1 + A_2) \frac{\partial \bar{\Phi}}{\partial x} - A_1 \frac{1}{L} \int_0^L \epsilon \frac{\partial \rho_1(x,t)}{\partial x} dx &= A_1 \Sigma_1(x) + A_2 \Sigma_2(x) - \Phi(t) \left(\frac{A_1 \sigma_1(L,t)}{L} + \frac{A_2 \sigma_2(L,t)}{L} \right) \\ - \frac{A_2}{L} \int_0^L \epsilon \frac{\partial \rho_2(x,t)}{\partial x} dx &= A_1 \Sigma_1(x) + A_2 \Sigma_2(x) - \Phi(t) \left(\frac{A_1 \sigma_1(L,t)}{L} + \frac{A_2 \sigma_2(L,t)}{L} \right) \\ &+ \frac{A_1}{L} \int_0^L \epsilon \rho_1(x,t) dx + \frac{A_2}{L} \int_0^L \epsilon \rho_2(x,t) dx \end{aligned}$$

Φ

$$\begin{aligned} \frac{\partial \bar{\Phi}}{\partial x} + \frac{1}{A\epsilon} (A_1 \sigma_1(L,t) + A_2 \sigma_2(L,t)) \bar{\Phi}(x) &= \frac{L}{A\epsilon} (A_1 \Sigma_1(x) + A_2 \Sigma_2(x)) \\ &+ \frac{1}{A\epsilon} \left[A_1 \int_0^L \epsilon \rho_1(x,t) dx + A_2 \int_0^L \epsilon \rho_2(x,t) dx \right] \end{aligned} \quad (11)$$

2. A conducting column "Connecting" the Outer Conductor.

If the geometry is shown as in Fig. 5, we can use the parallel plate approximation, but with the boundary condition

$$\Phi(0,t) = \Phi(L,t) = 0$$

Fig. 5

For the parallel plate at the same potential we set $\Phi(t) = 0$

in (9a) to get

$$D(L,t) = \int_0^L \frac{\epsilon}{L} \rho(x,t) dx \quad (12a)$$

$$\text{and} \quad D(0,t) = D(L,t) - \int_0^L \rho(x,t) dx = \int_0^L \left(\frac{\epsilon}{L} - 1 \right) \rho(x,t) dx \quad (12b)$$

Appendix V

The Flat Plate Approximation with Time Dependent Conductivity

Symbols Used

- ρ : the total charge density
 φ : Electric Field Potential
 ϵ : the constant dielectric constant taken to be $\epsilon = 3\epsilon_0$
 ρ_s : the source charge density
 $\hat{\rho}_s$: the total charge density at the end of charge deposition
 τ : the time period of charge deposition
 W : the surface charge density at the plate at $x = L$
 $Q(t)$: the total surface charge at the plate at $x = L$
 σ : the conductivity, assumed to be a function of t alone
 Σ_s : current density into the plate at $x = L$
           (net source charge deposited per unit area)

For all the approximations that follow we have

$$\sigma(x,t) = \begin{cases} \sigma & \text{for } x \text{ in the conducting medium and } 0 \leq t \leq \tau \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial \rho_s}{\partial t} = \dot{\rho}_s(x,t) = \begin{cases} \hat{\rho}_s / \tau & \text{for } x \text{ in the conducting medium and } 0 \leq t \leq \tau \\ 0 & \text{otherwise} \end{cases}$$

Equations to be solved:

$$(1a) \quad -\nabla^2 \varphi = \frac{\rho}{\epsilon} ; \quad \frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon} \rho = \frac{\partial \rho_s}{\partial t} + \nabla \sigma \cdot \nabla \varphi \quad (1b)$$

$$\text{with auxiliary quantities} \quad \vec{E} = -\nabla \varphi ; \quad \vec{D} = \epsilon \vec{E}$$

plus boundary conditions

In our approximations we have to solve

$$-\frac{\partial^2 \phi}{\partial x^2} = \frac{\rho}{\epsilon}$$

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon} \rho = \dot{\rho}_s$$

plus boundary conditions

A. Two Simple Parallel Plates

$$\rho(x,t) = \frac{\rho_s}{\tau} \frac{\epsilon}{\sigma} \left\{ 1 - \exp\left(-\frac{\sigma}{\epsilon} t\right) \right\} \quad 0 \leq x \leq L$$

$$\rho(x,t) = \frac{\rho_s}{\tau} \frac{\epsilon}{\sigma} \left\{ 1 - \exp\left(-\frac{\sigma}{\epsilon} \tau\right) \right\} \quad t > \tau$$

General Observation: If $\dot{\rho}_s$ is of one sign then ρ will be of one sign.

Then $\frac{\partial}{\partial x} D = \epsilon \frac{\partial E}{\partial x} = \rho$ will also be of one sign. For a given t the electric field is a monotonically increasing or a monotonically decreasing function of x . Therefore $|\vec{E}|$ occurs at either $x = 0$ or $x = L$.

For our problem we have

a. Two plates grounded

$$D(0,t) = E(0,t) = -\frac{L}{2} \hat{p}_s \frac{\epsilon}{\tau \sigma} \left\{ 1 - \exp\left(-\frac{\sigma}{\epsilon} t\right) \right\} \quad \text{for } 0 \leq t \leq \tau'$$

$$D(L,t) = -D(0,t)$$

$$D(0,t) = D(0,\tau') \quad \text{for } t \geq \tau'$$

Elsewhere $D(x,t)$ is a linear function of x .

b. Only the plate at $x = 0$ is grounded, $\Sigma_s = 0$.

$$D(L,t) = 0.$$

$$D(0,t) = -L \frac{\epsilon}{\tau \sigma} \hat{p}_s \left\{ 1 - \exp\left(-\frac{\sigma}{\epsilon} t\right) \right\} \quad \text{for } 0 \leq t \leq \tau'$$

$$D(0,t) = D(0,\tau') \quad \text{for } t \geq \tau'$$

B. Two Parallel Plates with Partial conducting volume Between

$$D_0(x,t) = D_0(L,t)$$

is the displacement in the nonconducting region.

Conducting Area: As
Total Cross Sectional Area: A

$D(x,t)$ is the displacement in the conducting region.

In the source region:

$$p(x,t) = \frac{\hat{p}_s}{\tau} \frac{\epsilon}{\sigma} \left\{ 1 - \exp\left(-\frac{\sigma}{\epsilon} t\right) \right\} \quad \text{for } 0 \leq t \leq \tau'$$

$$p(x,t) = p(x,\tau') \quad t \geq \tau'$$

The same general observation as in Case A holds. Namely: the maximum $|\vec{E}|$ occurs at either $x = 0$ or $x = L$.

a. Two plates grounded:

Same answer as in Part a of Case A.

b. Only the plate at $x = 0$ is grounded: Potential at $x = L$ is $\bar{\Phi}(t)$

$w_0(t)$ is the surface charge density at $x = L$ in the nonconducting region.

$w(t)$ is the surface charge density at $x = L$ in the conducting region.

$$D_0(L, t) = -w_0(t)$$

$$D(L, t) = -w(t)$$

$$w_0(t) = \frac{\epsilon}{L} \bar{\Phi}(t) = w(t) + \frac{1}{L} \int_0^L x \rho(x, t) dx \quad (1)$$

$$D_0(L, t) = D(L, t) - \int_0^L \rho(x, t) dx \quad (2)$$

(1) and (2) imply:

$$D(L, t) > D_0(L, t) > D_0(0, t) \quad (3)$$

(3) implies that $|\vec{E}|$ occurs at the ends of the conducting region. For our particular problem we find that

$$\frac{dw(t)}{dt} + \frac{A_s}{A} \frac{\sigma(L, t)}{\epsilon} w(t) = \frac{A_s}{A} z_s(t) - \frac{(A - A_s)}{A} \frac{1}{L} \int_0^L x \frac{\partial \rho(x, t)}{\partial x} dx \quad (4)$$

Substituting $\frac{\partial \rho}{\partial x} = \frac{\hat{\rho}_s}{\tau} \exp\left(-\frac{\sigma}{\epsilon} x\right)$ for $0 \leq x \leq \tau$ into (4) we find that

$$u(t) = \frac{V}{k_0} \left\{ \frac{1}{\pi} \left[1 - \exp\left(-\frac{k_0}{h} \frac{\sigma}{\epsilon} t\right) + \frac{1}{2} \frac{\epsilon}{\pi} \sigma \exp\left(-\frac{\sigma}{\epsilon} t\right) - \exp\left(-\frac{h_0}{A} \frac{\sigma}{\epsilon} t\right) \right] \right\}$$

$$\vartheta(t) = \vartheta(t, t) = \int_0^L \vartheta(t, t) dt, \quad w(t) = \frac{1}{\pi} \frac{\epsilon}{\sigma} \left[1 - \exp\left(-\frac{\sigma}{\epsilon} t\right) \right]$$

where V is the irradiated volume of the plate at $x = L$.

APPENDIX VI

Maximum Electric Field for Fully Illuminated Conductors
With the Region Between Them Having Constant Conductivity
and a Spherically Symmetric Charge Density Source

Shown in Figure 1 are
two concentric spherical conductors
with the following data:

$$\begin{aligned} R_e &= 15 \text{ cm} & \dot{\rho}_s &= \frac{1}{4} \text{ C/cm}^2 \\ R_i &= 10 \text{ cm} & \sigma &= \text{time interval of} \\ & & & \text{charge deposition} \\ \tau &= 0.5 \text{ sec} \end{aligned}$$

Fig. 1

Between the conductors is a
dielectric with dielectric constant
 ϵ . The dielectric is fully
illuminated and hence is conducting. We are interested in the *maximum*
electric field at the end of the charge deposition.

The problem amounts to finding the electric potential ϕ satisfying

$$(1) \quad -\nabla^2 \phi = \frac{\rho}{\epsilon} \quad \text{for} \quad R_i \leq r \leq R_e$$

where ρ satisfies

$$(2) \quad \frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon} \rho = \frac{\partial \rho_s}{\partial t} + \nabla \sigma \cdot \nabla \phi = \dot{\rho}_s$$

where the last term drops out because of our assumption on conductivity.

Case 1. The outer conductor is grounded and the inner conductor is not.

The total surface charge in the inner conductor will be $Q(t)$

$$(3) \quad Q(t) = \epsilon \int_{a=R_i} dS \hat{n} \cdot (-\nabla\varphi) = -\epsilon \int_{a=R_i} \frac{\partial\varphi}{\partial n} dS$$

The rate of change of the total surface charge at the inner spherical conductor is

$$(4) \quad \frac{dQ}{dt} = \dot{Q}_s - \int_{a=R_i} \sigma \hat{n} \cdot (-\nabla\varphi) dS = \dot{Q}_s + \int_{a=R_i} \sigma \frac{\partial\varphi}{\partial n} dS$$

where $\dot{Q}_s$ is the rate of charge deposition in the inner conductor. It is a calculable or given function of t . For our problem it is

$$\dot{Q}_s = V \frac{\dot{\rho}_s}{r} \quad \text{where } V \text{ is the volume of the inner conductor}$$

$$\begin{aligned} V &= \frac{4\pi}{3} \left\{ R_i^3 - (R_i - T)^3 \right\} \approx 4\pi R_i^2 T \\ &= (.5 \times 10^{-2}) 4\pi (10^{-1})^2 \\ &= 2\pi \times 10^{-4} \text{ m}^3 \end{aligned}$$

From (2)

$$(5) \quad \rho(t) = \int_0^R \frac{\epsilon}{\sigma} \left[1 - \exp\left(-\frac{\sigma}{\epsilon} t\right) \right] r \, dr = \frac{\epsilon}{2} \frac{\sigma}{\epsilon} \left[1 - \exp\left(-\frac{\sigma}{\epsilon} t\right) \right]$$

From (3) and (4) we have

$$(6) \quad \frac{dQ}{dt} = \int_0^R \frac{\sigma}{\epsilon} r \, dr$$

which can be solved to yield

$$(7) \quad Q(t) = \int_0^R \frac{\epsilon}{\sigma} \left[1 - \exp\left(-\frac{\sigma}{\epsilon} t\right) \right] r \, dr$$

The surface charge density at the inner conductor is:

$$(8) \quad \omega(t) = \frac{Q(t)}{4\pi R_i^2} = \frac{1}{4\pi R_i^2} \int_0^R \frac{\epsilon}{\sigma} \left[1 - \exp\left(-\frac{\sigma}{\epsilon} t\right) \right] r \, dr$$

$$= \frac{V}{4\pi R_i^2} \rho(t) \approx .5 \times 10^{-2} \rho(t)$$

And the electric field at $r = R_i$ is

$$(9) \quad \frac{\omega(t)}{\epsilon} = \frac{V}{4\pi R_i^2 \epsilon} \rho(t) \approx .5 \times 10^{-2} \frac{\rho(t)}{\epsilon}$$

We are to solve (1), which is now in the form

$$(10) \quad -\frac{1}{n^2} \frac{d}{dn} \left(n^2 \frac{dq}{dn} \right) = \frac{\rho}{\epsilon}$$

subject to

$$(10a) \quad q = 0 \quad \text{at} \quad n = R_e \quad \text{and} \quad -\frac{dq}{dn} \Big|_{n=R_i} = \frac{u_0(\epsilon)}{\epsilon}$$

From (10) and the boundary condition (10a) we have

$$n^2 \frac{dq}{dn} + R_i^2 \frac{u_0(\epsilon)}{\epsilon} = -\frac{\rho}{\epsilon} \frac{1}{3} (n^3 - R_i^3)$$

Or

$$(11) \quad \frac{dq}{dn} = -\frac{R_i^2}{\epsilon} \frac{1}{n^2} u_0(\epsilon) - \frac{\rho}{3\epsilon n^2} (n^3 - R_i^3)$$

Making use of (9) and (4a), we find that the electric field is equal to

$$(12) \quad -\frac{dq}{dn} = \frac{1}{3} \frac{\rho}{\epsilon} \frac{1}{n^2} \left\{ n^3 - (R_i - T)^3 \right\} \quad R_i \leq n \leq R_e$$

where $T = 0.5 \text{ cm}$ is the thickness of the inner conductor.

From equation (12) we make the following observation:

The maximum electric field occurs at $n = R_e$. In case $\frac{R_i}{R_e} \approx 1$ it is
roughly equal to $\frac{\rho}{\epsilon} (R_e - R_i)$.

At the end of the charge deposition period ($t = \tau$) we have

$$\rho = \frac{\epsilon}{\sigma} \frac{\rho_s}{\tau} \left[1 - \exp\left(-\frac{\sigma}{\epsilon} \tau\right) \right]$$

and the maximum electric field at the end of the charge deposition period is

$$\begin{aligned} (13) \quad -\frac{1}{\epsilon} \left| \frac{\partial \phi}{\partial x} \right|_{x=0} &= \frac{1}{3 R_c^2} \cdot \frac{\rho_s}{\sigma} \cdot \frac{(R_c - \tau)^2}{\tau} \left[\frac{\rho_s}{\epsilon} \frac{\epsilon}{\sigma \tau} \left[1 - \exp\left(-\frac{\sigma}{\epsilon} \tau\right) \right] \right] \\ &= 3.72 \cdot 10^{-2} \frac{\rho_s}{\sigma \tau} \left[1 - \exp\left(-\frac{\sigma}{\epsilon} \tau\right) \right] \end{aligned}$$

TABLE I

$$\epsilon = 3\epsilon_0, \tau = 4 \cdot 10^{-8} \text{ sec}$$

Maximum Electric Field

ρ (col/cm)	C (cm)	$\frac{\rho_s}{\sigma}$ (col/cm ²)	Maximum Electric Field (V/cm)	
			Case 1	Case 2 *
1	$2 \cdot 10^{-5}$	$2 \cdot 10^{-3}$	$2.75 \cdot 10^6$	$2.16 \cdot 10^6$
10	$2 \cdot 10^{-4}$	$2 \cdot 10^{-2}$	$2.41 \cdot 10^7$	$1.90 \cdot 10^7$
100	$2 \cdot 10^{-3}$	$2 \cdot 10^{-1}$	$8.86 \cdot 10^7$	$6.95 \cdot 10^7$
1000	$2 \cdot 10^{-2}$	$2 \cdot 10^0$	$9.31 \cdot 10^7$	$7.30 \cdot 10^7$
10000	$2 \cdot 10^{-1}$	$2 \cdot 10^1$	$9.31 \cdot 10^7$	$7.30 \cdot 10^7$
1	$2 \cdot 10^{-3}$	$2 \cdot 10^{-2}$	$8.86 \cdot 10^6$	$6.95 \cdot 10^6$
1	$2 \cdot 10^{-5}$	$2 \cdot 10^{-2}$	$2.75 \cdot 10^7$	$2.16 \cdot 10^7$

Case 2. Both Conductors Are Grounded

Let $w(t)$ be the surface charge density at the inner spherical surface then Gauss law yields

$$(14) \quad 4\pi r^2 D(r,t) = 4\pi \int_{R_i}^r r'^2 \rho(r',t) dr' + 4\pi R_i^2 w(t)$$

Since both conductors are grounded $w(t)$ is determined so that

$$(15) \quad \int_{R_i}^{R_e} D(r,t) dr = 0$$

Imposing the condition (15) into (14) we get

$$(16) \quad w(t) = - \frac{\int_{R_i}^{R_e} dr \frac{1}{r^2} \int_{R_i}^r r'^2 \rho(r',t) dr'}{\frac{R_i}{R_e} (R_e - R_i)}$$

For our particular case $\rho(r,t) = \rho(t)$, $w(t)$ is given by

$$(17) \quad w(t) = -\rho(t) \frac{(R_e - R_i)}{R_i} \left\{ \frac{1}{6} R_e + \frac{1}{3} R_i \right\}$$

and the electric field elsewhere is given (through (14)) by

$$(18) \quad E(r,t) = \frac{\rho(t)}{\epsilon r^2} \left\{ \frac{1}{3} (r^3 - R_i^3) - R_i (R_e - R_i) \left(\frac{1}{6} R_e + \frac{1}{3} R_i \right) \right\}$$

The electric field is negative at $r = R_i$, monotonically increasing to $r = R_e$. At $r = R_e$ ($R_e = 15\text{cm}$, $R_i = 10\text{cm}$)

$$\begin{aligned} (19) \quad E &= -\frac{\rho(t)}{\epsilon} \frac{1}{R_e^2} \left\{ \frac{1}{3} (R_e^3 - R_i^3) - R_i (R_e - R_i) \left(\frac{1}{6} R_e + \frac{1}{3} R_i \right) \right\} \\ &= -\frac{\rho(t)}{\epsilon} (R_e - R_i) \frac{1}{R_e} \left\{ \frac{1}{3} R_e + \frac{1}{6} R_i \right\} \\ &= -\frac{4}{9} (R_e - R_i) \frac{\rho(t)}{\epsilon} \end{aligned}$$

at $r = R_i$

$$(20) \quad E = -\frac{\rho(t)}{\epsilon} \frac{R_e - R_i}{R_i} \left\{ \frac{1}{6} R_e + \frac{1}{3} R_i \right\} = -\frac{7}{12} (R_e - R_i) \frac{\rho(t)}{\epsilon}$$

The electric field (in absolute value) is the largest at the inner conductor. Values of the maximum electric field are shown in the last column of TABLE 1.

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