IMPROVEMENT OF THE DYNAMIC APERTURE IN CHASMAN GREEN LATTICE DESIGN
LIGHT SOURCE STORAGE RINGS

E. A. Crossbie
Argonne National Laboratory
9700 S. Cass Avenue
Argonne, IL 60439 USA

I. Introduction

The half cell shown in Fig. 1 illustrates a typical Chasman-Green electron or positron storage ring lattice specifically designed for photon beams from undulators and wigglers located in each dispersion-free straight section. The need for a small particle beam emittance requires that the horizontal phase advance per cell should be in the neighborhood of $0.9 \times 2\pi$. Necessary chromaticity correcting sextupoles, $S_p$ and $S_p'$, located in the dispersion straight section introduce non-linear perturbations which limit the dynamic aperture because of amplitude dependent tune shifts. Two families of sextupoles, $S_1$ and $S_2$, can be introduced into the dispersion-free region to moderate the more harmful effects of $S_0$ and $S_0'$. The following sections discuss the nature of the perturbations and provide some guidelines for the adjustment of $S_1$ and $S_2$.

II. Distortion Functions

The Hamiltonian used to describe the motion of charged particles in a lattice containing sextupole fields may be written as:

$$H = \frac{1}{2} \sum J \sum \left( \frac{1/2}{B} \left( J_x \phi_x + J_y \phi_y \right) \right) + V(J_x, J_y, \phi_x, \phi_y, s) \tag{1}$$

where

$$V = -f(s) J_x \frac{3}{2} \left( \cos \frac{3\phi_x}{2} + 3 \cos \frac{\phi_x}{2} \right) - 3f(s) J_y \frac{3}{2} \left( 2 \cos \frac{\phi_x}{2} + \cos \frac{\phi_x}{2} + \cos \frac{3\phi_x}{2} \right);$$

$$f(s) = \frac{\sqrt{2}}{S_k} \delta(s-s_k);$$

$$\beta_x = \frac{1/2}{E_p} \left( \frac{B_x}{B_p} \right);$$

$$\phi_x = \phi_x \delta x.$$

For $V=0$, $J_x$, $J_y$ are constants of the motion. Treating $V$ as a perturbation, one introduces the generating function

$$F(\phi_x, \phi_y, J_x, J_y, s) = \exp \left( \int_{s_0}^{s} f(s') \frac{3}{2} \left( \sin \frac{3\phi_x}{2} + \sin \frac{\phi_x}{2} \right) \right) \tag{2}$$

to produce new variables

$$\phi_x = G_x, J_x = G_x, G_y = 3G_x \tag{3}$$

If $G$ satisfies the equation

$$G + G \frac{x}{y} + G_x + G_y = 0 \tag{4}$$

the new Hamiltonian is

$$H_s = \frac{1}{2} \sum J \sum \left( \frac{1/2}{B_x} \left( \frac{3}{2} \sin \frac{3\phi_x}{2} + \sin \frac{\phi_x}{2} \right) + \frac{1}{2} \left( \frac{3}{2} \sin \frac{3\phi_y}{2} + \sin \frac{\phi_y}{2} \right) \right) \tag{5}$$

and $J_{1x}$, $J_{1y}$ are constants of the motion to the first order in $V$.

The function that satisfies Eq. (3) is

$$\chi \frac{3}{2} \left( \sin \frac{3\phi_x}{2} + \sin \frac{\phi_x}{2} \right)$$

where

$$\alpha_x = \phi_x + \psi(s) - \frac{3}{2} \sin \frac{\phi_x}{2} - \frac{3}{2} \sin \frac{\phi_x}{2}$$

$$\alpha_y = \phi_y + \psi(s) - \frac{3}{2} \sin \frac{\phi_y}{2} - \frac{3}{2} \sin \frac{\phi_y}{2}$$

$$\psi(s) = \int_{s_0}^{s} \chi \left( \frac{3}{2} \sin \frac{3\phi_x}{2} + \sin \frac{\phi_x}{2} \right) \left( \frac{3}{2} \sin \frac{3\phi_y}{2} + \sin \frac{\phi_y}{2} \right) \left( \frac{3}{2} \sin \frac{3\phi_x}{2} + \sin \frac{\phi_x}{2} \right) \chi$$

$C$ a cell length

From Eq. (6), one can derive directly

$$G_x = -(2J_{1x})^{3/2} \left( C_3 \cos(\phi_x + \alpha_x) + C_1 \cos(\phi_x + \alpha_1) \right) + \left( 2J_{1x} \right)^{1/2} 2J_{1y} \left( 2 \cos(\phi_y + \alpha_y) \right)$$

$$+ C_4 \cos(\phi_x + \alpha_x) + C_5 \cos(\phi_y + \alpha_y) \tag{7}$$

$$G_y = -2J_{1x}^{1/2} 2J_{1y} \left( C_4 \cos(\phi_y + \alpha_y) - C_5 \cos(\phi_x + \alpha_x) \right) \tag{8}$$
where typically the various C's and a's are defined as
\[
C = \sqrt{A^2 + B^2} ; \tan \alpha = \frac{A}{B} ; J = 1, 3;
\]
and the corresponding A's and B's as
\[
B_j = \sum_k \frac{S_k}{16\sin \nu_k} \cos j(\phi \pm \phi_k) ; j = 1, 3;
\]
where
\[
B_j = \sum_k \frac{S_k}{16\sin \nu_k} \cos j(\phi \pm \phi_k) ; j = 1, 3;
\]
with \( \Lambda(Z) = B'(Z) \).

These functions have been called Distortion Functions. They are determined by the lattice structure and the distribution of sextupoles. The magnitude of the C's determines the maximum variations of amplitudes through the lattice about a constant value determined by \( I_x, J_y \). The first three distortion functions, \( B_3, B_1, B \) for the lattice of Fig. 1 with the chromaticity correcting sextupoles turned on are shown in the top half of Fig. 2.

The amplitude dependent tune shifts are determined by the magnitude of the B functions at the sextupole multiplied by the corresponding sextupole strengths. This can be shown as follows:

\[
\Delta v_x = \frac{1}{4\pi} \sum_{k} C_j \left[ j_{1x} S_k (B_j + B_j) + 2j_{1y} S_k (2B_1 - B_j + B_j) \right] \]

\[
\Delta v_y = \frac{1}{4\pi} \sum_{k} C_j \left[ 2j_{1x} S_k (2B_1 - B_j + B_j) - j_{1y} S_k (4B_1 + B_j + B_j) \right]
\]

To reduce these shifts additional sextupoles must be added to the lattice in such a way as to either reduce the B functions at all sextupoles or to achieve cancellations by taking advantage of their phases. Since the tune shifts depend quadratically on the sextupole strengths, this is not easy to do in practice. The solution is easier if one knows which harmonics are causing most of the problem.

### III. Harmonic Expansion

The expansion of \( V \) into harmonic components is accomplished by means of five sets of coefficients having the general form

\[
\frac{R}{\sqrt{\theta}} e^{i(\psi - 6\theta)} = \Lambda_{m} e^{-i(m\theta - 6\theta)} \]

The result (using a reflective symmetry point as reference) is

\[
v = \frac{R}{\sqrt{\theta}} \sum_{m} J_{x}^{3/2} \left( A_{m} \cos Q_{m} + 3A_{m} \cos Q_{m} \right)
\]

\[
-3J_{y}^{1/2} \left( 2B_{m} \cos Q_{m} + 3B_{m} \cos Q_{m} \right)
\]

where

\[
Q_{m} = \frac{J_{m} - J_{m}}{2} \cos Q_{m} ; j = 1, 3;
\]

\[
Q_{m} = \frac{\theta_{m} - \theta_{m}}{2} \cos Q_{m} ; j = 1, 3;
\]

\[
A_{m} = \sum_{k} \frac{S_k}{4\sin \nu_k} \cos j(\phi \pm \phi_k) ;
\]

\[
B_{m} = \sum_{k} \frac{S_k}{4\sin \nu_k} \cos j(\phi \pm \phi_k) ;
\]

Substitution of the components into Eq. (6) yields the following harmonic expansion for the Distortion Functions;

\[
B_j = \frac{3A_{m}}{J_{m}} \cos j(\phi \pm \phi_k) ; j = 1, 3
\]

\[
B_1 = \frac{3B_{m}}{J_{m}} \cos \phi \pm \phi_k ; j = 1, 3
\]

\[
B_3 = \frac{3B_{m}}{J_{m}} \cos \phi \pm \phi_k ; j = 1, 3
\]

(K. Y. Ng started from this form of the distortion functions and worked backwards to arrive at the Collins formula given by Eqs. (7) and (8)). Finally, by direct substitution of these expansions into Eqs. (10) and (11), and identification of the sums over \( k \), one arrives at the harmonic expansion of the amplitude-dependent tune shifts

\[
\Delta v_x = \frac{1}{4\pi} \sum_{k} C_j \left[ j_{1x} S_k (B_j + B_j) + 2j_{1y} S_k (2B_1 - B_j + B_j) \right]
\]

\[
\Delta v_y = \frac{1}{4\pi} \sum_{k} C_j \left[ 2j_{1x} S_k (2B_1 - B_j + B_j) - j_{1y} S_k (4B_1 + B_j + B_j) \right]
\]
\[ \Delta v_y = M_{21} J_{1x} + M_{22} J_{1y} \]

with:

\[ M_{11} = -36 \sum \frac{A_{3m}}{m} \left( \frac{3A_{1m}}{v_x - m} + \frac{3A_{1m}}{v_y - m} \right) \]

\[ M_{12} = M_{21} = 72 \sum \frac{B_{1m} A_{1m}}{m} \left( \frac{B_{1m}}{v_x - m} + \frac{B_{1m}}{v_y - m} + \frac{B_{1m}}{v_x - m} \right) \]

\[ M_{22} = -36 \sum \frac{B_{1m}}{v_x - m} \left( \frac{B_{1m}}{v_x - m} + \frac{B_{1m}}{v_y - m} \right) \]

IV. Predictions and Improvements by Harmonic Suppression

The lattice shown in Fig. 1 has tunes of \( v_x = 0.88 \) and \( v_y = 0.36 \) per cell. With the chromaticity correcting sextupole turned on, the calculated amplitude dependent tune shifts per cell are:

\[ \Delta v_x \text{ (units 10}^{-5} \text{)} = 7.18 N_x^2 + 1.5 N_y^2 \]

\[ \Delta v_y \text{ (units 10}^{-6} \text{)} = 3.02 N_x^2 + 0.98 N_y^2 \]

\[ N_x^2 = \frac{2J_x}{c} \quad N_y^2 = \frac{4J_y}{c} \quad c = \text{natural emittance} \]

Tracking programs show that the limit of stability on the median plan is at \( N_x = 41 \), where the tune per cell approaches 1. The tune shift increases rapidly as the one approaches the fundamental instability.

In fact, it is possible to predict the total dynamic aperture for this lattice using only the harmonic components \( A_{33}, A_{11}, \) and \( B_{11} \). It can be shown that for constant \( N_y \), the horizontal stable aperture should extend from \( N_{x1} \) to \( N_{x2} \) where

\[ N_{x1} = \frac{X}{3} \quad \text{(F+1);} \]

\[ N_{x2} = \frac{X}{3} \quad \text{(2F-1);} \]

\[ X_0 = \frac{(v_x - 1)}{2\sqrt{c(3A_{11} + A_{33})}} \]

\[ F = \sqrt{1 + \frac{36B_{11} c N_y^2 (3A_{11} + A_{33})}{(v_x - 1)^2}} \]

The prediction of Eq. (16) and the dynamic aperture obtained by tracking are shown in Fig. 3.

The amplitude dependent tune shifts for the lattice with the chromaticity correcting sextupoles turned on is much too large for stability in a ring containing uncorrected orbit errors. Inspection of Eqs. (15) and (16) suggest that the most efficient use of the harmonic suppression sextupoles \( S_1 \) and \( S_2 \) is to reduce \( A_{11} \) and \( B_{11} \). It turns out that if these sextupoles are tuned such that \( A_{11} \) and \( B_{11} \) are reduced to about 30% of their original values, the coefficient of \( J_{1x} \) in Eq. (10) is reduced to zero. The resultant increase in dynamic aperture is shown in Fig. 3.

References


