Tearing Modes in a Plasma with Magnetic Braiding

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ABSTRACT

We examine certain linear and nonlinear properties of macroscopic tearing modes driven by anomalous electron viscosity effects associated with magnetic braiding. It is shown that strong linear growth of $m = 1, 2$ tearing modes (respectively proportional to $\mu^{1/5}$ and $\mu^{1/3}$) can occur for rather modest values of μ . In the nonlinear phase, the island width grows in time as $t^{1/3}$. Some speculations regarding the disruptive instability in Tokamaks are also investigated.

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In recent years, the idea, that a Tokamak may suffer from broken magnetic surfaces and braided magnetic field lines, has attracted a lot of attention.¹⁻³ It is believed that magnetic braiding could result if a number of helical magnetic perturbations centered on different mode-rational surfaces are simultaneously excited in the plasma. The possible existence of braided field lines has variously been considered as an important candidate for explaining certain observed phenomena in Tokamaks, such as, the anomalous electron heat transport,¹⁻³ aspects of disruptive instability,¹⁻⁴ behavior of runaway electrons⁵ etc.

If field line braiding arises due to quasistatic, small-scale magnetic perturbations, one can show⁶ (following arguments similar to those in Refs. 1 and 2) that it will lead not only to an anomalous transport of electron heat but also to an equally strong transport of electron parallel momentum across unperturbed magnetic surfaces. This means that the Ohm's law must be modified to incorporate an anomalous electron viscosity coefficient $\bar{\mu}$, which leads to a smoothing out of perpendicular gradients of parallel current, i.e.,

$$E_{||} = \eta J_{||} - \bar{\mu} \frac{m}{ne^2} \nabla_{\perp}^2 J_{||} ; \quad (1)$$

here η is the resistivity and the viscosity $\bar{\mu} \sim \chi$, the coefficient of anomalous perpendicular electron thermal conductivity in a braided field.^{1, 2} This result is consistent with

a simple quasi-linear estimate of the influence of microscopic magnetic perturbations on macroscopic transport.⁶ In this letter we investigate certain linear and nonlinear properties of long-scale tearing modes in a plasma with microscopic magnetic braiding. We postulate that the dominant influence of field line braiding on the development of macroscopic tearing modes enters through the anomalous electron viscosity coefficient $\tilde{\mu}$.

Consider a cylindrical model of the Tokamak magnetic field

$$\underline{B} = B_z \hat{z} + B_\theta(r) \hat{\theta}, \quad (2)$$

where the system is given periodicity $2\pi K$ in the z -direction and $B_z \gg B_\theta$. The basic equations describing the tearing mode are the equation of motion,

$$\rho \frac{d\underline{v}}{dt} = -\nabla p + \frac{1}{c} (\underline{J} \times \underline{B}) \quad (3a)$$

and the Ohm's Law

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B}) - \frac{4\pi}{c} \nabla \times \left[\left(\frac{c^2 \eta}{4\pi} \underline{J} - \mu \nabla^2 \underline{J} \right) \hat{e}_\parallel \right] \quad (3b)$$

where $\mu = (c^2/\omega_p^2)\tilde{\mu}$ and rest of the symbols have standard meanings. Equations (3a, b) are to be supplemented by the equations $\nabla \times \underline{B} = (4\pi/c)\underline{J}$, $\nabla \cdot \underline{v}_\perp = 0$ (incompressibility of $\underline{v}_\perp$ is due to large B_z) and the boundary conditions at $r = 0$ and at the conducting wall.

we first consider the linear problem with resistivity $\eta = 0$

and look for solutions of the form $f(r) \exp(\gamma t + im\theta + ikz)$ where $k \equiv n/R$. Tearing modes are characterized by a growth rate between the resistive and/or viscous diffusion rates (τ_R^{-1} and/or τ_V^{-1} respectively) and the hydrodynamic rate (τ_H^{-1} , i.e., $\gamma \tau_H \ll 1 \ll \gamma(\tau_V, \tau_R)$). The left inequality is broken in the vicinity of the "singular surface," $r = r_s$ where $\underline{k} \cdot \underline{B}(r_s) = 0$, which is surrounded by a narrow boundary layer in which inertia and dissipation are important and outside of which they have negligible effect. The governing vector equations (3a, b) can, upon linearization, be reduced to a coupled set for the radial fluid displacement $\xi = v_r/\gamma r_s$ and the flux function $\psi = iB_{1r}/[(dq/dr)r_s B_\theta]$ by the operations $\underline{B} \cdot \nabla \times$ and $\hat{r} \cdot$, respectively. The result is, for $r \neq r_s$,

$$\frac{d}{dr} \left[H \frac{d\bar{\psi}}{dr} \right] - \left[\frac{g}{F^2} + \frac{1}{F} \frac{d}{dr} \left[H \frac{dF}{dr} \right] \right] \bar{\psi} = 0 \quad (4)$$

where $\bar{\psi} = -r_s^2 (dF/dr) \psi$, $F = \underline{k} \cdot \underline{B}$, $H = r^3/m^2$, and

$$g = \frac{(m^2 - 1)rF^2}{m^2} + \frac{k^2 r^2}{m^2} \left[rF^2 + \frac{2F}{m^2} (krB_z - mB_\theta) \right].$$

The expression for g is derived for $kr \ll 1$ and Eq. (4) can be solved asymptotically in this limit. Near $r = r_s$, the equations are

$$\frac{d^2 \xi}{dx^2} = \frac{x}{\lambda^2} \frac{d^2 \psi}{dx^2} \quad (5a)$$

$$\psi = -x\xi - \frac{\varepsilon}{\lambda} \frac{d^4 \psi}{dx^4} \quad (5b)$$

where $x \equiv (r - r_s)/r_s$, $\varepsilon \equiv \tau_H/\tau_V$, $\lambda = \gamma\tau_H$,
 $\tau_H = [(dq/qdr)B_0/(4\pi\rho)]^{1/2}]^{-1}$ is the poloidal Alfvén time and the
 viscous decay time $\tau_V \approx (\mu r_s^4)^{-1}$. Note that Eq. (5b) is fourth
 order, rather than second order as in the resistive problem.

For $m \geq 2$, $g(r_s)$ is finite and Eq. (4) yields a finite
 discontinuity Δ' in $\lim_{\delta \rightarrow 0} (d \ln \psi / dr) \Big|_{r_s - \delta}^{r_s + \delta}$. Equation (5) can be solved
 in closed form if $\psi \approx \text{constant}$. Noting that, for this
 case $d^n \psi / dx^n \sim \psi / x^{n-1}$, and balancing terms in (5a, b), we find
 $\lambda \sim x^{3/2} \sim \varepsilon^{1/3}$. Detailed calculations⁷ yield

$$\gamma\tau_H = \left\{ \frac{m}{\pi} \frac{\Delta' g}{dq/dr} \right\}^{2/3} \left(\frac{\tau_H}{\tau_V} \right)^{1/3} \quad (6)$$

For $m = 1$, $g(r_s) \sim (kr)^2 \ll 1$, and ψ varies across the
 singular layer. Eliminating ψ in Eq. (4) in favor of $\xi = \tilde{\psi}/Fr_s$,
 we obtain

$$\frac{d}{dr} \left(r^3 F^2 \frac{d\xi}{dr} \right) - g\psi = 0 \quad (7)$$

The solution, correct to first order in g , is $\xi = \xi_0 + \xi_1$, where

$$\xi_0 = \xi_\infty, \quad \frac{d\xi_1}{dr} = \frac{\xi_\infty}{r^3 F^2} \int_0^r g \, dr, \quad r < r_s, \quad (8a)$$

$$\xi_0 = 0, \quad \frac{d\xi_1}{dr} = \frac{\xi_\infty}{r^3 F^2} \int_0^{r_s} g \, dr, \quad r > r_s. \quad (8b)$$

The discontinuity in ξ_0 is permitted at $r = r_s$ because $F(r_s) = 0$.
 An ideal MHD $m = 1$ instability ($\mu = 0$) is obtained if an inner
 region with $x^2 \sim \lambda^2$ is used to smoothly connect (8a) to (8b). The
 ideal growth rate is

$$\lambda = \lambda_H = -\pi / (B_\theta q r)^2 \Big|_{r_s} \int_0^{r_s} dr \, g \lesssim \left(\frac{r_s}{R} \right)^2 \quad (9)$$

The viscous terms become important when $\epsilon^{1/5} > (r_s/R)^2$, requiring a solution of the complete set (5a, b) in that case. Of the six solutions, the one of interest matches smoothly to Eq. (8), viz

$$\lim_{|x| \rightarrow \infty} \frac{1}{\xi_\infty} \frac{d\xi}{dx} = -\frac{1}{\pi} \frac{\lambda_H}{x^2} \quad (10)$$

The analytical theory of this mode is very complicated. An analysis in Fourier space has been attempted and will be presented elsewhere.⁶ Here, we present the results of numerical calculations and simple scaling arguments. For $m = 1$, we have $\Delta\psi/\psi \sim 1$ in the inner region. Balancing terms in Eq. (5) now gives $\lambda \sim x \sim \epsilon^{1/5}$. The value of $\lambda/\epsilon^{1/5}$ as a function of $\lambda_H/\epsilon^{1/5}$ is obtained by numerically solving the eigenvalue problem as follows: we define the odd functions $\xi_{\text{odd}} = \xi - \xi_\infty/2$ and the even function $\chi = \psi''$ and solve the three coupled second-order differential equations in ξ_{odd} , ψ and χ subject to the asymptotic boundary conditions

$$\left[1 + (\hat{x}/\hat{\lambda})^2\right] \chi + 2 \xi'_{\text{odd}} = 0 \quad (11a)$$

$$\psi' + \hat{x} \xi'_{\text{odd}} + \xi_{\text{odd}} = 0 \quad (11b)$$

$$\psi' - \frac{\hat{x}}{2} \hat{x} = 1 \quad (11c)$$

where we have introduced the normalized variables $\hat{\lambda} = \lambda/\epsilon^{1/5}$, $\hat{x} = x/\epsilon^{1/5}$. The right hand side of Eq. (11c) sets the (arbitrary) size of the solution. The remaining boundary conditions follow from the assumed symmetry. Given $\hat{\lambda}$, the solution is completely determined and yields $\hat{\lambda}(\lambda_H)$ through Eq. (10). The results are

summarized in Fig. (1). we conclude that $m = 1$ modes driven by electron viscosity have a typical growth rate

$$\gamma\tau_H \approx (\tau_H/\tau_V)^{1/5} \quad (12)$$

we shall now describe the nonlinear evolution of the $m \geq 2$ instability. The analysis closely follows that of previous workers for the resistive problem.⁹ The basic physical idea is that as the island width exceeds the linear resistive layer thickness, nonlinear $\tilde{J} \times \tilde{B}$ terms in the equation of motion dominate the $\rho d\tilde{V}/dt$ terms. These nonlinear terms are stabilizing and slow down the growth of the island. The outer region is still treated linearly and in terms Δ' . The nonlinear equations describing the inner region dynamics, may now be written

$$\frac{\partial \psi}{\partial t} + \tilde{v} \cdot \nabla \psi = (-\eta J + \frac{m}{ne} \tilde{\mu} \nabla^2 J + E) c \quad (13a)$$

$$0 \approx (\tilde{B} \cdot \nabla) J \quad (13b)$$

where E is the imposed electric field and we have the defining equations

$$\tilde{B} = \nabla \psi \times \hat{z} - (kr/m) B_z \hat{\theta} + B_z \hat{z}, \quad (14a)$$

$$\frac{4\pi}{c} J = -\nabla^2 \psi - 2kB_z/m \quad (14b)$$

The problem has been made two-dimensional by imposing helical symmetry viz θ, z entering only through the combined variable $m\theta + kz$. Equation (13b) shows that in the nonlinear phase, $J \equiv J(\psi)$. we next derive a differential equation for $J(\psi)$ from

(13a). Let us assume $\psi = \psi_0''(x^2/2) + \tilde{\psi}(t) \cos ky$ where $\psi_0'' = (r_S/m) F'(r_S)$, $x = r - r_S$, and $\tilde{\psi}$ is taken independent of x because of the constant- ψ approximation. Changing variables from x, y, t to ψ, y, t and eliminating the $y \cdot \nabla \psi$ term by operating on the resulting equation with $\int_{-\pi/k}^{\pi/k} (dy/x)$, we obtain

$$\begin{aligned} \frac{\partial}{\partial \psi} \langle (\psi - \cos ky)^{1/2} \rangle \frac{\partial J}{\partial \psi} - \delta \langle 1/(\psi - \cos ky)^{1/2} \rangle J \\ = -\alpha \langle \cos ky / (\psi - \cos ky)^{1/2} \rangle - \epsilon \langle 1/(\psi - \cos ky)^{1/2} \rangle \end{aligned} \quad (15)$$

where $\langle F \rangle = (2\pi/k) \int_{-\pi/k}^{\pi/k} dy F$, $\psi = \psi/\tilde{\psi}$, $\delta = c^2 \eta / (8\pi \mu) (\tilde{\psi}/\psi_0'')$, $\alpha = (\tilde{\psi} \partial \tilde{\psi} / \partial t) / 8\pi \mu \psi_0''$, $\epsilon = E \tilde{\psi}^{1/2} / 8\pi \mu \psi_0''$ and $\partial/\partial x \gg k$ is assumed in the inner region. We have also retained resistivity and assumed that it dominates in the equilibrium $E = \eta J$. The solution of (15) is used to generate a Δ' , which is then matched to the external region Δ' (same as in linear theory) giving

$$\Delta' = \beta \int_{-1}^{\infty} d\psi (J - E/\eta) \langle \cos ky / (\psi - \cos ky)^{1/2} \rangle \quad (16)$$

where $\beta = 8\pi (2/\psi_0'' \tilde{\psi})^{1/2}$. Equations (15) and (16) describe the nonlinear temporal evolution of the $m \geq 2$ modes. For $\mu \rightarrow 0$, we recover the resistive results

$$w_R \equiv \left(\frac{16 \tilde{\psi}}{\psi_0''} \right)^{1/2} = \frac{\eta c^2}{4\pi (2)^{1/2} K_R} \Delta' t \quad (17a)$$

where $K_R = \int_{-1}^{\infty} [\langle \cos ky / (\psi - \cos ky)^{1/2} \rangle^2 / \langle 1/(\psi - \cos ky)^{1/2} \rangle] d\psi \approx 3.6$.

For $\eta = 0$, we get the new viscous result

$$w_V \equiv \left(\frac{16\tilde{\psi}}{\psi_0''} \right)^{1/2} = \left(\frac{24(2)^{1/2}}{\kappa_V} \Delta' t \right)^{1/3} \quad (17b)$$

where $\kappa_V = \int_{-1}^{\infty} d\psi \langle \cos ky (\psi - \cos ky)^{1/2} \rangle^2 / \langle (\psi - \cos ky)^{1/2} \rangle = 2.5$. Thus the width scales as $t^{1/3}$ in the nonlinear viscous case in contrast to t in the nonlinear resistive case. It is also instructive to note that as the island becomes bigger, δ increases and so the resistive limit is recovered whenever $\tilde{\psi}/\psi_0'' \gg 8\pi\mu/\eta c^2$; this is consistent with the fact that viscosity loses its punch as the current gradient becomes weaker. In Fig. 2 we have plotted $\dot{w}/\dot{w}_R$ versus time for several values of $[8\pi\mu/\eta c^2 w^2(t=0)]$ as obtained from a direct numerical integration of Eqs. (15) and (16). The results confirm the above two limiting cases. A model calculation similar to that of white et al.,⁹ can also be carried out for island saturation. It indicates⁶ that the saturation width for viscosity driven modes is the same as that for the resistive case; this is reasonable since the anomalous electron viscosity taps no new free energy sources.

We now point out some implications of the results derived above. If we assume $\bar{\mu} \sim \chi \sim a^2/\tau_E$ where τ_E is the measured energy confinement in a Tokamak discharge, we get $\tau_V \sim \tau_E (\omega_p^2 r_s^4 / a^2 c^2)$. For a plasma with $n \sim 10^{14} \text{ cm}^{-3}$, $a \sim 50 \text{ cm}$, $\tau_E \sim 10^2 \text{ msec}$, $\tau_H \sim 10^{-6} \text{ secs}$, $r_s/a \sim 1/2$, this then gives the linear growth times for viscosity driven $m = 1$ and $m \geq 2$ tearing modes to be hundreds of microseconds and a msec., respectively. For $T_e > 1 \text{ keV}$, this

growth is faster than the typical resistive growth. Note that this growth is independent of e-i collisions and is hence likely to be as virulent for hotter plasmas and that because of the $1/5$, $1/3$ powers, it is relatively insensitive to plasma parameters. It is also of interest to note that if we use the Rechester - Rosenbluth formula² for anomalous transport $\chi_e \sim D_{st} v_e$, then the above results would require only $|\tilde{b}/B|^2 \sim 10^{-8}$. When the island thickness exceeds the thickness of the linear boundary layer, nonlinear effects dominate and we get $w/r_s \sim (t/\tau_V)^{1/3}$. Electron viscosity dominates the nonlinear growth up to an island width $w_c \sim [32 \mu / (nc^2/4\pi)]^{1/2} \sim 4(2)^{1/2} c/\omega_p (\tau_R/\tau_E)^{1/2}$ where $\tau_R \approx 4\pi a^2 / nc^2$ is a resistive decay time; for island width $w > w_c$, the nonlinear growth is dominated by resistivity. Since $\tau_R \approx 40 a_m^2 T_{keV}^{3/2}$ where a_m is the plasma radius in meters and T_{keV} , the electron temperature in keV, one finds $w_c \approx$ few cm for $T_e = 1$ keV (other parameters same as above) and scaling up with temperature as $T^{3/4}$. We may conclude that if τ_V is determined by physical processes which control the gross energy confinement of a Tokamak plasma, then the initial phase of growth of macroscopic tearing modes (island widths up to a few cm.) should be dominated by electron viscosity effects. Experimentally, it may be interesting to look for a $w \sim t^{1/3}$ phase between the linear $\exp(\gamma t)$ and the resistivity dominated $w \sim t$ phases.

We finally examine the speculation^{1,10} that the sudden onset of a large anomalous electron viscosity during the nonlinear resistive growth of a magnetic island may be responsible for a disruptive instability. It is quite likely that either because of

nonlinear couplings⁴ or toroidal effects¹, a large magnetic island might generate very stochastic regions of magnetic field in the plasma (especially close to the separatrix). If these are modeled with an anomalous electron viscosity $\mu(t) = \mu_0[1 + \tanh t/t_0]$, we can study their influence on a nonlinearly growing island. The results of a numerical calculation are shown in Fig. 3 (a) where we see the island width going through 3 phases $w \propto t$, $w \propto t^{1/3}$ and $w \propto t$ again. The net increase in the island width due to the essentially "impulsive" viscous effect is (as shown by a model calculation⁶) given by

$$\Delta w = \int_0^{\infty} (\dot{w} - \dot{w}_R) dt = (16\pi^3 \mu_{KR}/\eta c^2 K_V)^{1/2}. \quad (18)$$

This is consistent with numerical results Fig. 3(b). If $\Delta w \sim a$, the disruptive instability may be triggered. This requires $\mu/\chi_e \sim 10^2$ where χ_e is the measured anomalous thermal conductivity for the normal discharge. It is not unreasonable to speculate that just prior to disruption, $\bar{\mu}$ is as large as above (especially if the large scale magnetic islands are themselves generating stochasticity). Note that the required $\bar{\mu}$ is much less than the maximum plausible value $\bar{\mu}_{\max} \sim a^3 \omega_p^2 / v_A$. We conclude that the sudden onset of electron viscosity is an interesting candidate for an explanation of the disruptive instability.

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Fig. 1. Viscous growth rate λ versus ideal kink mode growth rate λ_H . The small parameter $\epsilon = \tau_H/\tau_V$, the ratio of hydrodynamic to viscous decay times.

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Fig. 2. Rate of increase of island width versus $\tau \equiv (c^2 \eta \Delta / 8 \pi^2 2^{1/2} w_0) t$, relative to the resistive rate, Eq. (17a), for several different viscosities, $\theta \equiv 8 \pi \mu / \eta c^2 w_0^2$. Here w_0 is the initial island width.

Fig. 3. (a) Time dependence of $\dot{w}$ when the viscosity parameter increases rapidly from $\theta \ll 1$, to $\theta = 10$ at $\tau = 5$. (b) Increase in island width due to viscosity. The analytic estimate, Eq. (18), is in good agreement for $\theta \geq 5$.

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