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**ANYONS IN A MAGNETIC FIELD:
LANDAU LEVELS AND VERTEX OPERATOR REPRESENTATION**

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ABSTRACT

We construct exact many-body eigenstates of both energy and angular momentum for the N -anyon problem in an external magnetic field. Such states span the full ground-state eigenspace and have a natural representation in terms of the Fubini-Veneziano vertex operators of string theory.

Anyons^{1,2} — particles with fractional statistics in $2+1$ dimensions — have come to play an important role in our understanding of certain planar condensed matter phenomena,³ most notably the fractional quantum Hall effect.^{4,5,6} The general many-body quantum mechanics of anyons is not fully understood, but significant contributions have been made in few-body models⁷⁻⁹ in a mean-field approach,⁹ in a supersymmetric N -body model^{10,11} and for anyons in a harmonic potential.¹² Here we

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report on some recent progress on the quantum mechanics of N anyons in an external magnetic field.¹³ Specifically we present the construction of *exact* many-body eigenstates of both energy and angular momentum. We also show how these states arise naturally as expectation values of Fubini-Veneziano vertex operators between excited states, extending earlier observations of Fubini¹⁴ and others¹⁵ regarding the analogies between anyonic ground state wave functions and string theory vacuum expectation values. This vertex operator representation might prove useful for constructing new higher energy states.

The Hamiltonian for a collection of N identical anyons of mass m and charge e (labeled by Greek indices $\alpha, \beta, \dots$) moving on a plane in the presence of an external, constant, uniform magnetic field B (perpendicular to the plane) is given by

$$H = \sum_{\alpha=1}^N -\frac{1}{2m} \left(\vec{\nabla}_{\alpha} - i\vec{a}_{\alpha} - ie\vec{A}(\vec{x}_{\alpha}) \right)^2, \quad (1)$$

where we choose the external vector potential $\vec{A}$ in the symmetric gauge $A^i = -\frac{B}{2}\epsilon^{ij}x^j$ (without loss of generality we assume B is positive). The non-local effective gauge potential

$$a_{\alpha}^i(\vec{x}_1, \dots, \vec{x}_N) = \nu \sum_{\beta \neq \alpha} \epsilon^{ij} \frac{x_{\alpha}^j - x_{\beta}^j}{|\vec{x}_{\alpha} - \vec{x}_{\beta}|^2} \quad (2)$$

describes the statistical interactions. The associated statistical magnetic field b_{α} is

$$b_{\alpha}(\vec{x}_1, \dots, \vec{x}_N) = -2\pi\nu \sum_{\beta \neq \alpha} \delta^2(\vec{x}_{\alpha} - \vec{x}_{\beta}). \quad (3)$$

Each particle sees the $(N-1)$ others as vortices of strength $(-2\pi\nu)$, and the statistical phase $\pi\nu$ under interchange of two particles is acquired via the familiar Aharonov-Bohm effect. Note that since the statistical phase is $\pi\nu$ we may restrict ν to values in the interval $[0, 2)$ — any even integer part of ν may be considered as contributing to the angular momentum, as we discuss in more detail later. Equation (3) clearly shows the absence of statistical self-interactions and Lorentz forces (for hard-core particles).¹⁶ For quantum mechanical point particles the statistical interaction can also be represented by a local interaction with a gauge field whose dynamics is governed by a Chern-Simons term.³ Note however that the effects of such Chern-Simons interactions in (both relativistic and non-relativistic) field theories go beyond the mere transmutation of spin and statistics; indeed the Chern-Simons term supports solitons which would not exist in its absence.¹⁰

An alternative formulation for the many-body quantum mechanics of anyons is based on the observation that the statistical gauge field $\vec{a}_{\alpha}$ is pure gauge except at the locations of the particles, as is clear from (2) and (3). By a (singular) gauge transformation we can remove $\vec{a}_{\alpha}$ from the Hamiltonian at the expense of redefining the wave functions by a phase,

$$\psi_{\text{new}}(\vec{x}_1, \dots, \vec{x}_N) = e^{i \sum_{\alpha < \beta} \nu \theta_{\alpha\beta}} \psi_{\text{old}}(\vec{x}_1, \dots, \vec{x}_N), \quad (4)$$

where $\Theta_{\alpha\beta} = \arctan \left((x_\alpha^2 - x_\beta^2) / (x_\alpha^1 - x_\beta^1) \right)$. In this formulation the Hamiltonian becomes the usual sum of one-body Hamiltonians and the statistical interaction is hidden in the boundary conditions which make the wave functions *multi-valued*. Consequently, the many-body Hilbert space is not a simple direct product of one-body Hilbert spaces. Throughout this paper we shall work with these multi-valued wave functions, an approach originally advocated by Wilczek and Wu² (we will also drop the subscript "new" from the wave functions).

Using complex notation ($z = x^1 + ix^2$; $\bar{z} = x^1 - ix^2$; $\partial = \partial/\partial z$; $\bar{\partial} = \partial/\partial \bar{z}$) the gauge transformed Hamiltonian is

$$H = \sum_{\alpha=1}^N \left(-\frac{2}{m} \bar{\partial}_\alpha \partial_\alpha + \frac{e^2 B^2}{8m} |z_\alpha|^2 \right) - \frac{eB}{2m} J \quad (5)$$

where J is the total (canonical) angular momentum operator

$$J = \sum_{\alpha=1}^N (z_\alpha \partial_\alpha - \bar{z}_\alpha \bar{\partial}_\alpha) \quad (6)$$

Without the last term in (5), the Hamiltonian coincides with the harmonic potential problem previously considered by Wu⁷ and Chou.¹²

Before discussing many-body anyonic wave functions we briefly review some elementary facts concerning Landau levels.¹⁷ The one-body Landau levels are

$$\begin{aligned} \psi_n^J(z, \bar{z}) &= z^k \bar{z}^\ell |z|^{(|k-\ell|-(k+\ell))} e^{-\frac{eB}{4}|z|^2} L_n^{|k-\ell|} \left(\frac{eB}{2} |z|^2 \right) \quad , \\ J &= k - \ell \quad , \\ E_n^J &= \frac{eB}{m} \left(n + \frac{1}{2} + \frac{1}{2} (|J| - J) \right) \quad , \quad k, \ell \in \mathbb{Z}_+ \quad , \end{aligned} \quad (7)$$

where the $L_n^{|k-\ell|}$ are generalized Laguerre polynomials. The energy levels have an infinite degeneracy [for samples of *finite* area this degeneracy is finite] given by all states with the same quantum number n and arbitrary angular momentum aligned parallel to the external magnetic field. States with angular momentum aligned antiparallel to the external magnetic field have instead energies linearly proportional to the angular momentum. Many-body wave functions are obtained by forming totally symmetrized (bosons) or antisymmetrized (fermions) products of one-body wave functions. Thus, the ground state has energy and angular momentum eigenvalues:

$$\begin{aligned} \text{bosons:} \quad E &= \frac{eB}{m} \cdot \frac{N}{2} \quad , \quad J = d_+ \quad , \\ \text{fermions:} \quad E &= \frac{eB}{m} \cdot \frac{N}{2} \quad , \quad J = \frac{1}{2} N(N-1) + d_+ \quad , \end{aligned} \quad (8)$$

where d_+ is a non-negative integer. States with $n = 0$ and angular momentum below the minimal ground state values (0 for bosons, $\frac{1}{2}N(N-1)$ for fermions) necessarily have higher energy.

It is not known in general how to construct N -body anyonic wave functions. Here we take the approach of Laughlin⁴ and Wu,⁷ which is to decompose the multi-valued wave function in (4) into the product of a multi-valued “statistical prefactor” encoding *all* the statistical correlations, and a single-valued factor $\tilde{\psi}$. For $\nu \in [0, 2)$ this may be achieved in two ways,

$$\psi_I(z_1, \dots, z_N, \bar{z}_1, \dots, \bar{z}_N) = \prod_{\alpha < \beta} (z_\alpha - z_\beta)^\nu \tilde{\psi}(z_1, \dots, z_N, \bar{z}_1, \dots, \bar{z}_N) \quad (9a)$$

$$\psi_{II}(z_1, \dots, z_N, \bar{z}_1, \dots, \bar{z}_N) = \prod_{\alpha < \beta} (\bar{z}_\alpha - \bar{z}_\beta)^{2-\nu} \tilde{\psi}(z_1, \dots, z_N, \bar{z}_1, \dots, \bar{z}_N) \quad (9b)$$

[Note that in (9b) we write the exponent as $(2-\nu)$ in order to avoid singularities in the prefactor and to maintain the totally symmetric nature of $\tilde{\psi}$.] We shall refer to these as states of type-I and type-II, respectively (see also Ref. [8]). The crucial difference between these states is that the contribution to the angular momentum J coming from the statistical prefactor is positive for states of type-I but negative for type-II states. This difference is reflected in the corresponding spectra [see Eqs. (11a,b) below].

Having discussed the statistical prefactors (of type-I and type-II) which give rise to the multi-valued phase in (4), we now consider which forms of $\tilde{\psi}$ in Eqs. (9a,b) produce simultaneous eigenstates of the (commuting) angular momentum and energy operators. Recall that since the multi-valuedness of the full wave function is to be encoded *solely* in the statistics prefactor, $\tilde{\psi}$ must be single-valued and totally symmetric (under permutation of the $1, \dots, N$ labels). It is convenient to further decompose $\tilde{\psi}$ as $\tilde{\psi} = S\phi(\sum_\alpha |z_\alpha|^2)$. In this decomposition S will determine the contribution to the angular momentum from $\tilde{\psi}$ while ϕ will characterize the analogue of the Landau levels. The *most general* forms of S and ϕ which make the wave functions in (9) simultaneous eigenstates of energy and angular momentum are not known. However, several infinite classes of simultaneous eigenstates may be written down by making suitable choices for S and ϕ . In particular, choosing $S \equiv S_+^{(d_+)}(z_1, \dots, z_N)$ (any totally symmetric homogeneous polynomial of integer degree d_+ depending only on the z variables) for states of type-I makes (9a) a simultaneous eigenstate of H and J . Similarly, choosing $S \equiv S_-^{(d_-)}(\bar{z}_1, \dots, \bar{z}_N)$, depending only on the $\bar{z}$ variables, produces simultaneous eigenstates of type-II. [We shall comment below on more complicated choices for S .] In each case ϕ is required to satisfy the following differential equation

$$x\phi'' + \left(d_\pm + \frac{\delta}{2}N(N-1) + N\right)\phi' + \left(\frac{2m}{\epsilon B}E - x \pm d_\pm \pm \frac{\delta}{2}N(N-1)\right)\phi = 0 \quad (10)$$

where $x \equiv \frac{\epsilon B}{4} \sum_\alpha |z_\alpha|^2$, E is the energy eigenvalue, and the $+$ ($-$) refers to the type-I (type-II) states, for which $\delta \equiv \nu$ or $(2-\nu)$, respectively. Regular solutions to (10) give ϕ as a decaying exponential multiplied by a generalized Laguerre polynomial which introduces a “Landau level” index n . We are thus led to two distinct “branches” of states:

Type-I:

$$\begin{aligned}\psi_I(z_1, \dots, z_N, \bar{z}_1, \dots, \bar{z}_N) &= \prod_{\alpha < \beta} (z_\alpha - z_\beta)^\nu S_+^{(d_+)}(z_1, \dots, z_N) L_n^{d_+ + \frac{\nu}{2}N(N-1) + N-1}(2x) e^{-x} , \\ E_I &= \frac{eB}{m} \left(n + \frac{N}{2} \right) , \\ J_I &= \frac{\nu}{2} N(N-1) + d_+ ,\end{aligned}\tag{11a}$$

Type-II:

$$\begin{aligned}\psi_{II}(z_1, \dots, z_N, \bar{z}_1, \dots, \bar{z}_N) &= \prod_{\alpha < \beta} (\bar{z}_\alpha - \bar{z}_\beta)^{2-\nu} S_-^{(d_-)}(\bar{z}_1, \dots, \bar{z}_N) L_n^{d_- + \frac{2-\nu}{2}N(N-1) + N-1}(2x) e^{-x} , \\ E_{II} &= \frac{eB}{m} \left(n + \frac{N}{2} + d_- + \frac{2-\nu}{2} N(N-1) \right) , \\ J_{II} &= -\frac{(2-\nu)}{2} N(N-1) - d_- .\end{aligned}\tag{11b}$$

Note that type-I states have positive angular momentum bounded below by $\frac{\nu}{2}N(N-1)$ while type-II states have negative angular momentum bounded above by $[-\frac{(2-\nu)}{2}N(N-1)]$. The negative angular momentum of type-II states raises their energy (as expected), introducing a ν and N^2 dependence which is not present for type-I states. For type-I states the angular momentum is parallel to the external magnetic field which cancels the statistical frustration, thereby producing energy eigenvalues independent of the statistical parameter ν and which increase linearly with N .

It is instructive to examine the $\nu = 0$ (boson) and $\nu = 1$ (fermion) limits in (11). The ground state is described by type-I states with $n = 0$. Setting $\nu = 0$ (1), the type-I eigenvalues in (11a) give the correct energy and angular momentum eigenvalues for the N -body boson (fermion) problem — see Eq. (8). Furthermore, the $n = 0$ type-I wave functions in (11a) generate *all* the bosonic and fermionic N -body ground state wave functions when $\nu = 0$ and $\nu = 1$, respectively. To see this we first note that the general symmetric homogeneous polynomials $S_+^{(d_+)}$ in (11a) may be expanded in any one of a number of particular bases.¹⁸ One natural such basis is given by the permanents

$$B_{\{d_1, \dots, d_N\}}^{(1)} \equiv \sum_{\text{perms } \pi} z_{\pi(1)}^{d_1} z_{\pi(2)}^{d_2} \dots z_{\pi(N)}^{d_N} ,\tag{12}$$

while another is given by the Schur functions

$$B_{\{d_1, \dots, d_N\}}^{(2)} \equiv \frac{\sum_{\text{perms } \pi} (-1)^{\text{sgn } \pi} z_{\pi(1)}^{d_1 + N-1} z_{\pi(2)}^{d_2 + N-2} \dots z_{\pi(N)}^{d_N}}{\prod_{\alpha < \beta} (z_\alpha - z_\beta)} ,\tag{13}$$

where $\{d_1, \dots, d_N\}$ is any unordered set (e.g. $\{0, 1\}$ is identified with $\{1, 0\}$) of non-negative integers with $\sum_k d_k = d_+$. The number of basis elements for a given d_+ is simply the number of unrestricted partitions of d_+ into N elements. The (linear)

transformation between the two bases $\{B^{(1)}\}$ and $\{B^{(2)}\}$ involves coefficients which are characters of the symmetric group S_N .¹⁸ Remarkably the ratio of determinants defining the Schur functions (13) produces homogeneous polynomials. Now when $\nu = 0$ we see that the $\{B^{(1)}\}$ basis is more convenient [we are obviously free to choose whichever basis we wish] since then the $n = 0$ ψ_I wave functions in (11a) are *precisely* all the Slater permanents spanning the ground-state of the N -body bosonic problem. When $\nu = 1$ the $\{B^{(2)}\}$ basis is more convenient because in the $n = 0$ ψ_I wave functions the statistical prefactor $\prod_{\alpha < \beta} (z_\alpha - z_\beta)$ cancels the denominator of the Schur functions in (13), leaving the numerators which are *precisely* all the Slater determinants spanning the ground-state of the N -body fermionic problem. Thus, the eigenspace spanned by the $n = 0$ ψ_I wave functions is the complete ground-state of the N -anyon problem.

It is also interesting to note that the Laughlin⁴ wave functions which play an important role in the fractional quantum Hall effect,

$$\psi_L(z_1, \dots, z_N, \bar{z}_1, \dots, \bar{z}_N) = \prod_{\alpha < \beta} (z_\alpha - z_\beta)^{2k+1} \exp\left(-\frac{eB}{4} \sum_\alpha |z_\alpha|^2\right) \quad (14)$$

[where $k \in \mathbb{Z}_+$] appear as *particular* fermionic type-I states with $n = 0$ and higher angular momentum, generated by taking $S_+^{(d_+)} = \prod_{\alpha < \beta} (z_\alpha - z_\beta)^{2k}$. Then $d_+ = kN(N-1)$ and $J = \frac{2k+1}{2}N(N-1) = \frac{1}{2}N(N-1) + d_+$ as expected from (11a). In fact, we see directly from (14) that the Laughlin state ψ_L is a sum of Slater-determinant fermionic states (all in the lowest Landau level). Indeed, one of these states is the antisymmetrized sum of $z_1^0 z_2^{(2k+1)} z_3^{2(2k+1)} \dots z_N^{(N-1)(2k+1)} \exp\left(-\frac{eB}{4} \sum_\alpha |z_\alpha|^2\right)$, which corresponds to filling every $(2k+1)^{\text{th}}$ single-particle angular momentum state — illustrating the interpretation of the Laughlin state as one of filling fraction $\frac{1}{2k+1}$.

While the $n = 0$ type-I states in (11a) generate the *full* ground state eigenspace, the $n > 0$ type-I states do not cover *all* the higher energy states. This is because higher energy states may involve negative angular momentum. Many, but by no means all, of these higher energy states arise from the type-II states in (11b), which have negative total angular momentum. However, even in the bosonic and fermionic N -body problems there are wave functions not included in the states in (11). These arise when the totally symmetric angular momentum factor S in $\tilde{\psi}$ (see (9)) involves *both* the z variables *and* the $\bar{z}$ variables. In (11) we chose S to depend either only on the z 's (type-I states) or only on the $\bar{z}$'s (type-II states). To generate the missing states one should consider "mixed" S functions. The *general* solution to this problem which still produces eigenstates of energy and angular momentum is not known. However, we have found the following special "mixed" eigenstates:

Type-I:

$$\psi_I(z_1, \dots, z_N, \bar{z}_1, \dots, \bar{z}_N) = \prod_{\alpha < \beta} (z_\alpha - z_\beta)^\nu (z_1 + \dots + \bar{z}_N)^{d_-} L_n^{\frac{1}{2}N(N-1)+d_-+N-1}(2r) e^{-r} \quad ,$$

$$E_I = \frac{eB}{m} \left(n + \frac{N}{2} + d_- \right) \quad ,$$

$$J_I = \frac{\nu}{2} N(N-1) - d_- \quad , \quad (15a)$$

Type-II:

$$\begin{aligned} \psi_{II}(z_1, \dots, z_N, \bar{z}_1, \dots, \bar{z}_N) &= \prod_{\alpha < \beta} (\bar{z}_\alpha - \bar{z}_\beta)^{2-\nu} (z_1 + \dots + z_N)^{d_+} L_n^{\frac{2-\nu}{2} N(N-1) + d_+ + N-1}(2x) e^{-x} \quad , \\ E_{II} &= \frac{eB}{m} \left(n + \frac{N}{2} + \frac{2-\nu}{2} N(N-1) \right) \quad , \\ J_{II} &= -\frac{(2-\nu)}{2} N(N-1) + d_+ \quad . \end{aligned} \quad (15b)$$

Note that these states have angular momentum in the gap not covered by the states (11): $-\frac{2-\nu}{2} N(N-1) \leq J \leq \frac{\nu}{2} N(N-1)$.

We now show how the wave functions (11) (we concentrate on type-I states for ease of presentation) may be expressed using a vertex operator representation, thereby extending to higher angular momentum earlier observations of Fubini¹⁴ and others.¹⁵ We introduce the vertex operator

$$V_\nu(z) \equiv : e^{i\sqrt{\nu}\varphi(z)} : \quad , \quad (16)$$

where $\varphi(z)$ is the Fubini-Veneziano field of string theory,¹⁹

$$\varphi(z) = q - ip \ln z + i \sum_{n \neq 0} \frac{a_n}{n} z^{-n} \quad . \quad (17)$$

The modes of φ satisfy the commutation relations

$$[q, p] = i \quad , \quad [a_n, a_m] = n \delta_{n+m, 0} \quad . \quad (18)$$

In (16) the $: \quad :$ symbol denotes normal ordering with respect to these modes, i.e. p to the right of q and a_n ($n > 0$) to the right of a_{-n} . In string theory the a_n ($n > 0$) and a_{-n} are destruction and creation operators for oscillation modes of the string, which correspond to higher angular momentum states. The operators q and p are the center of mass position and momentum of the string. It has recently been pointed out that the vertex operators (16) can be used to reproduce Laughlin's ground state wave functions as correlation functions.^{14, 15}

First we note that by a simple reordering the product of N vertex operators can be rewritten as

$$\begin{aligned} \prod_{\alpha=1}^N V_\nu(z_\alpha) &= \prod_{\alpha < \beta} (z_\alpha - z_\beta)^\nu \exp \left[\sqrt{\nu} \sum_{n=1}^{\infty} \frac{a_{-n}}{n} \left(\sum_{\alpha} z_\alpha^n \right) \right] \\ &\times \exp [i\sqrt{\nu} N q] \prod_{\alpha} z_\alpha^{\sqrt{\nu} p} \exp \left[-\sqrt{\nu} \sum_{n=1}^{\infty} \frac{a_n}{n} \left(\sum_{\alpha} z_\alpha^{-n} \right) \right] \quad . \end{aligned} \quad (19)$$

This has the immediate consequence that the vertex operators are good candidates for anyonic operators, since

$$V_\nu(z_1) V_\nu(z_2) = V_\nu(z_2) V_\nu(z_1) e^{\pm i\nu\pi} \quad , \quad (20)$$

where the signs in the exponent correspond to the two topologically distinct ways of interchanging z_1 and z_2 . Also, the vertex operator (16) behaves as a fractional spin operator under rotations

$$\begin{aligned} [L_0, V_\nu(z)] &= z \frac{d}{dz} V_\nu(z) + \frac{\nu^2}{2} V_\nu(z) , \\ e^{iL_0\theta} V_\nu(z) e^{-iL_0\theta} &= V_\nu(ze^{i\theta}) e^{i\frac{\nu^2}{2}\theta} , \end{aligned} \quad (21)$$

where the rotation generator L_0 has the oscillator representation

$$L_0 = \frac{p^2}{2} + \sum_{n=1}^{\infty} a_{-n} a_n . \quad (22)$$

We now define the vacuum $|0\rangle$ by

$$p|0\rangle = 0 , \quad a_n|0\rangle = 0 \quad (n > 0) , \quad (23)$$

and the dual vacuum $\langle 0|$ by

$$\langle 0|q = 0 , \quad \langle 0|a_{-n} = 0 \quad (n > 0) . \quad (24)$$

Note that if $\langle x|$ satisfies $\langle x|q = \langle x|x$, then $\langle x| = \sum_p \exp(-ixp) \langle p|$, where $\langle p|$ is a state of momentum p ; thus for $x = 0$, we have $\langle 0| = \sum_p \langle p|$. It now follows immediately from (19) that

$$\langle 0| \prod_{\alpha=1}^N V_\nu(z_\alpha) |0\rangle = \prod_{\alpha < \beta} (z_\alpha - z_\beta)^\nu \quad (25)$$

which is precisely the relevant part of Laughlin's wave function for anyons, the exponential factor being regarded as a measure factor. Notice that in (25), momentum conservation selects the momentum state $(\sqrt{\nu}N)$ from the dual vacuum $\langle 0|$. We now show how this operator formalism may be used to represent the relevant part of *all* the $n=0$ type-I wave functions in (11) (similar arguments hold for the type-II states in the antiholomorphic sector).

In the dual Fock space of the φ field, consider the state at level d_+ defined by

$$|(\lambda_k)\rangle = \langle 0| \left(\frac{a_1}{\sqrt{\nu}} \right)^{\lambda_1} \left(\frac{a_2}{\sqrt{\nu}} \right)^{\lambda_2} \cdots \left(\frac{a_k}{\sqrt{\nu}} \right)^{\lambda_k} , \quad (26)$$

where $\{\lambda_k\}$ is an ordered set (e.g. $\{0,1\}$ has to be distinguished from $\{1,0\}$) of non-negative integers such that $\sum_k k\lambda_k = d_+$. A straightforward computation then shows that

$$\langle (\lambda_k)| \prod_{\alpha=1}^N V_\nu(z_\alpha) |0\rangle = B_{\{\lambda_k\}}^{(3)}(z_1, \dots, z_N) \prod_{\alpha < \beta} (z_\alpha - z_\beta)^\nu , \quad (27)$$

where $B_{\{\lambda_k\}}^{(3)}(z_1, \dots, z_N)$ is the following symmetric homogeneous polynomial of degree $d_+ = \sum_k k\lambda_k$

$$B_{\{\lambda_k\}}^{(3)}(z_1, \dots, z_N) = \left(\sum_{\alpha_1=1}^N z_{\alpha_1} \right)^{\lambda_1} \left(\sum_{\alpha_2=1}^N z_{\alpha_2}^2 \right)^{\lambda_2} \cdots \left(\sum_{\alpha_k=1}^N z_{\alpha_k}^k \right)^{\lambda_k} . \quad (28)$$

The $B_{\{\lambda_k\}}^{(3)}$ provide a third natural basis for the symmetric homogeneous polynomials of degree d_+ ; this basis is connected to the previously mentioned basis $B_{\{d_k\}}^{(1)}$ and $B_{\{d_k\}}^{(2)}$ (Eqs. (12) and (13)) by linear transformations with coefficients being respectively the compound and simple characters of the symmetric group S_N .¹⁸ Therefore we can represent *all* $n = 0$ type-I wave functions in (11) as linear combinations of vertex operator correlation functions (20). Thus the insertion of the oscillators a_k in the correlation function (25) is seen to increase the angular momentum of the Laughlin state; like in string theory, the state $(\{\lambda_k\})$ is a higher angular momentum state.

The relevant part of Type-I wave functions in (11) with $n \geq 1$ may be represented by means of the step operator

$$R^\dagger = -\frac{2}{\epsilon B} \sum_a \left(\delta_a - \frac{\epsilon B}{2} z_a \right) \left(\partial_a - \frac{\epsilon B}{2} \bar{z}_a \right) ,$$

which raises the energy without affecting the angular momentum. Thus

$$\begin{aligned} & \frac{1}{n!} (R^\dagger)^n (\{\lambda_k\}) \prod_{a=1}^N V_a(z_a) |0\rangle \\ &= \prod_{\alpha < \beta} (z_\alpha - z_\beta)^\nu B_{\{\lambda_k\}}^{(3)}(z_1, \dots, z_N) L_n^{d_+ + \frac{1}{2}N(N-1) + N-1} \left(\frac{\epsilon B}{2} \sum_a |z_a|^2 \right) . \end{aligned} \quad (29)$$

While the vertex operator formulation provides an interesting way to rewrite known wave functions, we expect it to prove useful for generating new eigenstates. Research in this direction is in progress. We hope that the convergence of ideas from solid state theory, particle theory and mathematics, which is reflected in the participation at this workshop will prove fruitful.

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