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HYDROMAGNETIC WAVES IN
HIGH β PLASMAS

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9. The ninth part is a letter from the

I. INTRODUCTION

In nearly all theories of wave propagation in magnetized plasmas, only those plasmas for which the particle pressure, p , is less than the magnetic pressure, $B^2/8\pi$, have been considered. Thus, $\beta = 8\pi p/B^2$ is taken to be less than unity, because the principal interest has been in those laboratory plasmas which are confined by such a magnetic field. However, for transient laboratory plasmas confined by their own inertia, or for astrophysical plasmas confined by inertia or gravity, the parameter β can be extremely large, and little basic theory has yet been done on such plasmas.

In this paper we study the propagation of waves in a large β , homogeneous, collisionless plasma embedded in an uniform magnetic field. We restrict our investigation to waves whose frequency is below the ion-cyclotron frequency, ω_{ci} , that is to hydromagnetic waves. This has already been done to some extent by Stepanov (1958), and our results are an extension of his work. To evaluate the dispersion relation for wave propagation we develop several numerical iteration schemes, based on the usual asymptotic methods for the solution of the Vlasov-Maxwell equations. In this investigation several surprising results emerge.

For parallel propagation the frequencies of the two circularly polarized waves are significantly different above a characteristic frequency of order ω_{ci}/β , rather than at ω_{ci} as in the low β theory. Above this frequency the right circularly polarized wave propagates much faster than the Alfvén speed, $v_A = B/(4\pi\rho)^{1/2}$, while the left circularly polarized wave propagates much slower than v_A . Even below this characteristic frequency, there is

considerably greater dispersion than for a low β plasma.

The physical reason for this frequency splitting can be understood as follows. For a hydromagnetic wave in a low β plasma the $\underline{E} \times \underline{B}$ drift currents of the electrons and ions, the Hall currents, cancel to a relative accuracy of $(\omega/\omega_{ci})^2$. The only significant current in the wave is the polarization current of the ions, parallel to $\underline{E}$, which is smaller than the Hall current of either species by (ω/ω_{ci}) . But if β is much larger than one, the ion thermal motion, v_i , is large compared to the Alfvén speed, v_A . Because of their rapid motion through the wave, the ions do not have time to develop their full $\underline{E} \times \underline{B}$ motion and there is a lack of cancellation between the ion and electron Hall currents by a factor of order $k^2 v_i^2 / \omega_{ci}^2$, where k is the magnitude of the wave vector. Thus, when $k^2 v_i^2 / \omega_{ci}^2$ is of order ω / ω_{ci} the Hall current is comparable with the polarization current and we expect the wave propagation to change its character. At this point we may estimate as $k v_A$; this gives $\omega \sim \omega_{ci} v_A^2 / v_i^2 = \omega_{ci} / \beta$.

The second result of interest is the strength of the damping when the propagation vector $\underline{k}$ makes even a small angle θ with $\underline{B}$. At finite angles of propagation the modes are approximately linearly polarized, as for a low β plasma, and one can identify one polarization with the magnetosonic mode and the other with the shear Alfvén mode. The magnetosonic mode has a perturbation in the magnetic field strength as in a low β plasma, but because there are many thermal particles moving at the wave speed the damping of the wave by resonant coupling to the perturbed magnetic field strength is very strong. Further, because the Hall terms are quite large, there is substantial linear coupling between the two modes and the

shear Alfvén mode is appreciably damped also.

The propagation speed of the magnetosonic mode decreases to zero at an angle θ of order $\beta^{-1/4}$, while its damping rate steadily increases. At larger angles it merges with the magnetosonic mode propagating in the opposite direction. Together they become a heavily damped nonpropagating mode and a less heavily damped nonpropagating mode. For small wave numbers, the shear Alfvén mode propagates normally as a weakly damped mode. But for larger wave numbers, this mode and the corresponding oppositely propagating mode change into two nonpropagating modes, one weakly damped and one strongly damped. It is to be emphasized that all this happens for very small angles of propagation of order $\beta^{-1/4}$.

The bulk of the paper is primarily concerned with the derivation of the frequency and damping rate of these modes as a function of the wave number, k , and β . Sections 2 and 3 are devoted to reducing the general electromagnetic dispersion relation to a manageable form, and developing the numerical iteration methods for its solution. The results are presented in section 4, and some analytic approximations to these results are given in section 5. In section 6 two applications of this work are discussed, and the whole paper is summarized in section 7.

II. THE DISPERSION RELATION

We consider a homogeneous plasma consisting of electrons and singly charged ions each with a Maxwell velocity distribution at the same temperature, T , and density, n . This plasma is embedded

in a uniform magnetic field $\underline{B}$. We take a linearized electromagnetic plane wave with all quantities varying as the real part of $\text{const} \times \exp(-i\omega t + i\mathbf{k} \cdot \mathbf{r}) = \text{const} \times \exp(-i\omega t + ik \cos \theta z + ik \sin \theta x)$ where x, y and z are Cartesian coordinates chosen so that the z axis lies along $\underline{B}$ and the wave number vector $\underline{k}$ is in the $x-z$ plane. θ is the angle between $\underline{k}$ and $\underline{B}$ (figure 1). We wish to determine the angular frequency of the wave ω as a function of the plasma parameters n, T and $\underline{B}$ and the wave parameters k and θ . We do this by solving the plasma dispersion relation as given by Akhiezer et al. (1967):

$$D = \begin{vmatrix} 4\pi K_{11} - \frac{c^2 k^2 \cos^2 \theta}{\omega^2} & 4\pi K_{12} & 4\pi K_{13} + \frac{c^2 k^2 \sin \theta \cos \theta}{\omega^2} \\ -4\pi K_{12} & 4\pi K_{22} - \frac{c^2 k^2}{\omega^2} & 4\pi K_{23} \\ 4\pi K_{13} + \frac{c^2 k^2 \sin \theta \cos \theta}{\omega^2} & -4\pi K_{23} & 4\pi K_{33} - \frac{c^2 k^2 \sin^2 \theta}{\omega^2} \end{vmatrix} \quad (1)$$

where the components of the dielectric tensor K_{ij} are listed in Appendix A. We do this for large β , keeping correction terms of order $1/\beta$, and for frequencies well below the ion cyclotron frequency ω_i . This solution will give us the behavior of hydro-magnetic waves in large β plasmas.

We reduce the number of independent parameters by introducing new dimensionless variables. We choose these new variables so that the dispersion relation becomes independent of β as β approaches infinity. From the discussion in the introduction, one sees that the proper unit of frequency should be ω_{ci}/β , while that of wave

number should be $\omega_{ci}/\beta v_A$. We thus define variables ν , ℓ and β by

$$\nu \equiv \frac{\omega}{k_0 v_A}, \quad \ell \equiv \frac{k \cos \theta}{k_0}, \quad \beta \equiv \beta_i \equiv \frac{8\pi nT}{B^2} = \frac{v_i^2}{v_A^2}, \quad (2)$$

where

$$k_0 \equiv \frac{\omega_{ci} v_A}{v_i^2}, \quad v_i^2 = \frac{2T}{m}, \quad v_A = \frac{B}{(4\pi nm)^{1/2}}, \quad (3)$$

$\omega_{ci} = eB/m_c$ is the ion-cyclotron frequency, and m is the ion mass. (Note: β refers to the ion β). We also treat the electron-ion mass ratio as infinitely small. Thus, we wish to solve for ν as a function of ℓ , θ and β . It will appear shortly that ν is a finite function of ℓ and $\theta\beta^{1/4}$ as β approaches infinity. We will solve for this function accurate to $1/\beta$.

Anticipating this result we may transform the various terms of the determinant D , expressing them in terms of the new variables and expanding them as series in $1/\beta$, or $1/\sqrt{\beta}$. This produces considerable simplification of these terms. For example, the argument of all the Bessel functions I_n , which occur in the matrix elements of the dielectric tensor K_{ij} in Appendix A, is $\mu \equiv k_{iv}^2/2\omega_i^2 = (\tan^2 \theta) \ell^2/\beta = (\theta\beta^{1/4})^2 \ell^2/\beta^{3/2}$. Thus, for finite ℓ and $\theta\beta^{1/4}$ this argument is of order $\beta^{-3/2}$ and we need keep only the leading terms in the expansion of the Bessel functions,

$$I_n(\mu) \approx \frac{\mu^n}{2^n n!} \left(1 + \frac{\mu^2}{4(n+1)} \right). \quad (4)$$

Further, we may also neglect the higher harmonic terms in the summation over n , the order of the Bessel function. We never

need terms with n greater than 2. A similar remark applies to the arguments which occur in the plasma dispersion function Z . For the argument occurring in the $n=0$ term we have,

$$z_0 = \frac{\omega}{k_{\parallel} v_i} = \frac{v}{v \sqrt{\beta}} \quad , \quad (5)$$

which is small compared to one. In this term we may employ the small argument expansion of the Z function (Fried and Conte, 1961),

$$Z(z_0) = i\sqrt{\pi} - 2z_0 \quad , \quad z_0 \ll 1 \quad . \quad (6)$$

Further, in the higher harmonic terms, $n \neq 0$, the argument z_n is given by:

$$z_n = z_0 - \frac{n\omega_i}{k_{\parallel} v_i} = \frac{v}{v \sqrt{\beta}} - \frac{n\sqrt{\beta}}{\ell} \quad , \quad (7)$$

which is large compared to one. For these terms we may employ the large argument expansion of the Z function (Fried and Conte, 1961),

$$Z(z_n) \sim -\frac{1}{z_n} - \frac{1}{2z_n^3} \quad , \quad z_n \gg 1 \quad . \quad (8)$$

With these simplifications we may expand all the terms in the dispersion relation D as a series in $1/\beta$, to a sufficiently high order that we may solve for v correct to order $1/\beta$. These expansions are,

$$D_{11} = 4\pi K_{11} - \frac{k_{\parallel}^2 c^2}{\omega^2} = \frac{c^2}{v_A^2} \left[1 + \frac{v^2}{\beta^2} + \frac{\ell^2}{\beta^2} \left(\frac{3}{2} - \frac{3}{8} \tan^2 \theta \right) - \frac{\ell^2}{\beta^2} \right], \quad (9)$$

$$D_{12} = 4\pi K_{12} = i \frac{c^2}{v_A^2} \left[\frac{v}{\beta} + \frac{\ell^2}{\beta^2} \left(\frac{1}{2} - \frac{3}{4} \tan^2 \theta \right) \right], \quad (10)$$

$$D_{22} = 4\pi K_{22} - \frac{k_{\perp}^2 c^2}{\omega^2} = \frac{c^2}{v_A^2} \left[1 + \frac{v^2}{\beta^2} + \frac{\ell^2}{\beta^2} \left(\frac{3}{2} - \frac{11}{8} \tan^2 \theta \right) + \frac{1}{\beta^2} \frac{1}{\sqrt{\beta}} \tan^2 \theta - 2 \tan^2 \theta - \frac{\ell^2}{\beta^2} \left(1 + \tan^2 \theta \right) \right], \quad (11)$$

$$D_{13} = 4\pi K_{13} + \frac{k_{\parallel} k_{\perp} c^2}{\omega^2} = \frac{c^2}{v_A^2} \left(-\frac{\ell^2}{\beta^2} \tan \theta + \frac{\ell^2}{\beta^2} \tan \theta \right), \quad (12)$$

$$D_{23} = 4\pi K_{23} = \frac{c^2}{v_A^2} \left[\frac{\sqrt{\pi} \beta^{1/2}}{\ell} - \frac{2i v}{\ell^2} + \frac{i \ell^2}{v} \left(\frac{1}{2} - \frac{3}{4} \tan^2 \theta \right) \right] \tan \theta, \quad (13)$$

$$D_{33} = 4\pi K_{33} - \frac{k_{\perp}^2 c^2}{\omega^2} = \frac{c^2}{v_A^2} \left(\frac{4\beta}{\ell^2} + \frac{2i \sqrt{\pi} \beta^{1/2} v}{\ell^3} - \frac{i \ell^2}{v^2} \tan^2 \theta \right). \quad (14)$$

We refer to the determinant D of equation (1) with elements given by equations (9) - (14) as the reduced dispersion relation (we have actually kept more terms than are strictly necessary in order that we be able to compare with the corresponding expressions for a cold plasma).

The largest term in the determinant is the 3-3 term, equation (14). $4\pi K_{33}$ is the parallel dielectric constant; because of its large magnitude, the parallel electric field is suppressed. Consequently, the dispersion relation to the lowest significant order in β is, as usual, simply the 2×2 determinant in the upper left-hand corner of D . For this reason, it is adequate to keep a smaller

number of terms in the 1-3 and 2-3 elements of the determinant in order to get ω correct to order $1/\beta$.

The first two terms of D_{11} and D_{22} , equations (9) and (11), are identical with those for $4\pi K_{11}$ and $4\pi K_{22}$ in a low- β plasma, while the last terms are the electromagnetic terms. The remaining terms are due to the large ion temperature. In these expressions the thermal corrections are relatively unimportant with the exception of the term $i\sqrt{\epsilon} \epsilon \beta^{1/2} \tan^2 \theta / \omega$ in equation (11). This is important enough to affect wave propagation when $\omega \beta^{1/4}$ is of order unity. This term represents the Landau damping of the waves by resonant ions travelling at the wave speed and interacting with the wave by means of the perturbation of the field strength, $\delta |B| = kc \sin \theta E_y / \omega$. In D_{12} , equation (10), the first term is identical with that for a low- β plasma. The second term represents the incomplete cancellation of the ion and electron $E \times B$ Hall currents due to the finite temperature of the ions. As anticipated by our choice of notation, this is important for wave propagation when ℓ and ω are of order unity.

III. SOLUTION OF THE DISPERSION RELATION

We wish to obtain the numerical solution of the dispersion relation, equation (1), with elements of the determinant D given by equations (9) - (14). This will give the dispersion relation, ω as a function of ℓ , θ and β , correct to order $1/\beta$.

Let us first write the dispersion relation to lowest order in $1/\beta$. This can be done by setting the 2×2 upper left-hand minor of D to zero;

$$\left(1 + \frac{i\sqrt{\pi}\ell\beta^{1/2}}{\nu} \tan^2 \theta - \frac{\ell^2}{\nu^2}\right) \left(1 - \frac{\ell^2}{\nu^2}\right) - \frac{\ell^4}{4\nu^2} = 0 \quad (15)$$

Rearranging terms we obtain the quartic equation,

$$(\nu^2 - \ell^2) + i\alpha(\nu^2 - \ell^2)\frac{\nu}{\ell} - \frac{\ell^4}{4\nu^2} = 0 \quad (16)$$

where

$$\alpha = \sqrt{\pi}\ell^{1/2}\nu^2 \tan^2 \theta \quad (17)$$

For $\theta = 0$, $\alpha = 0$ and the solution of equation (16) is

$$\nu = \pm \frac{\ell^2}{4} \pm \left(\ell^2 + \frac{\ell^4}{16}\right)^{1/2} \quad (18)$$

The roots clearly occur in positive and negative pairs. The two positive roots are plotted in Figure 3. If $\ell \ll 1$, these two roots are $\nu = \ell^2/4$ or,

$$\omega = kv_A \pm \frac{\beta}{4} \frac{(kv_A)^2}{\omega_{ci}} \quad (19)$$

Thus, in the small ℓ limit the wave propagation is essentially that of Alfvén waves in a cold medium with the difference that the dispersion in phase velocity is greater by a factor $\beta/4$.

For ℓ large compared to unity, the two roots are $\nu = 2$ and $\nu = \ell^2/2$, or in dimensional units,

$$\omega = 2\omega_{ci}/\beta \quad (20)$$

and

$$\omega = k^2 v_i^2 / 2\omega_{ci} \quad (21)$$

We shall call solution (20) the slow wave and solution (21) the fast wave.

From equations (19) - (21), it appears that there are three different classes of wave behavior and we shall employ three different numerical techniques to solve the dispersion relation corresponding to each of these cases.

A. Case 1 $l \ll 1$

For small v we see from equation (18) that $v \ll 1$ so we may write,

$$v = l + \delta v \quad (22)$$

Substituting this expression in the dispersion relation, we may write it in the form,

$$4l^2(\delta v)^2 + b_1 \delta v + c_1 = 0 \quad (23)$$

where

$$b_1 = \left(i\alpha + B_1 + \frac{\Gamma_1^2}{F_1} + A_1 - \frac{D_1^2}{F_1} \right) \quad (24)$$

$$c_1 = \left(A_1 - \frac{D_1^2}{F_1} \right) \left(i\alpha + B_1 \right) + C_1^2 + \frac{A_1 \Gamma_1^2 + 2C_1 D_1 E_1}{F_1} \quad (25)$$

and the expressions for A_1 through F_1 are listed in Appendix B. The notation is chosen so that A_1 and B_1 represent corrections to $v^2 v_{A11}^2 / c^2$ and $v^2 v_{A22}^2 / c^2$ after the leading terms have been removed. C_1 is $v^2 v_{A12}^2 / c^2$ and equals $il^3/2$ to lowest order; D_1 and E_1 are $v^2 v_{A13}^2 / c^2$ and $v^2 v_{A23}^2 / c^2$. F_1 is $v^2 v_{A33}^2 / c^2$, which

is very large. The expressions $A_1 - F_1$ may be computed once an estimate of δv is known. They may be introduced into equations (24) and (25) to find b_1 and c_1 , which in turn may be employed in equation (23) to get a better estimate of δv .

This is our basic iteration scheme. It clearly could be used to solve the full dispersion relation as well as our reduced form. To get started we take A_1, B_1, D_1 and E_1 zero and replace C_1^2 by $-\ell^6/4$, so the equation for the first iterate is

$$4\ell^2 \delta v_0^2 + 2i\ell\alpha\delta v_0 - \frac{\ell^6}{4} = 0 \quad . \quad (26)$$

There are two roots of equation (26) and each one gives rise to a different set of iterates, which converge to different branches of the dispersion relation.

The iteration scheme converges rapidly for small values of ℓ and the convergence is moderately good even for finite values of ℓ so that our iteration scheme actually overlaps that of the other cases as we shall see.

B. Case 2: $\ell \gg 1$, The Slow Mode

For this mode we choose our basic iteration equation again as a quadratic equation,

$$\frac{1}{4}v^2 + b_2v + c_2 = 0 \quad , \quad (27)$$

where

$$b_2 = \left[\left(1 - A_2 + \frac{D_2^2}{F_2^2} \right) \alpha - C_2 - \frac{D_2 E_2}{F_2^2} \right] i , \quad (28)$$

$$c_2 = -1 + A_2 + B_2 - A_2 B_2 - C_2^2 + \frac{E_2^2}{F_2^2} - \frac{A_2 E_2^2}{F_2^2} - \frac{2C_2 D_2 E_2}{F_2^2} + \frac{D_2^2 B_2}{F_2^2} - \frac{D_2^2}{F_2^2} . \quad (29)$$

Again the expressions $A_2 - F_2$ are defined in Appendix B. They are analogous to $A_1 - F_1$ of case 1 but with different leading terms removed. They can all be evaluated once v is known and lead to an equally convenient iteration scheme. To get started we take only the leading terms of b_2 and c_2 , so the equation for the first iterate is,

$$\frac{1}{4} v_0^2 + i \alpha v_0 - 1 = 0 . \quad (30)$$

We take $v_0 = -2i\alpha + 2(1-\alpha^2)^{1/2}$ as our first iterate. The iteration scheme converges well for ℓ large and moderately well for ℓ of order unity.

C. Case 3: $\ell \gg 1$, The Fast Mode

For this case we choose as the iteration equation the quadratic equation,

$$\frac{1}{\ell} v^2 + b_3 v + c_3 = 0 , \quad (31)$$

where

$$b_3 = \left(i\alpha + B_3 + \frac{E_3^2}{F_3} + \Lambda_3 - \frac{D_3^2}{F_3} \right) \frac{1}{2} \quad (32)$$

$$c_3 = \left(\Lambda_3 - \frac{D_3^2}{F_3} \right) \left(i\alpha + B_3 \right) + C_3^2 + \frac{\Lambda_3 E_3^2}{F_3} + \frac{2C_3 D_3 E_3}{F_3} \quad (33)$$

and where again $\Lambda_3 - F_3$ are given in Appendix B. They have a definition analogous to $\Lambda_1 - F_1$ and $\Lambda_2 - F_2$.

For this case, our equation for the first iterate is,

$$\frac{1}{4} \omega^2 + \frac{i\alpha}{2} \omega - \frac{1}{4} = 0 \quad (34)$$

Again the iteration scheme converges well for even moderate size ω 's.

It should be emphasized that each of our three iteration equations for ω , equations (23), (27), and (31), with appropriate definitions of the coefficients is fully equivalent to our original reduced dispersion relation. Thus, the solutions obtained by iterating them to convergence are exact solutions of the dispersion relation. The essential point is that for each parametric range, we rearrange the dispersion relation in a different manner in order to obtain a convergent iteration scheme. To do this we are guided by the infinite β equation, equation (16), in choosing dominant terms. The choice of the leading terms in the iteration equation, as exhibited in our equations for the initial iterates, equation (26), (30), and (34), is crucial for convergence. The distribution of the remaining terms between the linear and constant terms is arbitrary.

We have chosen a quadratic equation for our basic equation rather than a linear equation, as one might do for Newton's method,

because often the two roots are quite close together as in the case corresponding to equation (19).

As an indication of the convergence rate, we show in Figure 2 the various iterates as a function of iteration step for $\theta = 0$, and two values of β , 1.0 and 3.0, using the iteration method of case 1.

IV. RESULTS

The iteration methods of section 3 were used to derive the propagation and damping rates of hydromagnetic modes for large β plasmas. When $\theta = 0$ the waves propagate parallel to $\underline{B}$ and are undamped under our approximation, $\omega \ll \omega_{ci}$ [the number of ions that can cyclotron damp the wave is of order $\exp(-\beta/\ell^2)$]. For $\theta = 0$, we have derived ω (real) as a function of ℓ for various values of β . The two solutions are plotted in Figure 3 for $\beta^{1/2} = 1000$ (solid line) and $\beta^{1/2} = 10$ (dashed line).

For β less than unity, the waves propagate at the Alfvén speed v_A , but there is considerable dispersion. The upper branch is a right circularly polarized wave, while the lower curve is left circularly polarized. For β larger than unity, the speeds of propagation of the two waves are very different from the Alfvén speed. For the left circularly polarized (slow) wave, the frequency approaches a constant value. By examining the solution to lowest order in $1/\beta$, we can see that the Hall current in this wave is much larger than the polarization current, and it is this current which produces the perturbed magnetic field. For this reason the wave propagates more slowly than the Alfvén speed (more current is produced for the same electric field, as though the plasma were cold

and denser). For the right circularly polarized (fast) wave, the two currents are of the same order of magnitude and essentially cancel. Since the net current is smaller for the same electric field, the plasma behaves the same as a lighter cold plasma. Thus, the wave propagates at a speed greater than the Alfvén speed.

The propagation curves for different values of B lie quite close to each other, with the exception of the upper $\beta^{1/2} = 10$ curve at large values of ℓ . But as pointed out above, this wave should be cyclotron damped if ℓ is of order $\beta^{1/2}$ so that this part of the curve is not strictly correct.

Next we consider propagation for θ nonzero. For these cases the damping rates of the waves are of primary interest. In Figure 4 we give the damping rates of the waves, $-\text{Im}(\nu)$, as a function of the scaled angle $\theta\beta^{1/4}$ for $\ell = 0.5$, a relatively small value of ℓ . As can be seen from the figure, the two roots have nearly the same damping rate up to $\theta\beta^{1/4} = 0.532$. At larger angles the modes become more nearly linearly polarized. The one with $\underline{E}$ polarized in the y direction, the magnetoacoustic mode, becomes considerably more strongly damped. The other shear mode, with $\underline{E}$ polarized primarily in the $x-z$ plane, becomes more weakly damped. It suffers some damping however, because it has a small y component.

As θ increases the damping effect on the wave propagation of the magnetoacoustic mode increases, until at $\theta\beta^{1/4} = 1.07$, the real part of ν vanishes. At this angle of propagation, the wave joins its partner, which propagates in the negative direction. At larger values of θ they become two nonpropagating damped waves. The behavior is more clearly seen in Figure 5, in which we plot ν in

the complex plane with parameter $\delta\beta^{1/4}$. Note that for very large $\delta\beta^{1/4}$ this behavior with $\delta\beta^{1/4}$ occurs at very small values of the actual propagation angle θ . For larger values of θ there is, in addition to a weakly damped shear Alfvén wave propagating at the Alfvén speed, also a weakly damped nonpropagating wave which is a remnant of the magnetoacoustic mode.

In deriving the curves of Figure 4, we were able to obtain the curves up to the cusp at $\delta\beta^{1/4} \approx 0.532$ with the iteration method of case 1. Near the cusp the iteration process converged very slowly, probably because the roots were so nearly equal. For larger angles the method of case 1 again converged reasonably rapidly. The entire lower curve (the shear Alfvén wave) was given by this method. For $\delta\beta^{1/4}$ larger than the value at the second cusp, 1.07, $\delta\omega$ became larger than ω and the method of case 1 failed to converge. We were able to get the more strongly damped branch by employing the method of case 3 and the less strongly damped branch by the method of case 2. Thus, by a correct choice of method we were able to obtain the entire diagram of damping curves for $\delta = 0.5$. However, in order to achieve a reasonable convergence rate, it was necessary to modify method 3 by treating the constant term as known, and to modify method 2 by treating the quadratic term as known. The actual method for each part of the curve is indicated by numbers 1, 2, and 3 in Figure 4. The modified versions of methods 2 and 3 are indicated by 2' and 3' respectively.

The actual curves in Figures 4 and 5 were computed for $\beta^{1/2} = 10^3$. For comparison we sketch, as a dashed line in Figure 4, the damping rates for $\beta^{1/2} = 10$. Again we see that the two curves lie close

together.

Finally, we consider the damping rates for waves with relatively large λ . In Figure 6 we give the imaginary part of ω for $\lambda = 10$. For this value of λ the two waves are quite distinct in their properties, as can be seen in Figure 3. The fast mode is the more strongly damped one. Both modes become nonpropagating at $\theta\beta^{1/4} = 2.36$. Above this value we have four nonpropagating waves with four different damping rates. At the angle $\theta\beta^{1/4} = 2.83$ two of the nonpropagating modes combine to form propagating damped modes again. Beyond this angle the damping rates of these propagating modes gradually decrease with $\theta\beta^{1/4}$. One of the nonpropagating modes becomes increasingly strongly damped with increasing θ , while the other mode becomes more weakly damped. The behavior of the modes with $\theta\beta^{1/4}$ is again displayed in the complex ω diagram of Figure 7.

For large wave numbers, there are some distinct differences in wave propagation from that for small wave numbers. The polarization is never linear but always elliptical. For this reason, the ratio of the damping rates to the wave frequencies, for the two polarizations, are more nearly comparable than those for small wave numbers. For the larger wave numbers, there is actually a range of angles for which no wave propagation occurs.

The actual iteration methods used in deriving the various parts of Figure 6 are again indicated by numbers on the curves.

V. ASYMPTOTIC RESULTS

The numerical results of section 4 and Figures (3) - (7) give a qualitative picture of the wave propagation properties. For values of λ other than 0.5 and 10, one may, of course, apply the

same iteration technique to derive numerical values for v . From the results for the two cases, $\ell = 0.5$ and $\ell = 10$, we can see that for β larger than 100 the results essentially agree with β infinite. The value of v for β infinite can be found from solution of the quartic equation, equation (16). We give here some asymptotic solutions of this equation for small and large values of ℓ , and for different ranges of α . The derivation of these results is given in Appendix C.

A. Small ℓ

We treat α as of order ℓ^p as ℓ approaches zero. If α is of order ℓ^3 , ℓ small, then,

$$v = \ell - \frac{i\alpha}{4\ell} \pm \frac{1}{4\ell} \left(\ell^6 - \alpha^2 + \frac{4i\alpha^3}{\ell^2} \right)^{1/2}. \quad (35)$$

The first cusp in Figure 4 corresponds to the point where the square root nearly vanishes, $\alpha = \ell^3$. Since the last term in the square root is small, the damping rates are nearly equal for $\alpha < \ell^3$. For $\alpha > \ell^3$ the square root is imaginary (neglecting the last term) and the damping rates are quite different. Equation (35) represents the solutions well for α smaller than its value at the second cusp.

If α is of order ℓ^2 , ℓ small, then,

$$v = -\frac{i\alpha}{2\ell} \pm \left(\ell^2 - \frac{\alpha^2}{4\ell^2} \right)^{1/2}, \quad (36)$$

or,

$$v = \ell - \frac{i\ell^5}{8\alpha}. \quad (37)$$

The first expression corresponds to the upper curve or curves of Figure 4 and the second expression corresponds to the lower curve. If the square roots are expanded out for small α/ℓ^2 , it is found that they overlap a similar expansion of equation (35) for large α/ℓ^3 . Thus, equations (35) - (37) represent the whole of Figure 4 smoothly. They are indicated in Figure 4 by short dashed lines; there is no noticeable difference from the numerical results represented by the solid curves.

B. Large ℓ

If α is of order ℓ^3 , large ℓ , then,

$$v = -\frac{i\alpha}{2\ell} \pm \frac{1}{2} \left(\ell^4 - \frac{\alpha^2}{2} \right)^{1/2}, \quad (38)$$

$$v = -\frac{2i\alpha}{\ell^3} \pm 2 \left(1 - \frac{\alpha^2}{\ell^3} \right)^{1/2}. \quad (39)$$

Both of the cusps in Figure 6 correspond to the vanishing of the square roots at $\alpha = \ell^3$.

If α is of order ℓ^4 , ℓ large, then,

$$v = -\frac{i\alpha}{2\ell} - \frac{1}{2} \left(\ell^4 - \frac{\alpha^2}{2} \right)^{1/2}, \quad (40)$$

$$v = -\frac{i\ell^5}{8\alpha} \pm \left(\ell^2 - \frac{\ell^{10}}{64\alpha^2} \right)^{1/2}, \quad (41)$$

$$v = -\frac{2i\alpha}{\ell^3} + 2 \left(1 - \frac{\alpha^2}{\ell^6} \right)^{1/2}. \quad (42)$$

Equations (40) and (42) are identical to equation (38) with a minus sign before the square root, and equation (39) with a plus sign. Equation (41) agrees with the other two roots in the intermediate region $\ell^3 \ll \alpha \ll \ell^4$. Thus, equations (38) - (42) represent the whole of Figure 6. The numerical values for $\ell = 10$ are plotted in Figure 6 as short dashed lines. It is seen that the agreement while fair is not as good as that found for the $\ell = 0.5$ case. This is because ℓ is actually compared with four rather than one when discussing its magnitude. On the basis of this comparison, ℓ is actually quite small when it equals 0.5, while ℓ is not really so large when it equals 10.

It is of interest to compare these asymptotic results with those of Stepanov (1958). Stepanov's equations (31) and (32), are

$$\left(\frac{Y}{\omega}\right)_{1,2} = \left(\frac{\pi}{8}\right)^{1/2} \beta_i n_A \theta^2 \quad \text{for} \quad \frac{1}{u_i} > \frac{\pi \theta^4}{2 \beta_i n_A^2}, \quad (43)$$

$$\left(\frac{Y}{\omega}\right)_{1,2} = \left(\frac{\pi}{8}\right)^{1/2} \beta_i n_A \theta^2 \pm \frac{1}{2} \left(\frac{\pi}{2} \beta_i^2 n_A^2 \theta^4 - \frac{\pi^2 n_A^4}{u_i^2} \right) \quad \text{for}$$

$$\frac{1}{u_i} < \frac{\pi \theta^4}{2 \beta_i n_A^2}, \quad (44)$$

and when reduced to our notation, agree with our equation (35) and equation (37) respectively for $\alpha < \ell^3$. Thus, Stepanov's results for high β plasmas can be summarized as analytic approximations to the lowest branch of Figure 4. He has also given the behavior of this branch for θ of order unity, a result not included in our paper.

VI. TWO APPLICATIONS

As an illustration of how to employ our results, we consider two problems. The first is the hydromagnetic wave instability for cosmic rays trapped around supernovae. The second is the possible effect of our damping processes on collisionless hydromagnetic turbulence.

A. Cosmic Rays in Supernovae Remnants

It has been suggested that cosmic rays are produced in supernova explosions (Ginzburg and Syrovatskii, 1964). An important objection to this proposal is that the cosmic rays could be trapped by plasma instabilities in the expanding bubble of shock heated interstellar plasma surrounding the supernova remnant (Kulsrud and Zweibel, 1975). The trapping along the interstellar magnetic field in the bubble is accomplished by hydromagnetic waves in cyclotron resonance with the cosmic rays. These waves scatter the cosmic rays in pitch angle, preventing them from freely flowing along the field lines and escaping. If the cosmic rays are actually trapped in the expanding bubble, they will lose essentially all their energy by the mechanism of adiabatic decompression.

In order to decide whether the cosmic rays are trapped, it is necessary to compare the amplification of the hydromagnetic waves, by any streaming of the cosmic rays, with the various damping processes to which the waves are subject. Because the shocked interstellar plasma has a very large pressure compared to that of the compressed interstellar field, the plasma is a high β one and the results of our paper may be used to determine the wave damping

rates.

The hydromagnetic waves of interest are those which satisfy the resonance condition $k \cos \theta \approx c/v_i$, for interaction with cosmic rays with energy γmc^2 . This growth rate Γ is roughly,

$$\Gamma_{\gamma} = \omega_{ci} \frac{n_{cr}(E > \gamma mc^2)}{n_0} \frac{v_D}{v_A} \quad v_D \gg v_A \quad (45)$$

for $k \cos \theta = \omega_{ci}/\gamma c$ (Kulsrud and Cesarsky, 1971) where $n_{cr}(E > \gamma mc^2)$ is the number density of cosmic rays whose energy exceeds γmc^2 , n_0 is the background plasma density and v_D is the bulk velocity of the cosmic rays along the magnetic field. If the cosmic rays escape, we must set v_D equal to the shock speed v_s , of the edge of the expanding bubble. The growth rate, equation (43), is roughly independent of the propagation angle of the wave.

In order to compare it with the collisionless damping rates, we convert equation (43) to our units. The value of Γ for the resonant waves is, by equations (2) and (3), $\Gamma = kv_i^2/\gamma v_i v_A = v_i^2/\gamma c v_A$. For the shocked plasma, v_i is of order v_s . We therefore have, with $v_D = v_s = v_i$,

$$\Gamma_{\gamma} \approx \frac{\Gamma}{k_0 v_A} \approx \beta^{3/2} \frac{n(E > \gamma mc^2)}{n_0} \quad (45)$$

and Γ_{γ} is to be compared directly with $-\text{Im}(\nu)$ from our results.

Because $-\text{Im}(\nu)$ is zero for $\theta = 0$, we see that parallel propagating waves are always unstable according to our criteria. Once the plasma properties and cosmic ray densities are specified, the range of propagation angles for unstable waves can be determined for

any radius of the bubble and for waves interacting with cosmic rays of any energy. We do not carry out these calculations, but illustrate them by two numerical examples.

We first consider a bubble which has expanded to $3 \text{ pc} = 10^{19} \text{ cm}$ and assume the shock speed v_s is 10^9 cm sec^{-1} , the interstellar density n_0 is 1 cm^{-3} and the compressed interstellar magnetic field B is 10^{-5} gauss at this time. Then $\beta = 2.5 \times 10^5$, $v_A = 2 \times 10^6 \text{ cm sec}^{-1}$ and ℓ , for waves interacting with a $\gamma = 1$ cosmic ray proton, is 10. Assuming there are 10^{52} cosmic rays with $\gamma > 1$ in the bubble (Kulsrud and Zweibel, 1975), we obtain from equation (43), $\Gamma = 2 \times 10^{-4} \text{ sec}^{-1}$, so $v_\gamma = \Gamma/k_0 v_A = 400$. Comparison with Figure 6 shows that all the modes are unstable except the uppermost branch. From our asymptotic equation, equation (40), for this branch we get instability for $\alpha < 8000$ or $\theta < 18^\circ$.

As a second example, let us determine the radius at which $\beta = 1/2$ for waves resonating with $\gamma = 1$ protons. From the Sedov solution, v_s decreases as $R^{-3/2}$ (Landau and Lifshitz, 1959). Then $v_s = 2 \times 10^8 \text{ cm sec}^{-1}$, $\beta = 10^4$ and $\ell = 1/2$ when $R = 7 \text{ pc}$. At this radius the growth rate $\Gamma_{\gamma=1} = 2 \times 10^{-6} \text{ sec}^{-1}$ and $v_{\gamma=1} = 0.14$. From Figure 4 we see that the lowest branch is always unstable, while the branch above the first cusp is unstable for $\theta \beta^{1/4} < 0.6$ or $\theta < 0.06$ radius. For larger θ a second branch becomes unstable for $\theta \beta^{1/4} > 1.45$ or $\theta > 0.15$ radius.

B. Collisionless Hydrodynamic Turbulence

Consider a turbulent plasma with a very weak field, too weak to exert any significant force on the fluid. The turbulent shear

motions may be considered as waves in the plasma subject to the collisionless damping processes of our paper. If these damping rates are as rapid as the transfer rate of energy from eddy to eddy, then energy will be converted to heat and the turbulent spectrum will be prevented from reaching short wavelengths.

To estimate the effect of wave damping, we employ the Kolmogoroff theory of turbulence as a series of eddies, which we consider to be waves whose wave numbers are of magnitude k , and whose energies (per logarithmic band width in k) are $\rho \tilde{v}_k^2$ (Chandrasekhar, 1949). The energy is assumed to be transmitted to smaller eddies at a rate $k \tilde{v}_k$. We wish to compare this with our wave damping rate. The actual effective value of B depends on the amount of field generated by the turbulence itself. It certainly cannot be taken to be uniform, but at the cutoff eddy size it is plausible to take it as the field of the next larger eddy size. For simplicity we will leave the field as an undetermined parameter.

For a Kolmogoroff spectrum $k \tilde{v}_k$ is proportional to $k^{2/3}$. However, from equations (36) or (40), the most rapidly damped mode has $-\text{Im}(\nu)$ increasing more rapidly than k , at fixed θ . Thus, we see that for any sufficiently small wavelength the damping rate must exceed the turbulent transfer rate for some range of angles of propagation. If one whole mode of propagation and a part of the other are damped more rapidly than the turbulent transfer rate, then it is plausible to assume that the transport of energy to smaller eddies is suppressed. We take as our cutoff criterion that the turbulent transport rate equals the damping rate at the first cusp, in Figure 4 or 6. From equations (2), (3), (35), and (39),

$-\text{Im}(\omega) = k^2 v_A / k_0$ for $k < k_0 \equiv \omega_{ci} v_A / v_i^2$ and $-\text{Im}(\omega) = 2k_0 v_A$ for $k > k_0$. Thus,

$$-\text{Im}(\omega) = \max\left(\frac{k^2 v_i^2}{\omega_{ci}}, \frac{2\omega_{ci}}{\rho}\right) \quad (47)$$

If B is constant (with k), then the damping increases monotonically with k as k^2 , while the turbulent transfer rate increases as $k^{2/3}$ (for a Kolmogoroff spectrum). Thus, the turbulent transport is cut off for $k > k_c$, where k_c satisfies $k_c \tilde{v}_c = -\text{Im}(\omega)_c$. Let us take the Kolmogoroff spectrum as,

$$\tilde{v}_k = \epsilon^{1/2} \left(\frac{K}{k}\right)^{1/3}, \quad (48)$$

where ϵ is the energy density per unit mass of the turbulence at the "longest" wave number K . Then from equations (47) and (48) we see a cutoff is possible if,

$$K r_i^4 \left(\frac{\epsilon}{v_i^2}\right)^{3/2} < 8, \quad (49)$$

where $r_i \equiv v_i / \omega_{ci}$ is the ion gyroradius. If condition (49) is satisfied, then the cutoff wave number is,

$$k_{\text{cutoff}} = K^{1/4} r_i^{-3/4} \left(\frac{\epsilon}{v_i^2}\right)^{3/8}. \quad (50)$$

There is some ordinary viscous damping of this hot turbulence, but the magnetic field suppresses the transverse viscosity so the damping rate produced by it is of order $k^2 r_i^2 v_i / \lambda$, where λ is

the mean-free path. This is smaller than that of equation (50) if $\lambda > r_i$. Thus, our wave damping is probably the important mechanism for cutting off turbulence in an essentially collisionless plasma.

VII. SUMMARY AND CONCLUSION

We have determined the propagation and damping characteristics of low frequency (hydromagnetic) waves in collisionless high β plasmas. Such plasmas occur in nature and in the laboratory under conditions where the confinement is inertial or gravitational rather than magnetic. The essential differences in behavior from cold plasmas are produced by Doppler shift effects, which destroy the rather close cancellation of ion and electron currents in cold plasmas. Also, the fact that many ions travel at the propagation speed leads to strong damping effects.

We express our results in dimensionless units and plot the propagation and damping rates for two characteristic wavelengths, one long and one short. These are given in Figures (3) - (7). These curves were obtained numerically from solutions of the full collisionless dispersion relation, by employing an iteration method based on ordinary asymptotic analysis. Results for other wavelengths are given by approximate formulas, equations (35) - (42).

We found that the behavior is essentially different from that of a cold plasma in many ways, so that considerable care must be taken in treating hydromagnetic wave propagation in such high β plasmas.

To illustrate our results two applications were considered. The first was the two-stream instability in a supernova remnant, caused by the possible presence and escape of cosmic rays produced by the supernova. It was found that the stability criterion was modified, and some hydromagnetic waves propagating at large angles to the magnetic field were stable. However, enough waves were still unstable, even under these modified conditions, to prevent escape of cosmic rays from the remnant.

The second application was an attempt to determine the effect that a weak magnetic field would have on the cutoff radius of collisionless hydromagnetic turbulence. Employing arguments similar to Kolmogoroff's to describe the turbulence, we showed that the turbulent energy is in the form of hydromagnetic waves which transform their kinetic energy to heat. This leads to a cutoff radius at an eddy size considerably in excess of the cutoff by ordinary molecular viscosity. An attempt was made to roughly estimate this cutoff radius.

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APPENDIX A: THE DIELECTRIC CONSTANTS

In this Appendix we list the quantities K_{ij} in the notation of Akhiezer et al. (1967) page 29. In these expressions the following definitions are employed, where α is a species label;

$$\omega_p^2 = \frac{4\pi n_\alpha e^2}{m_\alpha} \quad , \quad \omega_{ce} = \frac{|e_\alpha| B}{m_\alpha c} \quad , \quad (A1)$$

$$v_\alpha^2 = \frac{2T_\alpha}{m_\alpha} \quad , \quad \mu = \frac{(k_\perp v_\alpha)^2}{2\omega_\alpha^2}$$

$$z_n = y_n = \frac{\omega - n\omega_\alpha}{k_\parallel v_\alpha} \quad , \quad k_\parallel = k \cos \theta \quad , \quad k_\perp = k \sin \theta \quad . \quad (A3)$$

$Z(z)$ is the plasma dispersion function (Fried and Conte, 1961) defined by,

$$Z(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp(-x^2)}{x-z} dx \quad , \quad (A4)$$

in the upper half z plane and by analytic continuation on the axis and in the lower half plane. $I_n(\mu)$ is the modified Bessel function and ϵ_α is $e_\alpha/|e_\alpha|$. In this notation $K_{ij} = \sum_\alpha K_{ij}^\alpha$ where,

$$K_{11}^\alpha = \frac{i\Omega_\alpha^2 z_0}{4\sqrt{\pi}\omega} \exp(-\mu) \sum_{n=-\infty}^{\infty} n^2 I_n^2 \frac{Z(z_n)}{i\sqrt{\pi}} \quad , \quad (A5)$$

$$K_{22}^\alpha = \frac{i\Omega_\alpha^2 z_0}{4\sqrt{\pi}\omega} \exp(-\mu) \sum_{n=-\infty}^{\infty} \left[\left(\frac{n^2}{\mu} + 2\mu \right) I_n^2 - 2\mu I_n' \right] \frac{Z(z_n)}{i\sqrt{\pi}} \quad (A6)$$

$$K_{33}^{\alpha} = \frac{\Omega_{\alpha}^2}{2\pi\omega} \left(y_0^2 + i\sqrt{\pi} z_0 \exp(-\mu) \sum_{n=-\infty}^{\infty} y_n^2 I_n \frac{Z(z_n)}{i\sqrt{\pi}} \right) , \quad (A7)$$

$$K_{12}^{\alpha} = \frac{i\Omega_{\alpha}^2 z_0}{4\sqrt{\pi}\omega} \exp(-\mu) \sum_{n=-\infty}^{\infty} n(I_n - I_n') \frac{Z(z_n)}{i\sqrt{\pi}} , \quad (A8)$$

$$K_{13}^{\alpha} = \frac{i\Omega_{\alpha}^2 z_0}{(8\pi)\omega} \exp(-\mu) \sum_{n=-\infty}^{\infty} n I_n y_n \frac{Z(z_n)}{i\sqrt{\pi}} , \quad (A9)$$

$$K_{23}^{\alpha} = \frac{i\Omega_{\alpha}^2 z_0 y_0'}{(8\pi)\omega} \exp(-\mu) \sum_{n=-\infty}^{\infty} (I_n' - I_n) y_n \frac{Z(z_n)}{i\sqrt{\pi}} . \quad (A10)$$

APPENDIX B: THE COEFFICIENTS $A_1 - F_3$

In this Appendix we list the constants $A_1 - F_1$, $A_2 - F_2$ and $A_3 - F_3$ employed in the iteration equations (23) - (25), (27) - (29), and (31) - (33). As mentioned in the text, these coefficients essentially represent quantities proportional to the D_{ij} given by equations (9) - (14) with leading terms removed.

$$\underline{A_1 - F_1}$$

$$A_1 \equiv \frac{v_A^2}{c^2} v^2 D_{11} - 2\ell\delta v , \quad (B1)$$

$$B_1 \equiv \frac{v_A^2}{c^2} v^2 D_{22} - 2\ell\delta v - i\alpha , \quad (B2)$$

while C_1 , D_1 , E_1 , and F_1 are equal to $v_A^2 v^2 / c^2$ times D_{12} , D_{13} ,

D_{23} and D_{33} respectively.

$$\underline{A_2 - F_2}$$

$$A_2 = \frac{v^2}{\ell^2} \frac{v_A^2}{c^2} D_{11} + 1, \quad (B3)$$

$$B_2 = \frac{v^2}{\ell^2} \frac{v_A^2}{c^2} D_{22} + 1 - \frac{iv}{\ell^3}, \quad (B4)$$

$$C_2 = \frac{v^2}{\ell^2} \frac{v_A^2}{c^2} D_{12} - \frac{iv}{2}, \quad (B5)$$

while D_2 , E_2 , and F_2 are $v^2 v_A^2 / \ell^2 c^2$ times D_{13} , D_{23} , and D_{33} respectively.

$$\underline{A_3 - F_3}$$

$$A_3 = \frac{v}{\ell^2} \frac{v_A^2}{c^2} D_{11} - \frac{v}{\ell^2}, \quad (B6)$$

$$B_3 = \frac{v}{\ell^2} \frac{v_A^2}{c^2} D_{22} - \frac{v}{\ell^2} - \frac{ia}{\ell^3}, \quad (B7)$$

while C_3 , D_3 , E_3 , and F_3 are $vv_A^2 / \ell^2 c^2$ times D_{12} , D_{13} , D_{23} and D_{33} respectively.

APPENDIX C: THE APPROXIMATE SOLUTION OF THE

$B = \infty$ DISPERSION RELATION

In this Appendix we wish to give the derivation of equations (35) - (42). Solving the quartic equation, equation (16), for v

for various assumptions concerning the magnitude of ℓ and α .

First consider $\ell \ll 1$, $\alpha \sim \ell^3$. Equation (16) is

$$(\nu^2 - \ell^2)^2 + i\alpha(\nu^2 - \ell^2)\frac{\nu}{\ell} - \frac{\ell^4 \nu^2}{4} = 0 \quad (C1)$$

In this equation the first term is of order ℓ^4 , the second ℓ^5 and the third ℓ^6 , since we know ν is of order ℓ . Thus, to lowest order $\nu = \ell$. Writing,

$$\nu = \ell + \nu_2 + \nu_3 \quad (C2)$$

where ν_2 is of order ℓ^2 and ν_3 of order ℓ^3 we find

$$4\ell\nu_2^2 + 2i\alpha\nu_2 - \ell^5/4 = 0 \quad (C3)$$

or

$$\nu_2 = - \frac{2i\alpha(4\ell^6 - 4\alpha^2)^{1/2}}{8\ell} \quad (C4)$$

Proceeding to higher order and making use of (C3) to simplify expressions in ν_2 we find that

$$(8\ell^2\nu_2 + 2i\ell\alpha)\nu_3 + \left(4\ell\nu_2^3 + 3i\alpha\nu_2^2 - \frac{\ell^5\nu}{2}\right) = 0 \quad (C5)$$

and after some simplification

$$\nu_3 = - \frac{(8\alpha^2 - 4\ell^6)\nu_2 + i\ell^5}{16\ell^2(4\ell^6 - 4\alpha^2)^{1/2}} \quad (C6)$$

Now we only need v_3 to distinguish between the damping of the two modes for $\alpha \sim \ell^3$. Thus, we keep only the imaginary part of v_3

$$\text{Im}(v_3) = - \frac{\alpha^3}{8\ell^3(4\ell^6 - 4\alpha^2)^{1/2}} \quad (C7)$$

As α approaches ℓ^3 , v_3 becomes infinite. However, by absorbing v_3 under the square root of v_2 we may remove this singularity and we obtain equation (35).

Next consider $\ell \ll 1$, $\alpha \sim \ell^2$. For this case we may treat the last term in (C1) as small and drop it. Cancelling the factor $v^2 - \ell^2$, we get the quadratic equation

$$v^2 - \ell^2 = - \frac{i\alpha v}{\ell} \quad (C8)$$

whose solution gives equation (36). For the other root we keep the last term and write

$$(v^2 - \ell^2) = \frac{1}{4} \frac{\ell^4 v^2}{(v^2 - \ell^2)^2 + i\alpha v/\ell} \approx - \frac{i\ell^6}{4\alpha} \quad (C9)$$

where we have set $v = \ell$ on the right side of (C9). Equation (C9) easily gives us equation (37).

Next consider $\ell \gg 1$, $\alpha \sim \ell^3$. For the fast mode we know v is of order ℓ^2 . Writing

$$v = v_2 + v_1 \quad (C10)$$

where $v_2 \sim \ell^2$, $v_1 \sim \ell$ we have

$$v_2^4 + \frac{i\alpha}{4} v_2^3 - \frac{\ell^4}{4} v_2^2 = 0 \quad . \quad (C11)$$

Excluding the root $v_2 = 0$, we find the solution of the quadratic, equation (38). For the slow mode we know v is of order ℓ^0 .

Writing

$$v = v_0 + v_{-1} \quad (C12)$$

where $v_0 \sim \ell^0$, $v_{-1} \sim \ell^{-1}$ we find

$$\ell^4 - i\alpha v_0 \ell - v_0^2 \frac{\ell^4}{4} = 0 \quad (C13)$$

whose solution is equation (39).

Finally, if $\ell \gg 1$ and α is of order ℓ^4 , we may guess the size of v by "extrapolating" equations (38) and (39). The minus sign in equation (38) gives v of order ℓ^3 and one sees that the terms in (C11) are still the leading ones, so equation (38) with the minus sign is still a correct solution, this is equation (40). The plus sign in equation (38), as well as the minus sign in equation (39), gives v of order ℓ . Writing

$$v = v_1 + v_0 \quad (C14)$$

we find from equation (C1) that

$$v_1^2 + \frac{i v_1 \ell^5}{4\alpha} - \ell^2 = 0 \quad . \quad (C15)$$

The solutions of this quadratic give equation (41). The plus sign in equation (39) indicates that there is a solution with v of order $1/\lambda$. But if this is the case, the terms of (C13) are the leading terms and equation (39) with a plus sign is still asymptotically correct. This is equation (42).

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Cartesian coordinates.

convergence of iterates for Case 1, two examples.
 n^{th} iterate.

n relation for parallel propagating waves ($\theta = 0$),
 vs $\ell \equiv \beta k v_A / \omega_i$. The solid curves correspond to
 d the long dashed curves to $\beta = 10^2$, while the
 hed curves represent the approximate solutions of
 (19) and (21).

ng rate, $-\text{Im}(v)$, as a function of $\theta \beta^{1/4}$ for $\ell = 0.5$.
 nations are the same as Figure 3. The short dashed
 present equations (35) - (37). The numbers on the
 dicate the iteration method used (see text).

v in the complex v plane with varying $\theta \beta^{1/4}$ for
 d $\beta = 10^6$. The numbers along the curves are the
 $\theta \beta^{1/4}$.

ng rate, $-\text{Im}(v)$ as a function of $\theta \beta^{1/4}$ for $\ell = 10$.
 gnations as Figure 4. The short dashed curves
 equations (38) - (42).

v in the complex v plane with $\ell = 10$, $\beta = 10^6$.
 gnation as Figure 5.

Fig. 1. 782337

Fig. 2. 792020

Fig. 3. 782339

Fig. 4. 782335

Fig. 5. 782338

Fig. 6. 782336

Fig. 7. 782334

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