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COLLECTIVE MODES IN THERMONUCLEAR PLASMAS

By

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FEBRUARY 1986

PLASMA
PHYSICS
LABORATORY

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PREPARED FOR THE U.S. DEPARTMENT OF ENERGY,
UNDER CONTRACT DE-AC02-76-CO-3073.

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Printed in the United States of America

Available from:

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Springfield, Virginia 22161

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<u>*Pages</u>	<u>NTIS Selling Price</u>
1-25	\$7.00
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51-75	\$10.00
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HIGH ENERGY COMPONENTS AND
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PPPL--2294

DE86 007574

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ABSTRACT

The theory of a class of collective modes of a thermonuclear magnetically confined plasma, with frequencies in the range of the ion cyclotron frequency and of its harmonics, is presented. These modes can be excited by their resonant cyclotron interaction with a plasma component of relatively high energy particles characterized by a strongly anisotropic distribution in velocity space. Normal modes that are spatially localized by the inhomogeneity of the plasma density are found. This ensures that the energy gained by their resonant interaction is not convected away. The mode spatial localization can be significantly altered by the magnetic field inhomogeneity for a given class of plasma density profiles. Special attention is devoted to the case of a spin polarized plasma, where the charged products of fusion reactions are anisotropically distributed. We show that for the mode of polarization that enhances nuclear reaction rates the tritium will be rapidly depolarized for toroidal configurations with relatively mild gradients of the confining magnetic field.

MASTER

I. INTRODUCTION

In a magnetically confined plasma, high frequency collective modes (in the ion cyclotron range of frequency) may be driven unstable by resonant interaction with a high energy ion population. Energetic ion populations can arise in thermonuclear plasmas either as a result of the injection of fast neutral beams or, more fundamentally, as the charged products of the fusion reactions. The presence of nonthermal features in the energetic ion distribution functions, such as nonmonotonicity and/or anisotropy in velocity space or spatial inhomogeneity, can be the source of the excitation energy that is necessary to sustain the growth of the mode amplitude.

We note that an analysis of the mode spatial structure that takes into account the inhomogeneity of a realistic plasma configuration is necessary in order to ascertain the effective occurrence of this type of instability. In fact, the energy transfer rate between the energetic ions and the modes, as derived in the limit of a homogeneous plasma model, is typically a relatively small quantity. Thus energy convection, which is often an important factor in the instability of modes in inhomogeneous configurations, or damping processes due to the bulk plasma species can significantly alter the energy balance. We therefore arrive at the conclusion that, in a toroidal plasma configuration, only radially localized normal modes, weakly damped by the bulk plasma species, are likely to be excited through the resonant interaction with a high energy ion population.

The original intent of our work was to obtain theoretical estimates of the polarization relaxation rate due to collective processes for a spin polarized plasma. Thus, a discussion of this issue is also included in the present paper. The interest in the physics of fusing plasmas with polarized nuclear spins has been rekindled recently by the analysis presented in Ref. 1. In particular, this analysis suggested that a thermonuclear plasma with coherently polarized nuclear spins could maintain its polarization for a time the order of, or longer than, the particle fusion time. Thus, control over the spin polarization of the fusing nuclei would give a series of advantages, ranging from a 50% enhancement¹ of the fusion rate in a D-T plasma to a more efficient confinement of the charged fusion reaction products. Additional advantages would be a more convenient redistribution of the neutron flux over the chamber walls, of relevance, for example, to the process of tritium breeding² in a lithium blanket, or possibly to the achievement of nearly neutron-free D-He³ burning conditions,³ although this possibility depends on the actual spin dependence of the D-D fusion cross section which is still under debate.^{4,5} An effective mechanism for causing a fast depolarization of the nuclear spin would be the possible presence in the plasma of collective magnetic fluctuations at a level above thermal. These can resonate with the nuclear spin precession frequency and thus induce spin-flip transitions (see Sec. 6). It was pointed out in Ref. 6 and 7 that magnetic fluctuations of this type can be excited in a polarized D-T plasma by an anisotropy in velocity space of the fusion

produced α -particles, this anisotropy being a consequence of the coherent polarization of the reacting D-T nuclei.

High energy ion populations are also produced in present-day experiments where neutral beams are injected to provide auxiliary heating. In fact, enhanced fluctuations in the ion cyclotron range of frequency have been recently observed in various devices⁸⁻¹⁰ during neutral beam injection experiments. An application of our theory to this case, however, would require a detailed study of the hot ion distribution function that results from neutral beam ionization processes and of its subsequent evolution. We consider this task beyond the scope of this paper. Our presentation is organized as follows:

In Sec. 2, the relevant modes are discussed in the limit of a homogeneous plasma model. Two different frequency ranges are considered for a plasma with two bulk ion species. In the lower range, the mode frequency is the order of the ion cyclotron frequency, and the dispersion relation is complicated by the presence of a cutoff and of a resonance frequency. In the upper range, the mode frequency corresponds to higher harmonics of the ion cyclotron frequency, and the dispersion relation reduces to that of magnetosonic waves. Subsequently, the presence of mode-particle resonances with the high energy ions and with the bulk plasma species is considered, and the relevant expression for the growth (damping) rate is derived.

In Sec. 3, the spatial structure of the modes is analyzed initially from a heuristic dispersion equation derived from the local dispersion relation given in Sec. 2, and then by gradually increasing the level of complexity, first in a cylindrical and

then in a toroidal configuration. We find that the radially localized normal mode solutions derived in a cylindrical configuration are also possible in a toroidal one, provided that the density profile does not belong to a given class. For density profiles in this class (of which a parabolic density profile is a remarkable example), the associated modes tend to drift out of the plasma column.

In Sec. 4, we adopt the geometrical optics approximation to describe the normal mode solutions found in Sec. 3 and to give a relatively clear graphic representation of various significant cases.

In Sec. 5, we evaluate the growth rate of the relevant modes in the case where the deuterium and tritium components of the fusing plasma are coherently spin polarized and the distribution of the produced α -particles is anisotropic in velocity space.

In Sec. 6, we evaluate the rate of spin depolarization as a function of the mode amplitude, and in Sec. 7 we summarize our conclusions.

II. HOMOGENEOUS MODEL

A. Mode Features and Dispersion Relation

In this section, we refer to a homogeneous, magnetized plasma with two bulk ion species (for definiteness, we shall consider a deuterium-tritium plasma), in which a small population of high energy ions is also present (we shall use a subscript 'h' to denote quantities referring to this 'hot' population).

We look for electromagnetic modes that can interact resonantly with the high energy ions and that are not damped, or at most only weakly damped, by the bulk plasma. Thus, we consider modes with a frequency ω close to one of the harmonics of the hot ion cyclotron frequency Ω_h , i.e., $\omega = p\Omega_h$, where p is an integer, and with a wave vector component $k_\perp$ perpendicular to the equilibrium magnetic field B_0 , such that $k_\perp v_h / \Omega_h \approx p$. Here, $v_h = (2\epsilon_h / m_h)^{1/2}$ is a characteristic velocity of the hot ion distribution function, and ϵ_h is the corresponding energy. Further, in order to avoid significant parallel electron Landau damping and transit-time damping, we shall focus on modes which have a vanishing parallel electric field and which propagate almost perpendicularly to B_0 .

We take the perpendicular fluctuating electric field to be of the form:

$$\underline{E}_\perp(\underline{r}, t) = \underline{E} \exp[-i(\omega t - k_\parallel z - k_\perp y)] \quad (2-1)$$

and, neglecting the displacement current, obtain from Maxwell's equations:

$$\underline{k} \times (\underline{k} \times \underline{E}) = - \frac{4\pi i \omega}{c^2} \underline{g}(\omega, \underline{k}) \cdot \underline{E}, \quad (2-2)$$

where $\underline{g}$ is the plasma conductivity tensor. In the cold plasma approximation which applies to the bulk plasma, $\underline{g}$ is given by the 3×3 matrix:

$$\underline{g} = \begin{pmatrix} \sigma_\perp & 0 \\ 0 & \sigma_\parallel \end{pmatrix}, \quad (2-3)$$

where small terms of order $(k_{||}/k_{\perp})^2$, and m_e/m_i have been neglected, $\sigma_{||} = i\omega_{pe}^2/(4\pi\omega)$, and

$$\underline{\underline{g}}_{\perp} = -\frac{i\sigma_o}{\omega} \begin{pmatrix} 1 & i\lambda \\ -i\lambda & 1 \end{pmatrix}. \quad (2-4)$$

In Eq. (2-4), σ_o is defined by

$$\sigma_o = -\frac{en_e c \omega^2}{B_o} \sum_{i=D,T} \frac{\alpha_i \Omega_i}{\Omega_i^2 - \omega^2}, \quad (2-5)$$

and the polarization factor $\lambda(\omega) = -i\vec{E}_Y/\vec{E}_X$ is given by

$$\lambda(\omega) = \sum_{i=D,T} \frac{\alpha_i \omega}{\Omega_i^2 - \omega^2} / \left(\sum_{i=D,T} \frac{\alpha_i \Omega_i}{\Omega_i^2 - \omega^2} \right). \quad (2-6)$$

The coefficients $\alpha_i = (n_i/n_e)$ in Eqs. (2-5) and (2-6) denote the ratio between the density of the ion species and the electron density. Equation (2-2) can be cast in the form

$$\underline{\underline{y}}(\omega, \underline{k}) \cdot \vec{E} = 0, \quad \text{with} \quad \underline{\underline{y}}(\omega, \underline{k}) = (4\pi i/\omega) \underline{\underline{g}}(\omega, \underline{k}) - c^2/\omega^2 [k_{\perp}^2 \underline{\underline{I}} - \underline{k} \underline{k}].$$

The mode dispersion relation is $|\underline{\underline{y}}(\omega, \underline{k})| = 0$, which, with $\vec{E}_{||} = 0$, reduces to

$$4\pi\sigma_o(1-\lambda^2) = k_{\perp}^2 c^2, \quad (2-7)$$

and can be rewritten as^{6,11}:

$$K(\omega) \equiv \frac{\omega^2}{k^2 v_A^2} \frac{\omega^2 - \bar{\Omega}^2}{\omega^2 - \Omega_{Hy}^2} = 1. \quad (2-8)$$

This dispersion equation describes magnetosonic (compressional, fast Alfvén) waves modified by the so-called ion-ion hybrid resonance that occurs in a plasma with more than one ion species. The cutoff frequency $\bar{\Omega}$ and the resonance frequency Ω_{Hy} are defined by $\bar{\Omega} = \alpha_D \Omega_T + \alpha_T \Omega_D$ and $\Omega_{Hy} = (\Omega_D \Omega_T \bar{\Omega} / \bar{\Omega})^{1/2}$, with $\bar{\Omega} = \alpha_D \Omega_D + \alpha_T \Omega_T$. The Alfvén velocity is given by $\bar{v}_A = (\Omega_e \bar{\Omega})^{1/2} d_e$, where $d_e = c/\omega_{pe}$ is the electron inertial skin depth. In the case of a D-T plasma with equal ion species concentrations, $\alpha_D = \alpha_T = 1/2$, so that $\bar{\Omega}^2 = 25/36 \Omega_D^2$ and $\Omega_{Hy}^2 = 2/3 \Omega_D^2$. The components of the wave-perturbed magnetic field are related through the equations $\hat{B}_x = i\lambda \hat{B}_y$ and $k_{||} \hat{B}_{||} = -k_{\perp} \hat{B}_y$, with $\hat{B}_{||} = -ck_{\perp} / \omega E_x$.

When the mode frequency ω is above the ion cyclotron frequency Ω_D , (high frequency range), the dispersion relation simplifies into the MHD dispersion relation $\omega^2 \approx k_{\perp}^2 \bar{v}_A^2$, as shown in Fig. 1.

In this frequency range, the wave polarization is elliptical and the polarization factor λ , which can be rewritten as $\lambda(\omega) = \omega(\alpha_D \Omega_T^2 + \alpha_T \Omega_D^2 - \omega^2) / [\bar{\Omega}(\Omega_{Hy}^2 - \omega^2)]$, scales as $\lambda(\omega) \sim \omega / \bar{\Omega}$. In the low frequency range, $\omega \sim \Omega_D, \Omega_T$, the polarization factor λ varies rapidly from right circular $\lambda=1$ for $\omega=\Omega_D, \Omega_T$, to left circular $\lambda=-1$ for $\omega=\bar{\Omega}$, to linear, $\lambda=0$ for $\omega^2 = \alpha_D \Omega_T^2 + \alpha_T \Omega_D^2$ and $\lambda=\infty$ for $\omega=\Omega_{Hy}$. We observe that in the frequency interval between Ω_{Hy} and $\bar{\Omega}$, the waves are evanescent with $k^2=0$ for $\omega=\bar{\Omega}$ and $k^2 \approx \infty$ for $\omega=\Omega_{Hy}$.

B. Mode-particle resonances

The resonant interaction between the waves and the high energy ion population and the damping due to the bulk electrons and ions add a hermitian part to the conductivity tensor, which is small in comparison to the antihermitian part given by Eq. (2-4). To evaluate the resulting imaginary part of the frequency, i.e., the growth or the damping of the modes, we write the tensor $\underline{\underline{Y}}$ as $\underline{\underline{Y}} = \underline{\underline{Y}}_0 + \delta\underline{\underline{Y}}$, where $\delta\underline{\underline{Y}} = (4\pi i/\omega) \sum_{j=e,D,T,h} \delta\underline{\underline{g}}_j$ accounts for the resonant contributions to the conductivity tensors of the bulk plasma species and of the hot ions. Then, setting $\omega = \omega_0 + i\gamma$ with $\gamma \ll \omega_0$, we obtain

$$\gamma = - \operatorname{Im} \frac{\underline{\underline{E}}^* \cdot \delta\underline{\underline{Y}} \cdot \underline{\underline{E}}}{\left. \underline{\underline{E}}^* \cdot \frac{\partial \underline{\underline{Y}}_0}{\partial \omega} \cdot \underline{\underline{E}} \right|_{\omega=\omega_0}} \quad (2-9)$$

The electron contribution $\delta\underline{\underline{g}}_e$ is important when the parallel phase velocity $\omega/k_{\parallel}$ of the modes is not sufficiently larger than the electron thermal velocity v_{the} or, in the case of modes with vanishing parallel electric fields, for plasmas with finite values of the ratio $\beta_e = 8\pi n_e T_e / B^2$. From the Vlasov equation we find that for these modes [see also Eq. (2-15)], the main contribution to $\delta\underline{\underline{g}}_e$ is given by $\delta\sigma_{xx} = (1/4\pi) (\omega_{pe}^2/\omega) (k_{\parallel} v_{the}/\Omega_e)^2 \zeta_e \operatorname{Im} Z(\zeta_e)$, where $\zeta_e = \omega/|k_{\parallel}| v_{the}$ is the ratio of the parallel phase velocity of the wave to the electron thermal velocity, and $\operatorname{Im} Z(\zeta_e)$ is the imaginary part of the plasma dispersion function. Then the electron damping rate is given by

$$\frac{\gamma_e}{\omega} = - \frac{\pi^{1/2} \beta_e \zeta_e \exp(-\zeta_e^2)}{\omega \partial K / \partial \omega} \quad (2-10)$$

where

$$\omega \partial K / \partial \omega = (2\omega^2 / k_{\perp}^2 v_A^2) \{1 + \Omega_{HY}^2 (\bar{\beta}^2 - \Omega_{HY}^2) / (\omega^2 - \Omega_{HY}^2)^2\} \quad (2-11)$$

is a positive quantity. We recall that the physical mechanism that causes this damping, usually referred to as transit time damping,¹¹ is the interaction between the electron magnetic moment μ and the parallel gradient of the mode magnetic field.¹² This damping is the magnetic analogue of the standard Landau damping, as can be shown by considering the parallel electron momentum balance equation $m_e dv_{\parallel} / dt = -\mu \mathbf{b} \cdot \nabla B_{\parallel}$, which holds in the presence of waves with $\bar{E}_{\parallel} = 0$ and $\omega < \Omega_e$. This equation is, in fact, identical to that which occurs in the presence of a parallel electric field, with $-\mu$ playing the role of the charge and $(B_0 / B_0) \cdot \nabla B_{\parallel} \approx \mathbf{b} \cdot \nabla B_{\parallel}$ that of the electric field. The bulk ion contribution $\delta \underline{\sigma}_D + \delta \underline{\sigma}_T$ comes from harmonics of the cyclotron resonance of the mode with the ions. From the Vlasov equation and Eq. (2-9), assuming a Maxwellian distribution function for the ion species [see also Eq. (2-15) below], we obtain the damping rate due to the i-species

$$\begin{aligned} \gamma_i^p / \omega = & - \frac{n_i}{n_e} \frac{\pi^{1/2}}{\omega \partial K / \partial \omega} \frac{\Omega_i \bar{\Omega}}{k_{\perp}^2 v_A^2} \frac{\omega}{|k_{\parallel}| v_{thi}} \exp(-b_i^2 - v_{\parallel}^2 / v_{thi}^2) \\ & \times \left(\frac{b_i}{2} \right)^{p-1} \times \frac{1}{(p-1)!} [p\lambda(\lambda/2-1) + p+1] \end{aligned} \quad (2-12)$$

for the resonance with the p -th cyclotron harmonic. In Eq. (2-12), we have taken the limit $b_i \ll 1$, $b_i = (k_{\perp} v_{thi} / \Omega_i)^2 / 2$, with $v_{thi} = (2T_i / m_i)^{1/2}$ the ion thermal velocity. The parallel resonant ion velocity is defined as $v_{||R} = (\omega - p\Omega_i) / k_{||}$.

Since the density ratio n_h / n_e is small, only the resonant part of δg_h need be considered. We write the perturbed current density j_{h1} as $j_{h1} = q_h \int d^3v \, v \, f_{h1}$, with q_h the electric charge and f_{h1} the perturbed hot ion distribution function. Then from the linearized Vlasov equation we obtain

$$f_{h1}(\underline{r}, t) = - \frac{q_h}{m_h} \int_{-\infty}^t dt' \left[\underline{E}_1 + \frac{1}{c} \underline{v}' \times \underline{B}_1 \right] \cdot \underline{\nabla}_{\underline{v}'} f_{ho}, \quad (2-13)$$

where $\underline{E}_1$, $\underline{B}_1$ are the perturbed electric and magnetic field, f_{ho} is the equilibrium distribution function, and the integration is along the unperturbed particle trajectories. We shall compute the hot ion perturbed distribution function by including the effects of their density gradient and of their magnetic curvature drift. These effects are outside the homogeneous model but are important in the inhomogeneous analysis in the next section. To include these effects, it will be sufficient to consider a slab model with the equilibrium magnetic field along z , inhomogeneous in the x direction, and with an external force along x simulating the effect of magnetic curvature. The equilibrium distribution function of the hot ions can then be written as $f_{ho} = f_{ho}(\varepsilon, \mu, X)$, where $\varepsilon = m_h v^2 / 2$ and $\mu = m_h (\underline{v}_{\perp} - \underline{v}_{Dh})^2 / (2B_0)$ are the energy and magnetic moment, $\underline{v}_{Dh}$ is the magnetic curvature drift velocity, and $X = x + v_y / \Omega_h$ is proportional to the y -component of the canonical

momentum. Assuming perturbations of the form given in Eq. (2-1) and using Faraday's equation, Eq. (2-13) becomes:

$$\begin{aligned}
 f_{h1} &= -q_h \int_{-\infty}^t dt' \left[\left(1 - \frac{\mathbf{k} \cdot \mathbf{v}'}{\omega} \right) \mathbf{E}_1 + \frac{\mathbf{v}' \cdot \mathbf{E}_1}{\omega} \mathbf{k} \right] \cdot \left[\mathbf{v}' \frac{\partial}{\partial \mathbf{v}} + \frac{(\mathbf{v}'_1 - \mathbf{v}'_{Dh})}{B_0} \frac{\partial}{\partial \mu} + \frac{\hat{e}_y}{m_h \Omega_h} \frac{\partial}{\partial X} \right] f_{ho} \\
 &= -q_h \left\{ \frac{i}{\omega \Omega_h m_h} E_{1y} \frac{\partial f_{ho}}{\partial X} - \frac{i}{\omega} (\mathbf{v}_{Dh} \cdot \mathbf{E}_1 + v_{||} E_{1||}) \frac{1}{B_0} \frac{\partial}{\partial \mu} f_{ho} \right. \\
 &\quad \left. + \int_{-\infty}^t dt' (\mathbf{E}_1 \cdot \mathbf{v}') \left[\frac{\partial}{\partial \mathbf{v}} + \left(1 - \frac{k_{||} v_{||}}{\omega} + \frac{\mathbf{v}' \cdot \mathbf{v}'_{Dh}}{\omega} \right) \frac{1}{B_0} \frac{\partial}{\partial \mu} + \frac{k_y}{\omega \Omega_h m_h} \frac{\partial}{\partial X} \right] f_{ho} \right\}, \quad (2-14)
 \end{aligned}$$

where the identity $(i/\omega) (d\mathbf{E}_1/dt) = (1 - \mathbf{k} \cdot \mathbf{v}/\omega) \mathbf{E}_1$ has been used. The time integration in Eq. (2-14) is performed taking $\rho_h \ll r_n$, where ρ_h is the hot ion gyroradius and r_n is the characteristic equilibrium density scale length. The resulting conductivity tensor of the high energy ion population is

$$\begin{aligned}
 \delta \underline{\underline{\sigma}}_h &= i q_h^2 \sum_{p=-\infty}^{+\infty} \int d^3 v \frac{v_z^2 (\pi f_{ho})}{\omega - k_{||} v_{||} - p \Omega_h - i \omega_{Dh}} \\
 &\quad * \begin{pmatrix} J_p'^2(\xi_h) & \frac{i p}{\xi_h} J_p(\xi_h) J_p'(\xi_h) \\ -\frac{i p}{\xi_h} J_p(\xi_h) J_p'(\xi_h) & \frac{p^2}{\xi_h^2} J_p^2(\xi_h) \end{pmatrix}, \quad (2-15)
 \end{aligned}$$

where the operator Π is defined by

$$\Pi = \frac{\partial}{\partial \varepsilon} + \left(1 - \frac{k_{\parallel} v_{\parallel} + \omega_{Dh}}{\omega}\right) \frac{1}{B_0} \frac{\partial}{\partial \mu} + \frac{k_{\perp}}{m_h \omega \Omega_h} \frac{\partial}{\partial X} \quad (2-16)$$

$\omega_{Dh} = k_{\perp} v_{Dh}$ is the magnetic curvature drift frequency, and $J_p(\xi_h)$ and $J'_p(\xi_h) \equiv dJ_p(\xi_h)/d\xi_h$ are Bessel functions of argument

$\xi_h = k_{\perp} v_{\perp} / \Omega_h$. Taking the resonant part of $\delta g_{\pm h}$ in Eq. (2-15) and inserting it into (2-9), we obtain the hot ion contribution to γ :

$$\begin{aligned} \gamma_h &= \frac{n_h}{n_e} \frac{\omega^2}{\omega \partial K / \partial \omega} \frac{\Omega_h \tilde{\Omega}}{k_{\perp}^2 v_A^2} \pi \\ &\times \sum_{p=-\infty}^{+\infty} \left(\frac{p \Omega_h}{k_{\perp} v_h} \right)^2 \int d^3 v \delta(\omega - k_{\parallel} v_{\parallel} - p \Omega_h - \omega_{Dh}) \frac{2 \varepsilon_h}{n_h} \\ &\left[\lambda J_p(\xi_h) - \frac{\varepsilon_h}{p} J'_p(\xi_h) \right]^2 \Pi f_{ho} \quad (2-17) \end{aligned}$$

All coefficients in Eq. (2-17) multiplying Πf_{ho} are positive definite. In the homogeneous case, the operator Π reduces to $\partial/\partial \varepsilon + B_0^{-1} \partial/\partial \mu \equiv (2/m_h) \partial/\partial v_{\perp}^2$, and the portion of velocity space where $\Pi f_{ho} > 0$ (instability region) gives a positive contribution to γ_h . Positive values of $\partial f_{ho}/\partial v_{\perp}^2$ can occur either for isotropic but nonmonotonic distribution functions, or for anisotropic distribution functions with a positive slope in the perpendicular direction. A positive growth rate γ_h will result if the parameters ω and $k_{\parallel}$ can be chosen such that, roughly speaking, the resonant surface lies mainly inside the instability

region. For each cyclotron harmonic, the shape of the p resonant surface is determined by the resonance condition expressed in the δ function of Eq. (2-17). The magnetic curvature drift frequency depends on ϵ, μ through the combination $\epsilon - \mu / (2B_0)$. Thus, the resonant surfaces in velocity space are given by ellipsoids $(v_{\parallel} - v_s)^2 + v_{\perp}^2/2 = \text{const}$, with $v_s = k_{\parallel} \epsilon_h / (m_h \omega_{Dh}^0)$ and $\omega_{Dh}^0 = \omega_{Dh}(\epsilon = \epsilon_h, \mu = 0)$. If $\omega_{Dh}^0 \gg k_{\parallel} v_h$, the center of the ellipsoids approaches the origin in velocity space. In the opposite limit, the resonant surfaces reduce to a plane $v_{\parallel} = \text{const}$.

In the inhomogeneous slab model, assuming that only the density of the high energy ion population depends on x , f_{ho} contains a spatial factor which can be written as $(1 - x/r_{nh})$, with $r_{nh}^{-1} = (-d \ln n_h / dx)$ the characteristic scale length of the hot ion density. Defining a diamagnetic frequency $\omega_{*h} = k_y v_h^2 / (\Omega_h r_{nh})$, the operator Π can be written as $\Pi = (1 - \omega_{*h}/\omega) \partial/\partial \epsilon + B_0^{-1} \partial/\partial \mu \equiv 2/m_h [\partial/\partial v_{\perp}^2 - (\omega_{*h}/\omega) \partial/\partial v_{\parallel}^2]$. The presence of a strong density gradient (small r_{nh}) can then alter the instability region significantly.

III. NORMAL MODES IN AN INHOMOGENEOUS PLASMA CONFIGURATION

The ratio v_h/ω derived in the previous section scales as n_h/n_a , which is, by hypothesis, a small quantity. The rate at which energy is transferred to the modes from the high energy ion population is therefore slow. Thus, in addition to considering the changes introduced in the resonant responses

of the bulk species and the hot ions by the plasma inhomogeneity, we must ascertain whether the modes of interest exist as normal modes in a realistic plasma configuration. In fact, a relatively small rate of spatial convection could be sufficient to drain away the energy gained through the resonant interaction and stabilize the modes.

In order to deal with problems of increasing complexity in constructing the relevant eigenmodes, we shall first consider a simplified inhomogeneous model that can be viewed as referring to a torus of large aspect ratio and vanishing poloidal field. On the basis of the information obtained from this model, we shall later derive the full dispersion equation for a realistic toroidal configuration and solve it in the high frequency range, i.e., when $\omega^2 \gg \Omega_{Hy}^2$, $\bar{\Omega}^2$, by means of a combined expansion in inverse powers of the poloidal number and of the aspect ratio.

A. Localized modes

Referring to polar coordinates in the poloidal plane, we consider the perturbed parallel magnetic field, which we take in the form $B_{\parallel} = \tilde{B}(r, \theta) \exp[-i(\omega t - k_{\parallel} z)]$, as the appropriate physical variable.

Then we write the differential form of the dispersion relation $\omega^2 = k^2 \bar{v}_A^2$ heuristically as

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \tilde{B} - \frac{1}{r^2} \frac{\partial}{\partial \theta^2} \tilde{B} = - \frac{\omega^2}{\bar{v}_A^2} \frac{n(r)}{n_0} \frac{B_0^2}{B_0^2(r, \theta)} \tilde{B}, \quad (3-1)$$

where $\bar{v}_A$, n_0 , and B_0^0 are the Alfvén velocity, plasma density, and magnetic field at the magnetic axis, with $B_0^2/B_0^2(r, \theta) = R^2/R_0^2 =$

$[1+(r/R_0)\cos\theta]^2$ a constant to zero order in the inverse aspect ratio $\epsilon=a/R_0$ (R_0 and a being, respectively, the major and minor radius of the plasma torus). For modes in the ion cyclotron frequency range, $\omega^2/\bar{v}_A^2 \gg 1/d_i^2 \gg 1/r^2$, where $d_i=c/\omega_{pi}$ is the ion inertial skin depth, and Eq. (3-1) can be solved by an expansion in powers of d_i/a .

Localized normal mode solutions of Eq. (3-1) are found by considering modes with short poloidal wavelengths. Then, taking for the moment the limit of infinite aspect ratio, we write $\hat{B}(r, \theta) = \hat{B}(r)\exp(-im\theta)$, with m a large integer of order a/d_i , and obtain from Eq. (3-1)

$$\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} \hat{B} = \left[\frac{m^2}{r^2} - \frac{\omega^2}{\bar{v}_A^2} \frac{n(r)}{n_0} \right] \hat{B} \quad (3-2)$$

For the modes to be localized in r to avoid spatial convection, the effective potential in the square brackets in Eq. (3-2) must have two turning points. This sets a lower bound on ω^2 . For the sake of simplicity, we shall consider in particular modes for which $\frac{d}{dr} \ll \frac{m}{r}$, in which case the turning points are close together (see Fig. 2), $\omega = \omega_0 + \delta\omega$ with $\delta\omega \ll \omega_0$, $\omega_0^2 = \bar{v}_A^2 [n_0/n(r_0)] (m/r_0)^2$, and the radius r_0 is chosen so that the linear term in the expansion of the effective potential around r_0 vanishes. Thus, assuming for the sake of illustration a parabolic density profile $n(r) = n_0(1-r^2/a^2)$, Eq. (3-2) reduces to

$$\frac{d^2}{dr^2} \hat{B} = \frac{(r-r_0)^2}{a^4} \hat{B} - \lambda \hat{B} \quad (3-3)$$

where $r_0 = a/\sqrt{2}$, $\Delta^{-1} = [3m^2/r_0^4 + \omega_0^2/(\bar{v}_{A0} a)^2]^{1/4} = 2m^{1/2}/a$, and $\Lambda = \omega_0 \delta\omega/\bar{v}_{A0}^2$. From Eq. (3-3), we obtain $\hat{B} = \hat{B}_0 \exp[-(r-r_0)^2/(2\Delta^2)]$.

This normal mode solution is localized within a radial annulus. Its width is of order $(d_i a)^{1/2}$ and is large compared to the mode poloidal wavelength, which is of order d_i .

For a general monotonically decreasing density profile, r_0 and Δ in Eq. (3-3) are given by

$$r_0 = -2 \left[\frac{d \ln[n(r)/n_0]}{dr} \right]_{r=r_0}^{-1} \quad (3-4)$$

and

$$\Delta^{-1} = \frac{(2m)^{1/2}}{r_0} \left[\frac{3}{4} - \frac{1}{2} \frac{d \ln[n'(r)/n_0]}{d \ln[n(r)/n_0]} \right]_{r=r_0}^{1/4}, \quad (3-5)$$

with $n'(r) = d n(r)/dr$.

B. Toroidal Effects on the Localized Modes

The restriction to a configuration with infinite aspect ratio can be lifted by solving Eq. (3-1) in the eikonal form $\hat{B}(r, \theta) = \hat{B}_0 \exp[i\Phi(r, \theta)]$ and by using a combined expansion in powers of the two smallness parameters m^{-1} and the inverse aspect ratio $\varepsilon = a/R_0$.

Thus Eq. (3-1) becomes:

$$-i \left[\frac{1}{r} \frac{\partial}{\partial r} = \frac{\partial}{\partial r} \nabla^2 + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \nabla^2 \right] + \left(\frac{\partial \Phi}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \Phi}{\partial \theta} \right)^2 = \frac{\omega^2}{\bar{v}_{A0}^2} \frac{n(r)}{n_0} \left(1 - \frac{r}{R_0} \cos \theta \right)^2 \quad (3-6)$$

which, expanding Φ in powers of m^{-1} , gives, to lowest order,

$$\left(\frac{\partial P^{(0)}}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial P^{(0)}}{\partial \theta} \right)^2 = \frac{\omega^2}{v_{Ao}^2} \frac{n(r)}{n_0} [1 + r/R_0 \cos \theta]^2. \quad (3-7)$$

By a further expansion of $P^{(0)}$ in powers of ϵ , $P^{(0)}(r, \theta) = \sum_{s=0}^{\infty} \epsilon^s P^{(0,s)}(r, \theta, \epsilon)$, we obtain to zero order

$$\left(\frac{\partial P^{(0,0)}}{\partial \theta} \right)^2 = \frac{r^2 \omega^2}{v_{Ao}^2} \frac{n(r)}{n_0} [1 + r/R_0 \cos \theta]^2 + \Gamma(r) \quad (3-8)$$

with

$$\left\langle \left(\frac{\partial P^{(0,0)}}{\partial r} \right)^2 \right\rangle_{\theta} = -\Gamma(r) \quad (3-9)$$

and, to next order,

$$2 \left[r^2 \frac{\partial P^{(0,0)}}{\partial r} \frac{\partial P^{(0,1)}}{\partial r} + \frac{\partial P^{(0,0)}}{\partial \theta} \frac{\partial P^{(0,1)}}{\partial \theta} \right] = - \left(\frac{\partial P^{(0,0)}}{\partial r} \right)^2 r^2 + \left\langle \left(\frac{\partial P^{(0,0)}}{\partial r} \right)^2 \right\rangle_{\theta} r^2. \quad (3-10)$$

Here, $\langle \rangle_{\theta}$ denotes average over the poloidal angle θ , and $\Gamma(r)$ is an integration factor, small compared to terms of order m^2 , that makes the θ average of the l.h.s. of Eq. (3-8) independent of r , i.e.,

$$\left\langle \frac{r^2 \omega^2}{v_{Ao}^2} \frac{n(r)}{n_0} (1 + r/R_0 \cos \theta)^2 \right\rangle_{\theta} + \Gamma(r) = m^2 = \text{const.} \quad (3-11)$$

Equation (3-8) yields

$$P^{(0,0)} = \pm \left[\frac{r \omega_0}{v_{Ao}} \left(\frac{n(r)}{n_0} \right)^{1/2} + \frac{\Gamma(r)}{2} \frac{v_{Ao}}{r \omega_0} \left(\frac{n_0}{n(r)} \right)^{1/2} \right] (\theta + r/R_0 \sin \theta) + \chi(r), \quad (3-12)$$

where $\chi(r)$ is an integration function, constant with respect to θ . The term inside the square brackets has been approximated up to terms of order ϵ^2 , and to this order is constant in r and coincides with the poloidal number m . Thus, to lowest significant order in ϵ ,

$$p^{(0,0)} = -m [\theta + r/R_0 \sin\theta] + \chi(r). \quad (3-13)$$

The function $\chi(r)$ is determined by Eq. (3-9). Up to terms of order ϵ^2 , we obtain

$$\left(\frac{d\chi}{dr}\right)^2 r^2 = -\Gamma(r) = -\frac{4m^2}{r_0^2} \left[\frac{3}{4} - \frac{1}{2} \frac{d \ln[n'(r)/n_0]}{d \ln[n(r)/n_0]} \right]_{r=r_0} (r-r_0)^2. \quad (3-14)$$

Equation (3-14) gives $\chi = \pm i(r-r_0)^2/(2\Delta^2)$, with Δ as in Eq. (3-5) in agreement with Eq. (3-3).

The equation for $p^{(0,1)}$, to lowest order in ϵ , reads

$$\frac{m}{r^2} \frac{\partial p^{(0,1)}}{\partial \theta} - \frac{d\chi}{dr} \frac{\partial p^{(0,1)}}{\partial r} = -\frac{d\chi}{dr} \frac{m}{R_0} \sin\theta. \quad (3-15)$$

In the neighborhood of $r=r_0$, we look for a solution of the form

$$p^{(0,1)}(r, \theta) = \frac{(r-r_0)}{r_0} \hat{p}^{(0,1)}(\theta) \quad (3-16)$$

and obtain from Eq. (3-15)

$$\frac{im\Delta^2}{r_0^2} \frac{\partial \hat{p}^{(0,1)}}{\partial \theta} + \hat{p}^{(0,1)} = \frac{mr_0}{R_0} \sin\theta. \quad (3-17)$$

For $\Delta^{-2} \neq \pm m/r_0^2$, the solution to Eq. (3-15) gives

$$p^{(0,1)} = -i(r-r_0) \frac{m}{R_0} \left[\frac{\exp(i\theta)}{1-m\Delta^2/r_0^2} - \frac{\exp(-i\theta)}{1+m\Delta^2/r_0^2} \right]. \quad (3-18)$$

When $\Delta^{-2} = \pm m/r_0^2$, the solution $\hat{p}^{(0,1)}$ to Eq. (3-15) becomes linear in θ . Thus, $p^{(0,1)}$ ceases to be periodic in θ and the above derivation is invalid. It is easy to show that, if terms of higher order in ε are retained in the dispersion equation (3-7), the phase $p^{(0,s)}$ gives a secular contribution of order ε^s when $\Delta^{-2} = \pm (ms)/r_0^2$, with s an integer. In particular, a parabolic profile leads to a secularity of order ε^2 . Therefore, we see that toroidal effects not only deform the region where the mode is localized but can also lead, for a given class of bulk density profiles, to an outward drift of the mode. The circular annulus which is obtained in the cylindrical limit is shifted, $s=1$, and distorted, $s \geq 2$, but remains well confined inside the plasma provided

$$\left| \left[\frac{3}{4} - \frac{1}{2} \frac{d \ln[n'(r)/n_0]}{d \ln[n(r)/n_0]} \right]_{r=r_0}^{1/2} - \frac{s}{2} \right| \gg \frac{\varepsilon^s}{2} \quad (3-19)$$

for all integer values of s . For example, in the case of a quasi-parabolic density profile of the type $n(r) = n_0 [1 - (1-\delta)(r/a)^2 - \delta(r/a)^4]$, Eq. (3-17) gives for $s=2$

$$3 \gg \varepsilon^2 \quad (3-20)$$

and shows that, for a quasi-parabolic density profile, small deviations from an exact parabolic one are sufficient to recover radially localized modes.

C. Dispersion Equation and Quadratic Form

To derive the correct dispersion equation in a realistic toroidal configuration, including the low frequency modes for which $\omega \sim \bar{\Omega}$, Ω_{By} , and justify the results obtained from Eq. (3-1), we start from Maxwell's equations in the form

$$\underline{b} \cdot \nabla \times (\underline{g}^{-1} \cdot \nabla \times \underline{\tilde{B}}_{||}) = - \frac{4\pi}{c^2} \frac{\partial \underline{\tilde{B}}_{||}}{\partial t} \quad (3-21)$$

In deriving Eq. (3-21), which plays the role of Eq. (2-2) of the homogeneous model, we have used the fact that $\tilde{B}_{\perp} \sim (k_{||}/k_{\perp}) \tilde{B}_{||} \ll \tilde{B}_{||}$. In Eq. (3-21), $\underline{b} = \underline{B}_0/B$, and $\underline{g}^{-1}$ is the inverse of the conductivity tensor defined by Eq. (2-3), with the z axis locally along $\underline{b}$. The perpendicular coordinates need not be specified, since $\underline{g}$ is rotationally invariant around this axis. As consistent with the results obtained from Eq. (3-1), we shall retain at most terms of order $m/r^2 \gg 1/r^2$ in the solution of Eq. (3-21). Then Eq. (3-21) takes the simpler form

$$\tilde{B}_{||} = \frac{ic^2}{4\pi\omega} \nabla_{\perp} \cdot \underline{g}^{-1} \cdot \nabla_{\perp} \tilde{B}_{||} \quad (3-22)$$

We refer for simplicity to a toroidal magnetic configuration with circular concentric magnetic surfaces. Using polar coordinates in the poloidal plane, Eq. (3-22) becomes

$$\begin{aligned} \tilde{B}_{||} = - \frac{c^2}{8\pi} \left[\left(\frac{1}{r} \frac{\partial}{\partial r} r - \frac{i}{r} \frac{\partial}{\partial \theta} \right) \frac{1}{\sigma_0(1+\lambda)} \left(\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right) + \right. \\ \left. + \left(\frac{1}{r} \frac{\partial}{\partial r} r + \frac{i}{r} \frac{\partial}{\partial \theta} \right) \frac{1}{\sigma_0(1-\lambda)} \left(\frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \theta} \right) \right] \tilde{B}_{||}, \quad (3-23) \end{aligned}$$

where $\sigma_0(r, \theta)$ and $\lambda(r, \theta)$ are defined by Eqs. (2-5) and (2-6) and depend on r and θ through the spatial dependence of the equilibrium density and magnetic field. For low frequency modes, the coefficient $1/[\sigma_0(1+\lambda)]$ is singular for $\omega = \bar{\omega}$, i.e., for $\lambda = -1$, whereas the coefficient $1/[\sigma_0(1-\lambda)]$ is always regular. For $\omega = \omega_{Hy}$, i.e., for $\lambda = \infty$, the two coefficients are regular but equal and opposite. For real ω^2 , the operator on the r.h.s. of Eq. (3-23) is hermitian and the following quadratic form can be constructed

$$\int_0^{\theta} d\theta \int_0^a r dr \left\{ - |\hat{B}_{||}|^2 + \frac{\left| \left(\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right) \hat{B}_{||} \right|^2}{\sigma_0(1+\lambda)} + \frac{\left| \left(\frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \theta} \right) \hat{B}_{||} \right|^2}{\sigma_0(1-\lambda)} \right\} = 0. \quad (3-24)$$

In the high frequency case, the coefficients $1/[\sigma_0(1 \pm \lambda)]$ are slowly varying functions of r and θ . Then, retaining again at most terms of order $m/r^2 v (ad_i)^{-1}$, Eq. (3-23) becomes

$$\hat{B}_{||} = \frac{-c^2}{4\pi} \left[\frac{1}{\sigma_0(1-\lambda^2)} \left[\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] - \frac{i}{r} \left[\frac{\partial}{\partial r} \frac{\lambda}{\sigma_0(1-\lambda^2)} \right] \frac{\partial}{\partial \theta} \right] \hat{B}_{||}. \quad (3-25)$$

The first two terms in Eq. (3-25) reduce to Eq. (3-1) when the limit $\omega \gg \bar{\omega}, \Omega_{Hy}$ is taken in the definitions of σ_0 and λ . The third term is $1/m$ smaller and introduces a small frequency shift between modes with opposite poloidal mode numbers.

D. Low Frequency Case and Electron Transit Time Damping

In the low frequency case, the derivatives of the coefficients $1/[\sigma_0(1 \pm \lambda)]$ are large, and an equation of the type of Eq. (3-25) cannot be derived. In fact, in this frequency case, the dependence of B_0 on the distance from the axis of symmetry of the torus introduces, in the plasma column, two closely spaced surfaces where $\omega = \Omega_{Hy}$ and $\omega = \bar{\omega}$, respectively. Modes on the low field side of the column are reflected by the cutoff surface $\omega = \bar{\omega}$, with a fraction of their energy tunnelling to the ion-hybrid resonance surface $\omega = \Omega_{Hy}$ where it is converted into short wavelength modes.^{13,14} For large m-numbers, the fraction of the energy which is lost in this way is small, since the distance between the cutoff surface and the resonance surface is large compared to the poloidal wavelength. At the cutoff surface, the normal component of the wave vector in the poloidal plane changes sign. Thus, modes localized within a portion of an annulus, in analogy to the high frequency case, are possible only if the cutoff surface is approximately at the middle of the plasma column ($\theta = \pm \frac{\pi}{2}$). In this case, a localized standing normal mode may in fact be expected to exist. This conclusion can be inferred (see Sec. 4) by numerically

solving the mode ray equation obtained in the framework of geometrical optics from the dispersion Eq. (2-8). However, modes that are reflected by the cutoff surface have necessarily large values of the parallel wave number and are thus subject to significant electron transit time damping. To show this, we recall that in a toroidal configuration the parallel mode number is given by the operator

$$k_{||} \equiv i\mathbf{b} \cdot \nabla = \frac{n^0}{R_0} - \frac{i}{R_0 q(r)} \frac{\partial}{\partial \theta}, \quad (3-26)$$

where n^0 is the toroidal mode number and $q(r)$ is the inverse rationalized rotational transform. For radially localized modes, if n^0 can be chosen such that $n^0 q(r_0) = m$, an effective $k_{||}$ small enough for the electron transit time damping to be negligible is obtained. In the low frequency range, the incident and the reflected components of the mode at the cutoff surface have equal values of n^0 and opposite m . Thus, for at least one of the two components, $k_{||}$ is large. Most of the fusion reactors that have been considered so far operate under conditions where $\beta = 8\pi n(T_e + T_i)/B^2$ has finite values. We note that a representative numerical estimate at $r=r_0$ of the related parameter $\beta_e = 8\pi n T_e/B^2$ is given $\beta_e(r=r_0) \approx 0.15 \times \beta_e(r=0) \approx 1.4 \times 10^{-2} \times [T_{e0}/15\text{keV}] \times [n_0/4 \times 10^{14} \text{ cm}^{-3}] / [B_0/50\text{keV}]^2$, where a parabolic density profile and a Gaussian temperature profile have been considered. On this basis, we conclude from Eq. (2-10) that in the low frequency range the electron transit time damping becomes important, due to the large values of the effective $k_{||}$.

In the high frequency range instead, choosing $n^0 = m/q(r_0)$, we obtain from Eqs. (3-18) and (3-26) the toroidal correction to $k_{||}$

$$\Delta k_{||} = m r \cos \theta / (q R_0^2) \sim (\epsilon^2 / q) (r_0 / a)^2 \cos \theta k_{\perp} . \quad (3-27)$$

Then $(\omega / k_{||} v_{th,e}) \sim [q / (\epsilon^2 \beta_e^{1/2})] (a / r_0)^2 (m_e / m_i)^{1/2}$ is sufficiently larger than one so that the electron transit time damping is negligible in this case.

IV. GEOMETRICAL OPTICS APPROXIMATION

A different approach to the investigation of the normal mode solutions of the dispersion equation (3-23) consists of looking for trapped orbits of the ray propagation within the framework of geometrical optics.¹³ This approach is to some extent justified by the smallness of the mode poloidal wavelength, but clearly fails to describe accurately the mode behavior in the vicinity of the cutoff resonance pair.

We use the local dispersion equation (2-8) to express ω as a function of $k_{\perp}$, which we restrict to the poloidal plane, and of the coordinate dependent parameters $\bar{n}$, Ω_{Hy} , and $\bar{v}_A$. From Hamilton's equations, with ω the Hamiltonian, we obtain the ray equations

$$\dot{\mathbf{x}} = \frac{\partial \omega}{\partial \mathbf{k}} = \frac{\mathbf{k}}{k} \left(\frac{\omega^2}{k^2} \right) \left(\frac{\omega}{2} \frac{\partial K(\omega)}{\partial \omega} \right)^{-1} , \quad (4-1)$$

and

$$\dot{\mathbf{k}} = - \frac{\partial \omega}{\partial \mathbf{x}} = \left(\frac{\partial K(\omega)}{\partial \omega} \right)^{-1} \frac{\partial}{\partial \mathbf{x}} \ln \left[\frac{\bar{v}_A^2}{\omega^2} \frac{\omega^2 - \Omega_{Hy}^2}{\omega^2 - \bar{n}^2} \right] . \quad (4-2)$$

From Eq. (4-1), using Eq. (2-8) and the explicit form of $\partial K(\omega)/\partial \omega$ given by Eq. (2-11), we obtain the relationship

$$\frac{\hat{x}^2}{\bar{v}_A^2} = \frac{(\omega^2 - \bar{\Omega}^2)(\omega^2 - \Omega_{Hy}^2)^3}{[(\omega^2 - \Omega_{Hy}^2)^2 - (\Omega_{Hy}^2 - \bar{\Omega}^2)\Omega_{Hy}^2]^2}, \quad (4-3)$$

which is independent of k^2 and fixes the initial value of $\hat{x}^2$ once ω and the initial value of $\underline{x}$ are given. Combining Eqs. (4-1) and (4-2), we obtain

$$\begin{aligned} & \frac{d}{dt} \left[\frac{(\omega^2 - \Omega_{Hy}^2)^2 - \Omega_{Hy}^2(\Omega_{Hy}^2 - \bar{\Omega}^2)}{(\omega^2 - \Omega_{Hy}^2)^2 (B_O/B_O^O)^2 (n_O/n)} \frac{d}{dt} \hat{x} \right] \\ &= \frac{1}{2} \left[\frac{(\omega^2 - \bar{\Omega}^2)(\omega^2 - \Omega_{Hy}^2)}{(\omega^2 - \Omega_{Hy}^2)^2 - \Omega_{Hy}^2(\Omega_{Hy}^2 - \bar{\Omega}^2)} \frac{d}{d\hat{x}} \ln n - \frac{d}{d\hat{x}} \ln B_O^2 \right], \quad (4-4) \end{aligned}$$

where $\hat{x} = \underline{x}/a$, B_O^O , and n_O are reference values of the magnetic field and of the density, and the time t is normalized to the Alfvén time $\tau_A = a/\bar{v}_A(B_O, n_O)$.

Equation (4-4) has been integrated numerically as an initial value problem both in the high and in the low frequency range. In the former case, results that are in agreement with the normal mode analysis of Sec. 3 have been obtained.

We have chosen the relevant parameters as $\omega = 5\Omega_D(R_O)$, $\varepsilon = 1/4$, $R_O = 520$ cm., $t = 5000 \tau_A$, and at $t=0$, $\hat{x} = \hat{r}_O$, $\hat{y} = 0$, $\hat{\dot{x}} = 0$, $\hat{\dot{y}} = 1$. In Fig. 3, the secular behavior of the modes in the case of a parabolic density profile is clearly seen as an outward drift

of the ray trajectory. This trajectory asymptotically drifts out along the $\theta = \pi/4$ axis. This result is easily recovered analytically by solving Hamilton's equations perturbatively in the high frequency range. Toroidal effects modify the unperturbed circular ray trajectory $r=r_0$, $\theta=\omega_\theta t$, and the resulting radial displacement $\delta r = r - r_0$ is described by the equation:

$$\delta \ddot{r} + \omega_r^2 \delta r = \sum_{s=1}^{\infty} A_s \varepsilon^s \cos^s(\omega_\theta t) \quad , \quad (4-5)$$

where the frequency ω_θ , ω_r are given by

$$\omega_\theta = \bar{v}_{A0} \left[\frac{n_0}{n(r_0)} \right]^{1/2} \frac{1}{r_0} \quad \text{and} \quad (4-6)$$

$$\omega_r = \omega_\theta r_0^2 \left[\frac{3}{r^2} - \frac{1}{2} \frac{d \ln \{n'(r)/n_0\}}{d \ln \{n(r)/n_0\}} \right]^{1/2}_{r=r_0} \quad . \quad (4-7)$$

The radial displacement is "secular," i.e., $\delta r(t) = t \sin(s\omega_\theta t)$, when $\omega_r = s\omega_\theta$. In this case, the orbit equation becomes $\delta r(\theta) = \theta \sin(s\theta)$, and $\delta r(\theta)$ has its "extrema" at $s\theta = \pi/2$.

Asymptotically, the solution to this transcendental equation is given by $\theta_{ex} = \pi/(2s)$. For a parabolic density profile, $s=2$, and we have $\theta_{ex} = \pi/4$, in agreement with Fig. 3. Considering a quasi-parabolic density profile $n(\hat{r}) = n_0 [1 - (1-\delta)\hat{r}^2 - \delta\hat{r}^4]$, the condition for the confinement of the modes inside the plasma column is given by $\delta \gg \varepsilon^2$. In Fig. 4, δ is taken equal to 0.2,

and the resulting ray trajectory is well trapped in a region that is somewhat distorted with respect to a circular annulus, due to toroidal effects.

In the low frequency case, a trapped ray trajectory has been found (see Fig. 5) when the cutoff/resonance pair of singular surfaces falls in the middle of the plasma column, i.e., when $\omega = \bar{\omega}(R_0)$ so that $\theta = \pi/2$. However, the trapping of this trajectory is very sensitive to the condition $\theta = \pi/2$. A small displacement of the cutoff/resonance pair from this position gives rise to a detrapping of the ray trajectory (see Fig. 6). On the other hand, this low frequency trajectory is not subject to the outward drift which occurs for the high frequency trajectories when the density gradient is such that $\omega_r = s\omega_g$. This is a consequence of the reflection of the trajectory at $\theta = \pi/2$, as the secular contributions to δr on each half period are equal and opposite and cancel.

V. EXCITATION BY FUSION REACTION PRODUCTS IN SPIN POLARIZED PLASMAS

The presence of a net nuclear spin polarization in the fusing nuclei is a new nonthermal feature of the relevant plasmas. Specifically, in the case of a D-T plasma, if the deuteron and triton spins are coherently polarized, the velocity space distribution at birth of the fusion produced α -particles is anisotropic.¹ Since this anisotropy is preserved during the collisional slowing down of the α -particles due to the electrons, it can lead to the excitation of the modes described in the previous sections.

In a spin-polarized plasma, it is important to consider the excitation of modes that can induce spin-flip transitions, i.e., of modes with a nonvanishing left circular polarized component of the fluctuating magnetic field, and with a frequency ω such that the resonance condition with one of the nuclear spin precession frequencies is satisfied (see Sec. 6)

$$\omega = \Omega_{D(T)}^P + l\Omega_{D(T)} + \delta\omega_{D(T)} \quad (5-1)$$

Here, l is an integer, $l\Omega_{D(T)}$ is the Doppler shift due to the cyclotron motion of the bulk ions, $\delta\omega_{D(T)} = (k_{||}v_{||} + \omega_D)_{D(T)}$ is the shift due to their parallel motion and curvature drift, and $\Omega_{D(T)}^P$ are, respectively, the precession frequency of the deuterons, $\Omega_D^P = 0.86 \Omega_D$, and of the tritons, $\Omega_T^P = 5.94 \Omega_D$. We observe that for the relevant plasma conditions we have $\delta\omega_{D(T)} \ll \Omega_{D(T)}^P$, and that the relative weight of the cyclotron harmonics with $l \neq 0$ is small, since the mode perpendicular wavelength is typically much larger than the gyroradii of the fusing nuclei. Therefore, the resonance condition (5-1) effectively reduces to $\omega = \Omega_{D(T)}^P$. Modes that resonate with the deuteron spin have frequencies ω that fall in the low frequency range. On the basis of the analysis of Sec. 3-D, we conclude that these modes are unlikely to be excited by the α -particles, due to the stabilizing effect of the electron transit time damping. Thus, the polarization of the deuterons does not seem to be considerably affected by collective plasma fluctuations.

Modes that resonate with the triton spin have frequencies in the high frequency range, and an analysis of the

conditions for their excitation by the anisotropy of the α -particle distribution is appropriate.

A. Homogeneous Limit

The equilibrium α -particle velocity space distribution function can be written as $f_{\alpha 0}(\varepsilon, \mu) = f_{\alpha}^0(\varepsilon) F_{\alpha}^P(\varepsilon, \mu)$, where $f_{\alpha}^0(\varepsilon)$ is isotropic and has the characteristic slowing down dependence,¹⁶ $f_{\alpha}^0(\varepsilon) = (\varepsilon^{3/2} + \varepsilon_C^{3/2})^{-1}$ for $0.67 \text{ MeV} \approx \varepsilon_C \leq \varepsilon \leq \varepsilon_{\alpha} = 3.52 \text{ MeV}$ and vanishes for $\varepsilon > \varepsilon_{\alpha}$, while F_{α}^P depends on the relative state of polarization of the reacting D-T nuclei. If the spins are not coherently polarized, $F_{\alpha}^P = 1$. In the case in which the deuteron and triton spins are both either parallel or antiparallel to the equilibrium magnetic field, the α -particle distribution is proportional to $\sin^2 \psi$,¹ where ψ is the angle between the particle velocity and the magnetic field direction, so that $F_{\alpha}^P = \mu B_0 / \varepsilon$. For other types of coherent spin polarization, $F_{\alpha}^P = 1/4(1 + 3\cos^2 \psi)$ and $F_{\alpha}^P = 1 - 3\mu B_0 / (4\varepsilon)$. From the analysis of Sec. 2-B, a necessary condition for the excitation of the relevant magnetosonic modes is that $\Pi f_{\alpha 0} > 0$ in a significant region of velocity space, where $\Pi = \partial/\partial \varepsilon + B_0^{-1} \partial/\partial \mu$. Thus, the excitation of the considered modes by a resonant interaction with the α -particles is possible only for parallel coherent polarization of the D-T nuclei. In this case, the instability region in the $(v_{\parallel}, v_{\perp})$ plane is approximately given by $v_{\parallel}^2/v_{\perp}^2 \leq 3/2$. This picture, however, is complicated by the presence of a "jump" in the α -particle distribution function for $\varepsilon = \varepsilon_{\alpha}$ that, in general, provides a strong stabilizing contribution for $v = v_{\perp} = (2\varepsilon_{\alpha}/m_{\alpha})^{1/2}$. This contribution can be made

negligibly small if the resonance of the mode with the α -particles at $v=v_\alpha$ happens to occur near a zero of the "weight" function $[\lambda(\omega)J_p(\xi_\alpha) - \xi_\alpha/pJ'_p(\xi_\alpha)]^2$ of Eq. (2-16), where $\xi_\alpha = k_\perp v_\perp/\Omega_\alpha$.

In the limit of a homogeneous plasma configuration, the resonant surfaces in velocity space are planes $v_{||} = (\omega - p\Omega_\alpha)/k_{||} = \text{const.}$ The real part of the mode frequency is fixed by $\omega = 5.94\Omega_D = 5.94\Omega_\alpha$, and the relevant cyclotron harmonic is $p=6$. A sufficient condition for the occurrence of the instability can be determined by requiring that the resonant plane $v_{||} = \text{const.}$ lies completely inside the instability region, i.e., $|v_{||}| \geq (3/5)^{1/2} v_\alpha$, and that the first zero of the relevant weight function occurs for $v_\perp \leq (2/5)^{1/2} v_\alpha$. In terms of the physical parameter $B_0^2/(8\pi n_{e0})$, this condition is given by:

$$\frac{[B_0/(10\text{kG})]^2}{8\pi[n_{e0}/(10^{14}\text{ cm}^{-3})]} \leq 1/3. \quad (5-2)$$

A numerical integration of Eq. (2-16) shows that characteristically $\gamma_\alpha/\omega \approx 3 \times 10^{-1} n_\alpha/n_e$ for plasma parameters satisfying Eq. (5-2) and for modes with $|k_{||}/k_\perp| \approx 5 \times 10^{-3}$.

B. Inhomogeneous Plasma Model

We consider an inhomogeneous toroidal plasma configuration and analyze the resonant interaction of the radially localized modes discussed in Sec. 3 with the α -particles. Three effects of the inhomogeneity need to be considered. These are the modification of the mode- α -particle resonance condition due to

the particle perpendicular drift motion in an inhomogeneous magnetic field; the existence of a new source of free energy due to the α -particles density gradient; and the global nature of the resonant interaction. This makes the growth rate of the relevant modes to be given by:

$$\gamma_{\alpha}^{gl} = \frac{\int_0^a r dr \int_0^{2\pi} d\theta \gamma_{\alpha}^{loc.}(r, \theta) |\tilde{B}_{||}(r, \theta)|^2}{\int_0^a r dr \int_0^{2\pi} d\theta |\tilde{B}_{||}(r, \theta)|^2}, \quad (5-3)$$

where the expression for $\gamma_{\alpha}^{loc.}$ and $\tilde{B}_{||}(r, \theta)$ can be obtained from Eqs. (2-16) and (3-20), respectively. The first two effects were already included in the solution of the Vlasov equation for the perturbed α -particle distribution function given in Sec. 2-B. In the case of the radially localized modes of Sec. 3, the resonance condition is

$$\omega - k_{||} v_{||} - p \Omega_{\alpha}(R) - \omega_{D\alpha} = 0, \quad (5-4)$$

with $k_{||}$ given in Eq. (3-27). The magnetic drift frequency $\omega_{D\alpha}$ is approximated by

$$\omega_{D\alpha} = - \frac{k_{\theta} v_{\alpha}^2}{2 v_{\alpha} R} \cos \theta \left(\frac{1}{2} \frac{v_{\perp}^2}{v_{\alpha}^2} + \frac{v_{||}^2}{v_{\alpha}^2} \right) \approx \bar{\omega}_{D\alpha} \left(\frac{1}{2} \frac{v_{\perp}^2}{v_{\alpha}^2} + \frac{v_{||}^2}{v_{\alpha}^2} \right). \quad (5-5)$$

In fact, $\partial/\partial r \ll (1/r) \partial/\partial \theta$ for the relevant modes and we can identify $k_{\theta}^2 = -m/r_0$ with $k_{\perp}$ and neglect the geodesic contribution to $\omega_{D\alpha}$. The resonant surfaces in velocity space, at fixed values of the

spatial variables (r, θ) , are shifted ellipsoids (see Fig. 7). Considering a poloidal cross section of the plasma column, we notice that disconnected portions of the annulus of mode localization are involved in the mode particle resonances (5-4). Each portion corresponds to a resonance with a different harmonic of the α -particle cyclotron frequency (see Fig. 8). We consider modes with a frequency ω such that the $p=6$ resonance falls in the low field portion of the annulus. This choice focuses on the modes that can induce spin flip transitions of both the circulating and the trapped tritium nuclei. We adopt an inhomogeneous "slab" model in every disconnected portion of the resonance region, identifying the "slab" coordinate x with $r-r_0$ and keeping $\cos\theta = \text{const}$. We fix the real part of the mode frequency ω and evaluate the local growth rate as a function of x numerically. The global growth rate can then be estimated by averaging the "slab" local growth rates over the two-dimensional resonance region.

We find that, for realistic plasma parameters, the anisotropy of the α -particle distribution function is not sufficient to drive the modes unstable if the contribution due to the density gradient of the α -particles is switched off. This increased stability, relative to the homogeneous model, is due to the fact that the resonance surfaces in velocity space change with spatial position, thus causing strong cancellations between stabilizing and destabilizing contributions. When the α -particle density gradient is included, unstable modes are found for reasonable density profiles, i.e., for profiles of the α -particles that are somewhat more peaked than those of the bulk plasma.¹⁷

This 'widening' effect on the instability region in velocity space due to the α -particle density gradient is shown in Fig. 7.

In particular, we consider the case of a reference INTOR-like device, where $R_0 = 520$ cm, $a = 130$ cm, $n_{e0}(R_0) = 4 \times 10^{14}$ particles/cm³, and $B_0(R_0) = 50$ kG, assuming a quasi-parabolic electron density profile $n_{e0}(r) = n_0 [1 - (1-\delta)(r/a)^2 + \delta(r/a)^4]$, with $\delta = 0.2$ that corresponds to the radially well-localized modes of Fig. 4. We choose the real part of the frequency $\omega = 5\Omega_\alpha(R_0)$, so that the condition $\omega = 5.94 \Omega_\alpha(R_0 + r_0)$ is satisfied. Then the $p=6$ resonance occurs for $\cos\theta = 1$, the $p=5$ resonance for $\cos\theta = 0$ (so that it has a negligible spatial extension, as both the toroidal correction to $k_{||}$ and $\omega_{D\alpha}$ are proportional to $\cos\theta$), and the $p=4$ resonance for $\cos\theta = -1$ (see Fig. 8).

In Fig. 9, the oscillating behavior of the relevant sum $\gamma_\alpha^{\text{loc.}}(x, p=4) + \gamma_\alpha^{\text{loc.}}(x, p=6)$ is shown as a function of the "slab" coordinate x , for $\omega_{* \alpha} / \omega = 0.55$. This latter choice corresponds to an α -particle density scale length $r_{n\alpha} / a = 0.14$. From this, we can deduce that the global growth rate is positive, the instability being due to the combined effect of the α -particle anisotropy and density gradient. An estimate of the global α -particle growth rate can be obtained from averaging local values given above. Then we have approximately $\gamma_\alpha^{\text{gl.}} / \omega \sim 2.3 \times 10^{-1} (\Delta\theta / 2\pi) n_\alpha(r_0) / n_e(r_0)$, where $\Delta\theta$ is the poloidal width of the resonance region. Taking $n_\alpha(r_0) / n_e(r_0) \sim 0.2 n_{20} / n_{e0}$ with $n_{20} / n_{e0} \sim 2 \times 10^{-3}$ (see Sec. 6), we obtain the estimate $\gamma_\alpha^{\text{gl.}} / \omega \sim 3 \times 10^{-5}$.

Finally, we notice that a more consistent evaluation of the mode-particle resonances relevant to our problem would require

the use of the analytical formalism developed for the theory of trapped particle modes¹⁸ in toroidal geometries. However, given the complexity involved, we consider this undertaking to be beyond the scope of the present paper.

C. Bulk Ion Cyclotron Damping

Now we consider the bulk ion cyclotron damping rate in the case of the high frequency modes, and show that for the reference device it is small in comparison with the α -particle growth rate previously derived. Referring to Eq. (2-12), we have:

$$\gamma_i^p / \omega = - \frac{n_i}{n_e} \frac{\pi^{1/2}}{2} \frac{\Omega_i \tilde{\Omega}}{\omega^2} \frac{\omega}{|k_{\parallel}| v_{thi}} \exp(-b_i - v_{\parallel R}^2 / v_{thi}^2) \left(\frac{b_i}{2}\right)^{p-1} \frac{1}{(p-1)!} \\ \times [\lambda p(\lambda/2 - 1) + p + 1] \quad (5-6)$$

for the resonance with the p -th ion harmonic. The quantity b_i can be conveniently parametrized as $b_i = (3/5) (\omega/\Omega_i)^2 (m_p/m_i) \beta_i$, where $\beta_i = 8\pi(n_D T_D + n_T T_T) / B_0^2$ is the ion beta.

In the homogeneous case, $\omega = 5.94 \Omega_D$, and the strongest contribution to the damping rate is due to the deuterons at the $p=6$ harmonic. This damping, however, is negligibly small, since the resonant deuteron parallel velocity is much larger than their thermal velocity. In fact, for $|k_{\perp}|/k_A \sim 5 \times 10^{-2}$ we have $(v_{\parallel R} / v_{thD})^2 \sim [10^{-2} (k_{\perp}/k_{\parallel}) (\tilde{v}_{A0} / v_{thD})]^2 \sim 48^{-1} \gg 1$.

In the inhomogeneous case, resonances with different ion cyclotron harmonics are possible due to the spatial structure

of the modes. When the mode frequency is $\omega = 5\Omega_D(R_0)$, the deuteron resonances are similar to those of the α -particles shown in Fig. 8 (since $\Omega_D = \Omega_\alpha$), while the triton resonances occur at higher harmonics (typically, for $p=6,7,8$) and thus produce a weaker mode damping rate.

The $p=5$ deuteron resonance is always present, since $v_{||R} \approx 0$ at $R=R_0$. Around $\theta=\pi/2$, the toroidal correction to $k_{||}$ is small, and $k_{||}$ can be approximated by its value in the cylindrical case. We introduce a poloidal averaged deuteron damping rate in order to take the resonance width into account and obtain

$$\langle \gamma_D^{p=5} \rangle_\theta = \gamma_D^{p=5}(\theta=\pi/2, r=r_0) \frac{1}{2\pi} \int_0^{2\pi} d\theta \exp \left\{ -[\omega - 5\Omega_D(\theta)]^2 / (k_{||} v_{thD})^2 \right\} \approx$$

$$\approx \frac{1}{2\pi^{1/2}} \frac{|k_{||}| v_{thD}}{\omega} \frac{1}{[d \ln \Omega_D / d\theta]} \bigg|_{\theta=\pi/2, r=r_0} \gamma_D^{p=5}(\theta=\pi/2, r=r_0). \quad (5-7)$$

Then from Eq. (5-6), we have

$$\langle \gamma_D^{p=5} \rangle_\theta / \omega \approx -7.3 \times 10^{-3} \frac{n_0}{n_e} [s_{i0}]^4, \quad (5-8)$$

where s_{i0} is taken at the magnetic axis. For a fusing plasma, s_{i0} is characteristically 10%. Thus, the damping rate due to the $p=5$ resonance is smaller than the growth rate due to the α -particles. For inverse aspect ratios $\epsilon \gtrsim \sqrt{2}/5$, the $p=4$ and $p=6$ deuteron resonances are also present. The condition on ϵ is obtained by imposing $v_{||R} \approx 0$ for $R \gtrsim R_0 - r_0$ and $R \lesssim R_0 + r_0$.

respectively. For the considered reference device, $\epsilon=1/4$, and these resonances do not occur. When considering devices with larger values of ϵ , the damping rate due to the $p=4$ deuteron resonance may become comparable to the α -particle growth rate, due to the strong dependence on p of the ion damping rate [see Eq. (5-6)].

VI. SPIN DEPOLARIZATION RATE

A simple estimate of the depolarization rate of the triton spins induced by the fluctuating magnetic field of the unstable modes can be obtained by referring to the homogeneous model presented in Sec. 2. As shown in Sec. 5, the resonance condition between the spins of the tritons and the modes is given by $\omega = \Omega_T^p$. The transition probability per precession period between the triton spin states is

$$w = \left(\frac{2\pi}{\hbar} \right)^2 \left| \left\langle -\frac{1}{2} \mid \underline{\mu} \cdot \underline{\tilde{B}} \mid \frac{1}{2} \right\rangle \right|^2 \frac{\delta(\Omega_T^p - \omega)}{\Omega_T^p} \quad (6-1)$$

where $\underline{\mu}$ is the magnetic moment of the triton and the interaction Hamiltonian is $H = -\underline{\mu} \cdot \underline{\tilde{B}}$. Using the expression for the fluctuating magnetic field derived in Sec. 2, Eq. (6-1) becomes

$$w = \gamma^2 \Omega_T^p \left| \frac{\tilde{B}_\perp}{B_0} \right|^2 \left\{ \frac{(1 - \lambda(\Omega_T^p)^2)^2}{1 + \lambda(\Omega_T^p)^2} \right\} \delta(\Omega_T^p - \omega) \quad (6-2)$$

where $\tilde{B}_\perp$ is the amplitude of the perturbed perpendicular magnetic field. The term $[1 - \lambda(\Omega_T^p)^2]^2$ arises from the angular

momentum selection rule, which requires that only the left circularly polarized component of the fluctuating magnetic field has a nonvanishing matrix element. For $\alpha_D = \alpha_T = 1/2$, the factor in braces in Eq. (6-2) is approximately equal to 0.7. Then, within the homogeneous model, the depolarization rate $\nu_{\text{dep}} = \Omega_T^P \omega / (2\pi)$ of the triton spins can be parameterized as

$$\nu_{\text{dep}} = \frac{\pi}{2} \Omega_T^P \left(\frac{k_{\parallel}}{k_{\perp}} \right)^2 \left(\frac{\tilde{B}_{\parallel}}{B_0} \frac{n_e}{10^{-1} n_{\alpha}} \right)^2 \left(\frac{\Omega_T^P}{\Delta\omega} \right) \left(\frac{10^{-1} n_{\alpha}}{n_e} \right)^2 s^{-1}, \quad (6-3)$$

where we have used the relationship $k_{\perp} \tilde{B}_{\perp} = k_{\parallel} \tilde{B}_{\parallel}$, and $\Delta\omega$ is an effective frequency width. An assumption on the saturation level of the instability has been made by normalizing $\tilde{B}_{\parallel}/B_0$ to the ratio γ_{α}/ω , which we have shown to be characteristically of order $10^{-1} n_{\alpha}/n_e$. The width $\Delta\omega$ depends, in the homogeneous model, on the expected features of the unstable spectrum.⁶ However, $\Delta\omega$ can be suitably defined to account for the effects of the inhomogeneous configuration described in Sec. 3. In such a configuration, the triton spins interact with the fluctuating magnetic field only when the tritons cross the region around the intersection between the annulus where the mode is localized and the plane where $\Omega_T^P = \omega$. This region must fall in the outer part of the annulus in order that all circulating particles, and most of the trapped particles with orbits around the magnetic surfaces contained in the annulus, cross it. The interaction between the spins and the mode is localized because the relative phase between the wave and the spin precession is not constant, as is the case in the homogeneous model at resonance, where the

interaction takes place for an infinite time. The relative phase varies with time as $\omega t - \int^t \Omega_T^P(t') dt' \approx 1/2 (d\Omega_T^P/dt) t^2$, where the time dependence of Ω_T^P is due to the triton motion in the inhomogeneous magnetic field and t has been chosen so that $\Omega_T^P(t=0) = \omega$. Then the transition probability per crossing of the interaction region is given by Eq. (6-1), with $|2\pi/(d\Omega_T^P/dt)|$ substituted for $4\pi^2 \delta(\omega - \Omega_T^P)/\Omega_T^P$. Since the crossing rate is given by $2v_{||}/(qR)$ and $d\Omega_T^P/dt = d\Omega_T^P/dR [-v_{||} r_0 \sin\theta_0 / (qR)] = +\Omega_T^P v_{||} r_0 \sin\theta_0 / (qR^2)$, with θ_0 a small angle that corresponds to the poloidal width of the interaction region (see Fig. 8), the depolarization rate for particles within the annulus is given by Eq. (6-3) with

$$\Delta\omega = \Omega_T^P r_0 \sin\theta_0 / (2R). \quad (6-4)$$

The tritons cross the annulus where the mode is localized due to their radial diffusion through the plasma. This diffusion time is short compared to the time for which the polarization has to be maintained. It is thus convenient to define a spatially averaged depolarization rate $\langle v_{\text{dep}} \rangle$, which accounts for the radial width of the mode and is given by

$$\langle v_{\text{dep}} \rangle = (d_i/r_0)^{1/2} v_{\text{dep}}. \quad (6-5)$$

The ratio n_x/n_e in Eq. (5-4) can be estimated by setting $n_x \tau_x / \tau_s \approx 3n_e T / \tau_E$, where τ_s is the x -particle slowing down time and τ_E is the energy replacement time. Making use of these results and taking $\tilde{B}_{||}/B_0 \sim \gamma_\alpha/\omega$ for the amplitude of

the magnetic fluctuations, so that $\hat{B}_1/B_0 \sim 1.5 \times 10^{-3} n_a/n_e$, we obtain

$$\langle v_{\text{dep}} \rangle \tau_E = 10\pi \frac{R}{r_0} \left(\frac{T}{\epsilon_a} \right)^2 \left(\frac{d_i}{\sin^2 \theta_0 r_0} \right)^{1/2} \frac{1}{v_{ei}^2 \tau_E} \Omega_T^P, \quad (6-6)$$

where v_{ei} is the electron-ion collision rate. For reactor parameters, employing the Lawson criterion to estimate τ_E , we find $\langle v_{\text{dep}} \rangle \tau_D \sim 5$ for $\tau_D \sim 5\tau_E$, τ_D being the particle diffusion time. This shows that the depolarization time due to the interaction with the unstable magnetosonic waves is at least as short as the diffusion time. Since in practice we must consider many recycling times, the triton depolarization is significant.

VII. CONCLUSIONS

We have developed the analysis of an important class of high frequency modes that can be excited in a multispecies, magnetically confined plasma, when an additional component of high energy particles with a velocity space distribution with a positive perpendicular slope is present. This component can be produced under certain conditions by the occurrence of nuclear fusion reactions, or by the injection of fast neutrals for plasma auxiliary heating. We have analyzed in particular the case where the components of a fusing plasma are coherently spin polarized and the charged fusion reaction products have an anisotropic distribution in velocity space. In a multispecies

plasma, the relevant modes exhibit different properties, one for frequencies of the order of the ion cyclotron frequency and one for higher frequencies. In a realistic magnetic confinement configuration such as an axisymmetric torus, the existence of spatially localized normal mode solutions is proved for the higher frequency range, under certain restrictions on the allowed particle density profiles. This implies that the energy that these "circulating" modes can gain from the resonant interaction with the high energy population is not subject to spatial convection. In the low frequency range, instead, localized modes "trapped" in the low magnetic field side of the torus are obtained only when the cut-off surface occurs at the center of the plasma column. At finite β regimes, however, these low frequency modes tend to be stabilized by electron transit-time damping.

On the basis of the analysis presented, we conclude that in plasmas spin polarized to enhance the nuclear reaction rate, the tritium nuclei will be depolarized in toroidal plasma confinement configurations with relatively mild gradients of the confining magnetic field. In a low- β plasma this corresponds to aspect ratios larger than about four.

ACKNOWLEDGMENTS

We wish to thank F. Rubini for developing the numerical code used to determine the ray trajectory discussed in Sec. 4. Useful discussions with E. Canobbio and M. Brambilla are gratefully acknowledged.

This work was sponsored in part by the U. S. Department of Energy, under Contract DE-AC08-78ET51013, and by the Consiglio Nazionale delle Ricerche of Italy.

One of us (P.D.) was supported in part by the Comitato Ignitor of Turin.

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FIGURE CAPTIONS

- Fig. 1: Graphic representation of the dispersion relation (2-8). The relevant symbols are defined in Sec. 2. In particular, $d_i = (m_i/m_e)d_e$ is the ion inertial skin depth and $\bar{v}_A = \bar{v}_A \Omega_{Hy} / \bar{\Omega}$. The two branches correspond to the two "MHD" waves: $\omega^2 = k_\perp^2 \bar{v}_A^2$ for $\omega > \Omega_D$, and $\omega^2 = k_\perp^2 \bar{v}_A^2$ for $\omega < \Omega_T$.
- Fig. 2: Turning points determining the radial width of the modes for a parabolic density profile in a cylinder.
- Fig. 3: "Drifting" ray trajectory in a poloidal plane for a toroidal configuration. This trajectory results from the integration of Eq. (4-4) for the case of high frequency modes when a parabolic density profile is assumed. The plasma minor radius is $a=130$ cm, and the inverse aspect ratio is $\epsilon=1/4$.
- Fig. 4: Ray trajectory for high frequency modes in the case of the quasi-parabolic density profile $n(r)=n_0[1-(1-\delta)(r/a)^2-\delta(r/a)^4]$, with $\delta=0.2$. Notice the toroidal distortion of the region of ray trapping. The dashed curve is the circular trajectory obtained in a cylindrical configuration.
- Fig. 5: Ray trajectory for low frequency modes with the cutoff surface $\omega=\bar{\omega}(R)$ at $\theta=\pi/2$.

Fig. 6: Ray trajectory for low frequency modes with a small displacement of the cutoff surface from the position $\theta = \pi/2$.

Fig. 7: Resonant curve (solid portion of the "shifted" ellipse) in the α -particle velocity space for $k_{\parallel} v_{\alpha} \neq 0$. The shaded area corresponds to the part of velocity space where the resonant interaction is destabilizing when the contribution of the α -particle density gradient is neglected. The dotted area shows the widening of the destabilizing region when this effect is included for the specific case $\omega/\omega_{* \alpha} = 0.55$ with $\omega_{* \alpha} = k_{\perp} \epsilon_{\alpha} / (m_{\alpha} \Omega_{\alpha} r_{n \alpha})$.

Fig. 8: Resonance regions between the mode and the various harmonics of the α -particle cyclotron frequency in the poloidal plane. The mode is radially localized around the surface $r=r_0$, the radial width being Δ . The mode frequency is $\omega = 5\Omega_{\alpha}(R_0)$. The angle θ_0 corresponds to the poloidal width of the region of interaction between the mode and the triton spins.

Fig. 9: Local growth rate $\gamma_{\alpha}^{loc.}(x)$ as a function of the "slab" variable x . The parameters are as in Figure 8. The plotted values of $\gamma_{\alpha}^{loc.}(x)$ are given by the sum of the growth rates deriving from the resonances with the $p=4$ and the $p=6$ harmonics of the α -particle cyclotron frequency.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

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