CONF-8807167--1

ANL-HEP-CP-88-84

FERMIONIC CENTER IN THE SUPERCONFORMAL ALGEBRA ON THE SUPERTORUS * *R*

ANL-HEP-CP--88-84

Cosmas K. Zachos

DE89 013273

High Energy Physics Division
Argonne National Laboratory, Argonne, IL 60439, USA

Abstract

Krichever and Novikov (KN) have introduced a generalization of the Virasoro algebra which describes conformal transformations on a Riemann surface of arbitrary genus. We examine their genus one algebra, find explicit expressions for its center, and discuss connections and symmetries among the algebra constants. We then proceed to construct explicitly the corresponding superconformal algebra on the supertorus. An intriguing feature of the NS odd spin structure algebra is that it is enriched by a central element which is anticommuting.

1. I am outlining work done in collaboration with L. Mezincescu and R. Nepomechie [1]. My coverage here is eclectic, and I would refer readers to our paper for a more detailed discussion. Krichever and Novikov (KN) [2] have pioneered a framework for describing conformal transformations on Riemann surfaces of arbitrary genus, of relevance in string physics and critical systems. Their generalization of the Virasoro algebra to genus one, the torus, is:

$$[e_n, e_m] = \sum_s c_{nm}^s e_{n+m-s} + c \chi_{nm}, \quad (1)$$

where the indices are half-integers and the constants c_{nm}^s , χ_{nm} are specified below.

We generalize this algebra to the supertorus. Our most striking algebra is the odd spin structure NS algebra:

$$\begin{aligned} [e_n, e_m] &= \sum_{s=-3/2}^{3/2} c_{nm}^s e_{n+m-s} - \frac{\delta}{2} \sum_{s=-5/2}^{5/2} d_{nm}^{s+1} g_{n+m-s-1} + \frac{3\hat{c}}{2} \chi_{nm}, \\ \{g_n, g_m\} &= -2 \sum_{s=-1/2}^{1/2} f_{nm}^s e_{n+m-s} - 2\delta \sum_{s=-3/2}^{3/2} h_{nm}^{s+1} g_{n+m-s-1} + \frac{\hat{c}}{2} \psi_{nm}, \\ [e_n, g_m] &= \sum_{s=-3/2}^{3/2} u_{nm}^s g_{n+m-s} + 2\delta \sum_{s=-3/2}^{3/2} h_{nm}^{s+1} e_{n+m-s-1} - \frac{\hat{c}}{4} \rho_{nm}. \end{aligned} \quad (2)$$

The structure constants and cocycles are related among themselves and are specified below. This algebra bears some correspondence to the Neveu-Schwarz algebra. Nevertheless, observe that the

*Talk at the IV International Workshop on Recent Developments in High Energy Physics, Crete, July 1988. Supported by the U.S. Department of Energy, Division of High Energy Physics, Contract W-31-109-ENG-38.

fermionic modular parameter δ of the supertorus (which may be thought of as a gravitino zero mode) allows new operator terms on the right-hand-side of this algebra, as well as the above fermionic center, parameterized by the numbers ρ_{nm} .

Since functions on the torus are doubly-periodic, they are *elliptic* functions [3]. I will sketch below the convenient elliptic function basis introduced by KN; how it specifies the conformal algebra; and how we generalize both this basis and hence the KN algebra to the supertorus [4]. For questions not pertaining to eq.(2), you may further consult other relevant work [5].

2. The torus has two periods, $2\omega_1$ and $2\omega_2$, with $\text{Im}(\omega_2/\omega_1) > 0$. It may be defined by identifying the points on the complex plane $z \approx z + 2m\omega_1 + 2n\omega_2$, for arbitrary integers m and n . Henceforth refer all points of the plane to this fundamental cell. Without loss of generality, choose $2\omega_1 = 1$, $2\omega_2 = \tau$.

Meromorphic functions on the torus are doubly periodic and are called *elliptic* functions [3]. An elliptic function without poles has to be a constant; the order of elliptic functions (number of poles in a cell) is two or higher; the sum of the locations of the poles of an elliptic function equals the sum of the locations of the zeros (equal in number), properly weighted by their multiplicities; significantly, the residues of an elliptic function at its poles in each cell sum to zero [3].

The prototype elliptic function is the Weierstrass $\mathcal{P}(z)$ function, with a double pole at the origin. Weierstrass further defined the odd quasi-elliptic functions $\zeta(z)$ (not the Riemann zeta function) and $\sigma(z)$ through $\zeta'(z) = -\mathcal{P}(z)$ and $\sigma'(z)/\sigma(z) = \zeta(z)$; ζ has a simple pole with residue 1 at the origin, while $\sigma(z) \sim z$ for $z \sim 0$. The functions $\mathcal{P}$, ζ , and σ stand in correspondence to the simply periodic functions $-\text{cosec}^2(z)$, $\cot$, and $\sin$, respectively. They may be expressed in terms of the less handy but more standardized Jacobi theta functions [3].

In analogy with the genus 0 case, Krichever and Novikov [2] distinguish two points on the surface, (punctures), which can be mapped to $z = z_0$ and $z = -z_0$. They then provide a basis for all elliptic functions with poles only at $z = \pm z_0$. Such functions may be expanded in the complete basis of elliptic functions $\{A_n(z)\}$, with n half-integer :

BASIS FUNCTIONS	$z \sim z_0$	$z \sim -z_0$
$A_n(z) = \frac{\sigma^{n-\frac{1}{2}}(z-z_0)\sigma(z+2nz_0)}{\sigma^{n+\frac{1}{2}}(z+z_0)} \frac{\sigma^{n+\frac{1}{2}}(2z_0)}{\sigma((2n+1)z_0)}, \quad n \neq -\frac{1}{2}$	$(z-z_0)^{n-\frac{1}{2}}$	$a_n(z+z_0)^{-n-\frac{1}{2}}$
$A_{1/2} = 1$	1	1
$A_{-\frac{1}{2}}(z) = \frac{\sigma^2(z)}{\sigma(z+z_0)\sigma(z-z_0)\sigma^2(z_0)}$	$(z-z_0)^{-1}$	$-(z+z_0)^{-1}$

(3)

Note that the constants

$$a_n = (-1)^{n-\frac{1}{2}} \frac{\sigma^{2n}(2z_0)\sigma((2n-1)z_0)}{\sigma((2n+1)z_0)}; \quad (4)$$

are an artefact of the lopsided normalization which simplifies behavior at z_0 at the expense of behavior at $-z_0$. You may check that the zeros and poles of this basis satisfy the abovementioned summation relations. Generic negative and positive n 's have their poles at z_0 and $-z_0$, respectively, while $A_{1/2}$, which obeys the generic formula, has none, and $A_{-1/2}$ has to straddle these two domains

with one pole at each point, since one cannot have an elliptic function with a simple pole. This minor asymmetry is reflected in the orthonormality relations:

$$\oint_{z_0} \frac{dz}{2\pi i} A_n(z) A_m(z) = \delta_{n+m,0}, \quad n, m \neq -\frac{1}{2};$$

$$\oint_{z_0} \frac{dz}{2\pi i} A_n(z) A_{-\frac{1}{2}}(z) = \delta_{n, \frac{1}{2}}, \quad \text{for } n \neq -\frac{1}{2}; \quad \oint_{z_0} \frac{dz}{2\pi i} [A_{-\frac{1}{2}}(z)]^2 = 4\zeta(z_0) - 2\zeta(2z_0). \quad (5)$$

The integrations are performed along a counterclockwise path about the point z_0 . To prove these relations, use the fact that the sum of the residues of an elliptic function at its poles in any cell is zero. As a result, the integrals may be evaluated alternatively through clockwise integration about $-z_0$. In the generic case, the product $A_n(z)A_m(z)$ is regular at either $z = z_0$ or $z = -z_0$, and hence the integrals vanish, except for $n + m = 0$.

You may thus use the above conditions to evaluate the expansion coefficients of any elliptic function with poles at $\pm z_0$ in terms of the A_n . For example, since the derivative of an elliptic function is also an elliptic function, it too may be expanded in the same basis:

$$A'_n(z) = \sum_m \gamma_{n,-m}(z_0) A_m(z). \quad (6)$$

so that

$$\gamma_{nm}(z_0) \equiv \oint_{z_0} \frac{dz}{2\pi i} A'_n(z) A_m(z). \quad (7)$$

and hence

$$\gamma_{nm}(z_0) = 0, \quad |n+m| > 1, \quad (n, m) \neq (-1/2, -3/2); \quad (8a)$$

$$\gamma_{n,1-n}(z_0) = n - \frac{1}{2}, \quad (8b)$$

$$\gamma_{n,-n}(z_0) = \left(n + \frac{1}{2}\right) D^{(1)}\left(2z_0\left(n + \frac{1}{2}\right)\right) + \left(n - \frac{1}{2}\right) D^{(1)}\left(2z_0\left(\frac{1}{2} - n\right)\right), \quad (8c)$$

where $D^{(1)}$ is defined below ((18a)),

$$\gamma_{n,-1-n}(z_0) = \left(n + \frac{1}{2}\right) \left[\zeta'\left(2z_0\left(n + \frac{1}{2}\right)\right) - \zeta'(2z_0) \right], \quad (8d)$$

$$\gamma_{-1/2,-3/2}(z_0) = -a_{-3/2}(z_0). \quad (8e)$$

The above then provides a matrix representation for the derivative operator in this basis. The raising-preserving-lowering operation on the index of the operand may be symbolized by . This may be used successively in subsequent manipulations which involve derivations, in conjunction with multiplication of basis functions A_n , which also closes in the same basis $\{A_n\}$.

3. Consider now a conformal transformation $z \rightarrow z + \epsilon(z)$, where z is a local complex coordinate. The corresponding generator L_ϵ defines a vector field:

$$L_\epsilon = \epsilon(z) \frac{\partial}{\partial z}, \quad (9)$$

and satisfies the (centerless) conformal algebra

$$[L_{\epsilon_1}, L_{\epsilon_2}] = L_{\epsilon_1 \frac{\partial}{\partial z} \epsilon_2}, \quad (10)$$

where $\epsilon_1 \overleftrightarrow{\partial}_z \epsilon_2 \equiv \epsilon_1 \partial_z \epsilon_2 - (\partial_z \epsilon_1) \epsilon_2$. One can treat anti-analytic transformations $\bar{z} \rightarrow \bar{z} + \bar{\epsilon}(\bar{z})$ analogously; to avoid repetition, focus on the analytic ones.

Recall how this reduces in the genus 0 case. The coordinate z on the complex plane serves as a global coordinate on the sphere, by stereographic projection. Distinguish the points (punctures) at $z = 0$ and $z = \infty$, which lie in the infinite past and in the infinite future, respectively. Consider functions $\epsilon(z)$ which are analytic for $z \neq 0, \infty$: such functions can be expanded in the basis $\{z^{n+1}\}$, for n integer. These are the globally-defined functions on the doubly-punctured sphere. In light of the above basis, generalization to the torus follows naturally. I proceed to compare functions, bases for conformal transformation generators, their commutation structure, and centers for the sphere and the torus with punctures at $\pm z_0$, respectively:

VIRASORO

$$\epsilon(z) = \sum_{n \in \mathbb{Z}} \epsilon_n z^{n+1}$$

$$L_n = z^{n+1} \frac{\partial}{\partial z}$$

$$z^{n+1} \overleftrightarrow{\partial}_z z^{m+1} = (m - n) z^{n+m+1}$$

$$[L_n, L_m] = (m - n) L_{n+m}$$

$$L_n = - \oint_0 \frac{dz}{2\pi i} z^{n+1} T(z)$$

$$\frac{1}{12} \oint \frac{dz}{2\pi i} (z^{n+1})''' z^{m+1}$$

KRICHEVER-NOVIKOV

$$\epsilon(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \epsilon_n A_n(z)$$

$$e_n = A_n(z) \frac{\partial}{\partial z}$$

$$A_n(z) \overleftrightarrow{\partial}_z A_m(z) = \sum_{s=-3/2}^{3/2} c_{nm}^s(z_0) A_{n+m-s}(z)$$

$$[e_n, e_m] = \sum_{s=-3/2}^{3/2} c_{nm}^s(z_0) e_{n+m-s}$$

$$e_n = - \oint_{z_0} \frac{dz}{2\pi i} A_n(z) T(z)$$

$$\chi_{nm} \equiv \frac{1}{12} \oint_{z_0} \frac{dz}{2\pi i} A_n'''(z) A_m(z) . \quad (11)$$

The above KN expressions hold for the generic case $|n| \neq \frac{1}{2}, |m| \neq \frac{1}{2}, n+m \neq -2$. For special values of n and m (e.g., $|n| = \frac{1}{2}, |m| = \frac{1}{2}$, etc.), they are modified. $T(z)$ is an analytic energy-momentum tensor of a conformal field theory on the respective 2-d manifold, whose moments, as indicated, provide an alternative representation of the algebra: but now a non-trivial center is also specified, through the singularities of the standard short-distance expansion for z_1 close to z_2 ,

$$T(z_1)T(z_2) \sim \frac{c/2}{(z_1 - z_2)^4} + \frac{2}{(z_1 - z_2)^2} T(z_2) + \frac{1}{(z_1 - z_2)} \partial_2 T(z_2) + (\text{finite}), \quad (12)$$

with c a positive real number characterizing the particular two-dimensional quantum field theory. The counterclockwise paths of integration run along equal-time surfaces about the point z_0 . Standard manipulations relate equal-time commutators to this operator product:

$$\langle [e_n, e_m] \rangle = \oint_{z_0} \frac{dz_2}{2\pi i} A_m(z_2) \oint_{z_2} \frac{dz_1}{2\pi i} A_n(z_1) \langle T(z_1)T(z_2) \rangle . \quad (13)$$

The structure constants of KN may be evaluated directly from orthonormality:

$$c_{nm}^s(z_0) = \oint_{z_0} \frac{dz}{2\pi i} \left[A_n(z) \overleftrightarrow{\partial}_z A_m(z) \right] A_{-n-m+s}(z), \quad (14)$$

$$|n|, |m|, |n+m-s| \neq \frac{1}{2}; \quad n+m \neq -2.$$

In the generic case, they vanish for $|s| > 3/2$ and are [2]:

$$c_{nm}^{3/2} = m - n, \quad (15)$$

$$c_{nm}^{1/2} = (n-m)D^{(1)}(2z_0(n+m-1)) + (m-1/2)D^{(1)}(2z_0(m+1/2)) - (n-1/2)D^{(1)}(2z_0(n+1/2)),$$

where $D^{(1)}$ is defined in (18a). Note the contrast to the Virasoro structure constants: here they depend on z_0 and the modular parameter τ through their dependence on the functions σ and ζ . Likewise [2],

$$\chi_{nm}(z_0) \equiv \frac{1}{12} \oint_{z_0} \frac{dz}{2\pi i} A_n'''(z) A_m(z), \quad (16)$$

which we compute:

$$\chi_{n,m} = 0, \quad n+m > 3; \quad (17a)$$

$$\chi_{n,3-n} = \frac{1}{12} \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) \left(n - \frac{5}{2}\right), \quad (17b)$$

$$\chi_{n,2-n} = \frac{1}{12} \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) \left\{ \left(n + \frac{1}{2}\right) D^{(1)} \left(2z_0 \left(n + \frac{1}{2}\right)\right) + \left(n - \frac{5}{2}\right) D^{(1)} \left(2z_0 \left(\frac{5}{2} - n\right)\right) \right\}, \quad (17c)$$

$$\begin{aligned} \chi_{n,1-n} = \frac{1}{12} \left(n - \frac{1}{2}\right) \left\{ \frac{1}{2} \left(n + \frac{1}{2}\right) \left(n + \frac{3}{2}\right) D^{(2)} \left(2z_0 \left(n + \frac{1}{2}\right)\right) + \frac{1}{2} \left(n - \frac{3}{2}\right) \left(n - \frac{5}{2}\right) D^{(2)} \left(2z_0 \left(\frac{3}{2} - n\right)\right) \right. \\ \left. + \left(n + \frac{1}{2}\right) \left(n - \frac{3}{2}\right) D^{(1)} \left(2z_0 \left(n + \frac{1}{2}\right)\right) D^{(1)} \left(2z_0 \left(\frac{3}{2} - n\right)\right) \right\}, \end{aligned} \quad (17d)$$

$$\begin{aligned} \chi_{n,-n} = \frac{1}{12} \left\{ \frac{1}{6} \left(n + \frac{1}{2}\right) \left(n + \frac{3}{2}\right) \left(n + \frac{5}{2}\right) D^{(3)} \left(2z_0 \left(n + \frac{1}{2}\right)\right) + \frac{1}{6} \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) \left(n - \frac{5}{2}\right) D^{(3)} \left(2z_0 \left(\frac{1}{2} - n\right)\right) \right. \\ \left. + \frac{1}{2} \left(n + \frac{1}{2}\right) \left(n + \frac{3}{2}\right) \left(n - \frac{1}{2}\right) D^{(2)} \left(2z_0 \left(n + \frac{1}{2}\right)\right) D^{(1)} \left(2z_0 \left(\frac{1}{2} - n\right)\right) \right. \\ \left. + \frac{1}{2} \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) \left(n + \frac{1}{2}\right) D^{(2)} \left(2z_0 \left(\frac{1}{2} - n\right)\right) D^{(1)} \left(2z_0 \left(n + \frac{1}{2}\right)\right) \right\}. \end{aligned} \quad (17e)$$

We have introduced the notation

$$D^{(1)}(2z_0 m) \equiv \zeta(2z_0 m) - m\zeta(2z_0), \quad (18a)$$

$$D^{(2)}(2z_0 m) \equiv \zeta'(2z_0 m) - m\zeta'(2z_0) + \left[D^{(1)}(2z_0 m)\right]^2, \quad (18b)$$

$$D^{(3)}(2z_0 m) \equiv \zeta''(2z_0 m) - m\zeta''(2z_0) + 3D^{(2)}(2z_0 m)D^{(1)}(2z_0 m) - 2\left[D^{(1)}(2z_0 m)\right]^3. \quad (18c)$$

In this notation, the antisymmetry of the cocycle $\chi_{n,m} = -\chi_{m,n}$ is manifest. It is evident in eq.(16) that the cocycle vanishes when either index is $1/2$.

We have given the values of $\chi_{n,m}$ only for $n+m \geq 0$, and c_{nm}^s for positive superscripts. However, for $n+m < 0$, no further calculations are necessary, because of the identity

$$\chi_{-n,-m}(z_0) = \frac{-1}{a_n(-z_0)a_m(-z_0)} \chi_{n,m}(-z_0), \quad |n|, |m| \neq \frac{1}{2}, \quad (19)$$

which relates values of the cocycle for $n+m < 0$ to the known values for $n+m > 0$. This identity, as well as the corresponding one for the structure constants

$$c_{-n,-m}^s(z_0) = \frac{a_{n+m+s}(-z_0)}{a_n(-z_0)a_m(-z_0)} c_{nm}^{-s}(-z_0), \quad |n|, |m|, |n+m+s| \neq \frac{1}{2}; \quad n+m \neq -2, \quad (20)$$

follow from the reflection property of the basis:

$$A_n(z, -z_0) = A_{-n}(z, z_0) a_n(-z_0) \quad , \quad |n| \neq \frac{1}{2}. \quad (21)$$

To see this, consider the expression (16) for the cocycle, now written to display the parameter z_0 :

$$\chi_{nm}(z_0) = \frac{1}{12} \oint_C \frac{dz}{2\pi i} A_n'''(z, z_0) A_m(z, z_0), \quad (22)$$

where C is a closed counterclockwise path around z_0 . Sending $z_0 \rightarrow -z_0$ yields

$$\chi_{nm}(-z_0) = \frac{1}{12} \oint_D \frac{dz}{2\pi i} A_n'''(z, -z_0) A_m(z, -z_0), \quad (23)$$

where D is a closed counterclockwise path around $-z_0$. Since the integrand is elliptic and has poles only at $\pm z_0$, and by virtue of the fact that the sum of the residues of an elliptic function at its poles in any cell is zero, it follows by use of (21) that

$$\begin{aligned} \chi_{nm}(-z_0) &= -\frac{1}{12} \oint_C \frac{dz}{2\pi i} A_n'''(z, -z_0) A_m(z, -z_0) \\ &= -a_n(-z_0) a_m(-z_0) \frac{1}{12} \oint_C \frac{dz}{2\pi i} A_{-n}'''(z, z_0) A_{-m}(z, z_0) \end{aligned} \quad (24)$$

which is the identity (19).

Likewise, for $|n|, |m|, |n+m-s| \neq \frac{1}{2}; n+m \neq -2$,

$$c_{nm}^s(z_0) = \oint_C \frac{dz}{2\pi i} \left[A_n(z, z_0) \vec{\partial}_z A_m(z, z_0) \right] A_{-n-m+s}(z, z_0). \quad (25)$$

Now send $z_0 \rightarrow -z_0$, to obtain

$$\begin{aligned} c_{nm}^s(-z_0) &= \oint_D \frac{dz}{2\pi i} \left[A_n(z, -z_0) \vec{\partial}_z A_m(z, -z_0) \right] A_{-n-m+s}(z, -z_0) \\ &= -a_n(-z_0) a_m(-z_0) a_{-n-m+s}(-z_0) \oint_C \frac{dz}{2\pi i} \left\{ \left[A_{-n}(z, z_0) \vec{\partial}_z A_{-m}(z, z_0) \right] A_{n+m-s}(z, z_0) \right\} \\ &= -a_n(-z_0) a_m(-z_0) a_{-n-m+s}(-z_0) c_{-n, -m}^{-s}(z_0). \end{aligned} \quad (26)$$

Consequently,

$$c_{-n, -m}^s(z_0) = \frac{-1}{a_n(-z_0) a_m(-z_0) a_{-n-m-s}(-z_0)} c_{n, m}^{-s}(-z_0), \quad (27)$$

for $|n|, |m|, |n+m+s| \neq \frac{1}{2}, n+m \neq -2$; and, since $a_{-m}(z_0) = -1/a_m(z_0)$, the identity (20) results.

From the first identity, and from eq.(17a), it immediately follows that $\chi_{n, m} = 0$ for $n+m < -3$. That is,

$$\chi_{n, m} = 0 \quad \text{for} \quad |n+m| > 3, \quad (28)$$

so that the KN algebra has a "local" central extension, in parallel with the vanishing of the structure constants c_{nm}^s for $|s| > 3/2$. You may visualize these directly by successively applying (8) in the "trident" schematic introduced above. For instance, triple application of (6) on (16) yields

$$\chi_{nm} = \frac{1}{12} \oint_{z_0} \frac{dz}{2\pi i} A_m \sum_{k, l, p} A_p \gamma_{n, -k} \gamma_{k, -l} \gamma_{l, -p} = \frac{1}{12} \sum_{k, l} \gamma_{n, -k} \gamma_{k, -l} \gamma_{l, m}. \quad (29)$$

Evidently, the indices of the composite trident-tree span 7 levels. In particular, triple application of (8b) instantly produces the leading term (17b). Relations of this type are summarized in the table at the end of this talk.

For the non-generic cases involving $A_{1/2}$ and $A_{-1/2}$ whose behavior at $z \sim -z_0$ is not canonical, eq.(21) is modified and yields corresponding modified relations.

The identities (19) and (20) may also be used to exhibit an automorphism of the KN algebra. Recall that the Virasoro algebra

$$[L_n, L_m] = (m-n)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \quad (30)$$

has the automorphism $L_n \rightarrow -L_{-n}$, $c \rightarrow -c$. Similarly, the KN algebra has the automorphism

$$e_n \rightarrow a_n(-z_0)e_{-n}, \quad |n| \neq \frac{1}{2}; \quad e_{\frac{1}{2}} \rightarrow e_{\frac{1}{2}}, \quad e_{-\frac{1}{2}} \rightarrow -e_{-\frac{1}{2}}, \quad c \rightarrow -c, \quad (31)$$

provided that also $z_0 \rightarrow -z_0$ in all functions of z_0 such as the structure constants $c_{nm}^*(z_0)$ and the cocycle $\chi_{nm}(z_0)$. From $a_n(-z_0) = -a_n(z_0)$ and $a_n(z_0)a_{-n}(z_0) = -1$, note that the square of the transformation (31) is 1.

It is worth emphasizing that the characteristic form of the conformal algebra on a surface is predicated on the global basis on that surface, as illustrated above, rather than on the short distance expansion (12) which is universal.

4. For spinning (supersymmetric) strings and superconformal critical systems, a relevant symmetry is the $N = 1$ superconformal symmetry. It is best described in $N = 1$ superspace with local complex analytic coordinates (z, θ) , where θ is an odd (Grassmann) coordinate obeying $\theta^2 = 0$. An infinitesimal superconformal transformation is given by

$$z \rightarrow z + V + \frac{1}{2}(DV)\theta, \quad \theta \rightarrow \theta + \frac{1}{2}DV, \quad (32)$$

where

$$V = V(z, \theta) = \epsilon(z) + \theta\psi(z) \quad (33)$$

is an even infinitesimal super-analytic function, and the superspace covariant derivative D is

$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}, \quad D^2 = \frac{\partial}{\partial z}. \quad (34)$$

The corresponding generator $\mathcal{L}_V$ is

$$\mathcal{L}_V = V\frac{\partial}{\partial z} + \frac{1}{2}(DV)D, \quad (35)$$

and obeys the (centerless) superconformal algebra

$$[\mathcal{L}_{V_1}, \mathcal{L}_{V_2}] = \mathcal{L}_{V_1 \wedge V_2}, \quad (36)$$

where

$$V_1 \wedge V_2 \equiv V_1 \overset{\leftrightarrow}{\partial}_z V_2 + \frac{1}{2}(DV_1)(DV_2). \quad (37)$$

On the $N = 1$ supersphere, with global analytic coordinates (z, θ) , we distinguish the two points $(z = 0, \theta)$ and $(z = \infty, \theta)$; and we look for a suitable basis for the functions $\epsilon(z)$ and $\psi(z)$, which are the even and odd components of $V(z, \theta)$:

$$\epsilon(z) = \sum_{n=-\infty}^{\infty} \epsilon_n z^{n+1}, \quad \psi(z) = \sum_r \psi_r z^{r+\frac{1}{2}}, \quad (38)$$

where ψ_r are Grassmann parameters. The index r can take on either half-integer (“Neveu-Schwarz”) or integer (“Ramond”) values. We use the bases in (38) to construct a superfield basis for the superconformal algebra. For $V = z^{n+1}$, i.e. vanishing $\psi(z)$,

$$\mathcal{L}_V \rightarrow L_n = z^{n+1} \frac{\partial}{\partial z} + \frac{1}{2}(n+1)z^n \theta \frac{\partial}{\partial \theta}. \quad (39a)$$

For $V = \theta \kappa z^{r+\frac{1}{2}}$ (where κ is an odd parameter), i.e. vanishing $\epsilon(z)$,

$$\mathcal{L}_V \rightarrow \frac{\kappa}{2} G_r \quad \text{with} \quad G_r = z^{r+\frac{1}{2}} \left(\frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z} \right). \quad (39b)$$

In this basis, the superconformal algebra (36) reads

$$[L_n, L_m] = (m-n)L_{n+m}, \quad [L_n, G_r] = \left(r - \frac{n}{2}\right) G_{n+r}, \quad \{G_r, G_s\} = -2L_{r+s}, \quad (40)$$

which is the familiar (centerless) Neveu-Schwarz-Ramond algebra.

The supertorus is the supersymmetric generalization of the torus, and is parameterized by superelliptic functions [4]. A spinor field $\psi(z)$ on a bosonic torus can have one of four possible spin structures [6] characterizing its sign change under parallel transport around each cycle:

$z \rightarrow z + 1$	+	-	-	+
$z \rightarrow z + \tau$	-	+	-	+

The first 3 structures are called “even”, and the fourth spin structure (+,+) is called “odd”.

For the even spin structure (+,-), the requirement that $V(z, \theta) = \epsilon(z) + \theta \psi(z)$ be invariant under parallel transport around both cycles implies that $V(z, \theta)$ satisfy

$$V(z, \theta) = V(z+1, +\theta), \quad V(z, \theta) = V(z+\tau, -\theta). \quad (41)$$

These conditions also follow if we identify the following points on the superplane

$$(z, \theta) \approx (z+1, +\theta), \quad (z, \theta) \approx (z+\tau, -\theta). \quad (42)$$

In fact, (42) *define* a supertorus with (+,-) spin structure [4]. The definition for the two other even spin structures, (-,+) and (-,-) is analogous.

A supertorus with odd spin structure is defined by identifying the points

$$(z, \theta) \approx (z+1, \theta), \quad (z, \theta) \approx (z+\tau+\theta\delta, \theta+\delta), \quad (43)$$

where δ is a new odd modular parameter. A function $V(z, \theta)$ on a supertorus with odd spin structure satisfies

$$V(z, \theta) = V(z+1, \theta), \quad V(z, \theta) = V(z+\tau+\theta\delta, \theta+\delta). \quad (44)$$

A super-analytic superfield which satisfies (41) or (44) is a “super-elliptic” function [4].

The number of spin structures doubles upon introducing the two punctures: ($z = \pm z_0, \theta$). If the above cycles flip their characteristic signs for fermion transport when pulled over either of the punctures, they are in the Ramond sector; for no change of sign, they comprise the Neveu-Schwarz

sector. Here, I report on the odd spin structure NS sector, leaving the other, less dramatic, ones to be examined in our paper. Our solution to the problem of finding a suitable superfield basis for functions with the superperiodicity (43) is the following. Introduce a function $\phi(z)$, which satisfies

$$\phi(z+1) = \phi(z), \quad \phi(z+\tau) = \phi(z) - 1. \quad (45)$$

As a consequence, the quantity $\theta + \phi(z)\delta$ is super-elliptic, and it is therefore the desired generalization of θ . We further demand that $\phi(z)$ be single-valued and that its poles (if any) be at $\pm z_0$. We choose

$$\phi(z) = \frac{1}{2\pi i} [\zeta(z - z_0) - 2(z - z_0)\eta_1] \quad (46)$$

where $\eta_1 \equiv \zeta(1/2)$. Our choice is somewhat asymmetric, as $\phi(z)$ has a pole at z_0 and not at $-z_0$, and leads to some asymmetries in the expressions below.

Our basis for super-elliptic functions analytic at $z \neq \pm z_0$ is

$$\hat{A}_n(z, \theta) \equiv A_n(z) + \phi(z)A'_n(z)\theta\delta, \quad \hat{B}_n(z, \theta) \equiv A_n(z)[\phi(z)\delta + \theta], \quad (47)$$

for n half-integer. The corresponding basis of superconformal transformations is

$$e_n = \hat{A}_n \frac{\partial}{\partial z} + \frac{1}{2}(D\hat{A}_n)D, \quad g_n = -2 \left[\hat{B}_n \frac{\partial}{\partial z} - \frac{1}{2}(D\hat{B}_n)D \right]. \quad (48)$$

For a mnemonic of the half-integral indices of the fermionic generators g_n , compare (39) with the $O(\delta^0)$ expansion of the above, namely $e_n = A_n \frac{\partial}{\partial z} + \frac{1}{2}A'_n \theta \frac{\partial}{\partial \theta} + O(\delta)$ and $g_n = A_n(\frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z}) + O(\delta)$.

To specify the relevant superconformal algebra, we evaluate the components of (37):

$$\hat{A}_n \wedge \hat{A}_m = \hat{A}_n \overleftrightarrow{\partial}_z \hat{A}_m + \frac{1}{2}(D\hat{A}_n)(D\hat{A}_m) = \sum_{s=-3/2}^{3/2} c_{nm}^s \hat{A}_{n+m-s} + \delta \sum_{s=-5/2}^{5/2} d_{nm}^{s+1} \hat{B}_{n+m-s-1}, \quad (49a)$$

$$\hat{B}_n \wedge \hat{B}_m = \hat{B}_n \overleftrightarrow{\partial}_z \hat{B}_m - \frac{1}{2}(D\hat{B}_n)(D\hat{B}_m) = -\frac{1}{2} \sum_{s=-1/2}^{1/2} f_{nm}^s \hat{A}_{n+m-s} + \delta \sum_{s=-3/2}^{3/2} h_{nm}^{s+1} \hat{B}_{n+m-s-1}, \quad (49b)$$

$$\hat{A}_n \wedge \hat{B}_m = \hat{A}_n \overleftrightarrow{\partial}_z \hat{B}_m + \frac{1}{2}(D\hat{A}_n)(D\hat{B}_m) = \sum_{s=-3/2}^{3/2} u_{nm}^s \hat{B}_{n+m-s} - \delta \sum_{s=-3/2}^{3/2} h_{nm}^{s+1} \hat{A}_{n+m-s-1}, \quad (49c)$$

where we introduce the new structure constants d_{nm}^s , f_{nm}^s , h_{nm}^s , and u_{nm}^s , detailed presently. Observe that the derivative of $\phi(z)$ is $A_{-3/2}(z)$, up to constants:

$$\phi'(z) = -\frac{1}{2\pi i} [P(z - z_0) + 2\eta_1] = -\frac{1}{2\pi i} [A_{-3/2}(z) + P(2z_0) + 2\eta_1]. \quad (50)$$

Now introduce the structure constants f_{nm}^s :

$$A_n(z)A_m(z) = \sum_{s=-1/2}^{1/2} f_{nm}^s A_{n+m-s}(z). \quad (51)$$

Further define the structure constants u_{nm}^s :

$$A_n(z)A'_m(z) - \frac{1}{2}A'_n(z)A_m(z) = \sum_{s=-3/2}^{3/2} u_{nm}^s A_{n+m-s}(z). \quad (52)$$

Direct calculation of $\hat{A}_n \wedge \hat{A}_m$ yields

$$\hat{A}_n \wedge \hat{A}_m = (A_n \vec{\partial}_z A_m) + \left[\phi (A_n \vec{\partial}_z A_m) \right]' \theta \delta = \sum_{s=-3/2}^{3/2} c_{nm}^s \hat{A}_{n+m-s} + \theta \delta \sum_{s=-3/2}^{3/2} c_{nm}^s \phi' A_{n+m-s}. \quad (53)$$

Now

$$\phi' A_n = -\frac{1}{2\pi i} A_{-3/2} A_n - \frac{1}{2\pi i} [\mathcal{P}(2z_0) + 2\eta_1] A_n = \sum_{k=0}^2 \alpha_n^k A_{n-k}, \quad (54)$$

where we introduce the constants α_n^k :

$$\alpha_n^0 = -\frac{1}{2\pi i} [\mathcal{P}(2z_0) + 2\eta_1], \quad \alpha_n^1 = -\frac{1}{2\pi i} f_{-3/2,n}^{-1/2}, \quad \alpha_n^2 = -\frac{1}{2\pi i} f_{-3/2,n}^{1/2}. \quad (55)$$

We then find that the last term of eq.(53) is given by

$$-\theta \delta \sum_{s=-3/2}^{7/2} d_{nm}^s A_{n+m-s}, \quad (56)$$

where

$$d_{nm}^s = -\sum_k c_{nm}^{s-k} \alpha_{n+m+k-s}^k, \quad -3/2 \leq s \leq 7/2. \quad (57)$$

Consequently (49a) follows. Next, evaluate

$$\hat{B}_n \wedge \hat{B}_m = -\frac{1}{2} \{A_n A_m + \phi(A_n A_m)' \theta \delta\} + \phi'(A_n A_m) \theta \delta = -\frac{1}{2} \sum_{s=-1/2}^{1/2} f_{nm}^s \hat{A}_{n+m-s} + \theta \delta \sum_{s=-1/2}^{1/2} f_{nm}^s \phi' A_{n+m-s}. \quad (58)$$

As above, the last term is

$$-\theta \delta \sum_{s=-1/2}^{5/2} h_{nm}^s A_{n+m-s}, \quad (59)$$

where

$$h_{nm}^s = -\sum_k f_{nm}^{s-k} \alpha_{n+m+k-s}^k, \quad -1/2 \leq s \leq 5/2; \quad (60)$$

so that (49b) follows. Finally,

$$\hat{A}_n \wedge \hat{B}_m = (A_n A'_m - \frac{1}{2} A'_n A_m) (\theta + \phi \delta) + \phi' (A_n A_m) \delta = \sum_{s=-3/2}^{3/2} u_{nm}^s \hat{B}_{n+m-s} + \delta \sum_{s=-1/2}^{1/2} f_{nm}^s \phi' A_{n+m-s}. \quad (61)$$

For special values of n and m , these equations are slightly modified. As a consequence, (36) dictates that the generators e_n and g_n obey the algebra (2).

As in the bosonic case, the center is derived from the supercontour integral representation of $\mathcal{L}_V$ in terms of the super-energy-momentum superfield $T(z, \theta)$,

$$\mathcal{L}_V = -\oint_{z_0} \frac{dz}{2\pi i} \int d\theta V(z, \theta) T(z, \theta), \quad (62)$$

where the z integration is performed along an equal-time counterclockwise path about the point z_0 . As usual, integration over the odd coordinate θ is defined by

$$\int d\theta \theta = 1, \quad \int d\theta = 0. \quad (63)$$

The projection on our basis $\hat{A}_n$ and $\hat{B}_n$ is then

$$e_n = - \oint_{z_0} \frac{dz}{2\pi i} \int d\theta \hat{A}_n(z, \theta) T(z, \theta), \quad g_n = 2 \oint_{z_0} \frac{dz}{2\pi i} \int d\theta \hat{B}_n(z, \theta) T(z, \theta). \quad (64)$$

Equal-time commutators are expressed in terms of operator products:

$$[\mathcal{L}_{V_1}, \mathcal{L}_{V_2}] = \oint_{z_0} \frac{dz_2}{2\pi i} \int d\theta_2 V_2(z_2, \theta_2) \oint_{z_0} \frac{dz_1}{2\pi i} \int d\theta_1 V_1(z_1, \theta_1) T(z_1, \theta_1) T(z_2, \theta_2). \quad (65)$$

The operator product expansion for $T(z_1, \theta_1)T(z_2, \theta_2)$ for z_1 close to z_2 necessarily reduces to the operator product expansion on the superplane

$$T(z_1, \theta_1)T(z_2, \theta_2) \sim \frac{\hat{c}/4}{z_{12}^3} + \frac{3}{2} \frac{\theta_{12}}{z_{12}^2} T(z_2, \theta_2) + \frac{1}{2} \frac{1}{z_{12}} D_2 T(z_2, \theta_2) + \frac{\theta_{12}}{z_{12}} \partial_2 T(z_2, \theta_2), \quad (66)$$

where $\hat{c} \equiv \frac{2}{3}c$, $\theta_{12} \equiv \theta_1 - \theta_2$, and $z_{12} \equiv z_1 - z_2 - \theta_1\theta_2$. We recast this result as follows:

$$\begin{aligned} T(z_1, \theta_1)T(z_2, \theta_2) \sim & \frac{\hat{c}}{4} D_1 \left[\frac{\delta(\theta_1 - \theta_2)}{(z_1 - z_2)^3} \right] + \frac{3}{2} \frac{\delta(\theta_1 - \theta_2)}{(z_1 - z_2)^2} T(z_2, \theta_2) \\ & + \frac{1}{2} D_1 \left[\frac{\delta(\theta_1 - \theta_2)}{(z_1 - z_2)} \right] D_2 T(z_2, \theta_2) + \frac{\delta(\theta_1 - \theta_2)}{(z_1 - z_2)} \partial_2 T(z_2, \theta_2), \end{aligned} \quad (67)$$

where $\delta(\theta_1 - \theta_2) = \theta_1 - \theta_2$. This implies

$$[\mathcal{L}_{V_1}, \mathcal{L}_{V_2}] = - \oint_{z_0} \frac{dz}{2\pi i} \int d\theta V_1 \wedge V_2 T(z, \theta) + \frac{\hat{c}}{8} \oint_{z_0} \frac{dz}{2\pi i} \int d\theta V_2 D V_1''. \quad (68)$$

We may now specify $\mathcal{L}_{V_1}$ and $\mathcal{L}_{V_2}$ through eqs.(64). For instance, consider the case $V_1 = \hat{A}_n$, $V_2 = \hat{A}_m$. Recalling eq.(49a) and performing the θ integration in the second term of the right-hand-side of eq.(68), we obtain

$$[e_n, e_m] = \sum_s c_{nm}^s e_{n+m-s} - \frac{\delta}{2} \sum_s d_{nm}^{s+1} g_{n+m-s-1} + \frac{\hat{c}}{8} \oint_{z_0} \frac{dz}{2\pi i} A_n''' A_m, \quad (69)$$

which appears in eq.(2) above, given the definition (16). Similarly, for $\{g_n, g_m\}$, the center is

$$\frac{\hat{c}}{2} \oint_{z_0} \frac{dz}{2\pi i} \int d\theta \hat{B}_m D \hat{B}_n'' \equiv \frac{\hat{c}}{2} \psi_{nm}; \quad (70)$$

and for $[e_n, g_m]$ the center reads

$$-\frac{\hat{c}}{4} \oint_{z_0} \frac{dz}{2\pi i} \int d\theta \hat{B}_m D \hat{A}_n'' \equiv -\frac{\hat{c}}{4} \delta \rho_{nm}. \quad (71)$$

Upon performing the θ integrations, these centers reduce to the expressions

$$\psi_{nm}(z_0) \equiv \oint_{z_0} \frac{dz}{2\pi i} A_n'''(z) A_m(z), \quad (72)$$

$$\rho_{nm}(z_0) \equiv \oint_{z_0} \frac{dz}{2\pi i} \phi'(z) [A_n''(z) A_m(z) - A_n'(z) A_m'(z)]. \quad (73)$$

More specifically,

$$\psi_{n,m} = 0, \quad |n+m| > 2; \quad (74a)$$

$$\psi_{n,2-n} = (n - \frac{1}{2})(n - \frac{3}{2}), \quad (74b)$$

$$\psi_{n,1-n} = (n - \frac{1}{2}) \{ (n + \frac{1}{2}) D^{(1)} \left(2z_0(n + \frac{1}{2}) \right) + (n - \frac{3}{2}) D^{(1)} \left(2z_0(\frac{3}{2} - n) \right) \}, \quad (74c)$$

$$\begin{aligned} \psi_{n,-n} = & \left\{ \frac{1}{2} (n + \frac{1}{2}) (n + \frac{3}{2}) D^{(2)} \left(2z_0(n + \frac{1}{2}) \right) \right. \\ & + \frac{1}{2} (n - \frac{1}{2}) (n - \frac{3}{2}) D^{(2)} \left(2z_0(\frac{1}{2} - n) \right) \\ & \left. + (n + \frac{1}{2}) (n - \frac{1}{2}) D^{(1)} \left(2z_0(n + \frac{1}{2}) \right) D^{(1)} \left(2z_0(\frac{1}{2} - n) \right) \right\}. \end{aligned} \quad (74d)$$

All centers are local. It bears repeating that the center of the commutator $[e_n, g_m]$ is proportional to δ , and thus fermionic, in some contradistinction to usual centers of infinite algebras.

I summarize much of the above information and some of the inter-relations among constants in the following

Table

TERMS	CONSTANT	CONNECTING RELATIONS
3	$\gamma_{nm} \equiv \oint \frac{dz}{2\pi i} A_n' A_m$	
2	$f_{nm}^s \equiv \oint \frac{dz}{2\pi i} A_n A_m A_{-n-m+s}$	
7	$\chi_{nm} \equiv \frac{1}{12} \oint \frac{dz}{2\pi i} A_n''' A_m$	$= \frac{1}{12} \sum_{k,l} \gamma_{n,-k} \gamma_{k,-l} \gamma_{l,m} = \frac{1}{12} \sum_k \gamma_{n,-k} \psi_{km}$
4	$c_{nm}^s \equiv \oint \frac{dz}{2\pi i} [A_n \partial_z A_m] A_{-n-m+s}$	$= \sum_k \gamma_{m,-k} f_{nk}^{s+k-m} - \gamma_{n,-k} f_{mk}^{s+k-n}$
4	$u_{nm}^s \equiv \oint \frac{dz}{2\pi i} [A_n A_m' - \frac{1}{2} A_n' A_m] A_{-n-m+s}$	$= \sum_k \gamma_{m,-k} f_{nk}^{s+k-m} - \frac{1}{2} \gamma_{n,-k} f_{mk}^{s+k-n}$
5	$\psi_{nm} \equiv \oint \frac{dz}{2\pi i} A_n'' A_m$	$= \sum_k \gamma_{n,-k} \gamma_{k,m}$
3	$\alpha_n^k \equiv \oint \frac{dz}{2\pi i} \phi' A_n A_{k-n}$	$= -\frac{1}{2\pi i} [\delta_{k,0} (\mathcal{P}(2z_0) + 2\eta_1) + \delta_{k,1} f_{-3/2,n}^{-1/2} + \delta_{k,2} f_{-3/2,n}^{1/2}]$
7	$\rho_{nm} \equiv \oint \frac{dz}{2\pi i} \phi' (A_n'' A_m - A_n' A_m')$	$= \sum_{k,l} \alpha_l^{l+m} \gamma_{n,-k} \gamma_{k,-l} - \alpha_l^{l+k} \gamma_{n,-k} \gamma_{m,-l}$
6	d_{nm}^s	$\equiv -\sum_k c_{nm}^{s-k} \alpha_{n+m+k-s}^k$
4	h_{nm}^s	$\equiv -\sum_k f_{nm}^{s-k} \alpha_{n+m+k-s}^k$

Note that, by dint of the simplicity of the leading term in (8b), the leading term of all of these constants may be found essentially by inspection from the above, e.g.

$$u_{nm}^{3/2} = \left(m - \frac{n}{2} - \frac{1}{4} \right), \quad u_{nm}^{-3/2} = \frac{a_n a_m}{a_{n+m+3/2}} \left(\frac{n}{2} - m - \frac{1}{4} \right), \quad (75)$$

cf. eq.(20), except for special values of n and m . Similarly,

$$f_{nm}^{1/2} = 1, \quad f_{nm}^{-1/2} = \frac{a_n a_m}{a_{n+m+1/2}}, \quad (76)$$

except for special values of n and m . Likewise, the leading term in the cocycle of eq.(73) is

$$\rho_{n,4-n} = -\frac{1}{\pi i} \left(n - \frac{1}{2} \right) \left(n - \frac{5}{2} \right). \quad (77)$$

5. Corresponding bases and algebras obtain for the remaining three NS spin structures and the four Ramond ones. In the latter case, a branch cut is introduced between z_0 and $-z_0$, to yield the requisite extra sign changes, and the fermions now have integer indices. Since bases for higher genera have been considered by KN (unpublished) and the last of ref.[5], you might expect the above supersymmetrization procedure to go through in these cases as well. Further desirable extensions of our work include the higher N superconformal algebras, non-orientable surfaces, the investigation of the supermodular response features of our basis, and the representation theory of the algebras involved (including its relation to that of the sphere).

The algebras presented here should provide a framework for an operator formalism for loop string theory [2], and facilitate calculational techniques for correlation functions on the torus. They may also shed further light on the closed string interaction vertex.

I wish to thank the University of Miami and the Organizers of this Conference for the pleasant and stimulating environment in which this work was initiated and presented, respectively.

References

1. L. Mezincescu, R.I. Nepomechie, and C.K. Zachos, Nucl.Phys. *B*, in press (1989).
2. I.M. Krichever and S.P. Novikov, Funct.Anal.Appl. *21* (1987) 46; I.M. Krichever and S.P. Novikov, Funct.Anal.Appl. *21* (1987) 47.
3. E.T. Whittaker and G.N. Watson, *A Course in Modern Analysis* (Cambridge University Press, Cambridge, 1963); K. Chandrasekharan, *Elliptic Functions* (Springer-Verlag, 1985).
4. L. Crane and J.M. Rabin, Comm.Math.Phys. *113* (1988) 601; J.M. Rabin, in *Super Field Theories*, eds. H.C. Lee, V. Elias, G. Kunstatter, R.B. Mann and K.S. Viswanathan (Plenum, 1987); J.M. Rabin and P.G.O. Freund, Comm.Math.Phys. *114* (1988) 131.
5. L. Bonora, M. Bregola, P. Cotta-Ramusino, and M. Martellini, Phys.Lett. *205B* (1988) 53; L. Bonora, M. Martellini, M. Rinaldi, and J. Russo, Phys.Lett. *206B* (1988) 444; D. Fairlie, P. Fletcher, and J. Nuyts, Durham preprint DTP 88/9, J.Math.Phys., in press; M. Bowick, in preparation; J. Alberty, A. Taormina, and P. van Baal, Comm.Math.Phys., in press; L. Bonora, A. Lugo, M. Matone, and J. Russo, SISSA preprint, May 1988.
6. N. Seiberg and E. Witten, Nucl.Phys. *B276* (1986) 272.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.