A FOUR-WAY RF BEAM SEPARATOR

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V. K. Neil

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Lawrence Livermore National Laboratory

University of California

Livermore, California 94550

ABSTRACT

A method for separating a continuous beam of relativistic particles into four pulsed beams is investigated theoretically. The separation is periodic with period $2\pi/\omega$ so that each of the four beams consists of current pulses of duration π/ω . The separation is accomplished by a series of rf cavities in the beam line. The cavities operate in the TM_{110} and have frequencies ω , 3ω , 5ω , 7ω , etc. The transverse momentum imparted to the beam particles results in a time-dependent displacement of the beam centroid at a position downstream of the cavity array. The mathematical limitations imposed by truncating a Fourier series are discussed, and an expression derived for the necessary phase and amplitude of each cavity. The rf induced by the beam in the cavities is treated in detail, and does not appear to be a serious problem.

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I. Introduction

In this work we investigate theoretically a method for separating a continuous beam of relativistic electrons into four pulsed beams. The separation is periodic with period $2\pi/\omega$ so that each of the four beams consists of current pulses of duration $\pi/2\omega$. The separation is accomplished by a series of rf cavities in the beam line. The cavities operate in the TM_{110} mode, which in recent years has become infamous as the "beam breakup" mode. In earlier times it was referred to as the "beam separator" mode and used to separate the bunched beam from an rf linac. The cavity frequencies are ω , 3ω , 5ω , 7ω , etc. This concept was the subject of a preliminary investigation at the Second Joint DOE/ODD Accelerator Technology Workshop.¹

The cavities impart a transverse momentum to the particles and produce a time dependent displacement of the beam centroid on a screen located at some distance downstream from the cavities. In practice the beam would be focused to a waist at the screen by focusing magnets upstream of the cavities. Ideally the centroid is at position 1 ($x = d$, $y = 0$) during the time interval $-\pi/4 < \omega t < \pi/4$. At $\omega t = \pi/4$ the centroid moves rapidly to position 2 ($x = 0$, $y = d$) and remains there until $\omega t = 3\pi/4$, at which time it moves rapidly to position 3 ($x = -d$, $y = 0$). At $\omega t = 5\pi/4$ the centroid moves rapidly to position 4 ($x = 0$, $y = -d$) and remains there until $\omega t = 7\pi/4$ ($-\pi/4$). Each position is the entrance to a separate beam line. The purpose of this work is in part to obtain a practical definition of "rapidly."

In Sec. II we assume that the amplitude and phase of each cavity can be controlled exactly and investigate the mathematical limitations of the scheme. Results indicate that the displacement will approximate the ideal to within

about 10% if five cavities are employed. (Five is taken as a practical limit to the number of cavities, but one example is given for more than five.)

Section III is devoted to determination of the relative amplitudes and phases of the cavities necessary to achieve the mathematical model employed in Sec. II. Section III also contains a physical model and numerical example for a series of cavities to deflect a 50 MeV electron beam.

To reduce rf power, the cavities are high Q structures, and the electromagnetic fields induced by the beam must be considered. The beam front will excite TM_{010} modes in the cavities, resulting in a modulation of the particle's energy. Furthermore, since the beam centroid is off axis to some extent in all cavities, the beam front will also excite TM_{110} modes. These questions are discussed in Sec. IV, where it is shown that neither of these effects is serious.

II. Mathematical Limitations

At the screen the ideal displacement of the beam centroid as a function of time over one cycle is shown in Figure 1. The Fourier series representing this displacement can be written in terms of the complex displacement $\xi \equiv x + iy$ in the form

$$\xi \propto e^{i\omega t} + (1/3)e^{-3i\omega t} - (1/5)e^{5i\omega t} - (1/7)e^{-7i\omega t} + (1/9)e^{9i\omega t} + \dots \quad (2.1)$$

This form will be useful in the next section, but here we note that $y(\omega t) = x[\omega t - (\pi/2)]$ and consider first the series for x , namely

$$x = \frac{2\sqrt{2}d}{\pi} \left[\cos \omega t + (1/3) \cos 3\omega t - (1/5) \cos 5\omega t - (1/7) \cos 7\omega t + (1/9) \cos 9\omega t + \dots \right] \quad (2.2)$$

The exact series is, of course, infinite, but we wish to approximate the ideal displacement as best we can with a few terms. Each term embodies the effect of one cavity. The coefficients of the terms in the truncated series can be chosen arbitrarily by adjusting the excitation level (or the length along the beam line) of the individual cavities.

An optimization computer code was used to obtain the following results. The input to the code is the function to be represented by the series together with a number of points within a given interval. The code then finds the values of the Fourier coefficients that give the best fit to the function at these points. Because of the symmetry conditions in the present work, the behavior of the x and y displacements over the entire interval $-\pi < \omega t < \pi$ can be obtained from behavior of the x displacement in the interval $0 < \omega t < \pi/2$. Our optimization was performed in this interval.

However, the best representation of the infinite series (and therefore the ideal displacement) over the entire interval $-\pi < \omega t < \pi$ is achieved when the coefficients of the terms in the truncated series are the same coefficients of the terms in the infinite series. A plot of $x(\omega t)$ is shown in Fig. 2 for five terms with the ratios of coefficients the same as those in Eq. (2.2). The value of the first ($\cos \omega t$) coefficient is chosen such that $x(0) = 1$. These coefficients are listed in Table 1. The computer code will in fact determine these values if the optimization points are equally spaced in the interval $0 < \omega t < \pi/2$. We see that the value of x is constant over about 20 degrees of rf phase, and the "rapid" motion from $x = 1$ to $x = 0$ occurs over about 20 degrees of phase. The beam does not stop at $x = 0$, but oscillates twice about $x = 0$ with an amplitude of about 10% of the displacement. The plot of $y(\omega t)$ in Fig. 2 illustrates that while x is relatively

well behaved in the interval $-\pi/4 < \omega t < \pi/4$, the value of y is performing the same oscillations that x performs in the interval $\pi/4 < \omega t < 3\pi/4$.

Table I

	C_1	C_3	C_5	C_7	C_9
Fig. 2	0.90778	0.30259	-0.18156	-.12968	0.10086
Fig. 4	-0.04039	2.49624	-2.34559	1.13981	-0.25081
Fig. 5	0.90651	0.28133	-0.18523	-0.10212	0.10435

Increasing the number of terms (i.e., number of cavities) does change the results somewhat. A plot of $x(\omega t)$ is shown in Fig. 3 for 9 terms, the ratio of the coefficients again the same as in Eq. (2.2). There are now more oscillations in the interval $\pi/4 < \omega t < 3\pi/4$, but the amplitude of the oscillations is still about 10% of the displacement. Nine cavities are probably more than practical considerations will allow, but for some applications the results of five cavities may be tolerable. In order to separate the four beams, the displacement d need be about one beam diameter, and the oscillation amplitude is about one-tenth of the beam diameter.

Better results are obtained from the optimization code if the input function does not change abruptly at $\omega t = \pi/4$. Instead we choose a function that is unity for $0 < \omega t < \pi/5$, then drops to one half at $\omega t = \pi/4$, and drops to zero at $\omega t = 3\pi/10$. The function has zero derivative at $\omega t = \pi/5$ and $3\pi/10$. The actual function employed in the interval $\pi/5 < \omega t < 3\pi/10$ is

$$x(\omega t) = \frac{\eta}{4} (\eta^2 - 3) + \frac{1}{2}, \quad (2.3)$$

in which

$$\eta = [\omega t - (\pi/4)]/(\pi/20). \quad (2.4)$$

The following two examples employ this function.

If the ideal displacement is to be approximated over a limited interval, $-\pi/4$ to $\pi/4$ and $5\pi/4$ to $7\pi/4$, then an excellent fit can be achieved, with 5 terms as shown in Fig. 4. But look at the behavior in the interval $\pi/4$ to $3\pi/4$! The beam executes one oscillation about $x = 0$ with an amplitude over 5 times the displacement! The coefficients that result in Fig. 4 are determined by placing all of the optimization points in the interval $0 < \omega t < \pi/4$. These coefficients are also given in Table I. Notice that these coefficients are vastly different from those used to generate Fig. 3.

A reasonable fit can be achieved with five terms in the intervals $-\pi/4$ to $3\pi/4$ and $5\pi/4$ to $7\pi/4$ with the oscillations elsewhere still tolerable if one wishes to separate the continuous beam into two beams of current pulses as follows: Let the cavities deflect the beam in the x direction only, and let a screen intercept the beam during the oscillations. Such a reasonable fit is shown in Fig. 5, and the coefficients of the series are again shown in Table I. In this scheme the continuous beam is lost on the screen about half the time. The beam passes to one side of the screen about one-fourth of the time, and on the other side the same amount of time. The displacement must be somewhat greater than the beam diameter in this scheme.

III. Phase and Amplitude of The rf Cavities

To separate the continuous beam into four beams of current pulses, each cavity is driven in a rotating TM_{110} mode. In this section we treat the fields and particle dynamics in one cavity and from these results derive an expression for the beam displacement at the screen. In Eqs. (3.1) through (3.19), quantities appear which have different values for different cavities, but for neatness no subscripts are employed until the final result, Eq. (3.20). In particular,

the frequency of the cavity is Ω . Three of the five cavities have $\Omega = \omega, 5\omega$, and 9ω . The other two cavities have $\Omega = -3\omega$ and -7ω . That is, in order to achieve the series in Eq. (2.1), the mode rotates in the opposite direction in these two cavities.

All modes that we consider in this work have a vector potential with an axial component only, therefore, we dispense with vector notation and it is understood that $A(r, \theta, t)$ represents the axial vector potential of the mode. For the externally driven TM_{110} modes we have

$$A(r, \theta, t) = A_0 J_1(\rho r) \cos(\theta - \Omega t) \quad (3.1)$$

in which J_1 is the ordinary Bessel function of the first kind of order one. The quantity ρ in Eq. (3.1) is chosen such that $J_1(\rho b) = 0$, with b the outer radius of the cavity. We have $\rho b = 3.8317$ for the TM_{110} mode. The frequency of the cavity is such that $|\Omega| = c\rho$, c the speed of light, so that the desired frequency determines the radius of the cavity. Near the axis we may set $J_1(\rho r) = \rho r/2$ and obtain

$$A = (A_0 \rho/2)(x \cos \Omega t + y \sin \Omega t) \quad (3.2)$$

in which we have introduced the rectangular coordinates $x = r \cos \theta$ and $y = r \sin \theta$. From the relation $\vec{B} = \nabla \times \vec{A}$ we have

$$\vec{B} = B (\sin \Omega t \hat{x} - \cos \Omega t \hat{y}) \quad (3.3)$$

with $B \equiv A_0 \rho/2$.

To separate the continuous beam into two beams of current pulses, or to achieve one pulsed beam, the mode does not rotate and the vector potential is

$$A = A_0 J_1(\rho r) \cos \theta \cos \Omega t \quad (3.4)$$

with the resulting magnetic field on axis has only a y component given by

$$B_y = -B \cos \Omega t \quad . \quad (3.5)$$

Gaussian units are employed throughout this work. In the cavity the equations of motion of a particle with axial speed v and total energy γmc^2 are

$$\ddot{x} = -e v B_y / \gamma mc = e v \omega_c \cos \Omega t \quad , \quad (3.6)$$

$$\ddot{y} = e v B_x / \gamma mc = e v \omega_c \sin \Omega t \quad , \quad (3.7)$$

in which $\omega_c \equiv eB/\gamma mc$. In complex notation these equations become

$$\ddot{x} + i\ddot{y} = v\omega_c e^{i\Omega t} \quad . \quad (3.8)$$

We solve this equation with the initial conditions $x = y = 0$ and $\dot{x} = \dot{y} = 0$ at $t = t_i$, with t_i the time the particle enters the cavity. The desired solution is

$$x + iy = \frac{v\omega_c}{i\Omega} \left[\frac{(e^{i\Omega t} - e^{i\Omega t_i})}{i\Omega} - (t - t_i) e^{i\Omega t_i} \right] \quad . \quad (3.9)$$

Equation (3.9) describes the transverse position of a particle as a function of time. From this expression we obtain the complex displacement, ξ , of the beam as a function of t and z by the substitution $t_i = t - (z/v)$, with z measured from the entrance of the cavity. We have

$$\xi(t, z) = \frac{v\omega_c}{\Omega^2} e^{i\Omega t} \left[\left(1 + \frac{i\Omega z}{v} \right) e^{-i\Omega z/v} - 1 \right] \quad . \quad (3.10)$$

This solution holds in the region $0 < z < h$, where h is the length of the cavity. The complex quantity ξ is a Lagrangian variable obeying the equation

$$\left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial z}\right)^2 \xi = v \omega_c e^{i\Omega t} \quad (3.11)$$

This equation may be solved directly with the conditions $\xi = 0$ and $\partial\xi/\partial z = 0$ at $z = 0$ to yield Eq. (3.10). Beyond the exit of the cavity ($z > h$), ξ satisfies Eq. (3.11) with the right hand side equal to zero. The solution in the region $z > h$ must have ξ and $\partial\xi/\partial z$ continuous at $z = h$. The desired solution is

$$\xi(t, z > h) = \frac{v \omega_c}{\Omega^2} \left[\left(1 + \frac{i\Omega z}{v}\right) \left(1 - e^{i\Omega h/v}\right) + \frac{i\Omega h}{v} e^{i\Omega h/v} \right] \times \exp i\Omega[t - (z/v)] \quad (3.12)$$

After the beam has passed through N cavities the expression for ξ is the sum of N terms of the form of Eq. (3.12) with the proper values of Ω , h , ω_c , and z for each cavity inserted in the corresponding term. Some rewriting of Eq. (3.12) will be convenient. First we factor out $\exp i\Omega h/2v$ and define

$$u = \Omega h/2v. \quad (3.13)$$

Equation (3.12) can now be written in the form

$$\xi = -\frac{2iv\omega_c}{\Omega^2} \left[\sin u - u \cos u + i\left(\frac{\Omega z}{v} - u\right) \sin u \right] \times \exp i \left[\Omega \left(t - \frac{z}{v} \right) + u \right] \quad (3.14)$$

The form of Eq. (3.14) suggests a change of variable. We define L to be

$$L \equiv z - (h/2). \quad (3.15)$$

The length L is the distance from the center of the cavity to the screen. We will also include an arbitrary phase ϕ in the contribution to ξ from each cavity, which now becomes

$$\xi = -\frac{2i\omega_0 c}{\Omega^2} \left[\sin u - u \cos u + \frac{i\Omega L}{v} \sin u \right] \exp i \left\{ \Omega [t - (L/v)] + \phi \right\} \quad (3.16)$$

Next we need to express the distance L from the center of each cavity to the screen in terms of the distance from some arbitrary point to the screen. We choose the distance L_1 from the center of the first cavity to the screen. The cavities may be arranged in any convenient sequence in the beam line as long as they are properly phased. However, to avoid confusion in this discussion we take the sequence to be in ascending frequency. That is, the cavities are arranged so that the beam first passes through the $n = 1$ cavity, then the $n = 3$ cavity, then the $n = 5$ cavity, etc., as shown in Fig. 6. The cavities are adjacent to each other as shown in the figure, however, to be more general, we introduce a time T_n for each cavity by the definition

$$vT_n = L_1 - L_n \quad (3.17)$$

Inserting Eq. (3.17) into Eq. (3.16) we have

$$\begin{aligned} \xi = & -\frac{2i\omega_0 c}{\Omega^2} \exp i\Omega [t - (L_1/v)] \exp i(\Omega T + \phi) \\ & \times \left[\sin u - u \cos u + i\Omega \left(\frac{L_1}{v} - T \right) \sin u \right] \quad (3.18) \end{aligned}$$

Finally, we factor out $i(\Omega L_1 \sin u)/v$ and introduce the quantities R and ψ for each cavity by the definition

$$R e^{-i\psi} = 1 - \frac{vT}{L_1} - \frac{i v (\sin u - u \cos u)}{\Omega L_1 \sin u} \quad (3.19)$$

This definition of ψ is chosen so that ψ is a positive angle for positive n and a negative angle for negative n . The total complex displacement of the beam at the screen can now be expressed as

$$\xi = \frac{L_1}{v} \sum \left\{ \omega_c h R \left(\frac{\sin u}{u} \right) e^{i(\Omega T + \phi - \psi)} e^{i\Omega[t - (L_1/v)]} \right\}_n . \quad (3.20)$$

In this expression all quantities in $\{ \}$ are to be given the values appropriate to each cavity. Since we are concerned only with the relative phase of the cavities with respect to the first cavity, we choose $\phi_1 = \psi_1$. Recall also that $T_1 = 0$ by definition. The quantity $(\sin u)/u$ is the familiar transit time factor for a particle traversing an rf cavity.

Suppose now we wish to have Eq. (3.20) be of the same form as Eq. (2.1). The time t in Eq. (2.1) is the retarded time $t - (L_1/v)$ in Eq. (3.20). Agreement between these two expressions requires that the phases ϕ and the amplitude M of the cavities be chosen according to Table II, where

$$M \equiv (\omega_c h R \sin u)/u . \quad (3.21)$$

TABLE II

Frequency Ω	n	$\phi - \psi + \Omega T$	M
ω	1	0	M_1 (arbitrary)
-3ω	-3	0	$M_1/3$
5ω	5	$\pm\pi$	$M_1/5$
-7ω	-7	$\pm\pi$	$M_1/7$
9ω	9	0	$M_1/9$

The phase of each cavity must be chosen according to this table, but the amplitude can be selected (by selecting $\omega_c h$) so that different ratios of the coefficients, such as those ratios discussed in Sec. II, can be achieved.

If the beam is to be deflected in one plane only, the magnetic field in each cavity is given by Eq. (3.5). In this situation only the real part of Eq. (3.20) need be considered. Furthermore, n can take on all positive values 1, 3, 5, 7, 9, etc.

As a numerical illustration, and for use in Sec. IV, we introduce a physical model of a series of 5 cavities arranged as shown in Fig. 6. The cavities deflect a beam of 50 MeV electrons through an angle of 10 mrad in the x direction at $\omega t = 0$. The cavities operate with rotating modes so that separation into four beams is achieved. For simplicity, we take the lengths h of the cavities to be as shown in Table III, namely 18 cm, 6 cm, etc. Neglecting the thickness of the walls separating the cavities, the total length along the beam line is 34.2 cm, about one foot. This ratio of length is chosen so that the transit angle u is the same in all five cavities. For each cavity the amplitude of the corresponding coefficient in the series (3.20) can be adjusted by choice of ω_c (i.e., magnetic field amplitude in the cavity). The magnetic field in all cavities is about 100G. The radius b of each box is determined by the relation

$$|n|\omega = 3.83 c/b$$

Values of b for the five cavities are given in Table III for a fundamental frequency of 100 MHz ($\omega = 2\pi \times 10^8$). We must choose a distance L_1 from the center of the first cavity to the screen, so we choose $L_1 = 150$ cm. The deflection at the screen is thus 1.5 cm.

For all cavities we have $u = \Omega h/2v = 0.178$. (In these calculations we set $v = c = 3 \times 10^{10}$ cm/sec.) The quantities vT , ΩT , ψ , R , and ϕ for each cavity are given in Table III. The lengths vT are found from the geometry of Fig. 6. The values of R and ψ are found from Eq. (3.19), and the values

of ϕ are then determined by the conditions in Table II. The values of ψ are quite small. We shall see in Sec. IV the rf induced by the beam gives rise to a time-dependent phase shift in each cavity that is comparable to or greater than ψ .

Table III

n	h(cm)	b(cm)	vT(cm)	ΩT (rad)	ψ (mrad)	R	ϕ (rad)
1	18.	183	0	0	3.78	1	0
-3	6	61	12	-0.754	-1.37	0.920	0.753
5	3.6	36.6	16.8	1.759	0.85	0.888	1.383
-7	2.57	26.1	19.9	-2.915	-0.62	0.867	-0.227
9	2.	20.3	22.2	4.179	0.49	0.853	-4.179

We now calculate the rf power requirements for the cavities. Although it may drive an electrical engineer up the wall, we calculate the wall losses in Gaussian units. However, we will employ complex notation and write Eq. (3.1) in the form

$$A = (2B/p) J_1(\rho r) e^{i(\theta - \Omega t)} \quad , \quad (3.22)$$

from which it follows that

$$\vec{B} = 2B \left[i \frac{J_1(\rho r)}{\rho r} \hat{r} - J_1'(\rho r) \hat{\theta} \right] e^{i(\theta - \Omega t)} \quad . \quad (3.23)$$

The cavity extends axially from $z = 0$ to $z = h$. On the wall at $z = 0$ we have the electric field given by

$$E_r = - (1 - i) B_\theta (\Omega/8\pi\sigma)^{1/2} \quad , \quad (3.24a)$$

$$E_\theta = (1 - i) B_r (\Omega/8\pi\sigma)^{1/2} \quad , \quad (3.24b)$$

in which σ is the conductivity of the wall material in units of sec^{-1} . For copper walls, we have $\sigma = 5.13 \times 10^{17} \text{ sec}^{-1}$ (corresponding to 5.7×10^7 mhos/m). In the cylindrical wall at $r = b$ we have

$$E_z = - (1 - i) B_0 (\omega/8\pi\sigma)^{1/2} . \quad (3.25)$$

The time average power loss to the walls is given by

$$P = \frac{c}{8\pi} \int \text{Re} [\vec{E}^* \times \vec{B}] \cdot d\vec{a} . \quad (3.26)$$

Employing Eq. (3.23) and (3.24) we find that the power loss P_s to one side wall is

$$P_s = cB^2 \left(\frac{\Omega}{8\pi\sigma} \right)^{1/2} \int_0^b \left\{ \left[J_1'(\rho r) \right]^2 + \frac{J_1^2(\rho r)}{(\rho r)^2} \right\} r dr .$$

The integral is readily performed and yields

$$P_s = \frac{cB^2}{2} \left(\frac{\Omega}{8\pi\sigma} \right)^{1/2} b^2 J_0^2(\rho b) . \quad (3.27)$$

We note that $J_0^2(\rho b) = J_0^2(3.83) = 0.16$. We choose $\sigma = 5.13 \times 10^{17} \text{ sec}^{-1}$ and $B = 100G$ in all cavities. We have $\Omega = 2\pi|n| \times 10^8$ and $b^2 = (b_1/n)^2$.

The power loss to both side walls, $2P_s$, is then found to be

$$\begin{aligned} 2P_s &= 1.12 |n|^{-3/2} \times 10^{13} \text{ ergs/sec} \\ 2P_s &= 1.12 |n|^{-3/2} \text{ MW} . \end{aligned} \quad (3.28)$$

The power lost to the outer wall, P_w , is given by

$$\begin{aligned}
 P_w &= \frac{cbh}{4} \operatorname{Re} \left[E_z^* B_\theta \right]_{r=b} , \\
 &= cb^2 \left(\frac{\omega}{8\pi a} \right)^{1/2} h b J_0^2(\rho b) . \quad (3.29)
 \end{aligned}$$

We see that $P_w = 2(h/b)P_s$. In our example $h/b \approx 0.1$ for all boxes, so that the total power $P = 1.22|n|^{-3/2}$ MW.

IV. Beam Induced rf

We examine three effects involving excitation of cavity modes by the beam. The first effect takes place in each cavity and is independent of the presence of the other cavities. The beam excites the same TM_{110} mode being used to deflect the beam in that cavity and is analogous to beam loading in an rf accelerator. We refer to this phenomena as resonant excitation because the transverse position of the beam varies as $e^{i\Omega t}$ and drives the mode at resonance. Since the applied rf imparts no energy to particles in the beam, the induced rf is reactive (i.e., $\approx 90^\circ$ out of phase with the applied rf), and results in a time-dependent phase shift in the total rf fields acting on the beam.

The second effect is the excitation of modes in a cavity by the periodic transverse motion imparted to the beam in an upstream cavity. For example, with the cavity arrangement in Fig. 6, the motion imparted to the beam in the $n = 1$ cavity will excite modes in the other four cavities. We refer to this effect as non-resonant excitation because any TM_{ip0} mode in the downstream cavity is driven off resonance.

The third effect is the excitation of TM_{0P0} modes in the cavities by the time-dependent current of the beam head. Excitation of these modes leads to a time-dependent energy variation of particles in the beam.

For these calculations we represent the axial vector potential of a particular mode of the cavity by the complex quantity $A(r, \theta, t)$ of the form

$$A(r, \theta, t) = q_\ell(t) A_\ell(r, \theta) \quad (4.1)$$

Also we have

$$\nabla^2 A = \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -\frac{4\pi}{c} j_z \quad (4.2)$$

in which j_z is the axial current density in the beam. The quantity $A_\ell(r, \theta)$ is the axial (only) component of the vector potential of the cavity mode, and obeys the Helmholtz equation

$$\nabla^2 A_\ell + (\omega_\ell/c)^2 A_\ell = 0 \quad (4.3)$$

with ω_ℓ the eigenfrequency of the mode. The complex orthogonal quantities employed are normalized such that

$$\int A_\ell A_\ell^* dv = 4\pi c^2 \quad (4.4)$$

with the integral extending over the volume of the cavity. The subscript ℓ is shorthand for the three subscripts $1P0$ or $0P0$, with P an integer. We neglect energy loss in the cavity walls. This approximation is valid for times short compared to the filling time, $2Q/\omega_\ell$, for the mode, where Q is the usual quality factor. The approximation is valid for the entire duration of the continuous beam if that duration is much less than $2Q/\omega_\ell$, as is the situation of interest in this work. From the above relations one can derive the equation for $q_\ell(t)$:

$$\ddot{q}_\ell + \omega_\ell^2 q_\ell = \frac{1}{c} \int j_z A_\ell^* dv \quad (4.5)$$

In terms of the complex vector $A(r, \theta, t)$, the vector potential is $A + A^*$ or $2 \operatorname{Re} A$.

A. Resonant Excitation

The applied rf has a vector potential given by Eq. (3.1). The resulting transverse motion of the beam in the cavity is given by Eq. (3.10). We assume the rise time of the beam current is sufficiently long that the axial dependence of the current in the cavity may be neglected in our calculation. The current density j_z may be written as

$$j_z(x, y, t) = I(t) \delta[x - \xi_x(z, t)] \delta[y - \xi_y(z, t)] \quad (4.6)$$

with $I(t)$ the time-dependent beam current.

We take $A_\ell(r, \theta, t)$ to be

$$A_\ell(r, \theta, t) = N_\ell J_1(\rho_p r) e^{-i\theta} \quad (4.7)$$

The normalization factor N_ℓ is given by

$$N_\ell^2 = \frac{4c^2}{\hbar b^2 J_0^2(\rho_p b)} \quad (4.8)$$

In this section we consider only $p = 1$, and drop the subscripts on ρ and N . Also, in this section only, we have $\omega_\ell = \Omega = c\rho$. Near the axis we have $J_1(\rho r) \approx \rho r/2$ and

$$A_\ell^* \approx (N\rho/2) (x + iy) \quad (4.9)$$

With the use of Eqs. (4.6) and (4.9), the integral on the right hand side of Eq. (4.5) becomes

$$\frac{1}{c} \int j_z A_L^* dv = \frac{I(t)}{2c} N_0 \int_0^h \xi(z,t) dz \quad (4.10)$$

Inserting ξ from Eq. (3.10) and integrating we find

$$\int_0^h \xi(z,t) dz = 4v^2 \frac{\omega_c}{\Omega^3} (\sin u - u \cos u) e^{i(\Omega t - u)} \quad (4.11)$$

In this expression we have again employed $u \equiv \Omega h/2v$. In practice $u \ll 1$ and we expand $\sin u$ and $\cos u$ to obtain

$$\sin u - u \cos u \approx u^3/3. \quad (4.12)$$

Inserting Eqs. (4.11) and (4.12) into Eq. (4.10) we obtain

$$\frac{1}{c} \int j_z A_L^* dv = \frac{I(t) \rho N \omega_c h^3}{12cv} e^{i(\Omega t - u)} \quad (4.13)$$

Equation (4.5) is to be solved with the initial conditions $q_L = 0$ and $\dot{q}_L = 0$ at $t = t_1$, where t_1 is the time at which the front of the beam enters the cavity. We define $\tau = t - t_1$, so that $t = \tau + t_1$ in the exponential factor in Eq. (4.13), and the current I can be expressed as a function τ . Then Eq. (4.5) is of the form

$$\frac{d^2 q_L}{d\tau^2} + \Omega^2 q_L = f(\tau) \quad (4.14)$$

with $f(\tau)$ representing the right hand side of Eq. (4.13). The desired solution is

$$q_2(\tau) = \frac{1}{\Omega} \int_0^\tau f(\tau') \sin \Omega(\tau - \tau') d\tau' . \quad (4.15)$$

We insert $f(\tau)$ from Eq. (4.13) and employ Eqs. (4.1), (4.7) and (4.8) to obtain the expression for the complex quantity $A(r, \theta, t)$. We have

$$A(r, \theta, t) = \frac{\omega_c}{3v} \left(\frac{h}{b} \right)^2 \frac{G(\tau) J_1(\rho r)}{J_0^2(\rho b)} e^{i(\Omega t_1 - u - \theta)} \quad (4.16)$$

in which

$$G(\tau) = \int_0^\tau I(\tau') e^{i\Omega\tau'} \sin \Omega(\tau - \tau') d\tau' . \quad (4.17)$$

The exact form of $I(\tau)$ will not affect the results drastically, which is fortunate since the time variation of the beam current may not be well known or easily controlled. Throughout this section we employ $I(t)$ of the form

$$I(\tau) = I_0 (1 - e^{-v\tau}) , \quad (4.18)$$

and find

$$G(\tau) = \frac{I_0}{2i} \left\{ \frac{v \sin \Omega\tau}{\Omega(2i\Omega - v)} + e^{i\Omega\tau} \left[\tau - \frac{2i\Omega(1 - e^{-v\tau})}{v(2i\Omega - v)} \right] \right\} . \quad (4.19)$$

We choose v small enough so that we may neglect v with respect to $2i\Omega$.

This is a fair approximation even if $v = \omega/2\pi$ (i.e., the beam current exponentiates in one cycle of the fundamental frequency) in the $n = 1$ cavity and a very good approximation in cavities with higher frequency. Within the validity of this approximation, for $v\tau \gtrsim 3$, the dominant term in Eq. (4.19) is given by

$$2iG(\tau) \approx I_0 e^{i\Omega\tau} (\tau - v^{-1}) . \quad (4.20)$$

We set $\tau = t - t_1$ and insert Eq. (4.20) into Eq. (4.16) and obtain

$$2\text{Re } A(r, \theta, t) = \frac{I_0 \omega c}{3v} \left(\frac{h}{b} \right)^2 \frac{J_1(\rho r)}{J_0^2(\rho b)} (t - t_1 - v^{-1}) \sin(\Omega t - u - \theta). \quad (4.21)$$

We see that the induced rf is almost 90° out of phase with the applied rf, which varies as $\cos(\Omega t - \theta)$, and grows linearly with time during passage of the continuous particle beam.

If the duration of the continuous beam is longer than the filling time of the mode, $2Q/\Omega$ replaces $t - t_1$ in Eq. (4.21). The Q value for these cavities is of the order of 10^4 to 10^5 , so that the filling time is of the order of $10 \mu s$, much longer than the duration of a continuous beam that we will consider.

We now find a ratio of the amplitude of the right hand side of Eq. (4.21) to the amplitude of the applied vector potential, which we write as $(2B/\rho)J_1(\rho r)$. Notice that B , the applied magnetic field on axis, cancels. The ratio can be expressed in terms of the Alfvén current I_a given by

$$I_a = \gamma \beta m c^3 / e = 17 \gamma \beta \text{ kA} \quad , \quad (4.22)$$

and becomes

$$\text{Ratio} = \frac{I_0(\text{kA})}{102 \gamma \beta} \left(\frac{h}{b} \right)^2 \frac{\Omega(t - t_1 - v^{-1})}{J_0^2(\rho b)} \quad . \quad (4.23)$$

In our example, all cavities have the same value $h/b \approx 10^{-1}$ and the same $J_0^2(\rho b) = (-.403)^2 = 0.16$. Thus the ratio is largest for the $\Omega = 9\omega$ cavity. For $I_0 = 10 \text{ kA}$, $v^{-1} = 10 \text{ ns}$, beam duration $t - t_1 = 70 \text{ ns}$, we find that the ratio at the $t - t_1 - v^{-1} = 60 \text{ ns}$ is 2.12×10^{-2} . Because of the phase relation

between the induced and applied rf, this value implies a phase shift of about 20 mrad in the total rf over the duration of the beam. Such a small phase shift should be easily compensated by the driving circuit of the cavity.

8. Non-resonant Excitation

We refer to the cavity in which the applied rf generates periodic transverse motion on the beam as the "sending" cavity. The periodic motion then excites modes in a downstream cavity, which we refer to as the "receiving" cavity. Subscripts s and r designate quantities in the sending and receiving cavities respectively. The frequency of the applied rf in the sending cavity is Ω_s , and the frequency of the induced rf in the sending cavity is ω_k . The periodic motion will excite the mode with $\omega_k \equiv \Omega_r$, that is, the same mode as the applied rf in the receiving cavity. The periodic motion will also excite higher order modes (TM_{1p0} with $p > 1$) in the receiving cavity, but with the cavity arrangement in Fig. 6 the level of excitation is small compared to the excitation level of the mode with $\omega_k = \Omega_r$. Therefore if we demonstrate that the induced rf at $\omega_k = \Omega_r$ is unimportant, the induced rf at higher frequencies will be even less important. We will thus consider only $\omega_k = \Omega_r$ in the following treatment.

The complex vector potential of the mode being excited is given by Eq. (4.7). We choose the expression, Eq. (3.16) for ξ . We set $\sin u = u$ and employ Eq. (4.12) so that Eq. (3.16) becomes

$$\xi(L, t) = - \frac{2iv\omega_c u}{\Omega_s^2} \left(\frac{u^2}{3} + \frac{i\Omega_s L}{v} \right) e^{i\Omega_s [t - (L/v)]} \quad (4.24)$$

Since we are interested only in demonstrating that the effect considered here is negligible, we concentrate on the amplitude and do not keep track of the phase of the motion or of the induced rf. Recall that the distance L is measured from the center of the sending cavity. We define L_{sr} to be the center-to-center distance from the sending cavity to the receiving cavity, as shown in Fig. 7. We also employ $T_{sr} \equiv L_{sr}/v$.

The equation analogous to Eq. (4.10) is

$$\frac{1}{c} \int j_z A_k^* dv = \frac{I(t)}{2c} (N_k \rho)_r \int_{L_{sr} - (h_r/2)}^{L_{sr} + (h_r/2)} \xi(L, t) dL \quad (4.25)$$

Inserting Eq. (4.24) and integrating, we obtain

$$\int_{L_{sr} - (h_r/2)}^{L_{sr} + (h_r/2)} \xi(L, t) dL = - \frac{4iv^2 \omega_c u}{\Omega_s^3} \left[\sin \Delta \left(\frac{u^2}{3} + i\Omega_s T_{sr} \right) + \sin \Delta - \Delta \cos \Delta \right] e^{i\Omega_s(t - T_{sr})} \quad (4.26)$$

The quantity Δ is given by

$$\Delta = \Omega_s h_r / 2v \quad (4.27)$$

The quantity $u \equiv \Omega h / 2v$ is the same for all cavities and is a small quantity.

With the cavity arrangement in Fig. 6, we have $\Delta < u$ always, and we set $\sin \Delta = \Delta$ and neglect $\sin \Delta - \Delta \cos \Delta \approx \Delta^3/3$ in Eq. (4.26). Furthermore, the condition $\Omega_s T_{sr} \gg u^2/3$ holds for all receiving cavities. Retaining only the dominant term on the right hand side of Eq. (4.26) and inserting Eq. (4.27) we have

$$\int \xi(L, t) dL = \omega_c h_s h_r T_{sr} e^{i\Omega_s t} \quad (4.28)$$

in which we have again ignored the phase. In this calculation we assume that the applied magnetic field, and thus ω_c , is the same in all cavities.

Proceeding as in the previous calculation, we obtain

$$A(r, \theta, t) = 2\omega_c h_s T_{sr} \left[\frac{J_1(\rho r)}{b^2 J_0^2(\rho b)} \right]_r H(t) e^{-i\theta}, \quad (4.29)$$

in which

$$H(t) = \int_0^t I(t') e^{i\Omega_s t'} \sin \Omega_r (t - t') dt' . \quad (4.30)$$

If we again choose $I(t) = I_0 (1 - e^{-\nu t})$, the result of the integration may be written in terms of the function $\mathcal{H}$ defined as

$$\begin{aligned} \mathcal{H}(\Omega_s, \Omega_r, t) = I_0 \left[i\Omega_s \sin \Omega_r t + \Omega_r \cos \Omega_r t - \Omega_r e^{i\Omega_s t} \right] \\ \times (\Omega_s^2 - \Omega_r^2)^{-1} . \end{aligned} \quad (4.31)$$

In terms of this function we have

$$H(t) = \mathcal{H}[\Omega_s, \Omega_r, t] - \mathcal{H}[(\Omega_s + i\nu), \Omega_r, t] . \quad (4.32)$$

We are interested in this expression at a time greater than the rise time, ν^{-1} , of the current, but also much shorter than the filling time, $2Q/\Omega_r$ of the receiving cavity. The terms in H with frequency Ω_r will decay in a filling time, but are still present at times of interest. The term in $\mathcal{H}[(\Omega_s + i\nu), \Omega_r, t]$ with frequency Ω_s will decay for $t \geq 2\nu^{-1}$, so we neglect this term. To first order in ν/Ω_c , we write $H(t)$ in the form

$$\frac{H(t)}{I_0} = -\frac{\Omega_r e^{i\Omega_s t}}{(\Omega_s^2 - \Omega_r^2)} - \frac{(\Omega_s^2 + \Omega_r^2)}{(\Omega_s^2 - \Omega_r^2)^2} \sin \Omega_r t + \frac{2i\Omega_s \Omega_r}{(\Omega_s^2 - \Omega_r^2)^2} \cos \Omega_r t \quad (4.33)$$

Inserting Eq. (4.33) into Eq. (4.29) yields the expression for the complex vector potential of the induced rf. We will consider separately the terms with frequencies Ω_s and Ω_r . First we take the ratio R_1 defined by

$$R_1 = \frac{h_r \times 12 \text{Re } A(\Omega_s)}{h_s \times \text{applied } A(\Omega_s)} \quad (4.34)$$

This ratio is approximately the ratio of the transverse momentum imparted to the beam at frequency Ω_s by the induced rf in the receiving cavity to the transverse momentum at Ω_s imparted to the beam by the applied rf in the sending cavity. It is a measure of the perturbation of the Ω_s term in Eq. (3.20) caused by the induced rf. The amplitude of the applied vector potential is $[(28/\rho)J_1(\rho r)]_s$. We set $J_1(\rho r) = \rho r/2$ for both cavities, insert the term with frequency Ω_s in Eq. (4.33) into Eq. (4.29) to obtain

$$R_1 = \frac{2I_0(\text{kA})}{17 \gamma \beta} \frac{h_r}{b_r^2} \frac{v I_{sr}}{J_0^2(3.83)} \frac{\Omega_r^2}{(\Omega_r^2 - \Omega_s^2)} \quad (4.35)$$

For our model, with $I_0 = 10$ kA, the largest ratio R_1 is obtained for $\Omega_s = \omega$ and $\Omega_r = 9\omega$, for which $R_1 = 8.2 \times 10^{-3}$. So, as alleged, the effect treated here is negligible.

The form of Eq. (4.35) suggests reversing the order of the cavities to reduce R_1 even further, but there is the problem of higher order TM_{1p0} modes in the downstream cavities. For instance, the $\Omega = 5\omega$ cavity has a $p = 2$ mode with frequency 9.16ω . If the $n = 5$ cavity is placed downstream from the $n = 9$ cavity, this mode would be appreciably excited. In fact R_1

for this mode being excited in the $n = 5$ cavity by the $n = 9$ cavity is about .06, so that the $\Omega = 9\omega$ term in Eq. (3.20) would suffer a 6% perturbation.

To evaluate the effects of the terms with frequency Ω_r in Eq. (4.33), we form the ratio R_2 given by

$$R_2 = \frac{|2\text{Re}A(\Omega_r)|}{\text{applied } A(\Omega_r)} .$$

This ratio is a measure of the perturbation of the Ω_r term in Eq. (3.20) caused by the induced rf. Since $\Omega_s^2 + \Omega_r^2 > 2\Omega_r\Omega_s$, we treat only the $\sin \Omega_r t$ term in Eq. (4.33). Proceeding as in the derivation of R_1 , and noting $h_s \Omega_s = h_r \Omega_r$, we find

$$R_2 = \frac{v}{\Omega_s} \frac{(\Omega_r^2 + \Omega_s^2)}{(\Omega_r^2 - \Omega_s^2)} R_1 . \quad (4.36)$$

We again take $v = 10^8 \text{ sec}^{-1}$ and find that $R_2 < 10^{-3}$ for all combinations of Ω_s and Ω_r .

It is a little difficult to determine the scaling of R_1 with energy (i.e., γ). If the same angle of deflection is desired as γ increases, one could increase B in all cavities at the expense of additional power, which scales as B^2 if this is done, R_1 scales as γ^{-1} . But the electrical breakdown limit is only about three hundred Gauss so after B has reached its limiting value, it is necessary to increase the length h of all boxes in proportion to the increase in γ . As a consequence, T_{sr} will increase as γ_1 so that R_1 will scale as γ . However, for higher energy beams, a smaller angle may suffice.

C. Excitation of TM_{010} Modes

The cavity modes considered here are azimuthally symmetric with axial component of vector potential given by the real expression

$$A_z(r) = N_z J_0(\omega_z r/c) \quad , \quad (4.37)$$

in which J_0 is the ordinary Bessel function of the first kind of order zero.

The eigenfrequencies ω_z are determined by the condition $J_0(\omega_z b/c) = 0$.

In terms of the zeros x_z of J_0 we have

$$\omega_z = x_z c/b \quad . \quad (4.38)$$

the normalization factor N_z is given by

$$N_z^2 = \frac{4c^2}{\hbar b^2 J_1^2(x_z)} \quad . \quad (4.39)$$

We take the current density j of the beam to be uniform within a radius $r = a$ and zero for $r > a$, and again take $I(t) = I_0(1 - e^{-vt})$.

In the previous calculations of the effects of induced rf, we could concentrate on one mode in each cavity. However, in this section we must consider excitation of many modes in one cavity. The question of how many will be discussed below. The quantity of interest is the total voltage across the cavity that results from the passage of the time-dependent current at the beam head. From Eq. (4.1) the axial electric field of one mode, E_z , is given by

$$E_z = -\frac{1}{c} \ddot{q}_z(t) A_z(r) \quad , \quad (4.40)$$

and the total voltage is

$$V(t) = \hbar \sum E_z(0,t) \quad . \quad (4.41)$$

Proceeding as in the previous calculations, we find $V(t)$ to be given by the following expression:

$$V = \frac{4hI_0 v}{b^2} \sum_{x_\ell} \frac{2J_1(x_\ell a/c)}{(x_\ell a/c)} \frac{[\cos \omega_\ell t - (v/\omega_\ell) \sin \omega_\ell t - e^{-vt}]}{(\omega_\ell^2 + v^2) J_1^2(x_\ell)} \quad (4.42)$$

We can obtain an upper bound on the value of V by setting the time-dependent quantity in [] equal to unity for each term in the series. The quantity $2J_1(x_\ell a/c) / (x_\ell a/c)$ has a maximum value of unity so we employ this maximum in each term. Neglecting v^2 with respect to ω_ℓ^2 , we have

$$V < \frac{4hI_0 v}{c^2} \sum_{x_\ell} \frac{1}{x_\ell^2 J_1^2(x_\ell)} \quad (4.43)$$

The terms in the series in Eq. (4.43) are the same in all cavities, but the maximum x_ℓ (i.e., the number of terms in the series) is different for different cavities. We choose the upper limit of the sum to be such that the frequency of the highest mode excited is below the cutoff frequency of the TM_{010} mode in the beam pipe. For a pipe of radius r_p this cutoff frequency equals $x_1 c/r_p$, so that the maximum x_ℓ for each cavity is given by

$$x_\ell(\max) \leq x_1 b/r_p$$

The upper bound on the voltage is largest in the $n = 1$ cavity, first of all because of the factor h in Eq. (4.43), and also because this cavity has the largest number of terms in the series. We choose $r_p = 3$ cm. The radius $b = 180$ cm for the $n = 1$ cavity and $x_1 = 2.405$, so that $x_\ell(\max) < 144$.

For large ℓ , the roots x_ℓ are given approximately by $x_\ell = (\ell - 1/4)\pi$, so that the maximum x_ℓ is x_{46} . The x_{47} mode has a frequency above cutoff in the beam pipe. With $h = 18$ cm, $I_0 = 10$ kA, and $v = 10^8 \text{ sec}^{-1}$, we have $2hI_0 v/c^2 = 72$ kV. (In these units, $I/c = I(\text{A})/10$, and $v = 240$ Stat-volts = 72 kV.)

The sum in Eq. (5.8) can be evaluated numerically, but a good approximate value is obtained by employing the relations $x_\ell = (\ell - 1/4)\pi$ and $J_1^2(x_\ell) \approx 2/\pi x_\ell$ so that $[x_\ell J_1(x_\ell)]^2 \approx (2\ell - 1/2)$. This approximation is poorest for $\ell = 1$, where $[x_1 J_1(x_1)]^2 = 1.56$, so that the error in the first term is about 4%. For all $\ell > 1$, the error is less than 1%. Using this approximation we have

$$V_n = 1 < 72 \sum_{\ell=1}^{46} (2\ell - \frac{1}{2})^{-1} = 190 \text{ kV} \quad (4.44)$$

Because we have set the time-dependent factor in Eq. (4.42) equal to unity, we have grossly overestimated the induced voltage, and still the upper bound represents an energy modulation of less than 1/2% of the 50 MeV particle energy.

For the $n = -3, 5, -7$, and 9 cavities, the same procedure yields $V = 50$ kV, 26 kV, 17 kV, and 12 kV respectively. For particle energy such that h increases with γ , the induced voltage increases with γ , but the percentage of total particle energy remains constant. The induced voltage is directly proportional to v , consequently a relatively slow rise of the beam current is desirable.

Reference

- 1 R. Briggs, E. Cornet, A. Paul and J. Simpson in the Proceedings of the Second Joint DOE/DOD Accelerator Technology Workshop, Jan. 1981, W. J. Schafer Associates WJSA-B-LJ-38. Page 263. (Secret)

Figure Captions

1. Ideal x and y displacement of the beam centroid as a function of time over one cycle of the fundamental frequency.
2. x and y displacement of the beam centroid resulting from the first 5 terms of the Fourier series for the ideal displacement, Eq. (2.1). The coefficients are normalized so that $x(0) = 1$.
3. x displacement of the beam centroid resulting from the first 9 terms of the Fourier series for the ideal displacement, Eq. (2.1). The coefficients are normalized so that $x(0) = 1$.
4. x displacement of the beam centroid resulting from optimization to the rounded function, Eq. (2.3), employing 5 terms of the Fourier series with all optimization points in the interval $0 < \omega t < \pi/4$.
5. x displacement of the beam centroid resulting from optimization to the rounded function, Eq. (2.3), employing 5 terms of the Fourier series with optimization points equally distributed in the interval $0 < \omega t < \pi/2$.
6. Schematic of the cavity array considered in Sections III and IV.

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5

Figure 6

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