On Helical Equilibrium

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Abstract

A straight, helical plasma equilibrium equation is solved numerically for a plasma with a helical magnetic axis. As is expected, by a suitable choice of the plasma boundary, the vacuum configuration is made $\oint dl/B$ stable. As the plasma pressure increases, the $\oint dl/B$ criterion will improve (again as expected). There is apparently no limit on the plasma β from the equilibrium consideration. Thus helical-axis stellarator β will presumably be limited by MHD stability β , and not by equilibrium β .

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I. Introduction

Two types of equilibria of magnetized plasmas have been extensively studied. One type is azimuthally symmetric toroidal equilibria, and the other is helically symmetric equilibria. The latter does not admit closed magnetic surfaces, thus helical symmetry must be violated to form toroidally closed surfaces in order to confine plasmas. Nonetheless, when the aspect ratio is large, the magnetic surfaces retain their properties, and numerically and experimentally the surfaces are found to exist for practical purposes. When the aspect ratio is reduced, the magnetic surfaces tend to break up. Since for large aspect ratio, the properties of helically symmetric configurations are nearly preserved, it is worthwhile to study helically symmetric equilibria and stabilities. Understanding of these topics will shed light on the properties of almost-helical symmetric devices such as stellarators.

In this paper we shall discuss the equilibria of helically symmetric configurations whose axes are not straight, but are helically twisted. This configuration has been studied previously¹⁻³ because it offers the possibility of higher critical β for stability.

In deriving a helically symmetric equilibrium equation,⁴ it is important to introduce helical coordinates. This is done as follows. Cylindrical coordinates (r, θ, z) are rendered into a dimensionless form by letting $(hr, \theta, hz) = (x, \theta, hz)$. Here

obviously $1/h$ has the dimension of length. Helical symmetry manifests itself by the condition that all the spatial dependence is expressed in terms of x and u where

$$x = hr , \quad (1)$$

$$u = \ell\theta + hz , \quad (2)$$

here ℓ is an integer.

It is convenient to introduce orthogonal coordinates (but not normalized) $\vec{e}_x, \vec{e}_u, \vec{e}_v$ such that in cylindrical coordinates

$$\vec{e}_x = (1, 0, 0) , \quad (3)$$

$$\vec{e}_u = (0, \ell, x) , \quad (4)$$

$$\vec{e}_v = (0, -x, \ell) . \quad (5)$$

Then in terms of these quantities, the helical equilibrium equation⁴ for a magnetic surface ψ can be expressed as

$$\frac{1}{x} \frac{\partial}{\partial x} \left(\frac{x}{K} \frac{\partial \psi}{\partial x} \right) + \frac{1}{x^2} \frac{\partial^2 \psi}{\partial u^2} + \frac{2\ell B_v}{K^2 h} + \frac{1}{2} \frac{1}{Kh^2} \frac{dB_v^2}{d\psi} + \frac{\mu_0}{h^2} \frac{dp}{d\psi} = 0 , \quad (6)$$

$$K \equiv \ell^2 + x^2 \quad (7)$$

where B_v and p are functions of ψ only. The magnetic field component, B_v , is expressed as

$$B_v = l B_z - x B_\theta . \quad (8)$$

Equation (6) is equivalent to the equilibrium equation

$$\vec{J} \times \vec{B} = \vec{\nabla} p . \quad (9)$$

The process used to reduce from Eq. (9) to Eq. (6) is well-known. However, some interesting relations can be found in the process, and a brief sketch is attached as Appendix A. If $l = 0$ in Eq. (6), we recover the azimuthally symmetric equilibrium equation,

$$- \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} \right) + \frac{1}{r} \frac{\partial \psi}{\partial r} = r^2 \mu_0 \frac{dp}{d\psi} + \frac{1}{2} \frac{d}{d\psi} (r^2 B_\theta^2) . \quad (10)$$

II. Vacuum Solution

In the absence of current (so that $p = 0$), Eq. (6) is reduced to

$$\frac{1}{x} \frac{\partial}{\partial x} \left(\frac{x}{K} \frac{\partial \psi}{\partial x} \right) + \frac{1}{x^2} \frac{\partial^2 \psi}{\partial u^2} + \frac{2l B_v}{K^2 h} = 0 . \quad (11)$$

If B_v is constant, J_r and J_u are zero by Eqs. (A9) and (A10) and J_v is zero by Eq. (A19). Unlike the vacuum solution of Eq. (10), Eq. (11) admits closed (though infinite on the v axis) magnetic surfaces in the (x, u) plane even when the (closed) boundary is connected and assumes a constant value. One numerical example is shown in Fig. 1. The configuration is an $l = 1$ helical system, with the boundary set at $x = 0.2$,

$-\pi/2 \leq u \leq \pi/2$; $u = \pm\pi/2$; $0.2 \leq x \leq 1.0$; and $x = 1.0$,
 $-\pi/2 \leq u \leq \pi/2$. The ψ value is taken to be zero. In the numerical analysis $2B_v/h$ was taken as unity. This is equivalent to stating that ψ is normalized with ψ_0 such that $h\psi_0/2B_v$ equals unity. In this way ψ is made dimensionless in Eq. (11) and in subsequent equations. The numerical method is briefly discussed in Appendix B.

The solution of Fig. 1 does not have the (average) magnetic well. The $\int dl/B$ that is numerically calculated for a fixed distance in z increases towards the periphery. Thus for very low plasma pressures, the plasma is unstable except for shear. Since shear can be calculated and is found to be not large, the stability limit is expected to be low as in the case of classic stellarators.⁵

In Fig. 2 the numerical solution for the $\int dl/B$ stable case is shown. The outer boundary is moved to lower radius; thus the plasma cross section is pinched near $u = \pm\pi/2$. This configuration has some similarity to Fig. 4b of Ref. 1. In general (empirically), surfaces having these shapes tend to have a favorable $\int dl/B$ property. Of course, similar magnetic surfaces can be obtained by solving for $\nabla^2\phi = 0$, assuming $\phi = \phi(x,u)$, and then solving for ψ by the relation

$$\vec{\nabla}\phi \cdot \vec{\nabla}\psi = 0. \quad (12)$$

A solution of Eq. (12) with the same helical periodicity is given by

$$\psi = \frac{1}{2} A_0 x^2 + A_1 \log x + \sum_{n=1}^{\infty} C_n x I'_n(x) \sin nu + \sum_{n=1}^{\infty} D_n x K'_n(x) \sin nu . \quad (13)$$

The first term on the right gives rise to the B_z field, and the logarithmic term is the field created conductor located at $x = r = 0$. The third term represents the helical perturbations which are finite at the origin (i.e., $x = 0$). The fourth term represents the perturbations divergent at the origin.

III. Finite Pressure

When the pressure is finite, the complete Eq. (6) must be solved. Unlike an axisymmetric equilibrium equation [Eq. (10)], there is no readily available expansion parameter; thus numerical solution is the only convenient alternative. It turns out that inclusion of finite pressure does not significantly change the convergence speed of numerical relaxation methods.

In calculating Eq. (6), ψ dependences of p and B_v are taken as

$$p = \alpha\psi \equiv W \frac{h^2}{\mu_0} \psi, \quad (15)$$

$$B_v = B_{v0}(1 - 4CW\psi), \quad B_{v0} \equiv \frac{h}{2}, \quad (16)$$

where C and W are dimensionless constants, and ψ is made dimensionless as discussed in the previous section. Equation (6) is then reduced to

$$\frac{1}{x} \frac{\partial}{\partial x} \left(\frac{x}{K} \frac{\partial \psi}{\partial x} \right) + \frac{1}{x^2} \frac{\partial^2 \psi}{\partial u^2} + \frac{1}{K^2} - \frac{4CW}{K^2} \psi + W \left(1 - \frac{C}{K} \right) + \frac{4C^2 W^2}{K} \psi = 0. \quad (17)$$

The term $W(1 - C/K)$ is proportional to J_v , and in small x approximation, J_v is almost identical to the axial current, J_z .

Equation (17) is solved numerically for the cases shown in Figs. 1 and 2. In Fig. 3 the numerical solution for $W = 0.4$ and $C = 1.2$ is shown. As expected, $\oint dl/B$ becomes more favorable, but this is illusory as it is simply a reflection of the diamagnetic effect of the plasma.

The plasma peak β is defined as

$$\beta_{\text{peak}} = \frac{p_{\text{max}}}{B^2/2\mu_0} \quad (18)$$

where B is measured at the location of plasma pressure maximum in the absence of plasma. Then considering Eq. (15) and the definition of ψ , we arrive at

$$\beta_{\text{peak}} = \frac{1}{B^2/B_{v0}^2} \frac{2\mu_0 \alpha \psi}{B_{v0}^2} = \frac{8W\psi_{\text{max}}}{B^2/B_{v0}^2} \quad (19)$$

Numerically magnetic field, B , is calculated in units of B_{v0} , and B/B_{v0} is typically 0.8-0.9. (Note that because the coordinates x , u , v are not normalized, B/B_{v0} can be less than unity.) For Fig. 3 the calculated β_{peak} is approximately 19%. As one might expect, the numerical value of the magnetic field at the maximum pressure is depressed by approximately 10%.

As W increases, p , and hence β , increases. The relaxation method employed here tends to show numerical instability. For the maximum W calculated ($W = 1.6$), the peak β as defined in Eq. (19) is 142%. That β exceeds unity may be due to the extra confinement of field curvature. The magnetic field strength at the pressure maximum drops to 37% of the vacuum field. Thus one concludes that there is no practical limit for equilibrium β , and the limit of the plasma pressure will be dictated by MHD stability.

IV. Miscellaneous and Conclusions

If B_v is a function of ψ , but pressure is zero, the force-free current equilibrium can be obtained. The constant ψ surfaces are very much similar to those shown in Figs. 1, 2, and 3.

In the case of Fig. 2 (modified boundary) rotational transform, $1/2\pi$, per pitch ranges from 0.68 (edge) to 0.79 (edge).

Since for a toroidal device of coil aspect ratio A , the number of helical pitches, N , is approximately equal to A for $\ell = 1$ ($N = \ell A$ for other ℓ 's), the total rotational transform once around the torus is

$$\frac{l}{2\pi} \approx \left(\frac{l}{2\pi}\right)_p A. \quad (20)$$

Now we estimate the shift of the magnetic axis as if the stellarator in consideration were a classic stellarator. The net $\vec{v} \times \vec{B}$ and curvature drift current density is $2p/BR$ where R is the major radius. The secondary current density, $j_{s\parallel}$, is therefore⁶

$$j_{s\parallel} = \frac{2p}{BR} \frac{A}{1/2\pi}. \quad (21)$$

The magnetic field, B_1 , produced by $j_{s\parallel}$ at the magnetic axis may be approximated as

$$B_1 = \mu_0 \frac{\pi a^2 j_{s\parallel}}{2\pi a} = \frac{\mu_0 p}{B} \frac{1}{1/2\pi} \quad (22)$$

where a is the plasma minor radius.

The shift of the magnetic axis δ is then given as

$$\frac{\delta}{a} \frac{1}{2\pi} \frac{B}{A} = B_1 = \frac{\mu_0 p}{B} \frac{1}{1/2\pi}. \quad (23)$$

Thus

$$\beta = \frac{\delta}{a} \left(\frac{1}{2\pi} \right)^2 \frac{2}{A} . \quad (24)$$

For δ/a to be appreciable, say $1/2$, β must be $(1/2\pi)^2 (1/A)$. Even if A is 5, the configuration under discussion has $\beta = 2.5$, that is the toroidal shift of the magnetic axis is not a limiting factor.

In conclusion, the helical equilibrium Eq. (6) was solved numerically for the helical axis plasma configurations. The numerical analysis showed that by modifying the boundary, one can achieve a negative V'' stellarator configuration which should have a higher critical β for stability.

For a finite pressure plasma, equilibrium is not a limiting factor for plasma β . The MHD stability limit must be evaluated by using the equilibrium solution.

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APPENDIX A

The Equilibrium Equation

Equation (9) of the text,

$$\vec{J} \times \vec{B} = \vec{\nabla} p, \quad (A1)$$

along with

$$\mu_0 \vec{J} = \vec{\nabla} \times \vec{B}, \quad (A2)$$

$$\vec{B} = \vec{\nabla} \times \vec{A}, \quad (A3)$$

is solved subject to the condition that all variables are functions of x and u only. Then from Eq. (A1),

$$-(\vec{J} \times \vec{B})_\theta h + \frac{h}{r} (\vec{J} \times \vec{B})_z = 0. \quad (A4)$$

Equation (A4) can be reduced to

$$J_r B_u - B_r J_u = 0 \quad (A5)$$

if

$$B_z + \frac{h}{x} B_\theta \equiv B_u, \quad (A5)$$

$$J_z + \frac{l}{x} J_\theta \equiv J_u . \quad (A7)$$

We define B_v as in Eq. (8), that is

$$B_v = lB_z - xB_\theta . \quad (A8)$$

Then from Eq. (A2)

$$\mu_O J_r = \frac{\partial B_v}{r \partial u} , \quad (A9)$$

$$\mu_O J_u = - \frac{1}{x} \frac{\partial B_v}{\partial u} . \quad (A10)$$

Now magnetic field, $\vec{B}$, is expressed in terms of $\vec{A}$. In particular, if we define

$$A_v \equiv lA_z - xA_\theta , \quad (A11)$$

we get

$$B_u = - \frac{1}{x} \frac{\partial A_v}{\partial x} h \equiv - \frac{1}{x} \frac{\partial \psi}{\partial x} h , \quad (A12)$$

$$B_r = \frac{1}{x} \frac{\partial A_v}{\partial u} h \equiv \frac{1}{x} \frac{\partial \psi}{\partial u} h . \quad (A13)$$

The function ψ plays the role of the magnetic surface function.

In view of Eqs. (A9), (A10), (A12), and (A13), Eq. (A5) yields

$$\frac{\partial \psi}{\partial r} \frac{\partial B_v}{\partial u} = \frac{\partial \psi}{\partial u} \frac{\partial B_v}{\partial r} , \quad (\text{A14})$$

that is

$$B_v = B_v(\psi) . \quad (\text{A15})$$

From $\vec{B} \cdot \vec{\nabla} p = 0$, similarly

$$p = p(\psi) . \quad (\text{A16})$$

The radial component of Eq. (A1) is

$$J_\theta B_z - J_z B_\theta = \frac{\partial \psi}{\partial r} \frac{dp}{d\psi} . \quad (\text{A17})$$

If we define the helical current, J_v , as

$$J_v \equiv l J_z - x J_\theta , \quad (\text{A18})$$

Eq. (A17) is transformed

$$\mu_0 J_v = K \mu_0 \frac{dp}{d\psi} + \frac{1}{2} \frac{d}{d\psi} B_v^2 \quad (\text{A19})$$

where

$$K \equiv l^2 + x^2 . \quad (\text{A20})$$

The J_v term can then be expressed in terms of ψ and B_z to yield

$$-\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{x} \frac{\partial \psi}{\partial x} - \frac{K}{x^2} \frac{\partial^2 \psi}{\partial u^2} - \frac{2}{h} B_z = \frac{K\mu_0}{h^2} \frac{dB}{d\psi} + \frac{1}{2h^2} \frac{dB_v^2}{d\psi} \quad (A21)$$

Or in terms of B_v ,

$$0 = \frac{1}{x} \frac{\partial}{\partial x} \left(\frac{x}{K} \frac{\partial \psi}{\partial x} \right) + \frac{1}{x^2} \frac{\partial^2 \psi}{\partial u^2} + \frac{2B_v}{K^2 h} + \frac{1}{2} \frac{1}{Kh^2} \frac{dB_v^2}{d\psi} \quad (A22)$$

which is Eq. (16) of the main text.

APPENDIX B

Numerical Methods

Equation (6) was solved for a fixed boundary. The boundary value of ψ was taken as zero, and $2\ell B_v/h$ was taken as one in the absence of plasma currents. Constant meshes for both u and x were taken and the accelerated relaxation method was used. It usually takes about 200 to 300 iterations to arrive at a value of ψ constant to the sixth decimal.

The constant ψ surface could be constructed by interpolating mesh point values. In the figures shown (main text) instead, the magnetic field lines are calculated and projected onto the (x,u) plane. The magnetic field strength can be calculated from the gradients of ψ and the definition of B_v . The lines are calculated in the real space (x,θ,hz) and are projected back into the (x,u) plane by the relation $u = \theta + hz$ ($\ell = 1$). The ψ contours shown in the figures agree reasonably well with constant ψ contours.

Calculation of the magnetic field line allows computation of $d\ell/B = dz/B_z$. The obtained $\int dz/B_z$ is normalized by $\int dz$. Numerical integration ends when u again becomes zero. The length z , when the integration ends, is half the pitch length of the field line around the magnetic axis. The rotational transform $1/2\kappa$ is the inverse of the pitch.

The magnetic field is normalized with respect to B_{v0} . Since B_{v0} is $h/2$ (for $\ell = 1$), B_u for example, is given as [from Eq. (A12)]

$$\frac{B_u}{B_{v0}} = - \frac{2}{x} \frac{\partial \psi}{\partial x} \quad (E1)$$

which is dimensionless. The terms B_θ and B_z can be derived from B_u and B_v using Eqs. (A6) and (A8).

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X, U PLANE

1 0.1378851E-01
 2 0.2575082E-01
 3 0.3585324E-01
 4 0.4405152E-01
 5 0.5028724E-01
 6 0.5448270E-01
 7 0.5653307E-01

W = 0

C = 1.5

Fig. 1. Solutions of the helical equilibrium equations. Constant ψ contour maps. The pattern is symmetric with respect to the x-axis. In this case, the configuration is not $\int d^2/B$ stable.

U= 0.000000E+00
 C= 1.200000
 X,U PLANE
 1 0.6087229E-02
 2 0.1363328E-01
 3 0.2102644E-01
 4 0.2771247E-01
 5 0.3335480E-01
 6 0.3769535E-01
 7 0.4050313E-01

DL/B
 1 1.244399
 2 1.245349
 3 1.248659
 4 1.253889
 5 1.259214
 6 1.264095
 7 1.267302

Fig. 2. Same as Fig. 1 except the $\psi = 0$ contour was changed.
 This change allows for the $\int dl/B$ stable configuration as evidenced by the printout of the $\int dl/B$ value.

$Z = 0.400000$
 $C = 1.200000$
 X, U PLANE
 1 $0.7368009E-02$
 2 $0.1599170E-01$
 3 $0.2401257E-01$
 4 $0.3090532E-01$
 5 $0.3641201E-01$
 6 $0.4036632E-01$
 7 $0.4262989E-01$

Fig. 3. Same as Fig. 2 except that the finite plasma pressure was taken into account. Constant ψ contours are little changed by the finite plasma pressure. The absolute value of ψ increases with increasing plasma pressure.