Statistical Description of an Attractor

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Attracting solutions of one dimensional maps are described statistically by an invariant probability distribution. Expressions for the Fourier transform of this function and the Liapunov exponent are derived. Good agreement between the theory and numerical computations is found.

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Consider the equation:

$$x_{t+1} = F(x_t, \alpha) \quad (1)$$

Here $t = 0, 1, 2, \dots$ and $F(x, \alpha)$ is a single valued, continuous function of x and a parameter α . Equations of this form appear in various fields of science.^{1,2} They describe the dynamical evolution of the variable x with time t being discrete. The sequence $x_0, x_1, x_2, \dots$ we will call an orbit. Assume that there is an interval $a \leq x \leq b$ such that every point of it maps by Eq. (1) within the same interval. Then depending on the value α there are 3 types of behavior possible.^{1,2} First: Every orbit which starts at $a \leq x_0 \leq b$ converges to a stable fixed point. This point is a stationary attractor. Second: There is a stable fixed point only considering every v^{th} iteration of Eq. (1) ($v = 2, 3, \dots$). This is a periodic attractor because it defines v points, $x = c_i, i = 1, \dots, v$ between which the orbit alternates during v iterations. Finally, if the orbit does not exhibit any periodicity it is called a stochastic or "strange" attractor. In this case the attracting set of points is infinite.² We always assume that there is only one attractor present on the interval $[a, b]$. Define a Liapunov exponent λ

$$\lambda = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^T \ln \left| \frac{dF(x_t)}{dx} \right| \quad (2)$$

It can be shown that $\lambda < 0$ for a periodic attractor, while for a stochastic attractor² $\lambda > 0$. Positive λ means that an orbit diverges exponentially from nearby orbits. This phenomenon is called stochastic instability.³ The operation defined by Eq. (2) is a dynamic averaging; it requires following an orbit, and can be performed with any smooth function $f(x)$. Usually, it is

done numerically, but if we know exactly the position of the attracting points then it is easy to see that

$$\overline{f(x)} = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^T f(x_t) = \frac{1}{v} \sum_{i=1}^v f(c_i) \quad , \quad (3)$$

where we assume here the case of a periodic attractor. Equation (3) can be generalized to the case of a stochastic attractor. Let us introduce a statistical averaging over an invariant probability distribution $P(x)$.²

$$\langle f(x) \rangle = \int_a^b f(x) P(x) dx \quad . \quad (4)$$

$P(x)$ is nonzero only within the interval $[a, b]$. It is easy to see from Eqs. (3) and (4) that in the case of a periodic attractor $P(x) = \sum_{i=1}^v \delta(x - c_i) / v$. We will deal later mainly with the Fourier transform of the invariant probability. It is well defined because $P(x)$ is an integrable function.

In this paper we calculate the characteristic function

$$a(k) = \langle e^{-ikx} \rangle \quad , \quad (5)$$

the moments

$$\langle x^n \rangle = (i \frac{\partial}{\partial k})^n a(k=0) \quad , \quad (6)$$

and the Liapunov exponent

$$\lambda = \langle \ln |df(x)/dx| \rangle \quad , \quad (7)$$

and make comparisons with numerical computations. We use some ideas developed in previous publications.^{4,5}

We derive first an equation for the Fourier amplitudes of $P(x)$. Consider an ensemble of orbits which have an initial density distribution $P(x, t = 0)$. The evolution of $P(x, t)$ from time t to $t + 1$ is described by the equation:

$$P(x, t+1) = \int_a^b \delta(x - F(y)) P(y, t) dy. \quad (8)$$

If we normalize $P(x, t)$ to unity it has a meaning of a probability distribution. The invariant probability distribution $P(x)$ can be found as the limit $P(x, t \rightarrow \infty)$. We may expand $P(x, t)$ in a Fourier series on the interval $[a, b]$

$$P(x, t) = \frac{1}{l} \sum_{n=-\infty}^{\infty} a_n^t e^{ik_n x}, \quad (9)$$

here $l = b - a$, $k_n = 2\pi n/l$ and

$$a_n^t = \int_a^b P(x, t) e^{-ik_n x} dx. \quad (10)$$

Applying Eqs. (9) and (10) we can Fourier transform Eq. (8) to get:

$$a_n^{t+1} = \sum_{m=-\infty}^{\infty} A_{mn} a_m^t \quad (11)$$

$$A_{mn} = \frac{1}{l} \int_a^b \exp\left[i \frac{2\pi}{l} (nx - mF(x))\right] dx. \quad (12)$$

The normalization condition implies:

$$\int_a^b P(x,t) dx = 1, \quad a_0^t = 1. \quad (13)$$

Consider the limit $t \rightarrow \infty$ of Eq. (11). Let $a_n = \lim_{t \rightarrow \infty} a_n^t$ and we find:

$$\sum_{n=-\infty}^{\infty} (\lambda_{mn} - \delta_{mn}) a_n = -A_{m0}, \quad (14)$$

where the prime indicates that the sum does not include $n = 0$, δ_{mn} is a Kronecker function and we used $a_0 = 1$. The system of Eqs. (14) has a unique solution for a_n if the determinant $|\lambda_{mn} - \delta_{mn}| \neq 0$. In order to solve Eq. (14) we will truncate it at $1 \leq |m|, |n| \leq N$. Truncation may be interpreted as a coarse graining of $P(x)$. This is justified if the truncation parameter N is large enough. If $f(x)$ in Eq. (4) is a smooth function with characteristic scale $1/k_f$, then $N \gg 1/k_f$. Secondly the a_n for $n \leq 1/k_f$ must be independent of N . Conditions on the truncation parameter N will be discussed further when we consider the numerical results.

We can also iterate Eq. (14) to obtain a formal series expansion for a_n . Noting from Eq. (12) that $|\lambda_{mn}| < 1$ we treat λ_{mn} as small and iterate, finding

$$a_n = A_{n0} + \sum_{m=-\infty}^{\infty} \lambda_{nm} A_{m0} + \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \lambda_{nm} \lambda_{mp} A_{p0} + \dots \quad (15)$$

Eq. (15) can be derived also using the method of paths.⁵

One of the findings of this paper is that Eq. (15) converges rapidly in the case of well-developed stochasticity, $\lambda \gg 1$. In this case it is enough to keep just a few terms in Eq. (15), while in the case of a periodic attractor, convergence is poor. In order to find the characteristic function $a(k)$ we

have to substitute k for k_n in Eq. (10) and Eq. (15).

Consider now a specific example of Eq. (1)⁶

$$F(x, \alpha) = -\alpha \sin x. \quad (16)$$

Depending on the value of α this mapping exhibits the three types of attractors mentioned in the introduction.^{2,6} Assume first $0 < \alpha < \pi$ and take $a = -\pi$, $b = \pi$. The interval $[-\pi, \pi]$ maps onto itself and contains only one attractor. Thus the conditions for the application of our theory are satisfied. Using Eq. (12) and Eq. (16) we find

$$A_{mn} = J_n(m\alpha). \quad (17)$$

The characteristic function $a(k)$ and the Liapunov exponent λ can be written now in the form

$$a(k) = J_0(k\alpha) + 2 \sum_{n=1}^{\infty} J_{2n}(k\alpha) a_{2n} \quad (18)$$

$$\lambda = \ln \frac{\alpha}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} a_{2n}. \quad (19)$$

Thus all quantities of interest can be expressed through the coefficients a_{2n} . In order to find a_{2n} , we use Eq. (14) in its truncated form

$$\sum_{n=1}^N [2J_{2n}(2m\alpha) - \delta_{mn}] a_{2n} = -J_0(2m\alpha) \quad m = 1, 2, \dots, N. \quad (20)$$

The odd terms a_{2n+1} cancel due to the relation $J_{-n}(z) = (-1)^n J_n(z)$. They can be determined from Eq. (18). Another way to calculate a_{2n} is to use series

(15), which can be written in the form

$$a_{2n} = J_0(2n\alpha) + 2 \sum_{m=1}^N J_{2m}(2n\alpha) J_0(2m\alpha) + \\ + 4 \sum_{m=1}^N \sum_{p=1}^N J_{2m}(2n\alpha) J_{2p}(2m\alpha) J_0(2p\alpha) + \dots \quad (21)$$

From Eqs. (6) and (18) we can calculate the second moment

$$\langle x^2 \rangle = \frac{1}{2} \alpha^2 [1 - a_2(\alpha)] . \quad (22)$$

All odd moments are equal to zero because $p(x)$ is symmetric.

Consider now the case $-\pi < \alpha < -1$. Then there are two attractors present: orbits starting at $0 < x_0 < \pi$ or $-\pi < x_0 < 0$ will converge to the right or to the left attractor respectively. In order to use our theory we could divide the interval into the two parts $[-\pi, 0]$ and $[0, \pi]$, whereupon the system Eq. (14) becomes complex instead of real. More simply, let $P_+(x)$ and $P_-(x)$ be invariant probabilities for the two domains, respectively. It is easy to show that $P_+(-x) = P_-(x)$, and that $P_+(x) + P_-(x)$ satisfies the same Eq. (8) as $P(x)$ for $|\alpha| < \pi$. Thus the $P(x)$ we have calculated can be used to find $P_{\pm}(x)$ for $1 < \alpha < \pi$. For example, $P_+(x) = 2P(x)$ for $x > 0$ and $P_+(x) = 0$ for $x < 0$. For any symmetrical function $f(x)$ in Eq. (4) averaging by using $P(x)$ or $P_{\pm}(x)$ gives the same result. Thus, Eqs. (19) - (22) are still applicable to find λ and $\langle x^2 \rangle$. Equations (18) - (22) also can be used for $|\alpha| > \pi$ if there is only one attractor present for all orbits with $-\infty < x_0 < \infty$. In this case our expressions involve Fourier integrals instead of Fourier series.

We have investigated these results numerically for the example Eq. (16). The results for the Liapunov exponent $\lambda(\alpha)$ are shown in Fig. 1. The points were obtained numerically using dynamical averaging through Eq. (2). Here $T = 10,000$. The solid curve was obtained by using Eq. (19) and matrix inversion of Eq. (20) with $N = 30$. We find that the value of λ is given quite accurately. The exceptions are points where λ varies rapidly in α , and insufficient Fourier terms are retained. This is easily understood in terms of Eq. (20). Rapid variation in α can only occur through high order arguments of the Bessel functions, and hence N must be taken sufficiently large. In addition, we observe that as λ approaches zero the determinant of the system Eq. (20) also approaches zero. This takes place at $\alpha = 1$, $\alpha = 2.26$ and 2.62 where the first, second, and third period doublings occur. Higher order period doublings,⁷ which occur near $\alpha = 2.7$ are not sufficiently well represented by the Fourier series with $N = 30$. Also $N = 30$ is insufficient to describe a narrow region of periodic attractors imbedded in a stochastic region, e.g., $\alpha = 3$.

Also shown on Fig. 1, as a dashed line, is the expression for λ using Eq. (19) and Eq. (21) including up to three Bessel products ($P = 3$) and $N = 30$. Agreement is good for the case of well-developed stochasticity, $\lambda > 1$. The same study has been done for the second moment $\langle x^2 \rangle$, and the regions of applicability of the two methods are essentially the same.

The case of well-developed stochasticity is especially interesting. We have examined the convergence of Eq. (21) in the truncation parameter P , equal to the maximum number of Bessel products. Convergence in the parameter P is very rapid, in fact for $\lambda > 3$ the inclusion of terms with two Bessel functions decreases the error by a factor of 3 or 4, but the inclusion of terms with three Bessel functions ($P = 3$) does not sensibly decrease the error. For

fixed N and P the error in $a(k)$ decreases exponentially with λ .

In Fig. 2 is shown the result for $a(k)$ with $\alpha = 72.25$ and $\lambda = 3.62$, using Eq. (18) and the first term of Eq. (21), and $N = 20$.

Finally in Fig. 3 we have plotted the invariant probability $P(x)$ for the same value of α as in Fig. 2. The points were obtained by dividing the interval $-\alpha < x < \alpha$ into bins and numerically advancing Eq. (1) until the relative accumulated number of occurrences in each bin approached a limit. The solid line is given by the equation $P(x) = 1/\pi(\alpha^2 - x^2)^{1/2}$ and was obtained by using Eq. (9), and $a_{2n} = J_0(2n\alpha)$. It is remarkable that the lowest truncation of Eq. (21) gives a reasonably good description of the invariant probability.

We conclude that the solution of Eq. (14) gives an accurate description of the Fourier transform of the invariant probability function if an adequate number of Fourier harmonics is retained. Furthermore, in the case of well-developed stochasticity, $\lambda \gg 1$, Eq. (15) gives a very rapidly convergent series for the characteristic function.

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Fig. 1 The Liapunov exponent λ as a function of α . The solid line was obtained from the inversion of Eq. (18) with $N = 30$, the dashed line from Eq. (19) and (21) with terms of order J^3 ($P = 3$), $N = 30$, and the points from dynamical averaging with $T = 10,000$.

Fig. 2 The characteristic function $a(k)$. The solid line was obtained using Eq. (19) and $a_{2n} = J_0(2n\alpha)$, and truncating with $N = 20$. Here $\alpha = 72.25$ and $\lambda = 3.62$. The points were obtained by dynamical averaging with $T = 40,630$.

Fig. 3 The invariant probability function $P(x)$ for $\alpha = 72.25$, $\lambda = 3.62$. The points were obtained by stepping Eq. (1) and the solid line from Eq. (9) with the lowest order truncation of Eq. (21).