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KINETIC EXTENSIONS OF MAGNETOHYDRODYNAMIC MODELS
FOR AXISYMMETRIC TOROIDAL PLASMAS

BY

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KINETIC EXTENSIONS OF MAGNETOHYDRODYNAMIC MODELS

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Abstract

A nonvariational kinetic-MHD stability code (NOVA-K) has been developed to integrate a set of non-Hermitian integro-differential eigenmode equations due to energetic particles for axisymmetric toroidal plasmas in a general flux coordinate (ψ, θ, ζ) system with an arbitrary Jacobian. The NOVA-K code employs the Galerkin method involving Fourier expansions in the generalized poloidal angle θ and generalized toroidal angle ζ directions, and cubic-B spline finite elements in the radial ψ direction. Extensive comparisons with the existing variational ideal MHD codes show that the ideal MHD version of the NOVA-K code converges faster and gives more accurate results. The NOVA-K code is employed to study the effects of energetic particles on MHD-type modes: (1) the stabilization of ideal MHD internal kink modes and the excitation of "fishbone" internal kink modes; and (2) the alpha particle destabilization of toroidicity-induced Alfvén eigenmodes (TAE) via transit resonances. Analytical theories are also presented to help explain the NOVA-K results. For energetic trapped particles generated by neutral beam injection (NBI) or ion cyclotron resonant heating (ICRH), a stability window for the $n=1$ internal kink mode in the hot particle beta space exists even in the absence of the core ion finite Larmor radius effect (finite ω_{*i}). On the other hand, the trapped alpha particles are found to have negligible effects on the stability of the $n=1$ internal kink mode, but the circulating alpha particles can strongly destabilize TAE modes via inverse Landau damping associated with the spatial gradient of the alpha particle pressure.

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1. Introduction

Linear stability analysis of magnetohydrodynamic (MHD) modes for axisymmetric toroidal plasma equilibria has been employed to aid in the analysis of experimental data and in the design of new experiments for thermonuclear fusion research. The ideal MHD model treats the plasma as a conducting fluid and is thought to govern the gross behavior of the plasma. In particular, ideal MHD instabilities have been observed to play an important role in limiting the beta (β) values of tokamak plasmas [1,2] and causing tokamak disruptions [3]. The PBX was basically designed [4,5] to avoid the instabilities of ballooning modes and kink modes so that the plasma may reach the second β stability region. The designs of future toroidal fusion reactors have also been significantly guided by the ideal MHD equilibrium and stability analysis [6]. Exact solutions of MHD instabilities are impossible to obtain without the use of numerical computations. A number of two-dimensional MHD stability codes [7-12] have been developed extensively to study the dependence of low- n (n is the toroidal mode number) ideal MHD instabilities on a variety of parameters relating to the geometry as well as the pressure and current profiles.

With the introduction and increased use of powerful auxiliary heating techniques, such as ion cyclotron resonant heating (ICRH) and neutral beam injection (NBI), present-day tokamak experiments and future fusion reactors will involve a significant amount of energetic particles. The energetic particles usually have anisotropic pressures and will significantly modify the plasma equilibrium. They will also significantly affect the MHD stability because some of the basic assumptions of MHD theory become invalid for energetic particles. For the low frequency MHD modes with $\omega \ll \omega_d$ (ω_d is the energetic particle magnetic drift frequency), the energetic particle dynamics are no longer governed by the $(\vec{E} \times \vec{B})/B^2$ drift, but rather by the magnetic (∇B and curvature) drifts. For the high frequency shear Alfvén waves with $\omega = \omega_t$ (ω_t is the energetic particle transit frequency), the ideal MHD stable global Alfvén modes can be driven unstable by the circulating energetic particles resonating with the background waves. In the high-power, nearly perpendicular neutral beam injection experiments, bursts of large amplitude $m=n=1$ (m is the poloidal mode number) MHD fluctuations, dubbed "fishbones" were observed in the Mirnov coil and x-ray signals [13-15]. These fishbone bursts are found to be correlated with the reductions of neutron emissivity which correspond to significant losses of energetic beam ions, and thus play an important role in limiting the β -values of tokamak devices. On the other hand, ICRH-heated plasmas with an energetic minority ion component on JET [16,17]

seemed to have demonstrated a sawtooth-free domain of tokamak operation. The JET experiments are interpreted as the energetic particle stabilization of both the resistive internal kink mode and the fishbone mode. In future tokamak reactors, alpha (α) particles may destabilize the toroidicity-induced shear Alfvén eigenmodes (TAE) [18-20] through circulating particle resonance by tapping the free energy associated with the α -particle pressure nonuniformity. The consequences of these effects are yet to be determined.

Two major mathematical formulations of the linear ideal MHD stability analysis have been employed in developing two-dimensional MHD stability codes. One is to solve the two-dimensional Euler-Lagrange eigenmode equations for linearized perturbations and obtain the growth rates of the MHD instabilities, as was done in the NOVA code [7]. The other existing two-dimensional ideal MHD stability codes [8-12] utilize a variational Lagrangian formalism [21] for linearized perturbations about an equilibrium configuration and involve the use of carefully selected expansion functions which enables the problem to be reduced to the minimization of an algebraic quadratic form with respect to certain variational parameters. Both formulations involve the use of the Galerkin procedure, which approximates the perturbations by a linear superposition of a finite subset of a complete set of expansion functions, $\xi = \sum \alpha_m \phi_m$, with ϕ_m 's independent of the expansion parameters α_m 's. The stability calculation is then reduced to the determination of eigenvalues and eigenfunctions of a matrix eigenvalue problem. For the variational Lagrangian formulation, this procedure preserves the virtues of the Lagrangian system. Nonetheless, the inherent limitation of these variational codes [8-12] is that because of their variational nature, they cannot be extended to the stability calculations of the non-Hermitian eigenmode equations, such as in the cases of ideal MHD with equilibrium flows, resistive MHD and kinetic MHD, etc., where variational energy principles cannot be established. In particular, the energetic particle dynamics are governed by the gyrokinetic equations and the resultant kinetic-MHD eigenmode equations are non-Hermitian integro-differential equations. On the other hand, the nonvariational approach employed by the NOVA code can be easily applied to the non-Hermitian eigenmode analysis.

In this paper, we will review the kinetic extensions of the MHD model of low- n modes for axisymmetric toroidal plasmas. Kinetic modifications of high- n MHD modes have been studied extensively only after the high- n WKB-ballooning mode formalism [22,23] was developed. However, it is impossible to generalize the high- n kinetic-MHD formulation to the low- n

modes. It is only recent that the low- n kinetic-MHD stability analysis has made major progress. We will first present the formulation and numerical methods employed in a nonvariational kinetic-MHD stability code (NOVA-K). Then we discuss the important physics obtained by the NOVA-K code. The NOVA-K code is the kinetic extension of a previously developed ideal MHD NOVA code [7]. In general, a flux coordinate (ψ, θ, ζ) system with an arbitrary Jacobian, the NOVA-K code employs Fourier expansion in the poloidal angle θ direction, as in the PEST code [8], and cubic B-spline finite elements in the radial ψ direction. An arbitrary nonuniform ψ -mesh is set up to provide the option of zoning the mesh to allow more finite elements near rational surfaces, the plasma edge, and the magnetic axis. In comparison with the existing variational ideal MHD stability codes [8-12], the ideal MHD version of the NOVA-K code converges faster and gives more accurate results.

In the following, we first briefly describe in Sec. 2 the anisotropic MHD equilibrium and present a class of generalized toroidal coordinate systems which can greatly improve the representation of various MHD instabilities. In Sec. 3, we formulate the kinetic-MHD eigenmode equations for our nonvariational treatment. The numerical methods which involve cubic-B spline finite elements, integrations of circulating and trapped particle integrals and vacuum solutions, are described in Sec. 4. Convergence studies of the ideal MHD version of the NOVA-K code are presented for the analytical Solovév equilibria [24], and detailed comparisons, as presented with other variational codes [7,25], are given in Sec. 5. In Sec. 6 we study the effects of neutral beam injections (NBI) and α -particles on the stability of the internal kink mode and the excitation of fishbone modes. We first derive and study an analytical dispersion relation. Then, we establish the correctness of the NOVA-K code by both convergence studies and qualitative agreements with numerical solutions of the analytical dispersion relation. For the NBI case, comparisons with numerical results involving analytical approximations indicate that the critical values of the energetic trapped particle beta for the stabilization of the ideal MHD internal kink and the destabilization of the resonant fishbone mode can be an order of magnitude different from those computed by the "non-approximated" NOVA-K code. In addition, a necessary condition for the excitation of the fishbone mode is found that the total plasma beta must be in the domain of the unstable ideal MHD internal kink modes. This is contrary to the analytical results of no total beta thresholds. For α -particles, the stability of the internal kink is only weakly affected. In Sec. 7, we present the NOVA-K results of the α -particle destabilization of TAE

modes via transit resonances. The principal conclusions of this work are summarized in Sec. 8.

2. MHD Equilibrium

2.1 Flux Coordinate System

In terms of the flux coordinate system (ψ, θ, ζ) , the equilibrium magnetic field with nested flux surfaces can be written as

$$\vec{B} = \nabla \zeta \times \nabla \psi + q(\psi) \nabla \psi \times \nabla \theta, \quad (2.1)$$

where $2\pi\psi$ is the poloidal flux within a magnetic surface, $q(\psi)$ is the safety factor, θ is the generalized poloidal angle varying between 0 and 2π , and ζ is the generalized toroidal angle varying between 0 and 2π . Since

$$\vec{B} \cdot \nabla = J^{-1} \left(\frac{\partial}{\partial \theta} + q \frac{\partial}{\partial \zeta} \right), \quad (2.2)$$

where $J = (\nabla \psi \times \nabla \theta \cdot \nabla \phi)^{-1}$ is the Jacobian, the magnetic field lines are straight in this coordinate system. For axisymmetric equilibria, $\vec{B}$ can also be expressed as

$$\vec{B} = \nabla \phi \times \nabla \psi + g \nabla \phi, \quad (2.3)$$

where $g = X B_\theta$, B_θ is the toroidal magnetic field, and ϕ is the toroidal angle in cylindrical (X, ϕ, Z) coordinates. Then, ζ is related to ϕ by

$$\zeta = \phi - q \delta(\theta, \psi), \quad (2.4)$$

where $\delta(\theta, \psi)$ is periodic in θ and is determined by

$$q(1 + \partial \delta / \partial \theta) = g J / X^2. \quad (2.5)$$

Along a flux surface in the poloidal plane we have

$$\frac{ds}{d\theta} = \frac{\mathbf{J} |\nabla\psi|}{X}, \quad (2.6)$$

where ds is the element of arc length along a constant (ϕ, ψ) line. The specification of $\mathbf{J}$, therefore, determines the r coordinate. In this paper, we choose the Jacobian in the form

$$\mathbf{J}(X, Z) = \frac{X^i}{\lambda(\psi) |\nabla\psi|^j B^k}, \quad (2.7)$$

where i, j, k can be freely specified and $\lambda(\psi)$ is given by the requirement that θ increases by 2π during one poloidal circuit. The form of $\mathbf{J}$ in Eq. (2.7) is a trivial generalization of that used in the PEST-2 code [26]. Thus, from Eq. (2.6) we have

$$\lambda(\psi) = \frac{2\pi}{\oint ds |\nabla\psi|^{j-1} B^k / X^{i-1}}. \quad (2.8)$$

For the choice $i = 2, j = k = 0$, the (ψ, θ, ζ) coordinate represents the PEST-1 coordinate. For $i = j = 1, k = 0$, we have the equal arc length coordinate system, and a Hamada-like coordinate system is obtained by letting $i = j = k = 0$. The choice $i = j = 0, k = 2$ is used for the Hamiltonian representation of the magnetic field [27].

The general flux coordinate system constructed here is not orthogonal and its metric is complicated because $\nabla\psi \cdot \nabla\theta \neq 0$, $\nabla\psi \cdot \nabla\zeta \neq 0$, $\nabla\phi \cdot \nabla\theta \neq 0$, and $|\nabla\zeta|^2 \neq 1/X^2$. However, ζ is still an ignorable coordinate for axisymmetric equilibria, and the perturbed quantities can be represented by a single mode varying as $\exp(-in\zeta)$.

2.2 Anisotropic MHD Equilibrium

We consider stationary anisotropic ideal MHD equilibria satisfying

$$\vec{J} \times \vec{B} = \nabla \cdot \vec{P} = \nabla P_{\perp} - \nabla \cdot [(P_{\perp} - P_{\parallel}) \hat{b} \hat{b}], \quad (2.9)$$

$$\nabla \times \vec{B} = \vec{J}, \quad (2.10)$$

$$\nabla \cdot \vec{B} = 0, \quad (2.11)$$

where $\hat{b} = \vec{B}/B$, and $\vec{J}$, $\vec{B}$ and $\vec{P}$ are the equilibrium current, magnetic field, and pressure tensor, respectively. $\vec{P}$ is obtained by specifying the particle guiding center distribution for each species $F(\epsilon, \mu, \psi_g)$, where energy $\epsilon = v^2/2$ and magnetic moment $\mu = v_\perp^2/2B$ are the constants of motions, $v_\perp$ is the component of the velocity perpendicular to $\vec{B}$, and ψ_g is the magnetic flux position of the guiding center. The parallel and perpendicular pressures are given by

$$\begin{pmatrix} P_\parallel \\ P_\perp \end{pmatrix} = \sum_{j, \sigma_\parallel} 2\pi M_j \int \frac{d\epsilon d\mu B F_j}{|v_\parallel|} \begin{pmatrix} 2(\epsilon - \mu B) \\ \mu B \end{pmatrix}, \quad (2.12)$$

where the summation is over the particle species j and $\sigma_\parallel$ which represents the direction of particle velocity parallel to $\vec{B}$, and M_j is the particle mass. The parallel velocity $v_\parallel$ has the form

$$v_\parallel = \sigma_\parallel \sqrt{2(\epsilon - \mu B)}. \quad (2.13)$$

$P_\perp$ and $P_\parallel$ are functions of magnetic flux function ψ and the magnitude of the magnetic field B . From Eq. (2.8), we obtain the perpendicular momentum balance equation

$$\nabla_\perp (B^2/2 + P_\perp) = \vec{\kappa} [B^2 + (P_\perp - P_\parallel)], \quad (2.14)$$

and the parallel momentum balance equation

$$\vec{B} \cdot \nabla P_\parallel = (P_\perp - P_\parallel) \hat{b} \cdot \nabla B, \quad (2.15)$$

where $\vec{\kappa} = \hat{b} \cdot \nabla \hat{b}$ is the magnetic field curvature. Note that Eq. (2.14) is automatically satisfied if $F(\epsilon, \mu, \psi_g)$ is used to compute $P_\perp$ and $P_\parallel$. It is convenient to introduce the functions

$$\sigma \equiv 1 - (1/B)(\partial P_\parallel / \partial B)_\psi \approx 1 + (P_\perp - P_\parallel) / B^2, \quad (2.16)$$

and

$$\tau \equiv 1 + (1/B)(\partial P_{\perp}/\partial B)_{\psi} . \quad (2.17)$$

If $\sigma > 0$ and $\tau > 0$ are satisfied everywhere in the plasma, the system is stable to the well-known "firehose" and "mirror" instabilities, respectively [28]. The numerical equilibrium is required to be stable to these modes.

The $\nabla\phi$ component of Eq. (2.9) gives

$$\vec{B} \cdot \nabla G = 0 . \quad (2.18)$$

Thus G is a function of ψ only. Then, the $\nabla\psi$ component of Eq. (2.9) leads to the anisotropic Grad-Shafranov equation [28]

$$\Delta^* \equiv \chi^2 \nabla \cdot \left(\frac{\nabla \psi}{\chi^2} \right) = - \left\{ \frac{\chi^2}{\sigma} \left(\frac{\partial P_{\parallel}}{\partial \psi} \right)_B + \frac{GG'}{\sigma^2} + \frac{\nabla \psi \cdot \nabla \sigma}{\sigma} \right\} , \quad (2.19)$$

which will be used to solve ψ if $P_{\parallel}(\psi, B)$, $G(\psi)$, and appropriate boundary conditions are specified. The shape of the current distribution is determined by choosing the function $G(\psi)$, and $P_{\parallel}(\psi, B)$ is obtained by providing a guiding-center particle distribution $F(\epsilon, \mu, \psi_0)$. The isotropic limit of equation (2.18) is obtained by setting $P_{\perp} = P_{\parallel}$, so that $\sigma = 1$, and $G = g$. Equation (2.18) differs little from the well-known isotropic form so that the modification of the isotropic equilibrium solver to tensor pressure is straightforward.

Properties of anisotropic tokamak equilibria have been studied by employing guiding-center particle distributions $F(\epsilon, \mu, \psi_0)$, which can be either prescribed analytically or obtained from the numerical solutions of a Fokker-Planck equation with a source term representing NBI or ICRH [29,30]. The effects of pressure anisotropy on plasma equilibrium can be best studied by NBI with different injection angles. It is found that [30] the surfaces of constant pressure are quite significantly shifted in relation to the surfaces of constant ψ , which vary little with changes in the injection angles. For parallel injection the constant pressure surfaces shift inward, and for perpendicular injection the surfaces shift outward. This can be understood by noting that in parallel injection the beam ions are nearly all circulating particles, and in perpendicular injection the beam ions are trapped in the outside region. The adiabatic invariance of

the magnetic moment, $\mu = v_{\perp}^2 / 2B$, dictates that circulating particles will have larger $v_{\perp}$ and smaller $v_{\parallel}$ in high magnetic field regions than in low field regions. Since the lower $v_{\parallel}$ in regions of high B allows ions to remain there longer, the pressure is enhanced accordingly to provide an inward shift of the constant pressure surfaces. On the other hand, the trapped ions in the perpendicular injection case will enhance the pressure more in the outside region leading to an outward shift of the surfaces.

3. Formulation of Kinetic-MHD Stability Equations

3.1 Model

We will consider an axisymmetric toroidal plasma consisting of the core isotropic and the hot anisotropic components with $n_h \ll n_c$ and $T_h \gg T_c$ so that $\beta_h < \beta_c$. Summing the collisionless equations of motion for each species, we obtain the linear momentum equation

$$\omega^2 \rho \vec{\xi} = \nabla \delta p_c + \nabla \cdot \delta \vec{p}_h + \vec{b} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{b}) , \quad (3.1)$$

where $\vec{\xi}$ is the usual fluid displacement vector, $\vec{b}$ is the perturbed magnetic field, δp_c is the perturbed core plasma pressure, $\delta \vec{p}_h$ is the perturbed hot plasma pressure tensor, and ρ is the total plasma mass density. The ideal MHD description is adopted for the core plasma and the following ideal MHD relations hold

$$\delta p_c + \vec{\xi} \cdot \nabla P_c + \Gamma_s P_c \nabla \cdot \vec{\xi} = 0 , \quad (3.2)$$

$$\vec{b} = \nabla \times (\vec{\xi}_{\perp} \times \vec{B}) , \quad (3.3)$$

and

$$\delta \vec{E} = i \omega \vec{\xi} \times \vec{B} , \quad (3.4)$$

where $\delta \vec{E}$ is the perturbed electric field, $\Gamma_s = 5/3$ is the ratio of specific heat, and P_c is the core plasma pressure. The gyrokinetic description neglecting the Finite-Larmor-radius correction is employed for the hot component. $\delta \vec{p}_h$ can be expressed as

$$\delta \vec{p}_h = \delta p_\perp \vec{I} + (\delta p_\parallel - \delta p_\perp) \hat{b} \hat{b} . \quad (3.5)$$

where $\delta p_\parallel$ and $\delta p_\perp$ are obtained from the perturbed hot plasma distribution function δf by

$$\begin{pmatrix} \delta p_\parallel \\ \delta p_\perp \end{pmatrix} = M_h \int d^3v \delta f \begin{pmatrix} 2(\varepsilon - \mu B) \\ \mu B \end{pmatrix} . \quad (3.6)$$

If we write δf as

$$\delta f = \frac{e}{M_h} \frac{\partial F}{\partial \varepsilon} \Phi - \frac{\mu b_\parallel}{B} \frac{\partial F}{\partial \mu} + g , \quad (3.7)$$

where e is the hot particle charge, and $b_\parallel$ is the parallel perturbed magnetic field, then g is governed by the drift-kinetic equation [31], with $g \propto \exp(-i\omega t)$, and is given by

$$\frac{d\mathbf{g}}{dt} \equiv \frac{\partial}{\partial t} + (\vec{v}_\parallel + \vec{v}_d) \cdot \nabla \mathbf{g} = -\frac{ieF}{T} \left(\vec{\omega} - \vec{\omega}_*^T \right) \left[\Phi - \frac{v_\parallel A_\parallel}{c} + \frac{M_h \mu b_\parallel}{e} \right] , \quad (3.8)$$

where $\vec{\omega} = -(T\omega/M_h)\partial \ln F / \partial \varepsilon$, $\vec{\omega}_*^T = -i(T/M_h\omega_c)(\hat{b} \times \nabla \ln F \cdot \nabla)$ only operates on the perturbed quantities, $\vec{v}_d = (\hat{b}/\omega_c) \times [\nabla(\mu B) + \vec{\kappa} v_\parallel^2]$ is the magnetic drift velocity, c is the speed of light, T is the average temperature of the hot particle, ω_c is the hot particle cyclotron frequency, Φ is the electrostatic potential, and $A_\parallel$ is the parallel vector potential.

3.2 Solutions of the Drift Kinetic Equation

Since we are interested in MHD-type modes, from Eq. (3.4) we have $\delta \vec{E}_\parallel = 0$ so that

$$i\omega A_\parallel = c \hat{b} \cdot \nabla \Phi . \quad (3.9)$$

For low β plasmas, we will also neglect the $A_\perp$ contribution to the perpendicular electric field and we get $\delta \vec{E}_\perp = -\nabla_\perp \Phi$. Then we find

$$\Phi - \frac{v_{\parallel} A_{\parallel}}{c} = \frac{i}{\omega} \left(\frac{d\Phi}{dt} + \vec{v}_d \cdot \delta \vec{E}_{\perp} \right). \quad (3.10)$$

By assuming that the perturbations vanish at $t' = -\infty$, Eq. (3.8) can be solved by the method of characteristics and we have

$$\mathbf{g} = \frac{eF}{T} \frac{(\tilde{\omega} - \omega_*)}{\omega} \Phi + \hat{\mathbf{g}}, \quad (3.11)$$

where the nonadiabatic particle distribution is given by

$$\hat{\mathbf{g}} = \int_{-\infty}^t dt' \left(-\frac{ieF}{T} \right) (\tilde{\omega} - \omega_*) \left[\frac{i \vec{v}_d \cdot \delta \vec{E}_{\perp}}{\omega} + \frac{M_H \mu b_{\parallel}}{e} \right]. \quad (3.12)$$

The time integration in Eq. (3.12) is along the unperturbed guiding-center trajectory. Note that to arrive at Eq. (3.11), we have taken $d[F(\tilde{\omega} - \omega_*)]/dt = 0$ so that typical particle radial excursions away from the flux surface (eg. trapped particle banana width) along the unperturbed trajectory are assumed to be small compared to the equilibrium scale length. From Eqs. (3.2) and (3.3) the integrand in Eq. (3.12) can be written as

$$\frac{i \vec{v}_d \cdot \delta \vec{E}_{\perp}}{\omega} + \frac{M_H \mu b_{\parallel}}{e} = \left(-\frac{M_H E}{e} \right) \left[\frac{\Lambda}{h} \nabla \cdot \vec{\xi}_{\perp} - 2 \left(1 - \frac{3\Lambda}{2h} \right) \vec{\kappa} \cdot \vec{\xi}_{\perp} \right]. \quad (3.13)$$

Then the total perturbed particle distribution is

$$\delta f = -\vec{\xi}_{\perp} \cdot \nabla F - \frac{\mu b_{\parallel}}{B} \frac{\partial F}{\partial \mu} + \hat{\mathbf{g}}. \quad (3.14)$$

From Eqs. (3.6) and (3.14), we obtain

$$\delta \vec{p}_H = -\vec{\xi}_{\perp} \cdot \nabla \vec{P} + b_{\parallel} \left(\frac{\partial \vec{P}}{\partial B} \right)_{\psi} + \delta \hat{p}_{\perp} \vec{I} + (\delta \hat{p}_{\parallel} - \delta \hat{p}_{\perp}) \hat{\mathbf{b}} \hat{\mathbf{b}}. \quad (3.15)$$

where $\widehat{\delta p}_{\parallel}$ and $\widehat{\delta p}_{\perp}$ are given by

$$\begin{pmatrix} \widehat{\delta p}_{\parallel} \\ \widehat{\delta p}_{\perp} \end{pmatrix} = M_h \int d^3 v \widehat{g} \begin{pmatrix} 2(\varepsilon - \mu B) \\ \mu B \end{pmatrix}. \quad (3.16)$$

In Eq. (3.16) the integral over velocity space can be expressed in terms of the new velocity space coordinates $(\varepsilon, \Lambda, \sigma)$

$$\int d^3 v = \sum_{\sigma} \sqrt{2} \pi \int_0^{\infty} d\varepsilon \varepsilon^{1/2} \int_0^{h(\psi, \theta)} \frac{d\Lambda}{h \sqrt{1 - \Lambda/h}}. \quad (3.17)$$

where $\Lambda = \mu B_0 / \varepsilon$ is the pitch angle, B_0 is the vacuum magnetic field at $X = R$, and $h = B_0 / B(\psi, \theta)$. On a flux surface, circulating particles correspond to $0 \leq \Lambda \leq h_{\min}(\psi)$, and trapped particles to $h_{\min}(\psi) \leq \Lambda \leq h$ at a given θ , where $h_{\min}(\psi) = \text{Min}[h(\psi, \theta)]$ on the ψ surface.

At this point it is appropriate to discuss the particle orbit in an equilibrium magnetic field. For axisymmetric tokamaks, the toroidal canonical angular momentum of a single particle is strictly conserved.

$$P_{\phi} = M X v_{\phi} - \frac{e X A_{\phi}}{c} = \text{constant}, \quad (3.18)$$

where A_{ϕ} is the toroidal component of the vector potential. In terms of the magnetic field representation Eq. (2.3), and performing a gyrophase average on Eq. (3.18), the trapped particle excursion away from a constant flux surface ψ_0 is given by

$$\psi = \psi_0 + \frac{M c X B_0 v_{\parallel}}{e B}. \quad (3.19)$$

For a small particle excursion away from ψ_0 , we expand $\psi = \psi_0 + \Delta r |\nabla \psi|$, and the trapped particle banana width is roughly given by

$$\Delta_b = \frac{2 M c v_{\parallel \max}}{e B_p} \left(\frac{B_0}{B} \right), \quad (3.20)$$

where $B_p = X |\nabla \psi|$ is the poloidal magnetic field, and v_{limax} is the parallel velocity at the field minimum $\theta = 0$. For circulating particles the radial excursion away from ψ_0 is smaller than the trapped banana width. In this paper we shall neglect the effects of the particle excursion away from ψ_0 . The bounce period τ_b for the trapped particles is given by

$$\tau_b = \oint \frac{d\theta}{|v_{\parallel}|} = \oint \frac{J B d\theta}{|v_{\parallel}|} \quad (3.21)$$

Likewise, the transit period τ_t for the circulating particles is given by

$$\tau_t = \int_{-\pi}^{\pi} \frac{J B d\theta}{|v_{\parallel}|} \quad (3.22)$$

The corresponding bounce and transit frequencies are defined as $\omega_b = 2\pi/\tau_b$ and $\omega_t = 2\pi/\tau_t$ respectively. Note that we take τ_b and τ_t to be positive by convention, and that τ_b is equivalent to $2\tau_t$ at the boundary between trapped and circulating particles.

To perform the time integration in Eq. (3.12), we first represent the perturbed quantities by the Fourier series in θ

$$\vec{\xi}(\psi, \theta, \zeta, t) = \sum_{m=-\infty}^{\infty} \vec{\xi}_m(\psi) \exp[i(m\theta - n\zeta - \omega t)] \quad (3.23)$$

then Eq. (3.12) can be written as

$$\hat{g} = \sum_m \int_{-\infty}^t dt' G_m e^{i[m(\theta' - \theta) - n(\zeta' - \zeta) - \omega(t' - t)]} e^{i(m\theta - n\zeta - \omega t)} \quad (3.24)$$

where the specific form of $G_m = G_m(\psi', \theta', \Lambda, \varepsilon)$ will be given in section 3.3 in terms of the final dependent variables of the eigenmode equations, ψ' , θ' , and ζ' are functions of t' . It is useful to define the following quantities:

$$\begin{aligned}\omega_d &\equiv -n \left(\frac{d\zeta}{dt} - q \frac{d\theta}{dt} \right) = -n \vec{v}_d \cdot (\nabla \zeta - q \nabla \theta) \\ &= \frac{nMcE}{e} \left\{ \frac{[\kappa_\psi + \kappa_s \nabla \psi \cdot \nabla (\zeta - q\theta)]}{|\nabla \psi|^2} \left(1 - \frac{\Lambda}{2h} \right) - \frac{\Lambda P_c'}{hB^2} \right\}\end{aligned}\quad (3.25)$$

is the instantaneous guiding center magnetic drift frequency, where $\kappa_\psi = 2 \vec{\kappa} \cdot \nabla \psi$ and $\kappa_s = 2 \vec{\kappa} \cdot (\vec{B} \times \nabla \psi / B^2)$,

$$S_m = m - nq, \quad (3.26)$$

$$W(t) = \int_0^t dt' (\omega_d - \langle \omega_d \rangle), \quad (3.27)$$

where $\langle A \rangle$ denotes the bounce or transit average of A and is given by

$$\langle A \rangle = \frac{1}{\tau_{b,t}} \oint \frac{A dl}{|v_{||}|}. \quad (3.28)$$

The first exponent appearing in Eq. (3.24) can now be written in the form

$$\begin{aligned}m(\theta' - \theta) - n(\zeta' - \zeta) - \omega(t' - t) &= -(\omega - \langle \omega_d \rangle - H \sigma_{||} S_m \omega_{b,t})(t' - t) \\ &\quad + [S_m(\theta' - \theta) - H \sigma_{||} S_m \omega_{b,t}(t' - t) + W(t') - W(t)],\end{aligned}\quad (3.29)$$

where $H = H(h_{\min} - \Lambda)$ is the Heaviside step function, so that H vanishes for trapped particles and is equal to one for circulating particles. Note that the second term in Eq. (3.29) is a periodic function of t' with period τ_b and τ_t for trapped and circulating particles, respectively. Therefore, it is useful to employ the decompositions

$$\begin{aligned}G_m(t') &= e^{i[S_m(\theta' - \theta) - H \sigma_{||} S_m \omega_{b,t}(t' - t) + W(t') - W(t)]} \\ &= \sum_{p=-\infty}^{\infty} a_{m,p}^\sigma e^{ip \omega_{b,t}(t' - t)},\end{aligned}\quad (3.30)$$

where the summation is over all integer values of p , and

$$a_{m,p}^{\sigma_{||}} = \frac{1}{\tau_{b,t}} \oint dt' G_m e^{i \left[S_m(\theta' - \theta) - (p + H \sigma_{||} S_m) \omega_{b,t}(t' - t) + W(t') - W(t) \right]} \quad (3.31)$$

By means of these decompositions the time integration in Eq. (3.24) can be done exactly and we have

$$\begin{aligned} \hat{g} &= \sum_{m,p} e^{i(m\theta - n\zeta - \omega t)} \int_{-\infty}^t dt' a_{m,p}^{\sigma_{||}} e^{-i \left[\omega - \langle \omega_d \rangle - (p + H \sigma_{||} S_m) \omega_{b,t} \right] (t' - t)} \\ &= \sum_{m,p} \frac{a_{m,p}^{\sigma_{||}} e^{i(m\theta - n\zeta - \omega t)}}{\left[\omega - \langle \omega_d \rangle - (p + H \sigma_{||} S_m) \omega_{b,t} \right]} \end{aligned} \quad (3.32)$$

Note that the form of the denominator in Eq. (3.32) is different for trapped and circulating particles. This is due to the fact that circulating particle orbits sample the entire field line and, therefore, their contributions depend on the rationality of the magnetic surface on which they move, i.e., the value of S_m . On the other hand, trapped particle orbits sample only a part of a field line and their contributions do not depend on the value of S_m .

To evaluate the time integration in Eq. (3.31), we have to treat trapped and circulating particles separately. For simplicity, we will assume the equilibrium has an up-down symmetry. The difference between $W(t')$ and $W(t)$ will also be ignored. It is useful to define a time-like variable $\hat{t}(\theta, \sigma_{||})$, which has a one-to-one correspondence with each point of the single particle orbit. For trapped particles $\hat{t}(\theta, \sigma_{||})$ is defined by

$$\hat{t}(\theta, \sigma_{||} = +1) = \int_0^\theta \frac{B d\theta'}{|v_{||}|} \quad (3.33)$$

where $\theta_T > \theta > -\theta_T$ corresponds to $(\tau_b/4) > \hat{t} > -(\tau_b/4)$, and $\pm\theta_T$ are the trapped particle turning points defined as the roots of the $v_{||}(\theta) \propto [1 - \Lambda/h(\theta)] = 0$. For $(3\tau_b/4) > \hat{t} > (\tau_b/4)$, we have $\hat{t}(\theta, \sigma_{||} = -1) = (\tau_b/2) - \hat{t}(\theta, \sigma_{||} = +1)$. Eq. (3.31) can be written as

$$a_{m,p}^{\sigma} = \frac{e^{-i[S_m\theta - p\omega_b\hat{t}]}}{\tau_b} \left\{ \int_{-\tau_b/4}^{\tau_b/4} d\hat{t}' + \int_{\tau_b/4}^{3\tau_b/4} d\hat{t}' \right\} G_m e^{i[S_m\theta' - p\omega_b\hat{t}']} \quad (3.34)$$

Making the change of variable $\hat{t}(\theta, -\sigma_{||}) = (\tau_b/2) - \hat{t}(\theta, \sigma_{||})$ in the second integral, Eq. (3.34) becomes

$$a_{m,p}^{\sigma} = \frac{e^{-i[S_m\theta - p\omega_b\hat{t}]}}{(\tau_b/2)} \int_{-\tau_b/4}^{\tau_b/4} d\hat{t}' G_m \times e^{iS_m\theta'} \left\{ \cos^2\left(\frac{p\pi}{2}\right) \cos(p\omega_b\hat{t}') - i \sin^2\left(\frac{p\pi}{2}\right) \sin(p\omega_b\hat{t}') \right\} \quad (3.35)$$

Note that the symmetry relation holds

$$a_{m,p}^{\sigma} = a_{m,-p}^{-\sigma} e^{ip\pi} \quad (3.36)$$

Summing up over $\sigma_{||}$, we obtain from Eqs. (3.32) and (3.35) the total nonadiabatic trapped particle distribution

$$\sum_{\sigma} \hat{g} = e^{-i[n(\zeta - q\theta) + \omega t]} \times \sum_m \left\{ \sum_{p=\text{even}} \frac{2 \cos(p\omega_b\hat{t})}{(\omega - \langle\omega_d\rangle - p\omega_b)} \int_{-\tau_b/4}^{\tau_b/4} d\hat{t}' \frac{G_m e^{iS_m\theta'} \cos(p\omega_b\hat{t}')}{(\tau_b/2)} + \sum_{p=\text{odd}} \frac{2 \sin(p\omega_b\hat{t})}{(\omega - \langle\omega_d\rangle - p\omega_b)} \int_{-\tau_b/4}^{\tau_b/4} d\hat{t}' \frac{G_m e^{iS_m\theta'} \sin(p\omega_b\hat{t}')}{(\tau_b/2)} \right\} \quad (3.37)$$

where "even" means all even integers and "odd" means all odd integers ranging from $-\infty$ to ∞ . Note that in deriving Eq. (3.37) we have made use of the fact that $\zeta - q\theta = \text{constant}$ along the particle orbit.

For circulating particles the time-like variable $\hat{t}(\theta, \sigma_{||})$ is defined by

$$\hat{t}(\theta, \sigma_{||}) = \sigma_{||} \int_0^\theta \frac{\mathbf{J} \cdot \mathbf{B} d\theta'}{|\mathbf{v}_{||}|}, \quad (3.38)$$

where $\pi > \theta > -\pi$ corresponds to $(\tau_t/2) > \hat{t} > -(\tau_t/2)$ for $\sigma_{||} = +1$. Then Eq. (3.31) becomes

$$a_{m,p}^{\sigma_{||}} = \frac{e^{-i[S_m\theta - (p+\sigma_{||}S_m)\omega_t\hat{t}]}}{(\tau_t/2)} \int_0^{\tau_t/2} d\hat{t}' G_m e^{i[S_m\theta' - (p+\sigma_{||}S_m)\omega_t\hat{t}']} \quad (3.39)$$

which has the symmetry relation

$$a_{m,p}^{\sigma_{||}} = a_{m,-p}^{-\sigma_{||}} \quad (3.40)$$

Then from Eqs. (3.32) and (3.39), we obtain the total nonadiabatic circulating particle distribution

$$\begin{aligned} \sum_{\sigma_{||}} \hat{\mathbf{g}} &= e^{-i[n(\zeta - q\theta) + \omega t]} \sum_{m, \tilde{p} = -\infty}^{\infty} e^{i(\tilde{p} - nq)\omega_t \hat{t}} \\ &\times \frac{2(\omega - \langle \omega_d \rangle)}{(\omega - \langle \omega_d \rangle)^2 - (\tilde{p} - nq)^2 \omega_t^2} \int_0^{\tau_t/2} d\hat{t}' \frac{G_m e^{i[S_m\theta' - (\tilde{p} - nq)\omega_t \hat{t}']}}{(\tau_t/2)} \quad (3.41) \end{aligned}$$

where $\tilde{p} = p + m$.

3.3 Integro-differential Eigenmode Equations

We decompose the displacement vector and perturbed magnetic field as

$$\vec{\xi} = \frac{\xi_\psi \nabla \psi}{|\nabla \psi|^2} + \frac{\xi_s (\vec{B} \times \nabla \psi)}{B^2} + \frac{\xi_b \vec{B}}{B^2} \quad (3.42)$$

and

$$\vec{b} = \frac{Q_\psi \nabla \psi}{|\nabla \psi|^2} + \frac{Q_s (\vec{B} \times \nabla \psi)}{|\nabla \psi|^2} + \frac{Q_b \vec{B}}{B^2}, \quad (3.43)$$

so that $\xi_\psi = \vec{\xi} \cdot \nabla \psi$, $\xi_s = \vec{\xi} \cdot (\vec{B} \times \nabla \psi / |\nabla \psi|^2)$, $\xi_b = \vec{\xi} \cdot \vec{B}$, $Q_\psi = \vec{b} \cdot \nabla \psi$, $Q_s = \vec{b} \cdot (\vec{B} \times \nabla \psi / B^2)$, and $Q_b = \vec{b} \cdot \vec{B}$. The three components of the induction equation, Eq. (3.3), obtained after taking the scalar products with $\nabla \psi$, $(\vec{B} \times \nabla \psi)$, and $\vec{B}$, can be written as

$$Q_\psi = \vec{B} \cdot \nabla \xi_\psi. \quad (3.44)$$

$$Q_s = \frac{|\nabla \psi|^2}{B^2} (\vec{B} \cdot \nabla \xi_s - \mathbf{S} \xi_\psi), \quad (3.45)$$

$$Q_b = -B^2 \nabla \cdot \vec{\xi}_\perp - \vec{\xi}_\perp \cdot (\vec{J} \times \vec{B} + \nabla B^2). \quad (3.46)$$

Similarly, the three components of the momentum equation Eq. (3.1), can be rewritten as

$$\begin{aligned} \nabla \psi \cdot (\nabla \delta p_1 + \nabla \cdot \delta \vec{p}_h) &= \omega^2 \rho \xi_\psi + |\nabla \psi|^2 \vec{B} \cdot \nabla \left(\frac{\vec{B} \cdot \nabla \xi_\psi}{|\nabla \psi|^2} \right) \\ &+ \left(|\nabla \psi|^2 \mathbf{S} - \vec{J} \cdot \vec{B} \right) \frac{|\nabla \psi|^2}{B^2} (\vec{B} \cdot \nabla \xi_s - \mathbf{S} \xi_\psi) + \frac{\nabla \psi}{B^2} \cdot (2 \vec{J} \times \vec{B} + \nabla B^2) Q_b, \end{aligned} \quad (3.47)$$

$$\begin{aligned} (\vec{B} \times \nabla \psi) \cdot (\nabla \delta p_1 + \nabla \cdot \delta \vec{p}_h) &= \omega^2 \rho |\nabla \psi|^2 \xi_s + (\vec{B} \cdot \vec{J}) \vec{B} \cdot \nabla \xi_\psi \\ &+ B^2 \vec{B} \cdot \nabla \left[\frac{|\nabla \psi|^2}{B^2} (\vec{B} \cdot \nabla \xi_s - \mathbf{S} \xi_\psi) \right] - \nabla \cdot \left(\frac{\vec{B} \times \nabla \psi}{B^2} \right) Q_b, \end{aligned} \quad (3.48)$$

$$\vec{B} \cdot (\nabla \delta p_1 + \nabla \cdot \delta \vec{p}_h) = \omega^2 \rho \xi_b + \vec{b} \cdot (\vec{B} \times \vec{J}), \quad (3.49)$$

where $\delta p_1 = \delta p_c + Q_b$ is the total perturbed pressure excluding the energetic particle contribution, and $\mathbf{S} = (\vec{B} \times \nabla \psi / |\nabla \psi|^2) \cdot \nabla (\vec{B} \times \nabla \psi / |\nabla \psi|^2)$ is the negative local shear.

$\nabla \cdot \vec{\xi}$ can be explicitly expressed as

$$\begin{aligned} \nabla \cdot \vec{\xi} = & \frac{\nabla \psi}{|\nabla \psi|^2} \cdot \nabla \xi_\psi + \nabla \cdot \left(\frac{\nabla \psi}{|\nabla \psi|^2} \right) \xi_\psi + \vec{B} \cdot \nabla \left(\frac{\xi_b}{B^2} \right) \\ & + \frac{\vec{B} \times \nabla \psi}{B^2} \cdot \nabla \xi_s + \nabla \cdot \left(\frac{\vec{B} \times \nabla \psi}{B^2} \right) \xi_s. \end{aligned} \quad (3.50)$$

Now ξ_b , Q_ψ , Q_s , and Q_b can be eliminated by using Eqs. (3.44-3.45) and (3.49). Then after some complicated algebra, we obtain from Eqs. (3.1) and (3.2) the following set of two-dimensional integro-differential eigenmode equations:

$$\begin{aligned} \nabla \psi \cdot (\nabla \delta p_1 + \nabla \cdot \vec{\delta p}_h) = & \left\{ \omega^2 \rho \xi_\psi + |\nabla \psi|^2 \vec{B} \cdot \nabla \left(\frac{\vec{B} \cdot \nabla \xi_\psi}{|\nabla \psi|^2} \right) \right. \\ & \left. - \left(|\nabla \psi|^2 \mathbf{S} - \vec{J} \cdot \vec{B} \right) \frac{|\nabla \psi|^2}{B^2} \mathbf{S} + \frac{\nabla \psi}{B^2} \cdot (2 \vec{J} \times \vec{B} + \nabla B^2) P_c' \right\} \xi_\psi \\ & + \frac{\nabla \psi}{B^2} \cdot (2 \vec{J} \times \vec{B} + \nabla B^2) \delta p_1 + \left(|\nabla \psi|^2 \mathbf{S} - \vec{J} \cdot \vec{B} \right) \frac{|\nabla \psi|^2}{B^2} \vec{B} \cdot \nabla \xi_s \\ & + \frac{\nabla \psi}{B^2} \cdot (2 \vec{J} \times \vec{B} + \nabla B^2) \Gamma_s P_c \nabla \cdot \vec{\xi}. \end{aligned} \quad (3.51)$$

$$\nabla \psi \cdot \nabla \xi_\psi = - |\nabla \psi|^2 \left[\frac{\nabla \psi}{|\nabla \psi|^2} \cdot \left(\frac{\vec{J} \times \vec{B} + \nabla B^2}{B^2} \right) + \frac{P_c'}{B^2} + \nabla \cdot \left(\frac{\nabla \psi}{|\nabla \psi|^2} \right) \right] \xi_\psi$$

$$\begin{aligned}
& - \frac{|\nabla\psi|^2}{B^2} \delta p_1 - \frac{|\nabla\psi|^2}{B^2} \Gamma_s P_c \nabla \cdot \vec{\xi} \\
& - |\nabla\psi|^2 \left[\frac{\vec{B} \times \nabla\psi}{B^2} \cdot \left(\frac{\vec{J} \times \vec{B} + \nabla B^2}{B^2} \right) + \nabla \cdot \left(\frac{\vec{B} \times \nabla\psi}{B^2} \right) + \frac{\vec{B} \times \nabla\psi}{B^2} \cdot \nabla \right] \xi_s,
\end{aligned} \tag{3.52}$$

$$\begin{aligned}
& \left[\omega^2 \rho \frac{|\nabla\psi|^2}{B^2} \xi_s + \vec{B} \cdot \nabla \left(\frac{|\nabla\psi|^2}{B^2} \vec{B} \cdot \nabla \xi_s \right) \right] - \Gamma_s P_c \nabla \cdot \left(\frac{\vec{B} \times \nabla\psi}{B^2} \right) \nabla \cdot \vec{\xi} \\
& = \vec{B} \cdot \nabla \left(\frac{|\nabla\psi|^2}{B^2} S \xi_\psi \right) + \nabla \cdot \left(\frac{\vec{B} \times \nabla\psi}{B^2} \right) P_c' \xi_\psi - (\vec{J} \cdot \vec{B}) \vec{B} \cdot \nabla \xi_\psi \\
& + \left[\nabla \cdot \left(\frac{\vec{B} \times \nabla\psi}{B^2} \right) + \frac{\vec{B} \times \nabla\psi}{B^2} \cdot \nabla \right] \delta p_1 + \left(\frac{\vec{B} \times \nabla\psi}{B^2} \right) \cdot \nabla \cdot \delta \vec{p}_h,
\end{aligned} \tag{3.53}$$

$$\begin{aligned}
& \left\{ \omega^2 \rho \left(1 + \frac{\Gamma_s P_c}{B^2} \right) \nabla \cdot \vec{\xi} + \vec{B} \cdot \nabla \left[\frac{\Gamma_s P_c}{B^2} \vec{B} \cdot \nabla (\nabla \cdot \vec{\xi}) \right] \right\} + \omega^2 \rho \left(\frac{\vec{B} \times \nabla\psi}{B^2} \right) \cdot \left(\frac{\vec{J} \times \vec{B} + \nabla B^2}{B^2} \right) \xi_s \\
& = - \left\{ \omega^2 \rho \left[\frac{\nabla\psi}{|\nabla\psi|^2} \cdot \left(\frac{\vec{J} \times \vec{B} + \nabla B^2}{B^2} \right) + \frac{P_c'}{B^2} \right] \xi_\psi + \vec{B} \cdot \nabla \left[P_c' - \frac{\nabla\psi \cdot \vec{J} \times \vec{B}}{|\nabla\psi|^2} \right] \frac{\vec{B} \cdot \nabla \xi_\psi}{B^2} \right\} \\
& - \omega^2 \rho \frac{\delta p_1}{B^2} + \vec{B} \cdot \nabla \left[\frac{1}{B^2} \vec{B} \cdot (\nabla \cdot \delta \vec{p}_h) \right],
\end{aligned} \tag{3.54}$$

where $P_c' = \partial P_c / \partial \psi$. Note that Q_b in $\delta \vec{p}_h$ can be expressed as

$$Q_b = \delta p_1 + P_c' \xi_\psi + \Gamma_s P_c \nabla \cdot \vec{\xi}, \tag{3.55}$$

and for general flux coordinate systems

$$\nabla\psi \cdot \nabla = |\nabla\psi|^2 \frac{\partial}{\partial\psi} + (\nabla\psi \cdot \nabla\theta) \frac{\partial}{\partial\theta} + (\nabla\psi \cdot \nabla\zeta) \frac{\partial}{\partial\zeta}. \tag{3.56}$$

In the ideal MHD limit, Eqs. (3.51)-(3.52) represent the toroidal generalization of the set of eigenmode equations derived by Appert et al., [32] for the circular cylindrical pinch.

In order to obtain $\delta\tilde{p}_h$ from Eqs. (3.15), (3.16), (3.37), and (3.41) we need to define $G_m(\psi', \theta', \Lambda, \varepsilon)$ in terms of the final dependent variables of the eigenmode equations. If we define the Fourier representation of the final dependent variables by

$$\begin{pmatrix} \xi_\psi \\ \delta p_1 \\ \xi_s \\ \nabla \cdot \vec{\xi} \end{pmatrix} = \sum_m \begin{pmatrix} \tilde{\xi}_{\psi m} \\ \tilde{\delta p}_m \\ \tilde{\xi}_{sm} \\ \tilde{\Delta}_m \end{pmatrix} e^{i(m\theta - n\zeta - \omega t)}, \quad (3.57)$$

then from Eqs. (31), (32), (43), (65), and (69), we have

$$G_m = \varepsilon \left(\omega \frac{\partial F}{\partial \varepsilon} + \omega_*^{(m)} \right) \hat{G}_m, \quad (3.58)$$

where

$$\hat{G}_m = A_m \tilde{\xi}_{\psi m} + B_m \tilde{\delta p}_m + C_m \tilde{\xi}_{sm} + D_m \tilde{\Delta}_m, \quad (3.59)$$

$$A_m = \left[\left(1 - \frac{\Lambda}{2h} \right) + \frac{\Lambda}{h} \left(\frac{P_\perp - P_\parallel}{2B^2} \right) \right] \frac{\kappa_\psi}{|\nabla\psi|^2} + \frac{\Lambda}{hB^2} \left(P_c' - \frac{\nabla\psi \cdot \nabla P_\perp}{|\nabla\psi|^2} \right), \quad (3.60)$$

$$B_m = \left[\left(1 - \frac{\Lambda}{2h} \right) + \frac{\Lambda}{h} \left(\frac{P_\perp - P_\parallel}{2B^2} \right) \right] \kappa_s - \frac{\Lambda}{h} \frac{(\vec{B} \times \nabla\psi \cdot \nabla P_\perp)}{B^2}, \quad (3.61)$$

$$C_m = \frac{\Lambda}{hB^2}, \quad (3.62)$$

$$D_m = \frac{\Gamma_s P_c \Lambda}{B^2 h}, \quad (3.63)$$

and

$$\omega_*^{(m)} = \left[m - (m - nq) \left(\frac{\vec{B} \cdot \nabla \zeta \times \nabla \psi}{B^2} \right) \right] \left(\frac{Mc}{eQ} \right) \frac{\partial F}{\partial \psi}. \quad (3.64)$$

For anisotropic equilibria, $\vec{B} \cdot \nabla \cdot \vec{\delta p}_h$ contains $\partial \xi_\psi / \partial \psi$ and $\partial \xi_s / \partial \psi$ terms which are proportional to $(P_\perp - P_\parallel)_h$, and $(\vec{B} \times \nabla \psi) \cdot \nabla \cdot \vec{\delta p}_h$ contains only surface operators. But $\nabla \psi \cdot \nabla \cdot \vec{\delta p}_h$ contains all first radial derivative terms $\partial \xi_\psi / \partial \psi$, $\partial \xi_s / \partial \psi$, $\partial \delta p_\parallel / \partial \psi$, and $\partial (\nabla \cdot \vec{\xi}) / \partial \psi$. The integral operators in Eqs. (3.51)-(3.54) are related to trapped particle bounce averages in the $\vec{B}$ direction. Since Eq. (3.53) does not have radial operators, Eqs. (3.51)-(3.54) are basically a third-order equation in $\partial / \partial \psi$. However, if the $(P_\perp - P_\parallel)_h$ terms can be neglected, such as in the case of low β_h (i.e., $\beta_h \ll \beta_c$), we have

$$\vec{B} \cdot \nabla \cdot \vec{\delta p}_h \approx \vec{B} \cdot \nabla (\delta \hat{p}_\parallel - \vec{\xi}_\perp \cdot \nabla P_{\parallel h}) - (\delta \hat{p}_\parallel - \delta \hat{p}_\perp) \hat{b} \cdot \nabla B, \quad (3.65)$$

$$(\vec{B} \times \nabla \psi) \cdot \nabla \cdot \vec{\delta p}_h \approx (\vec{B} \times \nabla \psi) \cdot \nabla (\delta \hat{p}_\perp - \vec{\xi}_\perp \cdot \nabla P_{\perp h}) - (\delta \hat{p}_\parallel - \delta \hat{p}_\perp) (\vec{B} \times \nabla \psi) \cdot \vec{\kappa}, \quad (3.66)$$

and

$$\nabla \psi \cdot \nabla \cdot \vec{\delta p}_h \approx \nabla \psi \cdot \nabla (\delta \hat{p}_\perp - \vec{\xi}_\perp \cdot \nabla P_{\perp h}) - (\delta \hat{p}_\parallel - \delta \hat{p}_\perp) \nabla \psi \cdot \vec{\kappa}. \quad (3.67)$$

Then Eq. (3.54) involves only surface operators and Eqs. (3.4)-(3.7) form a second-order equation in $\partial / \partial \psi$ as in the ideal MHD case [7,19]. For a given equilibrium Eqs. (3.51)-(3.54) can be formally solved by inverting Eqs. (3.53) and (3.54) to obtain ξ_s and $\nabla \cdot \vec{\xi}$ in terms of $\delta p_\parallel$ and ξ_ψ at each surface, and substituting them into Eqs. (3.51) and (3.52). Equations (3.51) and (3.52) can then be combined to form a second-order differential equation in $\partial / \partial \psi$. Admissible regular solutions must be periodic in both θ and ζ , and satisfy the appropriate radial boundary conditions. This inversion procedure will fail if the inverse of the surface operator does not exist for given values of ω at some ψ surface. Then only non-square-integrable solutions with spatial singularities at the singular surfaces are possible. If these radially singular solutions have nontrivial single-valued periodic behaviors in θ and ζ , the corresponding set of eigenvalues ω forms the continuous spectrum for the equilibrium. The continuous spectrum consists of the slow sound waves and shear Alfvén waves that may have

complex ω due to wave-particle resonances contained in the energetic particle perturbed pressures. In this paper, we shall consider $\beta_h < \beta_c$ and neglect the pressure anisotropy terms, i.e., the $(P_\perp - P_\parallel)_h$ terms. These pressure anisotropy terms will be included in a future work.

The boundary condition at the magnetic axis is $\xi_\psi = 0$. For fixed boundary modes the boundary condition is $\xi_\psi = 0$ at the plasma-wall interface. In general, the boundary condition at the plasma-vacuum interface is given by $\vec{b}_v \cdot \nabla \psi = \vec{B} \cdot \nabla \xi_\psi$, where $\vec{b}_v$ is the perturbed vacuum magnetic field which must be solved from the divergence-free equation $\nabla \cdot \vec{b}_v = 0$ with a proper wall boundary condition.

3.4 Quadratic Form

By taking an inner product of Eq. (3.1) with $\vec{\xi}^*$ and integrating over all the plasma volume with the assumption of a fixed conducting boundary, we obtain from Eq. (3.15) a quadratic form

$$D(\omega) = \delta W_f + \delta W_k - \delta K = 0, \quad (3.68)$$

where the inertial energy is given by

$$\delta K = \omega^2 \int d^3x \rho |\vec{\xi}|^2, \quad (3.69)$$

the total fluid potential energy due to both the core and hot components is

$$\begin{aligned} \delta W_f = \int d^3x \left\{ \Gamma_s P_c \left[\nabla \cdot \vec{\xi} \right]^2 + \left| \vec{b}_\perp \right|^2 + \left| \nabla \cdot \vec{\xi}_\perp + 2 \vec{\kappa} \cdot \vec{\xi}_\perp \right|^2 \right. \\ \left. + \left(\frac{\vec{J} \cdot \vec{B}}{B^2} \right) (\vec{b} \times \vec{B}) \cdot \vec{\xi}_\perp^* - 2 \left(\vec{\kappa} \cdot \vec{\xi}_\perp \right) (\vec{J} \times \vec{B}) \cdot \vec{\xi}_\perp \right\}, \end{aligned} \quad (3.70)$$

and the energetic particle potential energy is

$$\delta W_k = - \int d^3x \left\{ \nabla \cdot \vec{\xi}^* \delta \hat{p}_\perp + (\delta \hat{p}_\perp - \delta \hat{p}_\parallel) \left(\vec{\kappa} \cdot \vec{\xi}_\perp^* - \frac{\vec{B} \cdot \nabla \xi_{\parallel}^*}{B} \right) \right\} \quad (3.71)$$

In deriving Eqs. (3.68)-(3.71), we have neglected the pressure anisotropy terms, i.e., the $(P_\perp - P_\parallel)_h$ terms. The quadratic form is useful in providing the stability properties of the system in certain limits. In the low frequency limit $\omega \ll \omega_b$, the nonadiabatic trapped particle potential energy can be written as

$$\begin{aligned} \delta W_k^\dagger \approx & -2^{7/2} M \pi^2 \int d\psi \int_0^\infty d\varepsilon \varepsilon^{5/2} \int_{h_{\min}}^{h_{\max}} d\Lambda K_b \\ & \times \frac{\left(\omega \frac{\partial F}{\partial \varepsilon} + \hat{\omega}_* F \right)}{(\omega - \langle \omega_d \rangle)} \left| \left\langle \left(1 - \frac{3\Lambda}{2h} \right) \vec{\kappa} \cdot \vec{\xi}_\perp - \frac{\Lambda}{2h} \nabla \cdot \vec{\xi}_\perp \right\rangle \right|^2, \end{aligned} \quad (3.72)$$

where $K_b = \sqrt{\varepsilon/2} \tau_b$, τ_b is the trapped particle bounce period defined by Eq. (3.21) and is a function of ε and Λ , and $\hat{\omega}_* = (-i/\omega_c) (\hat{b} \times \nabla \ln F \cdot \nabla)$. Note that in the high- and low-frequency limits $\delta W_k^\dagger$ reduces to that of the collisionless [33,34] and the low-frequency kinetic energy principles [35-37]. In the low-frequency limit $\omega \ll \omega_d$, ω_* , the sign of the trapped particle potential energy is proportional to $(\hat{\omega}_*/\omega_d)$ and the energetic trapped particles have a stabilizing effect if $(\hat{\omega}_*/\omega_d) > 0$ and vice versa. On the other hand, if $\omega \approx \omega_d$ the energetic trapped particle can excite a resonant instability. This will be discussed in more detail in Section 6, where the energetic trapped particle effects on the $n=1$ internal kink mode were presented. In terms of the Fourier components, Eq. (3.72) can also be written as

$$\delta W_k^\dagger \approx -2^{7/2} M \pi^2 \int d\psi \int_0^\infty d\varepsilon \varepsilon^{5/2} \int_{h_{\min}}^{h_{\max}} d\Lambda K_b \sum_{m',m} \langle \hat{G}_{m'} e^{i(m'-nq)\theta} \rangle^*$$

$$\frac{\left(\omega - \frac{\partial F}{\partial \varepsilon} + \omega_*^{(m)}\right)}{\left(\omega - \langle \omega_d \rangle\right)} \left\langle \hat{G}_m e^{i(m-nq)\theta} \right\rangle, \quad (3.73)$$

where $\hat{G}_\pi$ is defined by Eq. (3.59). Similarly, the nonadiabatic contributions of energetic circulating particles on the potential energy is given by

$$\delta W_k^C = -2^{7/2} M \pi^2 \int d\psi \int_0^\infty d\varepsilon \varepsilon^{5/2} \int_0^{h_{\min}} d\Lambda K_1 \sum_{m, m', \tilde{p}=-\infty}^{\infty} \frac{\left\langle \hat{G}_{m', \tilde{p}} e^{i(S_{m'}\theta - S_{\tilde{p}}\omega_1 t)} \right\rangle^* \left(\omega - \langle \omega_d \rangle\right) \left(\omega - \frac{\partial F}{\partial \varepsilon} + \omega_*^{(m)}\right)}{\left(\omega - \langle \omega_d \rangle\right)^2 - (\tilde{p} - nq)^2 \omega_1^2} \left\langle \hat{G}_m e^{i(S_m\theta - S_{\tilde{p}}\omega_1 t)} \right\rangle, \quad (3.74)$$

where $K_1 = \sqrt{2\varepsilon} \tau_t$, τ_t is the circulating particle transit period defined by Eq (3.22) and is a function of ε and Λ , and $S_m = m - nq$. Eqs. (3.72)-(3.74) will be useful in understanding the numerical solutions of the NOVA-K code.

4. Numerical Methods

4.1 Galerkin Method

The eigenmode equations, Eqs. (3.51) - (3.54), are solved by the Galerkin method, where the eigenfunction is represented by a linear superposition of a finite subset of a complete set of basis functions. We first represent the perturbed quantities by a finite Fourier series in θ ,

$$\vec{\xi}(\psi, \theta, \zeta) = \sum_m \vec{\xi}_m(\psi) \exp[i(m\theta - n\zeta)], \quad (4.1)$$

where the summation over m is truncated to a total number of L poloidal harmonics, and n is the toroidal mode number. The elimination of ξ_s and $\nabla \cdot$

$\vec{\xi}$ proceeds by finding the algebraic Fourier matrix representation of the surface operators. Introducing the bracket notation for an operator $\mathbf{E}$,

$$\mathbf{E}_{m'm} \equiv \langle m' | \mathbf{E} | m \rangle \equiv \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} [\exp(i m' \theta)] \mathbf{E} [\exp(i m \theta)] , \quad (4.2)$$

Eqs. (3.53) and (3.54) reduce to

$$\mathbf{E}_{m'm} \begin{pmatrix} \xi_s \\ \nabla \cdot \vec{\xi} \end{pmatrix}_m = \mathbf{F}_{m'm} \begin{pmatrix} \delta p_1 \\ \xi_\psi \end{pmatrix}_m , \quad (4.3)$$

where $\mathbf{E}$ and $\mathbf{F}$ represent the operators of the left- and right-hand sides of Eqs. (3.53) and (3.54), respectively, and the Fourier indices m' and m have the same truncated domain. Note that the evaluation of the algebraic matrix operations, $\mathbf{E}_{m'm}$ etc., involves the following interchange of integration limits in θ and pitch angle Λ

$$\int_{-\pi}^{\pi} d\theta \int_0^{h(\theta, \omega)} d\Lambda = \int_{h_{\min}(\psi)}^{h_{\max}(\psi)} d\Lambda \int_{-\theta_1(\Lambda)}^{\theta_1(\Lambda)} d\theta + \int_0^{h_{\min}(\psi)} d\Lambda \int_{-\pi}^{\pi} d\theta , \quad (4.4)$$

where the first term on the right side represents the trapped particle integration and the second term represents the circulating particle integration. It also involves convolutions that occur in evaluating operator products. To avoid aliasing errors, a larger truncated Fourier series domain than L , say L^* , is imposed and is increased until the results are satisfactorily converged. $\mathbf{E}_{m'm}$ can be inverted to obtain ξ_{sm} and $(\nabla \cdot \vec{\xi})_m$ in terms of δp_{1m} and $\xi_{\psi m}$ and then by eliminating δp_{1m} in favor of $\xi_{\psi m}$. Equations (3.51) and (3.52) are reduced to a set of L second-order differential equations.

$$\frac{\partial}{\partial r} \left(H_{m'm}^{(1)} \frac{\partial \xi_m}{\partial r} \right) + H_{m'm}^{(2)} \frac{\partial \xi_m}{\partial r} + H_{m'm}^{(3)} \xi_m + \frac{\partial}{\partial r} \left(H_{m'm}^{(4)} \xi_m \right) = 0 . \quad (4.5)$$

where the H 's are algebraic $L \times L$ matrices and are functions of ω and r , where $r = (\psi/\psi_{\text{tot}})^{1/2}$ and $\xi_m = \xi_{\psi m}(r)$ is a vector of dimension L . The explicit

expressions of the H 's are too tedious and complicated to be presented in full detail here. The boundary conditions at the magnetic axis are now modified to $\xi_m = 0$ for $m^2 \neq 1$ and $\partial \xi_m / \partial r = 0$ for $m^2 = 1$.

4.2 Cubic B-Spline Finite Elements

All the variational ideal MHD stability codes [8-12] employ linear finite elements in the minor radius direction, which are the lowest order finite elements allowed for representing the displacement vector $\bar{\xi}$. Since they are in quadratic form, the numerical errors in the eigenvalues, ω , scale as N^{-2} , where N is the total number of the radial computational grid points. Therefore, our nonvariational approach requires higher order finite elements to achieve better accuracy and faster convergence. For example, with cubic B-spline finite elements [38-41] the errors in ω scale as N^{-4} .

Equation (4.5) is integrated by employing cubic B-splines [7,42] to approximate the eigenfunction $\xi_m(r)$. For a given partition of N -grid points, $D = \{r_1 < r_2 \dots < r_N\}$, the function $\xi_m(r)$ is approximated by piecewise cubic polynomials in each subinterval (r_i, r_{i+1}) , which is called a cubic spline because of an analogy between its theory and the linearized theory of the so called draftsman's spline. The polynomials and their first two derivatives are required to be continuous at the grid points. The cubic spline has the advantageous property of minimum curvature among all third order polynomials [42]. Since a cubic polynomial is determined by four coefficients and there are $N-1$ intervals, there are a total of $4(N-1)$ unknowns to define a cubic spline function. There are $3(N-2)$ constraints at the $(N-2)$ interior points so that the cubic spline function is determined by $N+2$ parameters. The most convenient cubic spline representation is the basis set of B-spline finite elements [38-41] $\{U_k | k=1, N+2\}$, which are local piecewise cubic polynomials occupying four grid intervals with continuity up to second-order derivatives and are shown in Fig. 1 for the uniform grid point case. It can be easily shown that every cubic spline can be expressed as a linear combination of B-spline elements. In order to define uniquely the $N+2$ B-spline elements, we assume that at each end of the grid partition there are three extra grid points extending outside the computational domain. This extended partition is unsatisfactory, and the problem is resolved by letting the exterior points coalesce at the end points, i.e., $r_{-2} = r_{-1} = r_0 = r_1$ and $r_{N+3} = r_{N+2} = r_{N+1} = r_N$. Typical sets of cubic B-spline finite elements for a uniform grid partition and for a nonuniform grid partition are shown in Fig. 2.

The eigenfunction $\xi_m(r)$ is interpolated by the representation

$$\xi_m(r) \approx \sum_{k=1}^{N+2} \xi_{m,k} U_k(r) \equiv S_m(r) , \quad (4.6)$$

where $S_m(r)$ is the cubic spline approximation of $\xi_m(r)$. For the given values of $\xi_m(r)$ at the N -grid points, Eq. (4.6) is not uniquely defined because there are $N+2$ parameters $\xi_{m,k}$. Uniqueness is obtained by imposing two boundary conditions for $S_m(r)$, and in this paper we require that $\xi_m'(r) = S_m'(r)$ at the end grid points. To implement these two extra boundary conditions, we interpolate the first and the last four grid points by cubic Lagrange polynomials $\tilde{S}_m(r)$ and require that $S_m'(r) = \tilde{S}_m'(r)$ at the end points $r = r_1$ and r_N [39]. At least for a uniform grid partition, this scheme preserves the asymptotic error scaling of N^{-4} [7,38,42] and has the advantage of not requiring an extra four function evaluations.

4.3 Matrix Form and Iteration Procedure

Operating on Eq. (4.5) with the projection operator

$$O_k = \int_{r_1}^{r_N} dr U_k , \quad (4.7)$$

and interpolating the known functions $H^{(1)}$, $H^{(2)}$, $H^{(3)}$, and $H^{(4)}$ in terms of the same cubic B-spline finite elements, we obtain a set of algebraic equations:

$$\sum_{m,k} M_{mk}^{m'k'} \xi_{m,k} = 0 , \quad (4.8)$$

where

$$M_{mk}^{m'k'} = \sum_{j=1}^{N+2} \left\{ h_{m'm}^{(1)j} \left[U_k U_j U_k' \right] \right|_{r_1}^{r_N} - \int_{r_1}^{r_N} U_k' U_j U_k' dr \right\}$$

$$\begin{aligned}
& + h_{m'm}^{(2)j} \int_{r_1}^{r_N} U_k' U_j U_k' dr + h_{m'm}^{(3)j} \int_{r_1}^{r_N} U_k' U_j U_k dr \\
& + h_{m'm}^{(4)j} \int_{r_1}^{r_N} U_k' \{ U_j' U_k + U_j U_k' \} dr \} ,
\end{aligned} \tag{4.9}$$

and

$$H_{m'm}^{(1)j} = \sum_{j=1}^{N+2} h_{m'm}^{(1)j} U_j(r) , \text{ etc.} \tag{4.10}$$

Note $M_{mk}^{m'k'}$ is a $(N+2)L \times (N+2)L$ matrix with nonvanishing elements along its L^2 7-banded diagonals. It is clear that by expanding the functions $H^{(1)}$, $H^{(2)}$, $H^{(3)}$, and $H^{(4)}$ in terms of the cubic B-spline finite elements, the integrals resulting from the projections of the operators O_k involve only the cubic B-spline finite elements and, therefore, may be done numerically once for all. The error associated with the Gibbs' phenomena, which could be present when using higher order elements to represent functions containing regions with singular-like behavior, can be reduced by using a nonuniform grid with packing near these regions. Because of the integral method employed here, the subsequent error in the matrix elements should be negligibly small when the projection operation is carried out.

The boundary conditions are then imposed to modify the matrix $M_{mk}^{m'k'}$. Since there is only one nonzero cubic B-spline finite element at either boundary, the boundary conditions $\xi_m = 0$ can be satisfied by requiring that the coefficient corresponding to that cubic B-spline finite element be zero. The boundary condition at the magnetic axis can be satisfied by $\xi_{m,1} = 0$ for $m^2 \neq 1$ and for the fixed boundary condition at the plasma-wall interface the boundary condition can be satisfied by setting $\xi_{m,N+2} = 0$ for all m . Since there are only two cubic B-spline finite elements that have nonzero first derivatives at either boundary, the boundary condition at the magnetic axis for $m^2 = 1$ modes, $\partial \xi_m / \partial r = 0$, can be satisfied by requiring $[\xi_{m,1} U_1'(r_1) + \xi_{m,2} U_2'(r_1)] = 0$. Similarly, at the plasma-vacuum interface the complicated boundary conditions, which can be written symbolically as $\partial \xi_m / \partial r = \sum d_{mm'} \xi_{m'}$, are satisfied by requiring $[\xi_{m,N+1} U_{N+1}'(r_{N+2}) + \xi_{m,N+2} U_{N+2}'(r_{N+2})] = \sum d_{mm'} \xi_{m',N+2}(r_{N+2})$.

The nontrivial solution of Eq. (4.8) can be obtained by requiring that the determinant of the matrix M_{mk}^{mk} to be zero,

$$f(\omega) \equiv \det |M(\omega)| = 0. \quad (4.11)$$

The eigenvalue problem is nonlinear in complex ω and its numerical solutions can be found by iterative complex root-finding procedures. The number of eigenvalues sought and the domain of complex ω -space that is considered depend on the nature of the physical problem being solved. Convergence of the eigenvalue is assumed if $|\{f(\omega_{p+1}) - f(\omega_p)\}/f(\omega_p)| < \epsilon_1$, and/or $|\omega_{p+1} - \omega_p| < \epsilon_2$, where ϵ_1 and ϵ_2 are appropriate small numbers, and p denotes the iteration step. When the eigenvalue iteration is converged to the required accuracy, Eq. (4.8) is used to construct the eigenvectors ξ_ψ and δp_1 by inverse iteration [43]. For this purpose, the matrices of Eqs. (3.51)-(3.54) for each surface are saved in disk files when they are computed during the calculation of the matrix elements of M_{mk}^{mk} . To make sure that the most unstable eigenvalue is obtained, several ways of searching for eigenvalues are employed. One way is to employ the Cauchy theorem to search for the number of zeros of $f(\omega)$ over a certain complex ω -domain, although at the cost of greater computer time. Another way is to plot the Stoke's diagram [44] of $[f(\omega)]^2$ over a certain complex ω -domain and look for turning points which correspond to zeros of $f(\omega)$. When a zero is located in a certain ω -domain, a higher order complex root finder using the Muller method can be applied to obtain the converged eigenvalue efficiently.

The iteration is computationally expensive because the matrix inversion must be carried out for Eqs. (72) and (73) at each iteration for each surface. Consequently, the execution time scales as the total number of radial grid points N as well as L^2 for the matrix inversion. Finally, to be consistent we have also employed cubic B-spline finite elements to obtain the equilibrium quantities from the mapping codes.

4.4 Trapped and Circulating Particle Integrals

To obtain the energetic particle contributions to the perturbed pressure we have to perform the velocity space integration as well as the trapped and circulating particle orbit averages of perturbed quantities. In this subsection we present the numerical methods employed in the NOVA-K

code. The trapped particle orbit averages as defined in Eq. (3.28) can be integrated by the Gauss-Legendre integration scheme [45] and are given by

$$\begin{aligned} \int_{\theta_{T1}(\Lambda)}^{\theta_{T2}(\Lambda)} \frac{d\theta f(\theta)}{\sqrt{1-\Lambda/h(\theta)}} &= \int_{\theta_{T1}(\Lambda)}^{\theta_{T2}(\Lambda)} \frac{d\theta F(\theta)}{\sqrt{(\theta-\theta_{T1})(\theta_{T2}-\theta)}} \\ &= \frac{\pi}{N} \sum_{i=1}^N F(\theta_i) + R_N, \end{aligned} \quad (4.12)$$

where

$$F(\theta) = \left(\frac{f(\theta) \sqrt{(\theta-\theta_{T1})(\theta_{T2}-\theta)}}{\sqrt{1-\Lambda/h(\theta)}} \right), \quad (4.13)$$

$$\theta_i = \frac{(\theta_{T2} + \theta_{T1})}{2} + \frac{(\theta_{T2} - \theta_{T1}) \cos \left[\frac{(2i-1)\pi}{2N} \right]}{2}, \quad (4.14)$$

and the remainder is given by

$$R_N = \frac{2^{4N+1}}{[4N+1]} \frac{[2N]!^3}{[4N]!^2} F^{(2N)}(x) \quad (0 < x < 1), \quad (4.15)$$

θ_{T1} and θ_{T2} are the trapped particle turning points for a given pitch angle Λ in the range $h_{\min}(\psi) \leq \Lambda \leq h_{\max}(\psi)$. $F(\theta)$ is an analytic function of θ , and N is the total number of grid points which is usually taken as 100. With this integration scheme the square root singularities near the turning points are properly taken care of. For the circulating particle orbit averages the Gauss-Legendre integration scheme can be employed by setting $\theta_{T1} = -\pi$ and $\theta_{T2} = \pi$.

For the pitch angle integration there is a square root singularity at $\Lambda = h_{\min}(\psi)$ and we use the Gauss-Legendre integration scheme. For circulating particles the integration formula is given by

$$\begin{aligned}
\int_0^{h_{\min}(\psi)} d\Lambda f(\Lambda) &= \int_0^{h_{\min}(\psi)} \frac{d\Lambda F(\Lambda)}{\sqrt{h_{\min}(\psi) - \Lambda}} \\
&= \sqrt{h_{\min}(\psi)} \sum_{i=1}^N w_i F(\Lambda_i) + R_N,
\end{aligned}
\tag{4.16}$$

where

$$F(\Lambda) = \sqrt{h_{\min}(\psi) - \Lambda} f(\Lambda), \tag{4.17}$$

$f(\Lambda)$ is assumed to be an entire function in the integration formula, $\Lambda_i = h_{\min}(\psi) (1 - x_i^2)$, and x_i is the i th positive zero of the Legendre polynomial of order $2N$, $P_{2N}(x)$. The weights w_i are the Gaussian weights of order $2N$ and are tabulated in Table 25.4 of Reference 45. Usually $N=48$ is used in the computation. R_N is defined in Eq. (4.15). Similarly, for the trapped particle pitch angle integration the scheme is given by

$$\begin{aligned}
\int_{h_{\min}(\psi)}^{h_{\max}(\psi)} d\Lambda f(\Lambda) &= \int_{-h_{\max}(\psi)}^{-h_{\min}(\psi)} \frac{dy F(y)}{\sqrt{h_{\min}(\psi) + y}} \\
&= \sqrt{h_{\max}(\psi) - h_{\min}(\psi)} \sum_{i=1}^N w_i F(\Lambda_i) + R_N,
\end{aligned}
\tag{4.18}$$

where

$$F(y) = \sqrt{h_{\min}(\psi) + y} f(-y), \tag{4.19}$$

$$y = -\Lambda, y_i = -h_{\max}(\psi) + [h_{\max}(\psi) - h_{\min}(\psi)] (1 - x_i^2).$$

4.5 Vacuum Solutions

Here, we consider the vacuum solution of $\nabla \cdot \vec{b}_v = 0$ with the vacuum region surrounded by a conducting wall. For the $n \neq 0$ modes, we represent $\vec{b}_v = \nabla \chi$. Then in terms of the Green's theorem we have

$$2\pi \chi(\vec{x}_s) = \int d\vec{s}_t \cdot \left[G(\vec{x}_t | \vec{x}_s) \nabla_t \chi(\vec{x}_t) - \chi(\vec{x}_t) \nabla_t G(\vec{x}_t | \vec{x}_s) \right], \quad (4.20)$$

where we have chosen the Green's function G to satisfy

$$\nabla_t^2 G(\vec{x}_t | \vec{x}_s) = 4\pi \delta(\vec{x}_t - \vec{x}_s), \quad (4.21)$$

and $G(\vec{x}_t | \vec{x}_s) = |\vec{x}_t - \vec{x}_s|^{-1}$. $\vec{x}_s$ can be either on the plasma-vacuum interface or on the wall, and the integral extends over both surfaces in Eq. (4.20). With the boundary conditions $\nabla \chi \cdot \nabla \psi = \vec{B} \cdot \nabla \xi_\psi$ on the plasma-vacuum interface and $\nabla \chi \cdot d\vec{s}_w = 0$ on the wall, Eq. (4.20) can be solved to obtain χ on both surfaces by the method of collocation [46].

For the $n = 0$ mode, the magnetic field cannot be described by a single-valued scalar potential. Instead, we follow the procedure of Lust and Martensen [47] and define

$$\vec{b}_v = \nabla \chi + a_1 \nabla \phi \times \nabla \alpha + a_2 \nabla \phi, \quad (4.22)$$

where a_1 and a_2 are constants related to the perturbed quantities. χ can be obtained by the same procedure as for the $n \neq 0$ modes with the additional constraint $\oint \chi d\theta = 0$ due to the singular nature of the matrix in Eq. (4.23) for the $n = 0$ mode. α can be solved by taking $\nabla \phi \cdot \nabla \times \vec{b}_v = 0$ and we have

$$\Delta \alpha \equiv \nabla \cdot \left(\frac{\nabla \alpha}{\chi^2} \right) = 0. \quad (4.24)$$

The boundary conditions are $\alpha = 1$ on the plasma-vacuum interface and $\alpha = 0$ on the wall. Equation (4.24) again can be solved in terms of the Green's theorem

$$\frac{4\pi\alpha(\vec{x}_s)}{X_s} = \oint \frac{d\vec{s}_t}{X_t^2} \cdot \left[\tilde{G}(\vec{x}_t|\vec{x}_s) \nabla_t \alpha(\vec{x}_t) - \alpha(\vec{x}_t) \nabla_t \tilde{G}(\vec{x}_t|\vec{x}_s) \right], \quad (4.25)$$

where the Green's function G satisfies

$$\Delta_t \tilde{G}(\vec{x}_t|\vec{x}_s) = -\frac{4\pi}{X_s^2} \delta(X_t - X_s) \delta(Z_t - Z_s), \quad (4.26)$$

and is given by

$$\tilde{G}(\vec{x}_t|\vec{x}_s) = -\frac{4\pi X_t}{r} P_{-1/2}^1(w), \quad (4.27)$$

where $P_{-1/2}^1$ is the associated generalized Legendre function,

$$r = \left[(X_s^2 - X_t^2)^2 + (Z_s - Z_t)^4 + 2(X_s^2 + X_t^2)(Z_s^2 - Z_t^2) \right]^{1/4}, \quad (4.28)$$

and

$$w = \frac{\left[(X_s^2 + X_t^2) + (Z_s - Z_t)^2 \right]}{r^2}. \quad (4.29)$$

To determine α_1 and α_2 we make use of the property

$$\int \nabla \chi \cdot \nabla \phi d^3x = \int \nabla \chi \cdot (\nabla \phi \times \nabla \alpha) d^3x = 0. \quad (4.30)$$

Then, with the aid of the boundary conditions $\vec{A} \times d\vec{s}_w = 0$ on the wall and $\vec{A} \times \nabla \psi = \vec{B} \cdot \nabla \xi_\psi$ on the plasma-vacuum interface, where $\vec{A}$ is the vector potential defined by $\vec{b}_v = \nabla \times \vec{A}$, we obtain

$$a_1 = \frac{\oint (\nabla\phi \times \nabla\alpha) \cdot (\hat{n}_p \times \vec{A}) ds_p}{\int |\nabla\phi \times \nabla\alpha|^2 d^3x}, \quad (4.31)$$

and

$$a_2 = \frac{\oint \nabla\phi \cdot (\hat{n}_p \times \vec{A}) ds_p}{\int |\nabla\phi|^2 d^3x}, \quad (4.32)$$

where s_p denotes the plasma surface and $\hat{n}_p = -\nabla\psi/|\nabla\psi|$. After we solve the vacuum magnetic field $\vec{b}_v$ in terms of ξ_ψ , we obtain the boundary condition for solving Eqs. (3.51)-(3.54) at the plasma-vacuum interface:

$$\delta p_1 = \vec{b}_v \cdot \vec{B} = \sum_{m,m'} \tilde{M}_{m,m'} \xi_{\psi m'} \exp[i(m\theta - n\zeta)]. \quad (4.33)$$

Note that δp_1 is related to ξ_ψ and $\partial\xi_\psi/\partial\psi$ in Eqs. (3.51)-(3.52). This concludes our discussion on the vacuum solutions.

5. Convergence Studies of the Ideal MHD NOVA Code

To illustrate the convergence properties of the NOVA code, we first consider the analytical Solovév equilibria [24] which have been used previously [7,25] for extensive comparisons of variational ideal MHD stability codes. Our ideal MHD results will be compared with the previous results, which provide a series of cross checks essential for validating such a large, complex code. The Solovév equilibria which satisfy the Grad-Shafranov equation, Eq. (2.19), for isotropic equilibria with $g' = 0$, are given by

$$\psi = \frac{\pi B_0}{ER^2 q(0)} \left[X^2 Z^2 + \frac{E^2}{4} (X^2 - R^2) \right], \quad (5.1)$$

$$P(\psi) = \frac{(1 + E^2) B_0}{2\pi E R^2 q(0)} (\psi_B - \psi), \quad (5.2)$$

$$q(\psi) = R B_0 \oint \frac{ds}{X |\nabla \psi|}, \quad (5.3)$$

where (X, ϕ, Z) is a cylindrical coordinate system, B_0 is the toroidal field at the magnetic axis $X = R$, $P(\psi)$ is the plasma pressure, and $q(\psi)$ is the safety factor with the contour of integration along a line of constant ψ and ϕ . The system is characterized by the following parameters: the ellipticity E , the inverse aspect ratio $\varepsilon_0 \equiv [\psi_B q(0)/\pi E R^2 B_0]^{1/2}$, and $q(0)$. The wall position is specified by $\Sigma = [\psi_w/\psi_B]^{1/2}$, and $2\pi\psi_B$ is the total poloidal flux in the plasma.

To achieve the correct eigenvalue and an accurate representation of the corresponding eigenfunction, in principle one requires a very large number of basis functions. Since this number is limited by the computer memory and computing time, it is necessary to extrapolate from the lower order representations to obtain the asymptotic eigenvalue. Fortunately, we need only a few points because the extrapolation formulas are simple.

Numerical convergence is shown below for the small aspect ratio, elliptical case with the parameters: $R = B_0 = 1$, $E = 2$, $\varepsilon_0 = 1/3$, $q(0) = 0.3$, $\Sigma = 1$, and $n = 2$. We employ a uniform r -mesh of N grid points and retain the poloidal harmonics $m = [-L_0, L_0]$. For the equal arc-length θ coordinate, the convergence curves of the eigenvalue ($\gamma^2 = -\omega^2$) are shown in Fig. 3. Here γ^2 is normalized in terms of $\omega_A^2 = B^2(0)/\rho(0)q^2(1)R^2$. The eigenvalue γ^2 scales as $\gamma^2 = \gamma_1^2 + C_1 \exp(-L_0/2)$ for fixed N and as $\gamma^2 = \gamma_2^2 + C_2 N^{-4}$ for fixed L_0 , where γ_1^2 and γ_2^2 are the converged values for fixed N and L_0 , respectively. The results from the PEST code show that γ^2 scales as $\gamma^2 = \gamma_3^2 + D_1 \exp(-L/2)$ for fixed N , where $L = 2L_0 + 1$, and for fixed L , as $\gamma^2 = \gamma_4^2 + D_2 N^{-2}$. Note that if we use the PEST θ -coordinate, the growth rate from our code also scales as $\gamma^2 = \gamma_1^2 + C_1 \exp(-L/2)$. Convergence curves from the PEST code are also shown in Fig. 3. Detailed comparison between the results of our code and those of the PEST code indicates that $|C_1| \approx |D_1|$ and $|C_2| \ll |D_2|$. Even with $N = 5$, our code converges in L_0 with an error of less than 1% of its converged value. On the other hand, comparable accuracy from the PEST code would require at least three times as many linear finite elements. The eigenfunction ξ_ψ and the plasma flow pattern for this case are shown

in Figs. (4a) and (4b), respectively. Comparisons of the converged values of the square of the growth rate from different Solovév equilibria obtained from various ideal MHD stability codes [7-12] are summarized in Table I. For most of the cases, our results are roughly between those of the PEST [8] and ERATO [9] codes. These small discrepancies may be due to the different mapping codes used in these stability codes.

6. Energetic Trapped Particle Effects on the Internal Kink Mode

The $n=1$ ideal MHD internal kink mode is usually unstable for tokamak plasmas with $\beta > \beta_{crit}$ and $q(0) < 1$ so that the $q = 1$ surface lies within the plasma. The growth rate of this mode is typically a factor of ϵ_0^2 smaller than the more dangerous external kink, where ϵ_0 is the inverse aspect ratio. The stabilization of the high- n ballooning modes by an energetic particle component mirror-trapped on the unfavorable-curvature side of the tokamak has been proposed by Rosenbluth et al. [48]. For the low frequency MHD modes with $\omega \ll \omega_d$, the energetic particle dynamics are no longer governed by the $(\vec{E} \times \vec{B})/B^2$ drift, but rather by the magnetic (∇B and curvature) drifts. Their stabilization mechanism requires that the hot particles drift across field lines rapidly, i.e., $\langle \omega_d \rangle \gg |\omega|$ and $(\omega_r / \langle \omega_d \rangle) > 0$. The stabilizing kinetic energy of the hot particles may bring the plasma into the ballooning mode second stability region, where the stability may improve with increasing β . They also argue that when $\langle \omega_d \rangle$ is not large enough, marginal stability occurs with a real frequency close to $\langle \omega_d \rangle$. The same arguments can be applied to the $n=1$ internal kink mode. Chen et al. [49] had shown that for $\beta_h > \beta_{crit}$ and $(\omega_r / \langle \omega_d \rangle) \gg 1$, the trapped energetic ions can resonantly destabilize a new branch of internal kink mode (called fishbone mode) with a real frequency comparable to $\langle \omega_d \rangle$ and that the growth rates are of the same order as the ideal internal kink values. This theory of fishbone mode was used to explain the experimental observations of large amplitude bursts of $m=n=1$ MHD fluctuations in tokamak experiments with high-power, nearly perpendicular neutral beam injections [13-15].

The previous analytical theories [49-53] of energetic particle effects on the internal kink mode were performed for large aspect ratio tokamak plasmas with circular, concentric magnetic surfaces, and the radial plasma displacement ξ_r was taken as the cylindrical solution of the

$m = n = 1$ mode with $\xi_r = \text{constant}$ for $q \leq 1$ (or $r \leq r_s$), and $\xi_r = 0$, otherwise. Other approximations made in calculating energetic particle contributions to δW_k involve setting $1-q(r)=0$, assuming $(\omega_+/\omega_-) > 0$ for the whole minor radius, and neglecting toroidal couplings of neighboring poloidal harmonics. In addition, an algebraic approximation was made in deriving the dispersion relation by cancelling the hot particle contributions from δW_f with δW_k so that δW_k is proportional to ω . Those approximations are quite erroneous for realistic finite aspect ratio tokamaks. In fact, although previous analytical theories [49-53] predicted the excitation of the fishbone mode, they not only failed to predict a stability window in β_h for the stabilization of both the internal kink and the fishbone mode, but also failed to give quantitative predictions of the critical β_h 's for both the stabilization of the internal kink and the excitation of the fishbone mode. Since the behavior of MHD modes depends strongly on both the energetic particle distributions and the plasma equilibria, it is important to have a numerical code (e.g., NOVA-K code) that computes the stability of low- n MHD modes for realistic energetic particle distributions and plasma equilibria.

In Section 6.1, we will first derive and study an analytic dispersion relation that does not cancel the hot particle contributions from δW_f with δW_k so that δW_k is not proportional to ω . Then in Section 6.2, we will present numerical studies of the effects of energetic particles on the $n=1$ internal kink mode by using the NOVA-K code for NBI energetic trapped particles with singular pitch angle distribution. We will first establish the numerical convergence of the NOVA-K code. By comparing the numerical results with the analytical theory, we will establish the correctness of the NOVA-K code and point out the usefulness of the numerical calculations. The effects of alpha particles with uniform pitch angle distribution on the $n=1$ internal kink mode is presented in Sec. 6.3.

6.1 Analytic Dispersion Relation for the $n=1$ Internal Kink Mode

To derive the analytic dispersion relation for the $n=1$ internal kink mode including energetic trapped particle contributions, we consider the large aspect ratio orderings with the small parameter $\epsilon_0 \equiv a/R \ll 1$. We will limit the consideration to the parameter range of the first stability boundary of the internal kink mode, and we order $\beta_{pc} \sim O(1)$, $\beta_h \sim O(\epsilon_0)$, where β_p is the poloidal beta. The relative temperatures of the core and

hot components is $T_c / T_h \sim O(\epsilon_o^2)$, which implies $n_h / n_c \sim O(\epsilon_o^3)$ and overall charge neutrality may be assumed. The usual internal kink ordering of $(\omega/\omega_A) \sim (\langle \omega_d \rangle / \omega_A) \sim O(\epsilon_o^2)$ is also assumed. With the present ordering we find from Eqs. (3.68)-(3.72) that $\delta K / \delta W_f \sim \delta W_k / \delta W_f \sim O(\epsilon_o^2)$. We note that the contribution of δK is significant only in a singular layer of a width Δ , which has the ordering $(\Delta/a) \sim (\omega/\omega_A) \sim O(\epsilon_o^2)$, but the contribution of δW_k is mainly from outside the singular layer. Therefore, outside the singular layer the contribution to the dispersion is $D_o = \delta W_f + \delta W_k$ and inside the singular layer the contribution to the dispersion is $D_i = \delta W_f - \delta K$. Since δW_f and δK are self-adjoint, we will variationally minimize the quadratic form, Eq. (3.68), to $O(\epsilon_o^2)$, by constructing an appropriate trial function, $\vec{\xi}_t$. For the case of circular cross sections, the minimization of δW_f outside the singular layer is nontrivial and can be done by choosing $\vec{\xi}_t$ as given by Bussac et al. [54], and to $O(\epsilon_o^2)$ we have

$$\delta W_f = 2\pi^2 R B_o^2 \left(\frac{r_s}{R} \right)^2 |\xi_{r0}|^2 \delta W_f^{(n)} + O(\epsilon_o^4) \quad , \quad (6.1)$$

where $\delta W_f^{(n)}$ is given by Hastie et al. [55], and near the first stability boundary it is proportional to $[\beta_p^2 - (\beta_{pcrit})^2]$, where β_{pcrit} is the critical β_p for the ideal MHD internal kink. To evaluate δW_k outside the singular layer, the lowest order trial function, $\vec{\xi}_\perp^{(0)}$, for $m = n = 1$ is sufficient and is given by

$$\vec{\xi}_\perp^{(0)} = \begin{cases} (\hat{e}_r + i\hat{e}_\theta) \xi_{r0} \exp[i(\theta - \zeta)] & , \text{ for } r < r_s \\ 0 & , \text{ for } r > r_s \end{cases} \quad (6.2)$$

Then, from Eq. (3.72) the hot trapped particle kinetic contribution is given by

$$\delta W_k = \frac{2\pi^2 R B_o^2 r_s q_s |\xi_{r0}|^2}{\omega_A} \delta \hat{W}_k + O(\epsilon_o^4) \quad , \quad (6.3)$$

where only the $m = n = 1$ contribution is kept and

$$\delta \widehat{W}_k = \frac{2^{3/2} \pi^2 M \omega_A}{R^2 B_o^2 r_s q_s} \int_0^{r_s} dr B \int_{1-\epsilon_o}^{1+\epsilon_o} d\Lambda \int_0^\infty d\epsilon \epsilon^{5/2} \frac{|K_2|^2}{K_b} \left(\frac{\omega \frac{\partial F}{\partial \epsilon} + \omega_*^{(1)}}{\langle \omega_d \rangle - \omega} \right) , \quad (6.4)$$

$$K_2 = \int_{-\theta_T}^{\theta_T} \frac{d\theta \cos(\theta) \exp[i(1-q)\theta]}{\sqrt{1-\Lambda/h}} . \quad (6.5)$$

from Eqs. (6.1) and (6.3) the total contribution of the dispersion relation from outside the singular layer is

$$D_o = \frac{2 \pi^2 R B_o^2 r_s q_s \left| \xi_{z,0} \right|^2}{\omega_A} \left(\gamma_B - \delta \widehat{W}_k \right) + O(\epsilon_d^4) , \quad (6.6)$$

where

$$\gamma_B = - \left(\frac{\pi \omega_A}{r_s q_s} \right) \left(\frac{r_s}{R} \right)^2 \delta W_1^{(7)} . \quad (6.7)$$

Inside the singular layer, $D_i = \delta W_i - \delta K$ is variational and can be minimized by choosing the trial function such that

$$\nabla \cdot \vec{\xi}_\perp = -2 \vec{\kappa} \cdot \vec{\xi}_\perp , \quad (6.8)$$

From which the lowest order trial function is determined by

$$\frac{\partial \xi_r}{\partial r} + \frac{\partial \xi_\theta}{\partial \theta} = 0 . \quad (6.9)$$

We should note that $\nabla \cdot \vec{\xi} \neq 0$ so that compressibility is finite in the singular layer. From Eqs. (3.43)-(3.45), we have

$$|\vec{b}_\perp|^2 = |\vec{B} \cdot \nabla \xi_\theta|^2 \approx B_o^2 (1-q)^2 \left| \frac{d\xi_r}{dr} \right|^2, \quad (6.10)$$

and from Eqs. (3.2), (3.44), (3.49), and (3.54) we have

$$\begin{aligned} & \int d^3x \left\{ \Gamma_s P_c |\nabla \cdot \vec{\xi}|^2 - \frac{\omega^2 \rho \xi_b^2}{B^2} \right\} \\ &= \int d^3x \Gamma_s P_c \nabla \cdot \vec{\xi}^* \left\{ \nabla \cdot \vec{\xi} + \vec{B} \cdot \nabla \left[\frac{\Gamma_s P_c}{\omega^2 \rho B^2} \vec{B} \cdot \nabla (\nabla \cdot \vec{\xi}) \right] \right\} \\ &\approx \int d^3x \Gamma_s P_c \nabla \cdot \vec{\xi}^* \frac{B \kappa_s \xi_\theta}{|\nabla \psi|} = \int d^3x \frac{B_o^2 \omega^2 \omega_T^2}{\omega_A^2 (\omega^2 - \omega_s^2)} |\xi_\theta|^2, \end{aligned} \quad (6.11)$$

where

$$\omega_T^2 = \left\langle \frac{\Gamma_s P_c B^2 \kappa_s^2}{\rho |\nabla \psi|^2} \right\rangle, \quad \omega_T^2 = \left\langle \frac{\Gamma_s P_c B^2 \kappa_s^2}{\rho |\nabla \psi|^2} \right\rangle, \quad \omega_A^2 = \left\langle \frac{1}{\rho J^2} \right\rangle, \quad (6.12)$$

and $\langle \dots \rangle$ means surface average at the $q = 1$ surface. Therefore, the total contribution from inside the singular layer is

$$D_i = \int_{r=r_s-\Delta}^{r=r_s+\Delta} d^3x B_o^2 \left\{ \frac{\omega^2}{\omega_A^2} \left[\frac{\omega_T^2}{(\omega^2 - \omega_s^2)} - 1 \right] + (1-q)^2 \right\} \left| \frac{\partial \xi_r}{\partial r} \right|^2. \quad (6.13)$$

The minimizing ξ_r , that satisfies the Euler-Lagrange equation obtained by varying D_i with respect to ξ_r and matches to the outside solution, is given by

$$\xi_r = \xi_{ro} \left[\frac{1}{2} - \frac{1}{\pi} \tan^{-1} \left(\frac{r-r_s}{\Delta} \right) \right], \quad (6.14)$$

where the singular layer width is

$$\Delta^2 = \frac{\omega^2}{(q_s \omega_A)^2} \left[\frac{\omega_T^2}{(\omega^2 - \omega_s^2)} - 1 \right] \xrightarrow{\omega \ll \omega_s, \epsilon_0 \rightarrow 0} \frac{-\omega^2(1 + 2q^2)}{(q_s \omega_A)^2} \quad (6.15)$$

Then D_i becomes

$$D_i = \frac{2\pi^2 R B_0^2 r_s q_s' |\xi_{ro}|^2}{\omega_A} (i\omega) \left[\frac{\omega_T^2}{(\omega_s^2 - \omega^2)} + 1 \right]^{1/2} \quad (6.16)$$

Finally, combining Eqs. (6.6) and (6.16), we arrive at the dispersion relation

$$-i\omega \left[1 + \frac{\omega_T^2}{(\omega_s^2 - \omega^2)} \right]^{1/2} = (\gamma_B - \delta \widehat{W}_k) \quad (6.17)$$

If we consider neutral-beam-injected (NBI) plasmas, the equilibrium hot particle distribution F_h is taken as a slowing-down distribution in energy and a delta function in pitch angle, i.e.,

$$F_h(\epsilon, \Lambda, \psi) = \begin{cases} \frac{n_h(\psi) \delta(\Lambda - \Lambda_0)}{\epsilon_c^{3/2} + \epsilon^{3/2}} & , \epsilon \leq \epsilon_b \\ 0 & , \epsilon > \epsilon_b \end{cases} \quad (6.18)$$

where ϵ_b is the neutral beam injection energy, and ϵ_c is the cutoff energy where the beam ions lose an amount of energy to the core ions equal to that lost to electrons. For simplicity, we take $\epsilon_c = 0$ and assume $|\omega| \ll \omega_s$, then the dispersion relation takes the form

$$-i \left(\frac{\omega}{\omega_A} \right) = \left(\frac{\gamma_B}{\omega_A} \right) - \hat{\beta}_h \left\{ \frac{\omega_*}{\omega_d} + \frac{\omega}{\omega_d} \left(\frac{\omega_*}{\omega_d} - 1 \right) \left[\ln \left(1 - \frac{\omega_d}{\omega} \right) + i \pi \sigma \right] \right\}, \quad (6.19)$$

where the hot trapped particle terms are volume averaged, and the detailed forms are not important in determining the qualitative stability properties. For $0 < \text{Re}(\omega) < \omega_d$, we set $\sigma = 0$ for $\text{Im}(\omega) > 0$, $\sigma = 1$ for $\text{Im}(\omega) = 0$, and $\sigma = 2$ for $\text{Im}(\omega) < 0$. ω_* and ω_d are the typical hot particle diamagnetic drift frequency and the bounce-averaged magnetic drift frequency, respectively. γ_B is the ideal MHD growth rate including both core and hot fluid contributions, and $\hat{\beta}_h$ is proportional to the hot particle beta. We should reiterate here that the first (ω_*/ω_d) term in Eq. (6.19) does not cancel with the hot particle contribution to γ_B . Equation (6.19) predicts that if $\gamma_B > 0$, the ideal branch will be stabilized for $\hat{\beta}_h > \hat{\beta}_{h1}$, where $\hat{\beta}_{h1} = (\gamma_B \omega_d / \omega_A \omega_*)$. At $\hat{\beta}_h = \hat{\beta}_{h1}$, $\omega = 0$. The resonant fishbone branch will be destabilized for $\hat{\beta}_h > \hat{\beta}_{h2}$, where $\hat{\beta}_{h2} = \omega_d / [\pi \omega_A (\omega_*/\omega_d - 1)]$. At $\hat{\beta}_h = \hat{\beta}_{h2}$, ω is purely real. Figure 5(a) shows the growth rate and Fig. 5(b) the real frequency versus $\hat{\beta}_h$ for the fixed parameter $\omega_*/\omega_d = 10$. Several values of $\bar{\omega}_D \equiv \omega_d/\gamma_B$ are used. The curves labeled $\bar{\omega}_D \geq 3$ correspond to the ideal branch. We see that for $(\omega_*/\gamma_B) \leq \pi(\omega_*/\omega_d - 1)$ (i.e., $\hat{\beta}_{h1} \geq \hat{\beta}_{h2}$), the ideal branch is unstable for all $\hat{\beta}_h$. For $\hat{\beta}_{h1} < \hat{\beta}_{h2}$, there is a stability window for $\hat{\beta}_{h1} \leq \hat{\beta}_h \leq \hat{\beta}_{h2}$. It is important to note that previous analytic studies [49-53] have concluded that $\hat{\beta}_{h1} \geq \hat{\beta}_{h2}$ so that there is no stability window. The erroneous conclusion is mainly due to the mistake of making exact cancellation of the first (ω_*/ω_d) term in Eq. (6.19) with the hot particle contribution to γ_B .

6.2 Effects of Neutral-Beam-Injected Particles on the Internal Kink Mode

We will assume $\beta_c \gg \beta_h$, so that the equilibrium is approximately isotropic. We first consider an equilibrium with circular plasma surface computed from a flux equilibrium code with the profiles

$$P(y) = P_0 (1 - y^\lambda)^\mu, \quad (6.20)$$

$$q(y) = q(0) + y \left\{ q(1) - q(0) + [q'(1) - q'(0) + q(0)] \frac{(1 - y_s)(y - 1)}{(y - y_s)} \right\}, \quad (6.21)$$

where $y_s = [q'(1) - q'(0) + q(0)] / [q'(0) + q'(1) - 2q(1) - q(0)]$, $y = (\psi - \psi_0) / \Delta\psi$, $\Delta\psi = \psi_{lim} - \psi_0$, ψ_{lim} is evaluated at the limiter, and ψ_0 is evaluated at the magnetic axis. The parameters for the "PDX05" equilibrium are $P_0 = 0.018$, $\lambda = 2$, $\mu = 2$, $\Delta\psi = 0.061$, $\langle\beta\rangle_{av} = 0.625\%$, $R = 1.43$, $R/a = 3.4$, $q(0) = 0.8$, $q(1) = 2.85$, $q'(0) = 13.82$ and $q'(1) = 106.6$, and $\langle\beta_{pol}\rangle = 2 \langle P \rangle / \langle B_{pol}^2 \rangle = 0.4278$. Figure 6 shows the flux surfaces of the "PDX05" equilibrium, and Figs. 7(a) - 7(c) show the radial dependencies ($r = y^{1/2}$) of the trapped particle pitch-angle space bounded by h_{min} and h_{max} , the hot particle diamagnetic drift frequency, and the bounce-averaged magnetic drift frequency $\langle\omega_d\rangle$ for $\Lambda_r = 1.1$, respectively. Also plotted in Fig. 7(a) is the curve of $\langle\omega_d\rangle = 0$. Between the $\Lambda = h_{max}(r)$ and the $\langle\omega_d\rangle = 0$ curves, we have $\langle\omega_d\rangle < 0$, and between the $\langle\omega_d\rangle = 0$ and the $\Lambda = h_{min}$ curves, $\langle\omega_d\rangle > 0$. Particles with $\Lambda < h_{min}$ are untrapped. For $\Lambda = h_{min}$ particles are barely trapped, and for $\Lambda = h_{max}$ particles are deeply trapped at $\theta=0$. It is clear that for a given pitch angle Λ_0 , $\langle\omega_d\rangle$ may change from being negative to positive as r changes, which means that $(\omega_*/\langle\omega_d\rangle)$ also varies from being positive to negative. If the trapped particles are destabilizing to a certain MHD mode for $(\omega_*/\langle\omega_d\rangle) > 0$ in a certain radial region, the trapped particles in the radial region with $(\omega_*/\langle\omega_d\rangle) < 0$ would be stabilizing. Thus, the net effect of hot trapped particles must be integrated over the entire hot particle population with proper weightings of different poloidal harmonics, which can only be achieved by numerical solution of nonvariational codes such as the NOVA-K code.

It should be emphasized that the NOVA-K can calculate stability for numerical hot particle distributions obtained from Fokker-Planck code calculations. The flux surface-averaged hot NBI particle density is typically of the order of 10^{-2} times the core density and $\mathcal{E}_b$ ranges between

30 and 100 keV. $n_h(\psi)$ is assumed to have the same functional dependence as $P_c(\psi)$ for simplicity. Since both ω_* and $\langle\omega_d\rangle$ are proportional to $\mathcal{E}_b$, we will introduce a temperature scaling factor C_T in ω_* and $\langle\omega_d\rangle$. If $C_T = 0$, $\omega_{*h} = \langle\omega_d\rangle = 0$ and the eigenmode equations, Eqs. (3.51) - (3.54), correspond to the Kruskal-Oberman energy principle [33]. For $C_T = \infty$, Eqs. (3.51) - (3.54) correspond to the case studied by Rosenbluth et al. [48]. From Eq. (3.16), we see that both $\delta\hat{p}_{\parallel}$ and $\delta\hat{p}_{\perp}$ are proportional to $n_h(\psi)T_h$. Therefore, we will set $n_h(\psi)\mathcal{E}_b = \alpha_h P_c(\psi)$ and vary α_h to change the hot particle pressure.

In the absence of energetic particles ($\alpha_h=0$), the $n = 1$ fixed boundary ideal MHD internal kink mode is unstable with the growth rate $\gamma/\omega_A = 1.195 \times 10^{-2}$. Because the eigenfunction ξ_{ψ} has a sharp radial variation near the $q=1$ surface, the computation was carried out by using a spatially tailored r -grid with dense grid packing near the $q = 1$ surface. Since the poloidal harmonics of ξ_{ψ} are dominated by $1 \leq m \leq 3$, we will use the equal arc-length r coordinate and keep 7 poloidal harmonics $m = [-2,4]$ in the computation to ensure convergence in m . We have also checked the numerical convergence in radial grid points by using $N=50, 75$, and 100 to obtain the converged growth rate. Figure 8 shows the poloidal components of the eigenfunction ξ_{ψ} versus r ; the q -profile is also shown in Fig. 8. We see that the $m=2$ harmonic can give a non-negligible contribution beyond the $q = 1$ surface.

In the following, we will present the results of the neutral-beam-injected (NBI) energetic particles on the stability of the $n=1$ internal kink modes and compare these results with the model dispersion relation calculations. The hot particle distribution is given by Eq. (6.18) with $\mathcal{E}_c = 0$, and $n_h(\psi)$ is taken to have the same c dependence as $P(\psi)$. The other fixed parameters for the energetic particles are $\mathcal{E}_b/T_c(0) = 10$, $R/\rho_h(0) = 100$, $M_h/M_c = 1$, $C_T = 1$, $\Lambda_0 = 1.1$, where M_c and M_h are the core and hot ion mass, respectively. $\rho_h(0)$ is the hot ion gyroradius at the magnetic axis, and Λ_0 is the pitch angle of the injected beam ions. For this equilibrium, the volume-averaged hot particle beta, $\beta_h = 2 \langle P_h \rangle / \langle B^2 \rangle$, is related to the total volume-averaged beta, $\beta = 2 \langle P \rangle / \langle B^2 \rangle$, by $\beta_h = 1.38 \alpha_h \beta$ where $\beta = 0.625\%$. In varying β_h (or α_h) we will keep the total β fixed so that as β_h is increased, the core plasma β_c is decreased by the same amount. Figure 9(a)

shows the α_h dependence of the growth rates, γ/ω_A , of the ideal and the resonant fishbone branches of internal kinks. The computations were carried out by performing convergence studies with $N = 50, 75$, and 100 nonuniform radial grid points. We see that the $N=100$ case gives approximately the converged solutions. For the ideal branch, the eigenfunction ξ_ψ becomes increasingly singular as $\alpha_h > 0.035$. For the resonant branch, ξ_ψ becomes more singular as $\alpha_h < 0.07$. One also realizes that the solutions with singular ξ_ψ cannot be trusted because our cubic B-spline finite elements do not give proper representation of the singular ξ_ψ . This is the most difficult problem in attempting the numerical solutions of MHD modes near marginal stability with $|\gamma/\omega_r| \ll 1$ or $|\gamma/\omega_A| \ll 1$ when the eigenfunction ξ_ψ becomes singular. The common practice of obtaining the behavior of the eigenvalue $\omega (= \omega_r + i\gamma)$ near marginal stability is to extrapolate from the ω with well-behaved eigenfunctions. The two broken curves in Fig. 9(a) show the extrapolated values of γ near marginal stabilities of the ideal and the resonant branches of the internal kink modes. We can now conclude from Fig. 9(a) that the ideal branch of the internal kink mode is stabilized by the injected energetic particles for $\beta_h > \beta_{h1} = 0.031\%$ ($\alpha_h = 0.036$), but for $\beta_h > \beta_{h2} = 0.0517\%$ ($\alpha_h = 0.06$) the fishbone branch of the internal kink mode is resonantly destabilized by the injected energetic particles. Figure 9(b) shows the negative real frequency, $-(\omega_r/\omega_A)$, versus α_h for the corresponding cases of Fig. 9(a). As $\alpha_h \rightarrow 0$, the real frequency of the resonant branch evolves into a slow sound mode of the continuous spectrum, which is below the shear Alfvén continuous spectra. Note that the shear Alfvén continuous spectrum does not go to zero at the $q=1$ surface due to finite values of β and the surface component of the magnetic curvature [19]. It is the $m=1$ slow sound branch of the continuous spectrum that goes to zero frequency at the $q=1$ surface and determines the singular behavior of the $n=1$ internal kink mode near marginal stability.

In Figs. 10(a) and 10(b), the growth rates and the negative real frequencies, respectively, versus the hot particle pressure scaling factor α_h are plotted for several values of the hot particle injection energy scaling factor C_T . The fixed parameters are the same as in Fig. 9. For $C_T < 0.4$, the hot particle pressure is stabilizing for small values of α_h , but becomes destabilizing as α_h becomes larger ($\alpha_h > 0.036$). The destabilizing effect is accompanied by an increase in $|\omega_r|$ and is associated with $\omega - \langle \omega_d \rangle \approx 0$ resonance. However, when C_T becomes larger with $C_T \approx 1$, the ideal branch is stabilized for $\alpha_h > \alpha_{h1} \approx 0.036$ (for $C_T=1$). But the resonant

fishbone branch is destabilized for $\alpha_h > \alpha_{h2} = 0.06$ for $C_T = 1$. For larger C_T , α_{h1} is smaller, but α_{h2} becomes larger. Note that the broken curves in Fig. 10(a) are obtained by employing the extrapolation scheme described in Fig. 9.

The results presented in Figs. 10(a) and 10(b) are qualitatively similar to Figs. 5(a) and 5(b) obtained from the analytical dispersion relation given by Eq. (6.19). Figure 5(a) clearly shows these behaviors through the variations in ω_d with ω_*/ω_d being held fixed. We should note that the real frequency shown in Fig. 5(b) is somewhat different from Fig. 10(b) for the resonant branch in the limit $\alpha_h < \alpha_{h2}$. This is because the analytical dispersion, Eq. (6.19), does not describe the MHD continuous spectrum that is contained in the NOVA-K code. The above comparison clearly demonstrates that the NOVA-K code gives correct results of the energetic particle effects on the $n=1$ internal kink mode.

Next, we examine the validity of the approximations made in the analytical dispersion, Eq. (6.17). Our calculations show that although the analytical dispersion gives qualitatively correct results, it fails to provide the correct values of the critical hot particle betas for both the stabilization of the ideal branch and the destabilization of the resonant fishbone branch. To make comparisons on the values of critical β_h , we impose similar approximations in computing the perturbed hot particle pressures δP_h given by Eq. (3.15), by retaining only the $m=1$ poloidal harmonics and taking $1-q(r) = 0$ in the nonadiabatic hot particle contribution. For the same parameters as in Figs. 9(a) and 9(b), the computed eigenvalues ω versus β_h are shown in Fig. 11. By comparing Figs. 9 and 11, we see that the qualitative behavior of the $n=1$ mode with analytical approximations is quite similar to that of Fig. 9 obtained without approximations. However, the horizontal β_h scale of Fig. 11 is about a factor of 4 smaller than that of Fig. 9. Thus, the analytical approximations have produced an error in β_{h1} and β_{h2} by roughly a factor of 4 in this particular example. The error is mainly due to the omission of higher poloidal harmonics. As seen in Fig. 8, the amplitude of the $m=2$ harmonic is not negligible outside the $q=1$ surface. Also from Fig. 7 with $\Lambda = 1.1$, $(\omega_*/\omega_d) < 0$ for $r > 0.85$, and thus the $m=2,3$ harmonics have opposite contributions to stability from the $m=1$ harmonic. This calculation tells us that we must employ the NOVA-K code to compute the correct values of the critical β_{h1} and β_{h2} .

Another important conclusion from the calculations of the NOVA-K code that is different from that of the analytical theory is that a necessary condition to excite the resonant fishbone mode is $\beta_2 > \beta > \beta_1$, where $\beta_{1,2}$ are the ideal MHD critical betas for the $n=1$ internal kink. Our conclusion is derived by computing the stabilities of several equilibria with different β . As β is closer to $\beta_{1,2}$, the growth rates of both the ideal branch and the resonant fishbone branch also decrease to zero. For $\beta \leq \beta_1$ and $\beta \geq \beta_2$, no instabilities are found by the NOVA-K code. On the other hand, it can be easily seen from the analytical dispersion relation, Eq. (6.19), that the critical β_{n2} for the resonant fishbone branch is independent of γ_B , and there is no constraint on the total plasma β . The breakdown of the analytical analysis for both the ideal and resonant fishbone branches in the ideally stable β domain is that the eigenfunction becomes singular near $q=1$ surface and the analytical dispersion is invalid. Finally, the effects of the neutral beam injection or ICRH heatings on the $n=1$ internal kink mode can be summarized schematically by the stability window in the β - β_n space shown in Fig. 12. This stability window provides an explanation of the experimental observations of the stabilization of the sawtooth in JET [16,17]. Here, we emphasize that unlike the previous analysis [51-53] the stability window exists even in the absence of the core ion finite Larmor radius effect (finite ω_{*i}).

6.3 Alpha Particle Effects on the Internal Kink Mode

The stabilization of the internal kink mode by trapped α -particles had been reported by several authors [50,56] who studied the analytic dispersion relations by employing large aspect ratio tokamak equilibria. However, this conclusion cannot be applicable to finite aspect ratio tokamaks because the analytical studies failed to take into account the pitch angle variations of the bounce-averaged magnetic drift frequency $\langle \omega_d \rangle$ and the contributions from $m=2$ and 3 poloidal harmonics. In fact, the numerical studies using the NOVA-K code indicated that the trapped α -particles have negligible effects on the stability of the $n=1$ internal kink mode.

In the NOVA-K code, we will employ the slowing-down distribution with $\mathcal{E}_c = 0$ for α -particles, but with uniform pitch angle distribution. We will study a Compact Ignition Tokamak (CIT) type equilibrium "ECIT02" which has a noncircular plasma surface defined by

$$X = R + a \cos [\theta + \delta \sin(\theta)] , \quad (6.22)$$

and

$$Z = \kappa a \sin(\theta) , \quad (6.23)$$

where the ellipticity is $\kappa = 2$, the triangularity is $\delta = 0.4$, the horizontal minor radius is $a = 0.65$, and the major radius is $R = 2.1$. The total pressure and q -profiles are given by Eqs. (6.20) and (6.21) with $P_0 = 1$, $\lambda = 1.05$, $\mu = 2$, $\langle \beta \rangle_{av} = 4.0159\%$, $q(0) = 0.8$, $q(1) = 3.2$, $q'(0) = 0.25$, and $q'(1) = 7$, and $\langle \beta_{pol} \rangle = 2 \langle P \rangle / \langle B_{pol}^2 \rangle = 1.0376$. Figures 13(a)-(c) show the flux surfaces of the equilibrium, the pressure profile, and the q -profile, respectively. Figure 14(a) shows the radial dependence of the trapped particle pitch-angle space bounded by h_{min} and h_{max} , and Figs. 14(b)-(f) show the pitch angle Λ dependence of the bounce-averaged magnetic drift frequency $\langle \omega_d \rangle$ at $r = 0.2, 0.4, 0.6, 0.8$, and 1 , respectively. Also plotted in Fig. 14(a) is the curve of $\langle \omega_d \rangle = 0$. It is clear that for a radial position, $\langle \omega_d \rangle$ may change from being negative to positive as Λ decreases, which means that $(\omega_s / \langle \omega_d \rangle)$ also varies from being positive to negative. As r increases, the pitch angle Λ domain of negative $(\omega_s / \langle \omega_d \rangle)$ widens. Thus, the net effect of the trapped α -particles must be integrated over the entire trapped α -particle population with proper weightings of different poloidal harmonics.

The parameters for the α -particles are $\mathcal{E}_\alpha / T_c(0) = 100$, $R / \rho_\alpha(0) = 100$, $M_\alpha / M_c = 4$, where M_c and M_α are the core and alpha mass, respectively, and $\rho_\alpha(0)$ is the α -particle gyroradius at the magnetic axis. The α -particle pressure is taken to be $P_\alpha(\psi) = \alpha_h P_0 [P(r)/P_0]^{7/2}$. For this equilibrium, the volume-averaged α -particle beta, $\beta_\alpha = 2 \langle P_\alpha \rangle / \langle B^2 \rangle$, is related to the total volume-averaged beta, $\beta = 2 \langle P \rangle / \langle B^2 \rangle$, by $\beta_\alpha = 0.6214 \alpha_h \beta$ where $\beta = 4.0159\%$. In varying β_α (or α_h) we will keep the total β fixed so that as β_α is increased, the core plasma β_c is decreased by the same amount. Figure 15 shows the poloidal components of the eigenfunction ξ_ψ versus r for $\beta_\alpha = 0.5\%$ ($\alpha_h = 0.2$). Figure 16 shows the α_h dependences of the growth rates, γ / ω_A , and the negative real frequency, $-(\omega_r / \omega_A)$, of the ideal internal kink. We see from Fig. 16 that the ideal internal kink mode is only weakly stabilized by the α -particles even with β_α as large as 1.3% . This result can be understood from the quadratic form Eqs. (3.68) and the nonadiabatic trapped α -particle potential energy δW_k defined in Eq. (3.72). Since $\omega \ll$

$\langle \omega_d \rangle$, δW_k , which is proportional to radial and pitch angle averages of $(\omega_*/\langle \omega_d \rangle)$, can be very small and thus has negligible effect on the stability of the $n=1$ internal kink mode.

7. Alpha Particle Destabilization of Toroidicity-Induced Alfvén Eigenmodes

7.1 Shear Alfvén Waves

Ignition tokamak experiments, which are being proposed for the near future, will contain alpha particles as fusion reaction products. These particles will have very high energies (e.g., the alpha particles in a D-T reaction are born at 3.5 MeV), and in neither coordinate nor velocity space will they be in thermodynamic equilibrium. Consequently, "thermonuclear" instabilities could be excited in a burning plasma by alpha particles. An example would be instabilities driven by the expansion-free energy associated with the spatial gradient of the alpha particle pressure, ∇P_α , which enters the dynamical equations that describe the plasma behavior through the alpha particle diamagnetic drift frequency $\omega_{* \alpha}$. It is expected that in a reacting tokamak plasma, the birth distribution of the alpha particles will be sharply peaked near the magnetic axis: typically, $L_\alpha/a = |a d(\ln P_\alpha)/dr|^{-1} < 0.4$ for ignition parameters, where L_α is the alpha particle pressure gradient scale length and a is the plasma minor radius. In this section, we will focus on the destabilization of shear Alfvén waves by alpha particles. For typical ignition parameters the alpha particle velocity $V_\alpha = (\epsilon_\alpha/M_\alpha)^{1/2} = 9 \times 10^8$ cm/sec for an energy ϵ_α of 3.5 MeV is comparable to the Alfvén speed $V_A = B/(N_i M_i)^{1/2}$. Thus, the transiting alpha particles could destabilize shear Alfvén waves via inverse Landau damping through the $\omega = k_{||} v_{||}$ wave-particle resonance. Here, $k_{||} = (m - nq)/qR$ is the parallel wave number for linearized waves that are Fourier decomposed as $\exp[i(m\theta - n\zeta - \omega t)]$. To satisfy the resonance condition, it requires that $V_\alpha > V_A$. To overcome the Landau damping by the inverse Landau damping associated with $\omega_{* \alpha}$, it requires that $\omega_{* \alpha} > \omega_A = V_A/qR$.

Briefly, let us describe the various types of shear Alfvén waves in a tokamak plasma. The ideal MHD equation of motion, written in cylindrical geometry (the Hain-Lust equation), shows that the coefficient of the

highest-order radial derivative term, $d^2\xi_r/dr^2$, where ξ_r is the radial fluid displacement, vanishes at radial locations where $\omega^2 = (k_{\parallel}V_A)^2$. This corresponds to the shear Alfvén resonance condition, and frequencies ω that satisfy $\min[(k_{\parallel}V_A)^2] < \omega^2 < \max[(k_{\parallel}V_A)^2]$ lie in the shear Alfvén continuum. This resonance leads to a singular mode structure; however, if electron parallel dynamics and ion finite Larmor radius effects are included, one obtains a nonsingular solution known as the kinetic Alfvén wave (KAW). Its mode structure is fairly localized, and hence it experiences strong electron Landau damping.

There are two global types of shear Alfvén waves that have radially extended mode structure. Both types have low mode numbers n and m . The first type of global shear Alfvén wave is a regular, spatially nonresonant wave whose frequency lies just below the minimum of the continuum, i.e., $\omega < k_{\parallel}V_A$ and $k_{\parallel} \neq 0$. This wave is called the Global Alfvén Eigenmode (GAE) [57]. Previous theoretical analysis of this mode was limited to cylindrical geometry, where it was found that transit wave-particle resonant interaction with super-Alfvénic alpha particles could destabilize it, although with weak growth rates [58]. When finite toroidicity is included, GAE modes with different poloidal mode numbers will become coupled. Such toroidal mode coupling tends to stabilize the GAE modes completely [18,59].

There is another type of global shear Alfvén wave, one that exists only in toroidal geometry. Its frequency lies within "gaps" in the shear Alfvén continuum that are created due to toroidal coupling. The existence of this so-called Toroidicity-Induced Alfvén Eigenmode (TAE) was previously shown in the ideal MHD limit without alpha particles [19,20]. In this section, we show that this TAE mode can be strongly destabilized by alpha particles in a burning tokamak plasma.

7.2 Stability of Toroidicity-Induced Alfvén Eigenmodes

Finite toroidicity introduces a new global Alfvén mode, the TAE, which is strongly destabilized by alpha particles. This effect was studied [59] with a model profile by treating alpha particle effects perturbatively. Here, we self-consistently investigate the alpha particle destabilization of the TAE mode by employing numerically computed finite β equilibria with noncircular flux surfaces. The existence of the TAE mode was previously demonstrated in the ideal MHD limit [19,20], in the absence of alpha particles. This mode exists inside gaps, due to toroidal coupling, in

the shear Alfvén continuum spectrum. For example, modes (n,m) and $(n,m+1)$ couple at radial location r_0 , where $q(r_0) = (m + 1/2)/n$, to form a gap. For $n=m=1$, this gap is bounded by

$$\omega_{\pm}^2 = \omega_0^2 + \left\langle \frac{\Gamma_s P B^2 \kappa_s^2}{|\nabla \psi|^2} \right\rangle \pm 2 \omega_0^2 \left(\frac{r_0}{R} + \Delta'(r_0) \right), \quad (7.1)$$

where $\omega_0^2 = (V_A/2qR)^2$ at $r = r_0$, $\Delta(r)$ is the Shafranov shift of the nonconcentric flux surfaces, and $\Delta' > 0$.

Figure 17 shows the continuous spectrum for the $n=1$ mode as a function of the radius for a numerical equilibrium (EC42.3D) with a circular plasma surface. The pressure and q -profiles of the "EC42.3D" equilibrium are those defined by Eqs. (6.20) and (6.21) with the parameters: $P_0 = 0.45$, $\lambda = 2$, $\mu = 2$, $q(0) = 1.05$, $q(1) = 2.3$, $q'(0) = 0.75$, and $q'(1) = 3$, $\langle \beta \rangle_{av} = 1.893\%$. The other parameters of the "EC42.3D" equilibrium are $a = 0.25$, $R = 1$. The real frequency of the $n=1$ fixed boundary TAE mode is $\omega_r/\omega_A = -0.705$, which lies within the continuum gap, $1.25 \geq (\omega/\omega_A)^2 \geq 0.55$, formed by the toroidal coupling of the $m=1$ Alfvén mode and $m=2$ Alfvén mode at $q=1.5$. In Fig. 17 we also see that the $m \geq 3$ sound modes will move into the Alfvén gap and couple with the $m=1$ and 2 Alfvén modes. In fact, as β increases, the frequency of the discrete TAE mode may move downward into the continuum and suffer Landau dampings through wave-particle resonances. The eigenfunction ξ_ψ of the $n=1$ fixed boundary TAE mode versus r for the "EC42.3D" equilibrium is shown in terms of poloidal harmonics in Fig. 18, which clearly shows the dominant $m=1$ and 2 components.

The NOVA-K code is employed to examine the alpha particle destabilization of this mode. The alpha particle perturbed pressure tensor $\delta \tilde{p}_\alpha$ in the momentum balance equation Eq.(3.1) contains both transit and bounce resonance effects. Core plasma kinetic effects on this mode are neglected here. The alpha particle equilibrium distribution function, F_α , was taken to be isotropic in pitch angle L and slowing down in energy ε , as defined in Eq. (6.18). Figures (19)-(23) show the pitch angle dependences of (a) the transit frequency ω_t , (b) transit-averaged $\langle \omega_d \rangle$, (c) bounce frequency ω_b , and (d) bounce-averaged $\langle \omega_d \rangle$ at the alpha particle birth energy ε_α for the radial locations $r = 0.2, 0.4, 0.6, 0.8$, and 1, respectively. $\langle \omega_d \rangle$ is much smaller than ω_t and ω_b . At the boundary between circulating

and trapped particles, $\Lambda = h_{\min}$, $\langle \omega_d \rangle$ has a logarithmic singularity and both ω_i and ω_b vanish. For larger r , ω_i decreases and ω_b increases, but ω_i is about a factor of 2-4 larger than ω_b . Thus, for typical reactor parameters the transit resonances can more effectively destabilize the TAE mode than the bounce resonances.

Figure 24 shows the growth rate for the $n=1$ fixed boundary TAE mode as a function of the alpha particle diamagnetic drift frequency $\omega_{* \alpha}$ (evaluated at $r = 0.5$ and for $m=1$) normalized to $\omega_A = V_A(0)/q(a)R$, for the "EC42.3D" equilibrium with $\langle \beta_\alpha \rangle = 0.4\%$. Here ϵ_α is the alpha particle birth energy and is taken to be $100T_c(0)$, and $n(\psi)$ is proportional to the alpha particle density, which was taken to be functionally related as $n(\psi) \propto [P(\psi)]^{7/2}$ to the total plasma pressure $P(\psi)$. The wave functions for the $m=1$ and $m=2$ poloidal harmonics show that these modes peak near $r = 0.5$; hence $\omega_{* \alpha}$ is a good measure of the alpha particle free energy that can be tapped via inverse Landau damping. Figure 24 shows that $\omega_{* \alpha}$ is large enough to overcome the usual Landau damping when $\omega_{* \alpha}/\omega_A > 1.5$ and that beyond this threshold the growth rate γ is approximately linearly proportional to $\omega_{* \alpha}$. Note that the proportional constant depends on the toroidal mode number. From these results we find that for typical ignition parameters, the growth rate of the $n=1$ TAE mode can be of the order of 10^{-2} of the real frequency - which is about one order of magnitude larger than the GAE mode growth rates (evaluated at small a/R).

7.3 Theoretical Interpretation

The numerical results concerning the two types of global shear Alfvén waves can be interpreted from the quadratic form Eq.(3.68). For the mode frequency, we write $\omega = \omega_r + i\gamma$ and assume that the growth rate is small ($|\gamma| \ll |\omega_r|$). Then Eq. (3.68) yields

$$\omega_r^2 \approx \frac{\delta W_I}{\delta K}, \quad (7.2)$$

and

$$\gamma = -\text{sign}(\omega_r) \int d\psi d\Lambda \left(\frac{9\pi^2 \tau_t P(\psi)}{4 \delta K} \right) \times \sum_{m', m, p=-\infty}^{\infty} \left\{ \omega_r - \frac{\omega_{*a}(m) \omega_r^2}{[(p-nq) \omega_t]^2} \right\} \frac{\text{Re} \{ \langle G(m', p) \rangle^* \langle G(m, p) \rangle \}}{[(p-nq) \omega_t]^2}, \quad (7.3)$$

where $G(m, p) = \hat{G}_m \exp\{i[(m-nq)\theta - (p-nq)\omega_t \hat{t}]\}$; $\hat{G}_m$ is given by Eqs. (3.59)-(3.62) and depends on the pitch angle, the equilibrium ψ -variation, and the mode amplitude; $\hat{t}$ is the time-like variable measuring alpha particle position along a field line and is defined in Eq. (3.38), ω_t and τ_t are the alpha particle transit frequency and transit time, respectively; and $\langle G(m, p) \rangle$ is the transit average of $G(m, p)$. The transit harmonic number is p , where p values close to m contribute the most. We define $\omega_{*a}(m) = m\omega_{*a}$, where, as before, ω_{*a} is the diamagnetic frequency for $m=1$.

Equation (7.3) shows that the instability condition is $\omega_{*a}(m) > \omega_-$; here the alpha particle free energy drive overcomes usual Landau damping. This marginal stability condition agrees with the numerical result, shown in Fig. 24, from which $\omega_{*a}/\omega_A = 1.5$ is marginal. Above this threshold, Eq. (7.3) indicates that the growth rate γ will tend to scale linearly with ω_{*a} which again agrees with Fig. 24.

The form of Eq. (7.3) also shows why the $n=0$ GAE mode is stable in toroidal geometry. Since the alpha particle effects are perturbatively small, to a good approximation the mode structure is given by the ideal MHD toroidal wavefunctions. Being self-adjoint, these wave functions have the symmetry property $|G(m, p)|^2 = |G(-m, -p)|^2$ for $n=0$. Thus, the alpha particle excitation terms $\omega_{*a}(m)$ and $\omega_{*a}(-m)$ are seen to tend to cancel in Eq. (7.3), so that the $n=0$ GAE mode is stable.

The results described herein seem to indicate that the global Alfvén eigenmodes (GAE) [57] will not be problematic in ignition tokamaks. Toroidicity tends to stabilize the GAE modes. Therefore, primary attention - especially experimental - should be focused on the toroidicity-induced shear Alfvén eigenmodes (TAE) [18-20], which can be strongly destabilized

by alpha particles. It should be noted that if the edge plasma density value is reduced, the toroidicity-induced mode may possibly resonate with the shear Alfvén continuum near the plasma periphery, an effect which could be stabilizing and should be investigated.

8. Summary and Conclusion

In this paper we have presented a nonvariational kinetic-MHD stability code (NOVA-K) which provides accurate and efficient numerical solutions of the stability of axisymmetric toroidal plasmas involving energetic particles. Energetic particle dynamics are governed by the drift kinetic equation and have significant effects on the stability of low frequency MHD modes. In a general flux coordinate system (ψ, θ, ζ) , we have formulated the kinetic-MHD eigenmode equations in terms of a coupled set of non-Hermitian two-dimensional integro-differential equations in the (ψ, θ) space. The NOVA-K code makes use of cubic B-spline finite elements in the minor radius ψ -direction and Fourier expansions in the poloidal θ and toroidal ζ directions. Following the Galerkin procedure, the problem is reduced to solving a matrix equation with nontrivial solutions. Extensive comparisons with existing variational ideal MHD codes [8-12] show that the ideal version of the NOVA code [7] can produce more accurate results with less computational effort.

Employing the NOVA-K code, we first studied the problems of energetic trapped particle stabilization of the $n=1$ internal kink mode and the excitation of the resonant fishbone mode by resonating with the energetic trapped particle magnetic drifts. An analytical dispersion relation, Eq. (6.17), involving large aspect ratio orderings was derived and studied to help understand the numerical results of the NOVA-K code. Extensive comparisons with the results of the analytical dispersion for energetic trapped particles with singular pitch angle distribution, such as in the cases of NBI and ICRH heatings, showed that the analytical results are qualitatively similar to the NOVA-K results, which indicates the existence of a stability window in the β_h space even in the absence of the core ion finite Larmor radius effect (finite ω_{ci}). Imposing analytical approximations on the NOVA-K code would give incorrect values of critical β_h , which can be an order of magnitude different from the correct values. In addition, the results of the NOVA-K code indicate that a necessary condition to excite the resonant fishbone mode is $\beta_2 > \beta > \beta_1$, where $\beta_{1,2}$ are the ideal MHD critical total betas for the $n=1$ internal kink. This necessary

condition is not predicted by the analytical dispersion relation. For the trapped α -particles with a uniform pitch angle distribution, the numerical results of the NOVA-K code showed that the stability of the $n=1$ internal kink mode for CIT type equilibria is very little affected, which is contrary to the previous conclusion [56] of strong stabilization of the $n=1$ internal kink mode obtained by assuming large aspect ratio orderings, keeping only the $m=1$ poloidal harmonic, and failing to integrate over the pitch angle with self-consistent equilibria.

The NOVA-K code has also been employed to study the transit α -particle destabilization of the toroidicity-induced shear Alfvén eigenmodes (TAE) via inverse Landau damping associated with the spatial gradient of the α -particle pressure. The TAE modes are shown to be strongly unstable with the growth rate, γ , being approximately linearly proportional to $\omega_{* \alpha}$ and typically of the order of $10^{-2} \omega_A$. Other types of global Alfvén waves are stable in ignition tokamaks due to toroidal coupling effects. Therefore, primary attention - especially experimental - should be focused on the toroidicity-induced shear Alfvén eigenmodes (TAE), which can be strongly destabilized by alpha particles.

Finally, we believe that tokamak fusion research has evolved into a new era in that the global plasma behaviors will be greatly affected by energetic particles. The kinetic-MHD models, such as the one presented in this paper, will eventually replace the MHD model for understanding the global plasma behaviors. Therefore, the nonvariational kinetic-MHD stability codes would become indispensable tools for studying the global plasma behaviors. In the NOVA-K code we have laid out the groundwork for implementing energetic particle physics into a fluid MHD code. The energetic particle physics included in the present version of the NOVA-K code can still be improved. Some important energetic particle physics that need to be included in the future are finite banana width, pressure anisotropy, finite particle density, and more realistic particle distribution functions generated from the Fokker-Planck code. On the other hand, the non-ideal MHD core plasma effects such as finite core ion FLR, electron and ion Landau resonances, and plasma resistivity [60], should also be considered for certain problems.

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TABLE 1

Comparison of the Eigenvalues γ^2 for Different Solovév Equilibria from Various Ideal MHD Stability Codes

ϵ_0	E	Σ	q(0)	q(a)	n	Kerner	PEST-1	ERATO	Degtyarev	NOVA
1/6	1	2	1.791	2.0	1	0.202	0.204		0.211	0.208
1/6	1	2	2.2387	2.5	1	0.504	0.506		0.511	0.508
1/3	2	1	0.3	0.522	2	0.413	0.427	0.431	0.430	0.430
1/3	2	1	0.7	1.219	2	0.118	0.119	0.120	0.121	0.119
1/3	2	∞	1.2	2.0897	1		0.75	0.78		0.748
1/3	2	∞	2.0	3.4829	1		0.68	0.75		0.656
1/3	2	∞	0.6	1.0449	2		1.31	1.40	1.32	1.35
1/3	2	∞	1.0	1.7415	2		1.03	1.07	1.06	1.038

Figure Captions

- Fig. 1 Typical cubic B-spline finite element for a uniform grid.
- Fig. 2 Cubic B-spline finite elements for uniform and nonuniform partitions.
- Fig. 3 Comparisons of convergence results in both the radial and the poloidal directions for our nonvariational code and the PEST code. The Solovév equilibrium has the parameters $R = B_0 = 1$, $E = 2$, $\epsilon_0 = 1/3$, $q(0) = 0.3$, $\Sigma = 1$, and $n = 2$. The eigenvalue γ^2 is extrapolated numerically in both the number of poloidal harmonics and the number of radial finite elements.
- Fig. 4 (a) The poloidal harmonics of the eigenfunction ξ_ψ versus r , (b) the projection of the displacement vector onto the $\phi = 0$ plane for the converged solution as shown in Fig. 3. The q -profile is also shown in Fig. (3a).
- Fig. 5 (a) The growth rates γ/γ_B computed from the analytical dispersion expression, Eq. (6.19), versus $\hat{\beta}_h$ for several values of $\bar{\omega}_D = \langle \omega_d \rangle / \gamma_B$. The curve labelled with $\omega_D \geq 3$ represents the ideal branch. (b) The corresponding real frequencies (ω_r/γ_B) versus $\hat{\beta}_h$.
- Fig. 6 The flux surfaces of a circular tokamak equilibrium "PDX05" with average beta $\beta = 0.625\%$, $q(0) = 0.8$, $q(1) = 2.85$, $R/a = 3.4$, $R \approx 1.43$.
- Fig. 7 (a) The radial variations of the trapped particle pitch angle Λ space bounded by $h_{\max}$ and $h_{\min}$ for the "PDX05" equilibrium. The bounce average magnetic drift frequency $\langle \omega_d \rangle = 0$ curve is also shown, and above the $\langle \omega_d \rangle = 0$ curve, the hot ion $\langle \omega_d \rangle$ is negative for the $n = 1$ mode. $\Lambda = h_{\max}$ corresponds to deeply trapped particles and $\Lambda = h_{\min}$ corresponds to barely trapped particles. (b) The hot ion diamagnetic drift frequency ω_{*h} versus r for the $m = 1$ poloidal harmonic. (c) The hot ion bounce-averaged magnetic drift frequency $\langle \omega_d \rangle$ versus r for $\Lambda = 1.1$, $n = 1$.

- Fig. 8 The poloidal components of the converged $n = 1$ fixed boundary internal kink mode eigenfunction ξ_ψ versus r . It is computed with an equal arc length θ -coordinate for the "PDX05" equilibrium described in Fig. 5. The eigenvalue is $\gamma = 0.01195$ and the q -profile is also shown.
- Fig. 9 (a) The growth rates, (γ/ω_A) , of both the ideal branch (I.B.) and the resonant fishbone branch (R.B.) versus α_h for the equilibrium described in Fig. 5. The total β is fixed and the hot particle slowing-down distribution is chosen with $\Lambda_0 = 1.1$, $\mathcal{E}_b/T_c = 10$, $R/\rho_h(0) = 100$, $M_h/M_c = 1$, $C_T = 1$, and thus the volume-averaged $\beta_h = 1.38\alpha_h\beta$ with $\beta = 0.625\%$. The broken curves represent the extrapolated growth rates near marginal stability. (b) The corresponding negative real frequencies, $-(\omega_r/\omega_A)$, versus α_h .
- Fig. 10 (a) The growth rates, (γ/ω_A) , versus α_h for both the ideal and the resonant branches for several values of C_T which scales $\langle\omega_d\rangle$ and ω_* . The fixed parameters are the same as in Fig. 8. (b) The corresponding negative real frequencies, $-(\omega_r/\omega_A)$, versus α_h .
- Fig. 11 The growth rates, (γ/ω_A) , and negative real frequencies, $-(\omega_r/\omega_A)$, of both the ideal and the resonant branches versus α_h for the "PDX05" equilibrium and fixed hot particle parameters as in Fig. 8. The computations are performed with analytical approximations when computing $\delta\hat{p}_\parallel$ and $\delta\hat{p}_\perp$ defined by Eq. (3.16).
- Fig. 12 The schematic stability window in the β - β_h space, which exists even in the absence of the core ion finite Larmor radius effect (finite ω_{*i}), due to the NBI or ICRH-heated energetic trapped particles on the $n=1$ internal kink mode.
- Fig. 13 (a) The flux surfaces, (b) the pressure profile, and (c) the q -profile of a CIT-type equilibrium "ECIT02" which has a noncircular plasma surface, defined by Eqs. (6.22) and (6.23), with the ellipticity $\kappa = 2$, the triangularity is $\delta = 0.4$, the horizontal minor radius is $a = 0.65$, and the major radius is $R = 2.1$. The total pressure and q -profiles are given by Eqs. (6.20) and (6.21) with $P_0 = 1$, $\lambda = 1.05$, $\mu = 2$, $\langle\beta\rangle_{av} = 4.0159\%$, $q(0) = 0.8$, $q(1) = 3.2$, $q'(0) = 0.25$, and $q'(1) = 7$, and $\langle\beta_{pol}\rangle = 2 \langle P \rangle / \langle B_{pol}^2 \rangle = 1.0376$.

- Fig. 14 (a) The radial dependence of the trapped particle pitch-angle space bounded by $h_{\min}$ and $h_{\max}$, and (b)-(f) the pitch angle Λ dependence of the bounce-averaged magnetic drift frequency $\langle\omega_d\rangle$ at $r = 0.2, 0.4, 0.6, 0.8$, and 1 , respectively. Also plotted in Fig. 14(a) is the curve of $\langle\omega_d\rangle = 0$.
- Fig. 15 The poloidal components of the $n=1$ internal kink mode eigenfunction ξ_ψ versus r for the "ECIT02" equilibrium with $\beta_\alpha = 0.5\%$ ($\alpha_h=0.2$).
- Fig. 16 The α_h dependences of the growth rates, γ/ω_A , and the negative real frequency, $-(\omega_r/\omega_A)$, of the ideal $n=1$ internal kink for the "ECIT02" equilibrium.
- Fig. 17 The continuous spectrum for $n=1$ as a function of the radius for a equilibrium "EC42.3D" with a circular plasma surface. The pressure and q - profiles of the "EC42.3D" equilibrium are defined by Eqs. (6.20) and (6.21) with the parameters: $P_0=0.45$, $\lambda=2$, $\mu=2$, $q(0)=1.05$, $q(1)=2.3$, $q'(0)=0.75$, and $q'(1)=3$, $\langle\beta\rangle_{av}=1.893\%$, $\alpha=0.25$, $R=1$.
- Fig.18 The poloidal harmonics of eigenfunction ξ_ψ of the $n=1$ fixed boundary toroidicity-induced Alfvén eigenmode (TAE) versus r for the "EC42.3D" equilibrium.
- Fig. 19 The pitch angle dependences of (a) the transit frequency ω_t , (b) transit-averaged $\langle\omega_d\rangle$, (c) bounce frequency ω_b , and (d) bounce-averaged $\langle\omega_d\rangle$ evaluated at the alpha particle birth energy ε_α for the "EC42.3D" equilibrium at the radial locations $r = 0.2$.
- Fig. 20 The pitch angle dependences of (a) the transit frequency ω_t , (b) transit-averaged $\langle\omega_d\rangle$, (c) bounce frequency ω_b , and (d) bounce-averaged $\langle\omega_d\rangle$ evaluated at the alpha particle birth energy ε_α for the "EC42.3D" equilibrium at the radial locations $r = 0.4$.
- Fig 21 The pitch angle dependences of (a) the transit frequency ω_t , (b) transit-averaged $\langle\omega_d\rangle$, (c) bounce frequency ω_b , and (d) bounce-averaged $\langle\omega_d\rangle$ evaluated at the alpha particle birth energy ε_α for the "EC42.3D" equilibrium at the radial locations $r = 0.6$.

- Fig. 22 The pitch angle dependences of (a) the transit frequency ω_t , (b) transit-averaged $\langle\omega_d\rangle$, (c) bounce frequency ω_b , and (d) bounce-averaged $\langle\omega_d\rangle$ evaluated at the alpha particle birth energy ε_α for the "EC42.3D" equilibrium at the radial locations $r = 0.8$.
- Fig. 23 The pitch angle dependences of (a) the transit frequency ω_t , (b) transit-averaged $\langle\omega_d\rangle$, (c) bounce frequency ω_b , and (d) bounce-averaged $\langle\omega_d\rangle$ evaluated at the alpha particle birth energy ε_α for the "EC42.3D" equilibrium at the radial locations $r = 1$.
- Fig. 24 Growth rate γ for the $n=1$ toroidicity-induced Alfvén eigenmode (TAE) as a function of the alpha particle's diamagnetic drift frequency $\omega_{* \alpha}$ (normalized to the shear Alfvén frequency ω_A) for the "EC42.3D" equilibrium.

#85T0198

Typical B-Spline

FIG. 1

Fig. 2

Fig. 3

Fig. 4(a)

Fig. 4(b)

FIG. 5

Fig. 6

FIG. 7

#88T0138

Fig. 3

Fig. 9

Fig. 10

#88T0136

Fig. 11

#89X0072

Fig. 12

88T0131

Fig. 13

Fig. 14

Fig. 15

Fig. 16

#89X0070

Fig. 17

Fig. 18

Fig. 19

Fig. 20

#89X0065

Fig. 21

#89X0064

Fig. 22

#89X0063

Fig. 23

88T1075

Fig. 24

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