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ABSTRACT

A continuous random walk procedure for solving some elliptic partial differential equations at a single point is generalized to estimate the solution everywhere. The Monte Carlo method described here is exact (except at the boundary) in the sense that the only error is the statistical sampling error that tends to zero as the sample size increases. A method to estimate the error introduced at the boundary is provided so that the boundary error can always be made less than the statistical error.

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Introduction:

A continuous random walk procedure for solving some elliptic partial differential equations at a single point is generalized to estimate the solution everywhere. The Monte Carlo method described here is exact (except at the boundary) in the sense that the only error is the statistical sampling error that tends to zero as the sample size increases. A method to estimate the error introduced at the boundary is provided so that the boundary error can always be made less than the statistical error.

Monte Carlo methods have been suggested [1,2] for solving elliptic partial differential equations at a single point. The theory described in Ref. 1 can be generalized to estimate the solution everywhere.

I shall show how to solve some partial differential equations of the form:

$$u_{xx} + u_{yy} - \alpha^2 u = 0, \ \alpha^2 \text{ constant.} \tag{1}$$

This is an important class of partial differential equations because any elliptic partial differential equation with constant coefficients can be reduced, by suitable transformations,[3] to the canonical form of Eq. 1.

Dirichlet Problem in Polar Coordinates

We shall consider solving Eq. 1 with u specified on the boundary. A separation of variables in polar coordinates yields:[1]

$$u(r,\theta) = \frac{a_0}{2} I_0(\alpha r) + \sum_{n=1}^{\infty} I_n(\alpha r)(a_n \cos n\theta + b_n \sin n\theta) \tag{2}$$

where the I_n are the modified Bessel functions. The expansion coefficients are obtained in the usual manner by integrating with cos nθ and sin nθ to give:

$$a_n = \frac{2}{I_n(\alpha r)} \frac{1}{2\pi} \int_{-\pi}^{\pi} u(r,\theta) \cos n\theta \, d\theta \ , \tag{3}$$

$$b_n = \frac{2}{I_n(\alpha r)} \frac{1}{2\pi} \int_{-\pi}^{\pi} u(r,\theta) \sin n\theta \, d\theta \ . \tag{4}$$

However, these coefficients may be estimated by sampling θ_i uniformly on $[0, 2\pi]$ and using:

$$\hat{a}_n = \frac{2}{I_n(\alpha r)} \frac{1}{M} \sum_{i=1}^{M} u(r,\theta_i) \cos n\theta_i \tag{5}$$

$$\hat{b}_n = \frac{2}{I_n(\alpha r)} \frac{1}{M} \sum_{i=1}^{M} u(r,\theta_i) \sin n\theta_i \quad . \tag{6}$$

Now if we wish to solve Eq. 1 in the vicinity of a point P_0 (see Fig. 1) we draw the largest circle (with center at P_0) that lies entirely within D and proceed to sample

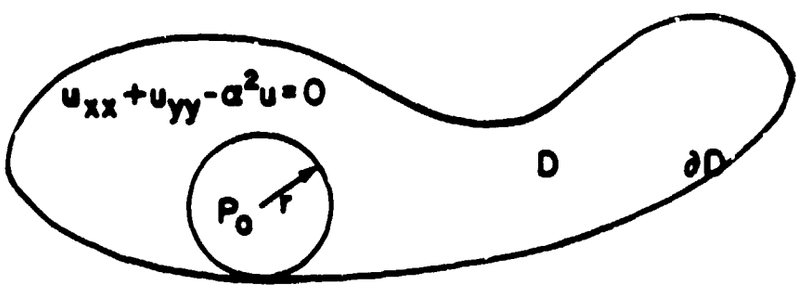

Fig. 1

the θ_i of Eqs. 5 and 6. We do not know $u(r,\theta_i)$ and so we use the single point theory[1] to get a one particle estimate $\hat{u}(r,\theta_i)$ and then use:

$$\hat{\hat{a}}_n = \frac{2}{I_n(\alpha r)} \frac{1}{M} \sum_{i=1}^{M} \hat{u}(r,\theta_i) \cos n\theta_i$$

$$\hat{\hat{b}}_n = \frac{2}{I_n(\alpha r)} \frac{1}{M} \sum_{i=1}^{M} \hat{u}(r,\theta_i) \sin n\theta_i \quad .$$

Thus, the solution at any point inside D may be estimated by:

$$\hat{u}(\rho,\phi) = \frac{\hat{\hat{a}}_0}{2} I_0(\alpha\rho) + \sum_{n=1}^{N} I_n(\alpha\rho)(\hat{\hat{a}}_n \cos(n\phi) + \hat{\hat{b}}_n \sin(n\phi)) \quad . \tag{9}$$

Example

Consider solving Eq. 1 in a rectangle of dimensions $\pi/2$ by 1 with u on the boundary equal to sin x. Using superpoition we may use an iterative approach to solve for the difference (or residual) between the true solution and the estimated solution. The approximate solution was taken to be:

$$\hat{u}(r,\theta) = \frac{\hat{a}_0}{2} J_0(r) + \sum_{n=1}^{8} J_n(r) (\hat{a}_n \cos n\theta + \hat{b}_n \sin n\theta) \quad (10)$$

where the coefficients were taken to be zero until the estimate was at least three times the standard deviation of the estimate. Furthermore, the sample size on each iteration was doubled if the estimated error was more than .7 times the estimated error on the previous iteration, otherwise the sample size was taken to be the same size as on the previous iteration. Table I shows how the error in $\hat{u}(.2\pi + .4, .5)$ decreases with the number of iterations.

Estimation of the Boundary Error

To estimate the boundary error we occasionally (say on every tenth sample) require the random walk to get within some smaller ε, say $\varepsilon/10$, of the boundary. We then score the difference in the estimates of the coefficients between applying the ε or $\varepsilon/10$ rule for termination to the same random walk. Thus we can estimate how different our answer would have been if the random walks had terminated within $\varepsilon/10$ of the boundary rather than within ε.

Application to Other Coordinate Systems

There is nothing special about the polar expansion coefficients. The expansion coefficiients can be generated for any coordinate system that Eq. 1 is separable in because the expansion coefficients always involve an integral that may be estimated by Monte Carlo.

TABLE I

Iteration	Number Samples	Total Samples	$\hat{u}(x_1, y_1)$	$\hat{\sigma}_u$ (error)	Last Non-Zero COEFF	$u(x_1, y_1) - \hat{u}(x_1, y_1)$
1	100	100	.80428	8.1E-2	--	5.2E-2
2	200	300	.87097	1.0E-2	A(1)	-1.5E-2
3	200	500	.85455	2.6E-3	A(2)	1.9E-3
4	200	700	.85640	4.6E-4	A(3)	3.3E-5
5	200	900	.85565	1.4E-4	A(3)	7.9E-4
6	200	1100	.85633	1.8E-4	A(3)	1.0E-4
7	400	1500	.85645	9.7E-5	A(3)	-2.0E-5
8	400	1900	.85651	7.8E-5	A(3)	-7.7E-5
9	800	2700	.85642	1.1E-5	A(4)	8.5E-6
10	800	3500	.85643	4.8E-6	A(4)	3.9E-6
11	800	4300	.85643	8.5E-7	A(5)	1.9E-7
12	800	5100	.85643	4.1E-7	A(5)	-2.4E-7
13	800	5900	.85643	5.6E-7	A(5)	-1.1E-6
14	1600	7500	.85643	3.5E-7	A(5)	1.6E-6
15	1600	9100	.85643	2.1E-7	A(5)	8.5E-7
16	1600	10,700	.85643	1.3E-7	A(6)	2.7E-7
17	1600	12,300	.85643	9.9E-8	A(6)	1.8E-8
18	3200	15,500	.85643	6.4E-8	A(6)	-2.2E-7
19	3200	18,700	.85643	7.9E-8	A(6)	2.2E-7
20	6400	25,100	.85643	1.7E-8	A(6)	-2.0E-8
21	6400	31,500	.85643	1.9E-8	A(6)	-2.1E-8
22	12,800	44,300	.85643	1.5E-8	A(6)	-5.4E-8
23	25,600	69,900	.85643	6.7E-9	A(6)	7.2E-9

TABLE I (cont)

Iteration	Number Samples	Total Samples	$\hat{u}(x_1, y_1)$	$\hat{\sigma}_u$ (error)	Last Non-Zero COEFF	$u(x_1, y_1) - \hat{u}(x_1, y_1)$
24	25,600	95,500	.85643	5.9E-9	A(7)	-4.0E-9
25	51,200	146,700	.85643	5.0E-10	A(7)	8.6E-10
26	51,200	197,900	.85643	2.8E-10	A(7)	-5.7E-10
27	51,200	249,100	.85643	1.2E-10	A(8)	1.8E-10
28	51,200	300,300	.85643	5.8E-11	A(8)	2.4E-11
29	51,200	351,500	.85643	2.1E-11	A(8)	5.9E-12

Note: mean time per sample is independent of the iteration index.

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3. P. R. Garabedian, "Partial Differential Equations," Wiley, New York, 1964.