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Applications of the Operational Method

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Abstract

New equations for continuous, operational interpolation on four points in rectangular array are derived. The equations can be applied to the interpretation of experimental data. Other topics include convergence of operational formulas and extrapolation by operational methods.

Introduction

Operational formulas for interpolating experimental data were introduced in 1985 [1]. The introductory paper contained formulas for data in unusual designs, such as the periphery of a rectangle. A larger collection of two-dimensional formulas was subsequently prepared [2]. These formulas can be developed in at least three ways:

1. Operational interpretation of a two-variable trigonometric identity containing only sines and cosines and homogeneous in sines.

Multiply the identity by some power of a function of two variables, $F^n(x,y)$. The power of $F^n(x,y)$ is such that every sine and cosine in the identity can be associated with one $F(x,y)$. That is, the identity is saturated with $F(x,y)$ terms. Every sine and cosine term acts as an operator on the adjacent function according to the rules described in References [1] and [2]. This procedure can be applied to identities in any number of variables.

Application of this technique to a variety of identities yields a variety of operational formulas. A particular design can be selected from the variety, or generated by combining selected equations. Multiplying a trigonometric identity to saturation has been the most common method used to generate formulas. The formulas developed in this manner are exact on linear data and certain transcendental functions applied to the linear data. These functions include cosine, simple exponentials, and compound exponentials such as $P^x + 1/P^x$, where P is some positive number. The hyperbolic cosine is an example of a compound exponential.

2. Apply the method of limits to operational formulas developed as in the first method.

The original operational formulas can be transformed into new interpolating formulas by the method of limits. A constant K is added to every term in the original operational equation. The equation is then expanded. Only terms containing K are retained, for only these terms keep their significance as the value of K is increased without limit. (Hence the name, the method of limits.) The constant is then removed by division, if that is possible. The geometric pattern represented by the new equation may or may not be the same pattern represented by the original equation. The formulas developed by the method of limits are usually exact on linear data in rectangular array as well as on the squares of the linear data.

3. A trigonometric identity is multiplied by some power of F(x,y) that is too low to allow each sine and cosine term to be associated with F(x,y). The identity is then expanded in the usual manner, momentarily ignoring the unassociated sine and cosine terms. The unassociated sines and cosines now act as operators on values to which they are adjacent, an operation that can sometimes be performed in more than one way. The sines and cosines chosen to remain unattached to F(x,y) are a matter of discretion, so several operational formulas can often be developed from one multiplication. F(x,y) can be raised to any power less than that required to saturate the identity, so more than one operational equation can be developed in this manner, too.

Configurations

An example of an equation developed by the third technique is Equation 46 in Reference [3]. A similar approach can be applied to

$$\sin^2(x)\cos^2(y) - \cos^2(x)\sin^2(y) = \sin^2(x) - \sin^2(y) \quad (1)$$

Multiply the identity by $F^3(x,y)$, sparing one $\sin(x)$ and one $\cos(x)$ on the left-hand side of the equation. The resulting operational equation

$$E = \frac{[(I + C - G - A)(H + B)(F - D) - (I - C + G - A)(H - B)(F + D)]}{4[(F - D)^2 - (H - B)^2]} \quad (2)$$

can be combined with a standard formula

$$E = [2(B + D + F + H) - (I + C + G + A)] / 4 \quad (3)$$

to yield a new formula that applies to points on the periphery of a rectangle

$$(I + A + C + G)[(H - B)^2 - (F - D)^2] + 2(I - A)(HD - BF) + 2(G - C)(HF - BD) - 2(B + D + F + H)(B + D - F - H)(B - D + F - H) = 0 \quad (4)$$

Operational formulas can be characterized as "algebraic" if they are exact on linear data and the squares of the linear data. Equation 4 is an example of the "algebraic" class. They contrast to the formulas that are exact on exponentials, characterized as "exponential" formulas. It appears to be true that the method of limits yields algebraic formulas from exponential formulas. The operational formulas derived by undersaturating a trigonometric identity with $F(x,y)$ also seem to belong to the algebraic class, although whether this is generally true is not known.

Another equation for the points on the periphery of a rectangle is

$$I(B - D + F - H) = A(B - D + F - H) + (G - C)(B + D - F - H) \quad (5)$$

How many equations in the algebraic category exist for this configuration? How many in the exponential category apply to the same configuration? The number of exponential equations could be equal to the number of trigonometric identities, homogeneous in sine terms, that also contain the same number of sines and cosines in every multiplicative group. When multiplied by $F^n(x,y)$, such identities do not leave any unassociated $F(x,y)$ terms.

G   H   I
D   E   F
A   B   C
Nine-Point
Rectangle

With reference to the nine-point rectangle shown on the left-hand side of the page, $F(x,y)$ represents center point E. If every $F(x,y)$ introduced into a trigonometric identity is operated on by sine or cosine, none remain to represent the center point E. Some of the identities are probably redundant, so the proposed answer may only represent an upper limit to the number of operational, exponential formulas that apply to the periphery of a rectangle.

In the operational method, sines and cosines translate into pairs of points. Is there any configuration of points, symmetric about center point E, that cannot be preceded by a trigonometric identity?

Continuous Interpolation

Two operational algorithms have been developed for the continuous interpolation into rectangles defined by data at their vertices. The algebraic algorithm can be executed by an iterative procedure, as shown in Table 2 of Reference [4].

The interpolation can also be accomplished by evaluating the responses at points B, D, E, F, and H and then applying the appropriate equations to fit a surface to all nine points [5] - Table 3 of Reference [4]. This process can be summarized by Equation 6a:

$$F(x,y) = T1(x^2) - T2(x) + T3(y) - T4(y) + T5(y^2) + A \quad (6a)$$

$$T1 = [A^2 - 2AC + C^2 - G^2 + I(2G - I)]/[2(A + C - G - I)] \quad (6b)$$

$$T2 = [3A^2 - 2A(C + G + I) - C^2 + 2C(G + I) - G^2 + I(2G - I)]/[2(A + C - G - I)] \quad (6c)$$

$$T3 = A - C - G + I \quad (6d)$$

$$T4 = [3A^2 - 2A(C + G + I) - C^2 + 2C(G + I) - (G - I)^2] / [2(A - C + G - I)] \quad (6e)$$

$$T5 = [A^2 - 2AG - C^2 + 2CI + (G - I)(G + I)] / [2(A - C + G - I)] \quad (6f)$$

In Equation 6a, the coordinates of points A, E, and I are (0,0), (0.5,0.5), and (1,1), respectively. Other points have coordinates consistent with this system.

An equation for the exponential case of bivariate, rectangular interpolation, as illustrated in Reference [1] and Table 4a in Reference [6], can be developed using Prony's method [7]. Prony's method of interpolation uses exponentials to represent equidistant data lying on a curve. It uses one of two forms for this representation: the sum of two exponentials, or the sum of a sine and a cosine. The exponential version takes the form

$$F(x) = (J)(P^x) + (K)(Q^x) \quad (7)$$

The trigonometric version takes the form

$$F(x) = R^x [S \cos(Tx) + U \sin(Tx)] \quad (8)$$

The appropriate version is determined by the computations. Prony's method reproduces the values predicted by the operational, exponential method of bivariate interpolation. That is, values operationally interpolated along any line in rectangle ACIG can be reproduced by Prony's method applied to the same line. The lines must be straight lines in the rectangle; typical examples are sides ABC and ADG, or diagonal AEI.

To apply Prony's method to rectangle ACIG, take four equidistant, colinear points. The values at these points are determined by the iterative, operational algorithm given in Table 4a in Reference [6]. If the values lie on diagonal AEI, for example, the coordinates of the required points are (0,0), (1/3,1/3), (2/3,2/3), and (1,1). If the values at these four points determine

a double exponential, the interpolating form will be that of Equation 7 and the constants will be supplied by Prony's technique. Only values along AEI can be interpolated, but these will agree with the operational, exponential method for that line. To interpolate data along another line, use equidistant data along the new line.

Prony's method seems never to have been extended into two dimensions. Preoccupation with the techniques of univariate interpolation is part of the reason for this deficiency. The operational method supplies adequate excuse for reexamination of Prony's old technique and extension of it into two dimensions.

Until a method is developed that directly yields the interpolating equation for a rectangle filled with exponential data, the following procedure may suffice. Consider rectangle sides ADG and CFI. If these sides can be expressed as a double exponential, then the rectangle can be interpolated by Equation 9

$$F(x,y) = JR^xP^{3y} + (K/R^x)Q^{3y} \quad (9)$$

The letters in Equation 9 follow Equation 7, while R is the ratio of the J value for side CFI to the J value for side ADG. Both J and K in Equation 9 are taken from side ADG. Some examples will make this formula clear.

Example 1. A table of perfect squares is constructed with A = 1, C = 9, G = 49, and I = 81. Sides ADG and CFI are represented by Equations 10 and 11, respectively. (The data for Prony's method were obtained from the operational exponential method, Table 4a in Reference [6].)

$$ADG = (8.269)(1.824^Y) - (7.269)(0.5482^Y) \quad (10)$$

$$CFI = (13.46)(1.824^Y) - (4.464)(0.5482^Y) \quad (11)$$

For Equations 10 and 11, J = 8.269, K = -7.269, P = 1.824, Q = 0.5482, and R = 1.628. These values are substituted into Equation 9 and the resulting

equation is found to reproduce the values predicted by the operational, exponential method. Center point E is a good place to begin the check.

This method is very sensitive to errors, so double precision should be used in all of the calculations. Nevertheless, the accuracy of the method suffices for most laboratory purposes. The example uses only four significant figures, not enough for most applications. The factor 3 in Equation 9 derives from the fact that the four equidistant data are numbered 0, 1, 2, and 3 in Prony's method, whereas the rectangle coordinates extend from zero to unity in the x- and y-directions.

Example 2. The rectangle in Example 1 is translated by 10 units. That is, 10 is added to the datum at each vertex. In this case $J = 13.41$, $K = -2.406$, $P = 1.644$, $Q = 0.6084$, and $R = 1.534$. Substituting these values into Equation 9 yields an interpolating equation that agrees with the operational, exponential method.

The essence of the demonstrated technique can also be applied to rectangle sides ABC and GHI. The proposed method has been tested on other rectangles, such as those represented by a table of a single exponential or a table of cubed numbers. However, it should always be verified in particular cases. An interpolating equation containing the sum of a sine and cosine is discussed in the Appendix.

Prony's method is a good way to determine whether the function underlying the rectangle behaves like a simple or compound exponential, or like the sum of a sine and a cosine. It is not known how Prony's method, used for this qualitative purpose, compares to the characteristic test [8].

Extrapolations

Like most numerical techniques, the operational method can be adapted to extrapolations. Operational extrapolations suffer the same deficiencies as most such techniques, but probably in no worse proportion. Because Equation 23

in Reference [1] (reproduced as Equation 16 below) fits more than one function [such as 2^x , $P\sin(x) + Q\cos(x)$, and $P\sinh(x) + Q\cosh(x)$], it might be as good as a low-order polynomial for extrapolations. Moreover, extrapolations with that equation are determined by three fixed points and one floating point. The floating point represents the underlying function, too, so its mobility may confer some advantage over methods that extrapolate from fixed points alone.

Derivatives at points beyond a specified three-point curve can be estimated, as shown by Equations 24 and 25 in Reference [4]. These derivatives are directly available; the extrapolated points do not require evaluation.

Let Equation 12 in Reference [8] be multiplied by $\sin(2x)$ instead of $\sin(x)$ as in that reference. The result is

$$\sin(2x)/x = 2\sin(x)\cos(x)[36 + 24\cos(x)]/[56\sin(x) + 4\sin(x)\cos(x)] \quad (12a)$$

Inserting the Euler identities into Equation 12a, and treating $1/(ix)$ as the integration operator, yields

$$\int_A^E F(x)dx = h(D + B)[36C + 12(D + B)]/(28C + D + B) \quad (12b)$$

That is, the area approximated extends one unit beyond curve BCD in both directions.

The concept of integration is itself subject to extrapolation in the reverse direction, that is, from the range B to D, to the midpoint, C. For example, consider the approximation

$$1/x = 3[2\cos(x) + 3]/[\sin(x)(\cos(x) + 14)] \quad (13)$$

Equation 13 can be multiplied by $F(x)/F^2(x) = C/C^2 = 1/C$. This leads to the integral

$$\int F(x)dx = [12hC^2(D + B + 3C)]/[(D - B)(D + B + 28C)] \quad (14)$$

which represents the integrated function at midvalue C. If the data are the exponentials 2, 4, 8, the value of the integral is 5.77, a good approximation to the true value of 5.77. If, however, the data are the squares 9, 16, 25, then the integral is predicted to be 32.66, a poor approximation to the true value of 21.33.

The reason for the failure of Equation 14 can be found by substituting functions for B, C, and D, and then assessing the value of the integral as the distance parameter, h, diminishes to zero. Let the substituted function be P raised to the powers (x-h), (x), and (x+h), for B, C, and D, respectively. The integral takes the value $P^x/\ln(p)$. This is the correct value. However, if the substituted functions are the squares of (x-h), (x), and (x+h), the integral takes the value $x^3/2$. The denominator is the wrong number, 2, instead of the correct number, 3. Substitution of other functions for B, C, and D shows that only the exponential function behaves properly, so Equation 14 is of no value as a practical tool. Whether some improvement could be made by manipulation of an introduced constant is not known. Advantages deriving from the operational method are determined by application, and in this case there seems to be no advantage.

A curious idea, derived by extrapolating the concept of differentiation from integral values to fractional values, is one-halfth derivatives. Formulas for fractional derivatives derive from the operational method as naturally as integer derivatives. The idea was first mentioned in Reference [6], section 10. An operational formula for the one-halfth derivative, based on Equation 12a in Reference [6] is

$$[C']^{(1/2)} = (C)\text{SQRT}[(28C(D - B) + D^2 - B^2) / (h(36C^2 + 12C(D + B)))] \quad (15)$$

The notation on the left-hand side of Equation 15 represents the one-halfth derivative, not the square root of the first derivative. If the sequence of data is the squares of 3, 4, and 5, so that $B = 9$, $C = 16$, $D = 25$, the one-halfth derivative at mid-value C is predicted by Equation 15 to be about 11.2. This value approaches 11.3 as the distance between the points diminishes. The analytic form of the one-halfth derivative at midvalue C is the square root of x when the data are linear. When the squares of the linear data are evaluated, the one-halfth derivative takes the form of the square root of 2 times $x^{3/2}$. (The data represent various functions performed on x ; at midpoint C , $x = 4$.) Operational formulas for the one-halfth integral may also be possible.

The integral of EdA , where E and A are related as in Equation 16, has already been mentioned in Reference [6].

$$E^2(C^2 - DB) + AE(D^2 + B^2 - 2C^2) + A^2(C^2 - DB) - [C(D - B)]^2 = 0 \quad (16)$$

Let the fixed points B , C , and D have values 32, 8, and 2, respectively. Then floating value A varies as $2^{(1+x)}$, where x varies between 0 and 4. Floating value E varies as $2^{(5-x)}$, and x also varies between 0 and 4. The integral of EdA can thus be described as

$$\int_0^4 2^{(5-x)} d[2^{(1+x)}] - \ln(4) \int_0^4 2^{(5-x)} 2^x dx = 177.4 \quad (17)$$

To obtain an accurate result by this procedure, the three-point echo function given in Equation 16 must fit the three fixed points exactly, as well as all intermediate points. Equation 16 is exact on simple exponentials, but is only approximate on the function x^2 . For example, the integral in Equation 17 would be only approximate if the three fixed points B , C , and D were the squares of 3, 4, and 5, respectively. In this case, the echo function given as

Equation 26 in Reference [8] gives a more accurate result. That echo function is exact on linear data and the squares of linear data. The integral EdA can be used for extrapolations.

Curvilinear Data

The Euler expression for the cosine function is

$$2\cos(p) = e^{ip} + e^{-ip} \quad (18a)$$

This equation is not of direct numerical use, because one side is real while the other side is imaginary. Assume that the exponentials on the right-hand side of the equation can be represented by some number, x . This representation has an interesting operational interpretation.

Thus,

$$2\cos(p) = x + x^{-1} \quad (18b)$$

Equation 18a can be generalized by DeMoivre's theorem

$$2\cos(np) = x^n + x^{-n} \quad (19)$$

If Equation 18b is multiplied by $F(p)$, denoted by the letter C , Equation 20 results

$$D + B = (x + x^{-1})C \quad (20)$$

Equation 20 can be made more general by adding a constant translational factor, Q , to the data

$$D + B + 2Q = (x + x^{-1})(C + Q) \quad (21)$$

An expression derived from $(2i)\sin(p)$ is similar to Equation 21

$$D - B = (x - x^{-1})(C + Q) \quad (22)$$

Solving Equations 21 and 22 yields

$$(D + Q)/(C + Q) = x \quad (23)$$

Equation 23 yields an expression for Q

$$Q = (C^2 - DB)/(B - 2C + D) \quad (24a)$$

Then x is found as

$$x = (C - D)/(B - C) \quad (24b)$$

Let J and K represent two symmetrically opposed points on either side of curve midpoint, C. Substituting J and K for D and B, respectively, in Equations 21 and 22, solving these equations, and then generalizing as in Equation 19, yields J as a function of x, n, C, and Q

$$J = x^n(C + Q) - Q \quad (25)$$

Equation 25 applies to many practical cases. For example, let the data be the sequence of squares 9, 16, and 25, as B, C, and D, respectively. Then Q is 31/2 and x is 9/7. If $n = -0.5$, the point midway between B and C has a value of 12.28; whereas, if $n = 0.5$, the point midway between C and D has a value of 20.22. When x is negative, substitute $[\text{abs}(x)]^n \cos[(n)(\pi)]$ for x^n .

Convergence

An enormous number of new interpolating formulas can be obtained by the operational method and its extension through the method of limits. By

comparison to standard formulas, some of the operational formulas are harder to use. Their predictive qualities are not necessarily superior to the standard formulas, some do not apply in certain circumstances, and some of them are already known.

The convergence properties of the new formulas have not been examined. These properties can be divided, qualitatively, into two categories: the rate at which a formula approaches its limiting value, and the proximity of a formula to its limiting value on first application. The first of these categories is convergence, but the second is related to both convergence and plasticity.

Scientists do not usually have the luxury of repeating their measurements at ever decreasing distances from the target. Where the measurements are taken at a substantial distance from the target, the importance of plasticity may surpass the importance of a high rate of convergence from a terrible first approximation. The definition of plasticity was given in Reference [9], section 3f. Plasticity is the logarithm of the number of functions exactly described by an interpolating formula.

An illustration of plasticity is the comparison of the initial prediction of the arithmetic average to the initial prediction of an operational formula. Three pertinent formulas are given as Equations 26, 27, and 28.

$$E = (A + C + G + I)/4 \quad (26)$$

$$E^2 = (GI - AC)(CI - AG)/[(I - A)^2 - (C - G)^2] \quad (27)$$

$$E = [(C + G)(I - A)^2 - (I + A)(C - G)^2]/[2((I - A)^2 - (C - G)^2)] \quad (28)$$

Table 3 in Reference [1] compares the initial predictions of Equations 26 and 27. It can be surmised from this table that the initial estimate predicted by Equation 27 is often superior to that predicted by the average. This is a reflection of the higher plasticity of the operational formula.

The order of convergence of a numerical formula can be denoted by ALPHA. The order is estimated by the following equation*

$$\text{ALPHA} = \ln[e(h)/e(h/2)]/\ln(2) \quad (29)$$

The term $e(h)$ represents the error of the formula applied to data at a distance, h , from the center point, and $e(h/2)$ is the error at half of this distance. The error is the difference between the true value at the center point and the value predicted by the center point formula from measurements at any distance from the center point.

Equations 26 and 27 can be applied to a test function like Equation 30

$$F(x,y) = 1/4 + 3(x + 1)^5(y + 1)^6/4 \quad (30)$$

Equation 30 is used to determine the values at the corner points of a rectangle with vertices A(-1,-1), C(1,-1), G(-1,1), and I(1,1). The coordinates are then successively halved as center point E(0,0) is approached.

The initial prediction of the arithmetic average, Equation 26, is 384.25, whereas the operational formula yields an initial prediction of 0.25. The operational formula gives a much better initial guess for the true center point value, $E = 1.000$. When the distance parameter, h , has been diminished to 0.0625, the average predicts 1.075 for E , whereas the operational formula predicts 0.984. The average has moved toward the limiting value of 1.000 faster than the operational formula, but this is partly because the average had much farther to go.

*Equations 29 and 30 were contributed by a correspondent.

The initial prediction of Equation 28 is also 0.25, and it is also 0.984 when the distance parameter, h , has diminished to 0.0625. The average and both operational formulas approach an order of convergence given by $ALPHA = 2$. For the average, $ALPHA$ starts out near 4.6, whereas $ALPHA$ is initially about 0.36 and 0.39 for Equations 27 and 28, respectively.

The operational formulas fail if $A = C = G = I$. The average asserts that $E = A = C = G = I$, in this case, that the five points lie in a plane. If $A = C = G = I$, there is no method for determining the height of the curved surface above the plane. Hence, there is no method for determining E , since it lies on this surface. The average asserts the height above the plane is zero, an arbitrary prediction. The operational formulas signal the insolvability of the problem by diverging. The point is, a high initial rate of convergence is not the only desirable property of a center point estimator.

The order of convergence of the operational derivative formulas is also a matter of interest. This order can be estimated by $ALPHA$, or qualitatively by expanding the original, trigonometric angle estimators into a Taylor series. The formula that is least disturbed by the presence of higher order terms in the series makes the best estimator, generally speaking. For example, Equation 31 expands to $x - x^7/2100$ plus higher terms, whereas Equation 32 expands to $x + x^5/20$ plus higher terms. The operational derivative formula obtained from Equation 31 should, therefore, make the better estimator for most purposes.

$$x = [56\sin(x) + 4\sin(x)\cos(x)]/[36 + 24\cos(x)] \quad (31)$$

$$x = [\tan(x) + 2\sin(x)]/3 \quad (32)$$

Operational derivative estimators are not exact, but they are usually acceptable on functions such as $\sin(x)$. Because of their iterative nature, they can be used to approximate higher order derivatives on three-point curves. Suppose the three points are the sines of 10, 20, and 30 degrees. The combination of endpoint and center point estimators that is most accurate at

some higher derivative is, therefore, the best combination of estimators. The argument may not truly be this simple, but the test is simple, and this simplicity recommends it. By the proposed standard, the following two combinations appear to be the best that have been discovered so far.

1. The first set derives from Reference [10]. Center point derivative estimator C' is Equation 10 therein. The difference $D' - B'$ is Equation 9, and the sum $D' + B'$ is Equation 8 in the same location. At the 20th derivative of $\sin(x)$, the values are 0.17365, 0.34202, and 0.5, at points B, C, and D, respectively. These values are correct to five digits. They were computed from ten-digit data. If B, C, and D are taken as $\exp(1)$, $\exp(2)$, and $\exp(3)$, respectively, the tenth derivatives are 2.7184, 7.3894, and 20.086 at B, C, and D. These numbers are wrong, but only in the last digit.

2. The derivative sum $D' + B'$ is given by Equation 20 in Reference [4], whereas the difference $D' - B'$ is obtained from Equation 23 in the same location. The center point formula is Equation 6 in Reference [10]. At the 20th derivative of $\sin(x)$, the values at points B, C, and D are 0.17365, 0.34202, and 0.5, respectively. The tenth derivatives of e^x at $x = 1$, $x = 2$, and $x = 3$ for B, C, and D, respectively, are 2.7299, 7.4205, and 20.171 for the points in the same order.

Derivative formulas can be combined in various ways: different combinations have different properties. It appears that the first group might be superior to the second group, but this is not necessarily true. For example, if $B = 10$, $C = 100$, and $D = 1000$, the fifth derivatives at B, C, and D are 1585, 7316, and 72307 for the equations in the first group, respectively; whereas, they are 795.31, 7953.1, and 79531 for the equations in the second group. The correct values are 647.26, 6472.6, and 64726, respectively. As the data diverge, the approximations worsen, and the superiority of the equations in the first group is no longer apparent.

ALPHA, computed from Equation 29, can also be obtained for derivative formulas. Let the test data be $B = 9$, $C = 16$, and $D = 25$, the squares of the linear data, 3, 4, and 5. Tests on a few operational first derivative formulas produced $ALPHA = 2$. The value of ALPHA can be higher on other functions, for example, $\exp(3)$, $\exp(4)$, and $\exp(5)$.

Bivariate Interpolation

The operational method has produced two techniques for bivariate interpolation on four points in rectangular array [1, 11]. Other methods for bivariate interpolation can also be developed operationally. Consider the definitions of the sums of sines and cosines, taken in the sense described in the section on curvilinear data.

$$2\cos(m + n) + 2\cos(m - n) = JK + 1/JK + J/K + K/J \quad (33a)$$

$$2\cos(m + n) - 2\cos(m - n) = JK + 1/JK - J/K - K/J \quad (34a)$$

$$(2i)\sin(m + n) + (2i)\sin(m - n) = JK - 1/JK + J/K - K/J \quad (35a)$$

$$(2i)\sin(m + n) - (2i)\sin(m - n) = JK - 1/JK - J/K + K/J \quad (36a)$$

Multiplying Equations 33a to 36a by $F(m,n)$, denoted E, yields

$$(I + A + C + G)/E = \text{right-hand side of Equation 33a} \quad (33b)$$

$$(I + A - C - G)/E = \text{right-hand side of Equation 34a} \quad (34b)$$

$$(I - A + C - G)/E = \text{right-hand side of Equation 35a} \quad (35b)$$

$$(I - A - C + G)/E = \text{right-hand side of Equation 36a} \quad (36b)$$

The "b" series of equations contains the four numbers at rectangle vertices, A, C, I, and G. The four equations contain only two unknowns, J and K. This leaves two unknowns that can be positioned at pleasure in the equations. Call these unknowns Q and T, and let them be put on the left-hand sides of the "b" series as follows

$$[T(I + A + C + G) + 4Q]/(E + Q) = \text{right-hand side of Equation 33a} \quad (33c)$$

$$T(I + A - C - G)/(E + Q) = \text{right-hand side of Equation 34a} \quad (34c)$$

$$T(I - A + C - G)/(E + Q) = \text{right-hand side of Equation 35a} \quad (35c)$$

$$T(I - A - C + G)/(E + Q) = \text{right-hand side of Equation 36a} \quad (36c)$$

The parameter Q acts as a translating factor, while T acts as a multiplicative factor on the rectangle corner point values. The unknowns T and Q might have been positioned elsewhere in the equations, but the selected locations suffice for the illustration.

The equations can be solved for all of the unknowns. The results are

$$J = \text{SQRT}[(C - I)/(A - G)] \quad (37)$$

$$K = \text{SQRT}[(G - I)/(A - C)] \quad (38)$$

$$Q = E(CG - AI)/[(A - G)(A - C)JK + AI - CG] \quad (39)$$

$$T = E(A - C - G + I)/[(A - G)(A - C)JK + AI - CG] \quad (40)$$

Multiplying Equations 41a and 42a by $F(m,n)$, using the notation of the "c" series of equations, yields Equations 41b and 42b, respectively.

$$2\cos(m + n) = JK + 1/(JK) \quad (41a)$$

$$(2i)\sin(m + n) = JK - 1/(JK) \quad (42a)$$

$$[T(P + M) + 2Q]/(E + Q) = JK + 1/(JK) \quad (41b)$$

$$T(P - M)/(E + Q) = JK - 1/(JK) \quad (42b)$$

Equations 41b and 42b can be solved to obtain an expression for any arbitrary point, P, inside rectangle ACIG. The argument of sine and cosine in Equations 41a and 42a can be changed from $(m + n)$ to $(xm + yn)$. By Demoiivre's theorem, this changes the product JK into $J^x K^y$. The response at any arbitrary point, P, inside rectangle ACIG is then given by Equation 43

$$P = [J^x K^y (E + Q) - Q]/T \quad (43)$$

Q and T are available from Equations 39 and 40, respectively. When the substitutions are made, the equation estimating the response at an arbitrary point, P, is

$$P = [J^{(x+1)} K^{(y+1)} (A - G)(A - C) + AI - CG]/(A - C - G + I) \quad (44)$$

Equation 44 does not contain E, so it represents an interpolating technique for rectangles defined by numbers at the four corner points only. It is invariant under translation and rotation provided that division by zero is not indicated. (This should be tested by rotating the rectangle in 90-degree increments. The equation does not apply if division by zero arises in any configuration of the data obtained by rotation.) Equation 44 fails for linear data in rectangular array because its denominator becomes zero. However, it is asymptotically exact on linear data--at least for most laboratory purposes.

Equation 44 applies to rectangles in which center point E is at (0,0), corner point A is at (-1,-1), and corner point I is at (1,1). It is exact on tables of simple exponentials, like A = 2, C = 8, G = 128, and I = 512. It also applies if a constant is added to these data. Using only four points in rectangular array, it provides an estimate of curvature in the x- and y- directions.

If the equations for J and K cannot be evaluated because the arguments of the square roots are negative, an artifice can be used. Take the absolute value of the ratio $(C-I)/(A-G)$ in Equation 37 and find its square root. This root is used in Equation 44 as J. However, substitute $(-x)J^{(x+1)}$ for $J^{(x+1)}$. Similarly, if the right-hand side of Equation 38 contains the negative ratio $(G-I)/(A-C)$, take the square root of its absolute value as K. In Equation 44, substitute $(-y)K^{(y+1)}$ for $K^{(y+1)}$. If both ratios are negative, make both substitutions. The programming should be checked to verify that the substitutions reproduce the original data.

The roots of negative numbers can be interpreted using trigonometric functions. If the ratio in Equation 37 is negative, take the square root of its absolute value as J, and substitute $(-1)J^{(x+1)}\sin[(\pi)(x/2)]$ for $J^{(x+1)}$ in Equation 44. Likewise, if the ratio in Equation 38 is negative, take the square root of its absolute value for K, and substitute $(-1)K^{(y+1)}\sin[(\pi)(y/2)]$ for $K^{(y+1)}$. If both the ratios in Equations 37 and 38 are negative, take the square roots of their absolute values as J and K, respectively, and multiply $J^{(x+1)}K^{(y+1)}$ in Equation 44 by $(-1)\cos[(\pi)(x+y)/2]$. In the last case, the surface may not be unique, as can be ascertained by rotating the data through a right angle. By this method, the surface is interpolated using the real parts of the complex numbers obtained by extracting roots of negative quantities.

Other possibilities for the interpolation include substituting the negative SQRT in Equations 37 and 38 into Equation 44. The square root of the sum or difference of the squares of the real and imaginary parts of the complex numbers generated in Equation 44 can also be used to estimate P.

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Appendix – Bivariate Interpolation, Trigonometric Case

No satisfactory method is known for analytically representing data in rectangular array if Prony's method indicates the data are best represented as the sum of a sine and cosine as in Equation 8. This problem awaits the extension of Prony's method into two dimensions, or the application of some equivalent procedure. The following technique can be considered illustrative of the problem.

Determine, by Prony's method for four variables, the equations of rectangle sides ADG and CFI. If these equations are in the form of Equation 7, use the method outlined in the text. If these equations are in the form of Equation 8, denote the value of the argument of the sine and cosine functions as ARG. Then obtain the equations of two vertical lines parallel to ADG and CFI and equally spaced between them. These equations should also be in the form of Equation 8, and ARG should be present in them. These four equations contain four different coefficients of the cosine and four different coefficients of the sine.

Arrange the cosine coefficients into a group ordered as the lines from left to right. That is, the cosine coefficient for ADG is first, while the cosine coefficient for CFI is last. By Prony's method, find an expression for the group of four cosine coefficients, and similarly find an expression for the group of four sine coefficients.

Any point, P, in the rectangle can be represented by Equation (A-1).

$$\begin{aligned} F = & [\text{equation of cosine coefficients}][\cos((3y)(\text{ARG}))] \\ & + [\text{equation of sine coefficients}][\sin((3y)(\text{ARG}))] \end{aligned} \quad (\text{A-1})$$

The equations of the coefficients of sine and cosine are expressed in terms of x only. The value of x, taken as a rectangle coordinate in the system that places A at (0,0) and I at (1,1), must be multiplied by 3 before it is used in

the equations of the coefficients. Similarly, the coordinate y must be multiplied by 3 before it multiplies ARG. This factor of 3 is already shown in Equation (A-1). The following two examples will illuminate the method.

Example 1. Data are measured at rectangle vertices A, C, G, and I. The data are A = 2, C = 6, G = 9, and I = 10. The equations of the vertical rectangle sides are

$$ADG = (2.000)\cos(0.3061x) + (9.798)\sin(0.3061x) \quad (A-2)$$

$$CFI = (6.000)\cos(0.3061x) + (8.000)\sin(0.3061x) \quad (A-3)$$

Two lines parallel to ADG and CFI, and equally spaced between them, are computed. These four parallel lines all have the same value of ARG, while the cosine coefficients are 2, 3.417, 4.760, and 6. The sine coefficients of the four vertical lines are 9.798, 9.398, 8.795, and 8. The coefficients are in the same order as the lines from ADG to CFI. The equations of the cosine and sine coefficients are

$$\text{cosine coefficients} = (2.000)\cos(0.1474x) + (9.798)\sin(0.1474x) \quad (A-4)$$

$$\text{sine coefficients} = (9.798)\cos(0.1474x) - (2.006)\sin(0.1474x) \quad (A-5)$$

Equations (A-4) and (A-5) are substituted into Equation (A-1), each x being changed into 3x. Equation (A-1) is now suitable for application to the aforementioned rectangle.

Example 2. The data in Example 1 are remeasured. They agree except at C, where remeasurement changes the value from 6 to 5. The equations of the vertical sides are

$$ADG = (2.000)\cos(0.2986x) + (9.928)\sin(0.2986x) \quad (A-6)$$

$$CFI = (5.000)\cos(0.2986x) + (8.807)\sin(0.2986x) \quad (A-7)$$

The equations of two lines parallel to ADG and CFI, and equidistant between them, are obtained. The four equations, taken from left to right, have cosine coefficients of 2, 3.038, 4.041, and 5. The equation of cosine coefficients is

$$\text{cosine coefficients} = 2\cos(0.1059x) + (9.928)\sin(0.1059x) \quad (\text{A-8})$$

The sine coefficients are 9.928, 9.661, 9.286, and 8.807. The equation of the sine coefficients is

$$\text{sine coefficients} = (9.928)\cos(0.1059x) - (1.982)\sin(0.1059x) \quad (\text{A-9})$$

After multiplying x by 3, Equations (A-8) and (A-9) are substituted into Equation (A-1). The value of ARG is 0.2986. Equation (A-1) is now suitable for interpolating points inside rectangle ACIG, in which center point E has coordinates (0.5,0.5).

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