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SQUARE FUNCTION ANALYSIS OF THE INDUCTOR-CONVERTER BRIDGE

by

Mehrdad Ehsani and Robert L. Kustom

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FUSION POWER PROGRAM

Argonne National Laboratory
9700 South Cass Avenue
Argonne, Illinois 60439

Prepared for the
U. S. Department of Energy
under Contract W-31-109-Eng-38

Distribution Categories:
Magnetic Fusion Energy (UC-20)
Plasma Systems (UC-20a)
Magnetic Systems (UC-20b)
Fusion Systems (UC-20d)

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ARGONNE NATIONAL LABORATORY
9700 South Cass Avenue
Argonne, Illinois 60439

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March 1979

TABLE OF CONTENTS

	<u>Page</u>
Abstract	v
Introduction	1
The Two-Phase ICB	2
Verification of the Two-Phase Results	8
The Three-Phase ICB	11
Conclusions.	17
References	18
Appendix	26

LIST OF FIGURES

<u>No.</u>	<u>Title</u>	<u>Page</u>
1	Circuit Diagram for the 3-Capacitor Model IC Bridge . .	19
2	Circuit Diagram for the Two-Phase ICB	20
3	(a) $S_q (\gamma + \gamma_o)$, (b) $Tr (\gamma + \gamma_o)$	21
4	One-Phase Equivalent Circuit of ICB	22
5	(a) Phase Current in a Three-Phase ICB, (b) Decomposition of the Phase Current into Two S_q Functions	23
6	The Per Phase Equivalent Circuit for the Three- Phase ICB	24
7	The Average Power vs γ_o in Three-Phase ICB	25

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ABSTRACT

The transfer of energy between two superconducting coils can be controlled with an Inductor-Converter Bridge (ICB) by changing the relative timing between the storage side bridge and the load side bridge. The average voltages on the coils have been previously derived as a function of relative timing between the two halves of the bridges and the relative currents in the two coils using a one-line, harmonic current source representation of each coil and its bridge network. Since the coil current in each of the ICB phases appears as bipolar rectangular pulses, an analysis based on unit step functions has been developed.

The unit step function analysis leads to simple polynomial expressions relating the average coil voltages to relative timing. The new derivations are shown to have the same harmonic representation as previously developed. The polynomial expressions are more compatible to microprocessor control than are the harmonic function expressions.

Introduction

The Inductor-Converter Bridge (ICB) is a DC-AC-DC converter, with solid state switching elements, for reversible energy transfer from a storage inductor to a load inductor. A circuit diagram is shown in Fig. 1. Conduction of current is commutated between thyristors at fixed intervals following the normal inverter sequence of 15, 16, 26, 24, 34, and 35. The active controllability of the rate of energy transfer renders these circuits especially suitable for inductive loads requiring random power or current levels. One such application is in the magnetic field coils of the tokamak fusion devices.

The details of operation and the characteristics of the ICB circuits have been reported in previous publications.^{1,2} For the purposes of this report, it is sufficient to indicate that the direction and the level of power flow is determined by the relative timing between the source and the load bridge switching cycles.

An analytical method for deriving the average load power flow as a function of the relative bridge phase differences and currents in the coils has been presented in a similar report.³ This method was based on the Fourier components of the fundamental square waveforms of ICB circuits. The results were in the form

$$\langle p \rangle = \sum_{n=1}^{\infty} f_n(\alpha) , \quad (1)$$

where α is the time-phase difference between the two bridges and the functions $f_n(\alpha)$ represent the Fourier components. The open form of this result makes the solution of α in terms of $\langle p \rangle$ tedious and time consuming, particularly in a computer algorithm for real time control.

This report presents the results of power calculation using specially tailored functions that are based on step functions symbolized by $Sq(x)$ and $Tr(x)$ representation. These waveforms, which are inherent in the operation of ICB circuits, are used to formulate the average power equation in closed form. Therefore, the exact evaluation of $\langle p \rangle$ in terms of α , and a simple inverse solution of α in terms of $\langle p \rangle$, is made possible.

For verification, the new results are expanded in their Fourier components over the interval of interest to reproduce the previously reported results.

The Two-Phase ICB

To illustrate the use of the Sq and Tr functions, the average power flow will be calculated for a two-phase ICB. This circuit is the simplest ICB configuration and can be shown to be physically realizable. A circuit diagram is shown in Fig. 2. The graphical form of the Sq and Tr functions is shown in Fig. 3. The Sq function can be analytically reproduced by the sum of unit step functions. They form a good mathematical representation of the currents $i_1(t)$ and $i_2(t)$. The Tr functions are the integrated form of Sq and are good representations of the capacitor voltage. A complete mathematical development of the Sq and Tr functions is presented in the Appendix.

It has been shown³ that an equivalent circuit for each phase of ICB can be represented by the diagram of Fig. 4. Since a two-line ac system can be viewed either as two phase with respect to a reference or one phase line-to-line, we have chosen the latter interpretation for drawing the equivalent circuit. Therefore, the calculated power from the equivalent circuit will be the total power of Fig. 2 as opposed to half power, which corresponds to one of the two phases.

The two current sources shown in Fig. 4 can be analytically described using the Sq functions described in the Appendix as follows:

$$\begin{cases} i_1(t) = I_1 \sum_{n=-\infty}^{\infty} \text{Sq}(t - nT - \tau_\phi - \tau_o) = I_1 \sum_{n=-\infty}^{\infty} \text{Sq}(\gamma - n - \gamma_\phi - \gamma_o) & (2) \\ i_2(t) = I_2 \sum_{n=-\infty}^{\infty} \text{Sq}(t - nT - \tau_\phi) = I_2 \sum_{n=-\infty}^{\infty} \text{Sq}(\gamma - n - \gamma_\phi) , & (3) \end{cases}$$

$$-\infty < t < \infty, 0 < \tau < T, 0 < \gamma < 1, 0 < \tau_\phi + \tau_o < T/2, 0 < \gamma_\phi + \gamma_o < 1/2^*.$$

*Note that for the case of $T/2 < \tau_\phi + \tau_o < T$ or $1/2 < \gamma_\phi + \gamma_o < 1$, the identity for $[T/2 - (\tau_\phi + \tau_o)]$ can be used, and the calculation will proceed in a similar manner.

In the above, τ_ϕ and γ_ϕ represent, respectively, the common time or phase delay of both i_1 and i_2 sources with respect to the reference time, and τ_o and γ_o represent the relative time difference or phase delay between i_1 and i_2 .

The capacitor voltage is*

$$\begin{aligned}
 v_c(t) &= \frac{1}{C} \int_0^t (i_1 + i_2) \, d\tau = \frac{I_1}{C} \int_0^t \sum_{n=-\infty}^{\infty} \text{Sq} (\tau - nT - \tau_\phi - \tau_o) \, d\tau \\
 &+ \frac{I_2}{C} \int_0^t \sum_{n=-\infty}^{\infty} \text{Sq} (\tau - nT - \tau_\phi) \, d\tau \\
 &= \frac{I_1}{C} \sum_{n=-\infty}^{\infty} \int_0^t \text{Sq} (\tau - nT - \tau_\phi - \tau_o) \, d\tau \\
 &+ \frac{I_2}{C} \sum_{n=-\infty}^{\infty} \int_0^t \text{Sq} (\tau - nT - \tau_\phi) \, d\tau \\
 &= \frac{I_1}{C} \sum_{n=-\infty}^{\infty} \text{Tr} (t - nT - \tau_\phi - \tau_o) + \frac{I_2}{C} \sum_{n=-\infty}^{\infty} \text{Tr} (t - nT - \tau_\phi) . \quad (4)
 \end{aligned}$$

*Initial voltage on the capacitor has been assumed equal to zero. It can be easily shown that this voltage will not contribute to the average power.

The power delivered by the source i_1 is

$$p_1(t) = i_1 v_c = \frac{I_1^2}{C} \left[\sum_{m=-\infty}^{\infty} S_q(t - mT - \tau_\phi - \tau_o) \right] \cdot \left[\sum_{n=-\infty}^{\infty} Tr(t - nT - \tau_\phi - \tau_o) \right] \\ + \frac{I_1 I_2}{C} \left[\sum_{m=-\infty}^{\infty} S_q(t - mT - \tau_\phi - \tau_o) \right] \cdot \left[\sum_{n=-\infty}^{\infty} Tr(t - nT - \tau_\phi) \right]. \quad (5)$$

In the above, only the terms in the same location in time, same period number, will produce non-zero products. Therefore,

$$p_1(t) = \frac{I_1^2}{C} \sum_{n=-\infty}^{\infty} S_q(t - nT - \tau_\phi - \tau_o) \cdot Tr(t - nT - \tau_\phi - \tau_o) \\ + \frac{I_1 I_2}{C} \sum_{n=-\infty}^{\infty} S_q(t - nT - \tau_\phi - \tau_o) \cdot Tr(t - nT - \tau_\phi). \quad (6)$$

The general form of the average power delivered by i_1 is

$$\langle p_1 \rangle = \frac{1}{T} \int_{n_1 T + \tau_1}^{(n_1+1)T + \tau_1} p_1 dt$$

(Equation continued on following page)

$$\begin{aligned}
 &= \frac{I_1^2}{TC} \int_{n_1 T + \tau_1}^{(n_1+1)T + \tau_1} \sum_{n=-\infty}^{\infty} S_q(t - nT - \tau_\phi - \tau_0) \cdot Tr(t - nT - \tau_\phi - \tau_0) dt \\
 &+ \frac{I_1 I_2}{TC} \int_{n_1 T + \tau_1}^{(n_1+1)T + \tau_1} \sum_{n=-\infty}^{\infty} S_q(t - nT - \tau_\phi - \tau_0) \cdot Tr(t - nT - \tau_\phi) dt \quad (7)
 \end{aligned}$$

Since the above S_q and Tr functions are periodic with period T , the product $S_q Tr$ is also periodic with period T . Therefore, the above averaging integrals can be evaluated over any one period as follows:

$$\begin{aligned}
 \langle p_1 \rangle &= \frac{I_1^2}{TC} \int_{\tau_\phi}^{T + \tau_\phi} \sum_{n=-\infty}^{\infty} S_q(\tau - nT - \tau_\phi - \tau_0) \cdot Tr(\tau - nT - \tau_\phi - \tau_0) d\tau \\
 &+ \frac{I_1 I_2}{TC} \int_{\tau_\phi}^{T + \tau_\phi} \sum_{n=-\infty}^{\infty} S_q(\tau - nT - \tau_\phi - \tau_0) \cdot Tr(\tau - nT - \tau_\phi) d\tau \quad (8)
 \end{aligned}$$

To further simplify the integrals, we will use a translation of axis of the kind $\tau = \tau' + \tau_\phi$.

$$\langle p_1 \rangle = \frac{I_1^2}{TC} \int_0^T \sum_{n=-\infty}^{\infty} S_q(\tau' - nT - \tau_0) \cdot Tr(\tau' - nT - \tau_0) d\tau'$$

(Equation continued on following page)

$$\begin{aligned}
 & + \frac{I_1 I_2}{TC} \int_0^T \sum_{n=-\infty}^{\infty} S_q (\tau' - nT - \tau_0) \cdot \text{Tr} (\tau' - nT) d\tau' \\
 & = \frac{I_1^2}{TC} \int_0^T S_q (\tau - \tau_0) \cdot \text{Tr} (\tau - \tau_0) d\tau \\
 & + \frac{I_1 I_2}{TC} \int_0^T S_q (\tau - \tau_0) \cdot \text{Tr} (\tau) d\tau .
 \end{aligned} \tag{9}$$

We will now normalize the variable τ using the relationships A-14, A-15, and A-16 given in the Appendix:

$$\begin{aligned}
 \langle p_1 \rangle & = \frac{I_1^2}{TC} \int_0^1 S_q (\gamma - \gamma_0) \cdot (T) \text{Tr} (\gamma - \gamma_0) d(\gamma T) \\
 & + \frac{I_1 I_2}{TC} \int_0^1 S_q (\gamma - \gamma_0) \cdot (T) \text{Tr} (\gamma) d(\gamma T) ,
 \end{aligned}$$

where $\gamma = \frac{\tau}{T}$ and $\gamma_0 = \frac{\tau_0}{T}$.

(Equation continued on following page)

$$\begin{aligned}
 \langle p_1 \rangle &= \frac{\pi_1^2}{c} \int_0^1 \text{Sq} (\gamma - \gamma_0) \cdot \text{Tr} (\gamma - \gamma_0) d\gamma \\
 &+ \frac{\pi_1 I_2}{c} \int_0^1 \text{Sq} (\gamma - \gamma_0) \cdot \text{Tr} (\gamma) d\gamma .
 \end{aligned} \tag{10}$$

Using the identities of A-5 and A-6 of the Appendix, Eq. (10) can be transformed into

$$\begin{aligned}
 \langle p_1 \rangle &= \frac{\pi_1^2}{c} \int_0^1 \text{Sq} [\gamma + (1/2 - \gamma_0)] \cdot \text{Tr} [\gamma + (1/2 - \gamma_0)] d\gamma \\
 &+ \frac{\pi_1 I_2}{c} \int_0^1 -\text{Sq} [\gamma + (1/2 - \gamma_0)] \cdot \text{Tr} (\gamma) d\gamma .
 \end{aligned} \tag{11}$$

Now using the integral relations, Eqs. (A-7), (A-8), and (A-9) of the Appendix with $\gamma_1 = \gamma_2 = (1/2 - \gamma_0)$ for the first integral and $\gamma_1 = (1/2 - \gamma_0)$, $\gamma_2 = 0$ for the second integral, we will get,

$$\langle p_1 \rangle = 0 - \frac{\pi_1 I_2}{c} [2 (1/2 - \gamma_0)^2 - 1/2 + \gamma_0] \tag{12}$$

Note that there is no net power from the interaction of i_1 and the capacitor voltage due to i_1 —a fact physically obvious. Therefore, the net power delivered by i_1 is due to the interaction of i_1 and the capacitor voltage generated by i_2 only. The final result may be simplified and presented in several forms:

$$\begin{aligned} \langle p_1 \rangle &= \frac{\pi I_1 I_2}{C} (\gamma_0 - 2 \gamma_0^2) = \frac{\pi I_1 I_2}{C} \left(\frac{t_0}{T} - \frac{2 t_0^2}{T^2} \right) \\ &= \frac{I_1 I_2}{\pi C} (T t_0 - 2 t_0^2) = \frac{I_1 I_2}{\omega C} \left(\alpha - \frac{\alpha^2}{\pi} \right) \end{aligned} \quad (13)$$

In the above, γ_0 is the timing difference between i_1 and i_2 in fractions of the unity period; t_0 is the timing difference between i_1 and i_2 in seconds; α is the phase difference between i_1 and i_2 in radians where the period is 2π radians and ω is the angular frequency of i_1 and i_2 . The radian units are given for comparison with the results from Fourier analysis and have no real relation to the present analysis other than interpreting the square functions by trigonometric terminology.

Verification of the Two-Phase Results

The average power flow from one inductor to another using Fourier analysis³ is given by

$$\langle p_1 \rangle_T = \sum_{n=1}^{\infty} \frac{4 I_1 I_2}{n^2 \pi \omega C} [1 - (-1)^n] \sin n \alpha \quad (14)$$

where I_1 and I_2 is the instantaneous current in the source and load inductor, respectively; ω is the angular frequency of bridge operation, C is the line-to-line capacitance, and α is the phase angle in radians by which the load

bridge leads the source bridge. The same average power calculated by square function analysis and expressed in radians was seen to be

$$\langle p_1 \rangle_{sq} = \frac{I_1 I_2}{\omega C} \left(\alpha - \frac{\alpha^2}{\pi} \right) . \quad (15)$$

Comparison of the two forms shows that for the power to be identical, we must have

$$\sum_{n=1}^{\infty} \frac{4}{n^3 \pi^2} [1 - (-1)^n] \sin n \alpha = \alpha - \frac{\alpha^2}{\pi} . \quad (16)$$

Expanding the right-hand side of Eq. (16) in its Fourier sine series in the interval $(0, \pi)^*$ gives

$$\alpha - \frac{\alpha^2}{\pi} = \sum_{n=1}^{\infty} b_n \sin n \alpha . \quad (17)$$

The values of the constants b_n are

$$b_n = \frac{2}{\pi} \int_0^{\pi} \left(\alpha - \frac{\alpha^2}{\pi} \right) \sin n \alpha \, d\alpha ,$$

(Equation continued on following page)

*Since the absolute value of average power delivered by the source inductor is periodic in α with period π , the equivalence between the two expressions in the interval $(0, \pi)$ only, is of interest.

$$\begin{aligned}
 &= \frac{2}{\pi} \left[\frac{1}{n^2} \sin n\alpha - \frac{\alpha}{n} \cos n\alpha - \frac{1}{\pi} \left(\frac{2\alpha}{n^2} \sin n\alpha \right. \right. \\
 &\quad \left. \left. - \frac{n^2 \alpha^2 - 2}{n^3} \cos n\alpha \right) \right]_0^{\pi}, \\
 &= \frac{2}{\pi} \left[\frac{1}{n^2} \sin n\pi - \frac{\pi}{n} \cos n\pi - \frac{2}{n^2} \sin n\pi \right. \\
 &\quad \left. + \frac{n^2 \pi^2 - 2}{n^3 \pi} \cos n\pi + \frac{2}{n^3 \pi} \right], \\
 &= \frac{2}{\pi} \left[-\frac{\pi}{n} \cos n\pi + \frac{\pi}{n} \cos n\pi - \frac{2}{n^3 \pi} \cos n\pi + \frac{2}{n^3 \pi} \right] \\
 &= \frac{4}{n^3 \pi^2} (1 - \cos n\pi) = \frac{4}{n^3 \pi^2} [1 - (-1)^n]. \quad (18)
 \end{aligned}$$

Therefore,

$$\alpha - \frac{\alpha^2}{\pi} = \sum_{n=1}^{\infty} \frac{4}{n^3 \pi^2} [1 - (-1)^n] \sin n\alpha \quad (19)$$

which is the equality needed to prove the equivalence of Fourier and square function analyses.

The Three-Phase ICB

It has been shown³ that the average power flow in the three-phase ICB may be calculated from the one-phase equivalent circuit. The total power is three times that of one circuit. It is possible to show that a three-phase ICB is the superposition of two simple equivalent circuits of the kind used for the two-phase ICB.

Figure 5(a) shows a typical waveform for a current in one of the 3- ϕ ICB phase equivalent circuits. Figure 5(b) shows the decomposition of this waveform into two Sq functions of appropriate phase difference. From this illustration, it is evident that the source and load current waveforms, respectively, can be shown as follows,

$$\left\{ \begin{aligned} i_1(t) &= \frac{I_1}{2} \sum_{n=-\infty}^{\infty} \left[\text{Sq} \left(\tau - nT + \frac{T}{6} - \tau_\phi - \tau_o \right) \right. \\ &\quad \left. + \text{Sq} \left(\tau - nT - \tau_\phi - \tau_o \right) \right] \\ i_2(t) &= \frac{I_2}{2} \sum_{n=-\infty}^{\infty} \left[\text{Sq} \left(\tau - nT + \frac{T}{6} - \tau_\phi \right) + \text{Sq} \left(\tau - nT - \tau_\phi \right) \right] . \end{aligned} \right. \quad (20)$$

Similar arguments can be made for the reduction of Eq.(20) as was in the 2- ϕ case for the average power calculation. Thus, with the summation and τ_ϕ eliminated, Eq. (20) becomes

$$\begin{cases} i_1(t) = \frac{I_1}{2} [Sq(\tau + \frac{T}{6} - \tau_0) + Sq(\tau - \tau_0)] \\ i_2(t) = \frac{I_2}{2} [Sq(\tau + \frac{T}{6}) + Sq(\tau)] \end{cases} ,$$

$$0 \leq t \leq T, 0 \leq \tau \leq T, 0 \leq \tau_0 \leq \frac{1}{2} . \quad (21)$$

Again, as in the 2- ϕ case, it can be shown that for average power delivered by i_1 , only v_2 due to i_2 need be calculated:

$$\begin{aligned} v_2(t) &= \frac{1}{C} \int_0^t i_2 d\tau = \frac{I_2}{2C} \int_0^t [Sq(\tau + \frac{T}{6}) + Sq(\tau)] d\tau \\ &= \frac{I_2}{2C} [Tr(\tau + \frac{T}{6}) + Tr(t)] . \end{aligned} \quad (22)$$

The power due to the interaction of i_1 and v_2^* is

$$\begin{aligned} P_{12}(t) &= i_1 v_2 = \frac{I_1 I_2}{4C} \left[Sq(\tau + \frac{T}{6} - \tau_0) \cdot Tr(\tau + \frac{T}{6}) \right. \\ &\quad + Sq(\tau + \frac{T}{6} - \tau_0) \cdot Tr(\tau) + Sq(\tau - \tau_0) \cdot Tr(\tau + \frac{T}{6}) \\ &\quad \left. + Sq(\tau - \tau_0) \cdot Tr(\tau) \right] . \end{aligned}$$

*This is not the total instantaneous power delivered by i_1 .

The average power delivered by i_1 is

$$\begin{aligned}
 \langle p_{12} \rangle &= \langle p_1 \rangle = \frac{1}{T} \int_0^T i_1 v_2 dt, \\
 &= \frac{I_1 I_2}{4TC} \int_0^T \left[S_q \left(\tau + \frac{T}{6} - \tau_0 \right) \cdot \text{Tr} \left(\tau + \frac{T}{6} \right) \right. \\
 &\quad + S_q \left(\tau + \frac{T}{6} - \tau_0 \right) \cdot \text{Tr} (\tau) \\
 &\quad + S_q (\tau - \tau_0) \cdot \text{Tr} \left(\tau + \frac{T}{6} \right) \\
 &\quad \left. + S_q (\tau - \tau_0) \cdot \text{Tr} (\tau) \right] d\tau. \quad (23)
 \end{aligned}$$

Using Eqs. (A-14), (A-15), and (A-16) of the Appendix, Eq. (23) becomes:

$$\begin{aligned}
 \langle p_1 \rangle &= \frac{II_1 I_2}{4C} \left[\int_0^1 S_q \left(\gamma + \frac{1}{6} - \gamma_0 \right) \cdot \text{Tr} \left(\gamma + \frac{1}{6} \right) d\gamma \right. \\
 &\quad \left. + \int_0^1 S_q \left(\gamma + \frac{1}{6} - \gamma_0 \right) \cdot \text{Tr} (\gamma) d\gamma \right.
 \end{aligned}$$

(Equation continued on following page)

$$+ \int_0^1 \text{Sq} (\gamma - \gamma_0) \cdot \text{Tr} (\gamma + \frac{1}{6}) d\gamma$$

$$+ \int_0^1 \text{Sq} (\gamma - \gamma_0) \cdot \text{Tr} (\gamma) d\gamma \Bigg]$$

$$\Delta \langle p_1 \rangle_1 + \langle p_1 \rangle_2 + \langle p_1 \rangle_3 + \langle p_1 \rangle_4 \quad (24)$$

Each of the above average power components is similar to a 2- ϕ average power calculation as performed previously. The four components will be calculated separately and the results will be summed to calculate the total power:

$$\langle p_1 \rangle_1 = \frac{\pi I_1 I_2}{4C} \int_0^1 \text{Sq} (\gamma + \frac{1}{6} - \gamma_0) \cdot \text{Tr} (\gamma + \frac{1}{6}) d\gamma \quad (25)$$

It was demonstrated in the transformation from Eq. (8) to Eq. (9) that a common phase shift in the two currents with respect to the reference does not contribute to the average power. Therefore, the integration of $\langle p_1 \rangle_1$ is accomplished by Eq. (A-8) of the Appendix, noting that γ_2 is zero and that $(1/2 - \gamma_1)$ is equal to $1/2 - \gamma_0$, a quantity always greater than zero.

$$\langle p_1 \rangle = \frac{\pi I_1 I_2}{4C} \int_0^1 \text{Sq} (\gamma - \gamma_0) \cdot \text{Tr} (\gamma) d\gamma \quad ,$$

$$= \frac{\pi I_1 I_2}{4C} (\gamma_0 - 2 \gamma_0^2) \quad (26)$$

Likewise, the integration of $\langle p_1 \rangle_4$ is accomplished by Eq. (A-8) of the Appendix; i.e.,

$$\begin{aligned} \langle p_1 \rangle_4 &= \frac{\pi I_1 I_2}{4C} \int_0^1 \text{Sq} (\gamma - \gamma_0) \cdot \text{Tr} (\gamma) d\gamma , \\ &= \frac{\pi I_1 I_2}{4C} (\gamma_0 - 2 \gamma_0^2) . \end{aligned} \quad (27)$$

The integration for $\langle p_1 \rangle_2$ is of the form

$$\langle p_1 \rangle_2 = \frac{\pi I_1 I_2}{4C} \int_0^1 \text{Sq} (\gamma + \frac{1}{6} - \gamma_0) \cdot \text{Tr} (\gamma) d\gamma . \quad (28)$$

The results of the integration of Eq. (28) depend on whether $1/6 - \gamma_0$ is greater than or less than zero. The polynomial in Eq. (A-4) provides a solution for Eq. (28) when $0 \leq \gamma_0 \leq 1/6$ with the choice of γ_1 equal to $1/6 - \gamma_0$ and γ_2 equal to zero. If $1/6 \leq \gamma_0 \leq 1/2$, the polynomial of Eq. (A-8) gives the solution for Eq. (28). The results are:

$$\langle p_1 \rangle_2 = \frac{\pi I_1 I_2}{4C} (2 \gamma_0^2 + 1/3 \gamma_0 - 1/9), \quad 0 \leq \gamma_0 \leq 1/6 \quad (29)$$

and

$$\langle p_1 \rangle_2 = \frac{\pi I_1 I_2}{4C} (-2 \gamma_0^2 + 5/3 \gamma_0 - 2/9), \quad 1/6 \leq \gamma_0 \leq 1/2 . \quad (30)$$

The equation for $\langle p_1 \rangle_3$ is

$$\langle p_1 \rangle_3 = \frac{\pi I_1 I_2}{4C} \int_0^1 \text{Sq} (\gamma - \gamma_0) \cdot \text{Tr} \left(\gamma + \frac{1}{6} \right) d\gamma \quad (31)$$

Equation (31) can be transformed by Eq. (A-5) of the Appendix into

$$\langle p_1 \rangle_3 = \frac{\pi I_1 I_2}{4C} \int_0^1 -\text{Sq} \left[\gamma + \left(\frac{1}{2} - \gamma_0 \right) \right] \cdot \text{Tr} \left(\gamma + \frac{1}{6} \right) d\gamma \quad (32)$$

Again, using the integration Equations (A-7) and (A-8) with $\gamma_1 = (1/2 - \gamma_0)$ and $\gamma_2 = 1/6$, two solutions result:

$$\langle p_1 \rangle_3 = \frac{\pi I_1 I_2}{4C} \left(-2 \gamma_0^2 + \frac{1}{3} \gamma_0 + \frac{1}{9} \right), \quad 0 \leq \gamma_0 \leq 1/3 \quad (33)$$

and

$$\langle p_1 \rangle_3 = \frac{\pi I_1 I_2}{4C} \left(2 \gamma_0^2 - \frac{7}{3} \gamma_0 + \frac{5}{9} \right), \quad 1/3 \leq \gamma_0 \leq 1/2 \quad (34)$$

The total power in the circuit is the sum of the four power components over three ranges of γ_0 since $\langle p_1 \rangle_2$ and $\langle p_1 \rangle_3$ are functionally different in these ranges:

$$\langle p_1 \rangle = \begin{cases} \frac{\pi I_1 I_2}{C} (2 \gamma_0 - 3 \gamma_0^2), & 0 \leq \gamma_0 \leq \frac{1}{6} \\ \frac{\pi I_1 I_2}{C} (3 \gamma_0 - 6 \gamma_0^2 - \frac{1}{12}), & \frac{1}{6} \leq \gamma_0 \leq \frac{1}{3} \\ \frac{\pi I_1 I_2}{C} (\gamma_0 - 3 \gamma_0^2 + \frac{1}{4}), & \frac{1}{3} \leq \gamma_0 \leq \frac{1}{2} \end{cases} \quad (35)$$

$$(36)$$

$$(37)$$

The resultant value of $\langle p_1 \rangle$ expressed in the Eqs. (35), (36), and (37) represents the closed form of the infinite harmonic series for 3- ϕ ICB, obtained from Fourier analysis.³

Several interesting remarks can be based on the preceding analysis:

1. The per phase equivalent circuit for the 3- ϕ ICB may be further decomposed as shown in Fig. 6. From this figure it is evident that four components of average power are at work. Depending on the size of γ_0 , some of these components may be negative.
2. The three segments of $\langle p_1 \rangle$ vs γ_0 curve are shown in Fig. 7. At the points of interface, the curves connect smoothly; i.e., both the value and the slopes at the interface points match.
3. In an argument similar to that given in the Fourier analysis, for ICB systems with low frequency response in comparison to the bridge frequency, the net power may be approximated by

$$\langle p_1 \rangle = \frac{T}{C} I_1(t) \cdot I_2(t) f(\gamma_0) ,$$

where $I_1(t)$ and $I_2(t)$ are relatively slow (compared to a switching sequence) time-varying inductor currents.

4. The γ_0 value for maximum net power can be calculated from differentiating the middle segment of $\langle p_1 \rangle$ vs γ_0 curve. The result is $\gamma_{0 \max} = 1/4$ which corresponds to 90° phase shift.

Conclusions

The calculations in this report illustrated some of the applications of the square function method. This method making use of the rectangular (Sq) and triangular (Tr) wave shapes is well suited to the ICB circuits, in a similar way that the trigonometric rules are used in the analysis of the circuits with sinusoidal waveshapes. The calculations relate more directly to the actual circuit waveforms and the complications of transforming the problem

to a form usable by other techniques; e.g., Fourier technique, were eliminated. Consequently, the results were simpler and easier to interpret.

Furthermore, expressing the actual circuit waveforms analytically allows other useful calculations such as actual capacitor voltages throughout the transfer cycle, the study of commutation throughout the cycle, and the actual coil voltages and instantaneous currents.

The simplicity of the functional dependence of the various circuit quantities should also help in the design process. This technique may also prove useful in the frequency studies of the ICB circuits. The results obtained by the square function method are simpler to use and more efficient to implement in an active control algorithm, especially, for microprocessor controllers where complex functional calculations are extremely time-consuming by comparison to simple multiplication and addition.

The calculation procedure used for the 3- ϕ ICB demonstrates the method for any n -phase ICB.

Finally, the 2- ϕ ICB presented in this report is not only itself a useful design candidate for energy transfer, but also was shown to be the building block of the more complicated ICB configurations.

References

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Fig. 1. Circuit Diagram for the 3-Capacitor Model IC Bridge.

TRIGGERING SEQUENCE $S_{11} S_{14}, S_{12} S_{13}, S_{11} S_{14}, S_{12} S_{13}, \dots$

Fig. 2. Circuit Diagram for the Two-Phase ICB.

Fig. 3. (a) $Sq(\gamma + \gamma_0)$. (b) $Tr(\gamma + \gamma_0)$

Fig. 4. One-Phase Equivalent Circuit of ICB.

Fig. 5. (a) Phase Current in a Three-Phase ICB,
 (b) Decomposition of the Phase Current
 into two Sq functions.

Fig. 6. The Per Phase Equivalent Circuit for the Three-Phase ICB.

Fig. 7. The Average Power vs γ_0 in Three-Phase ICB.

APPENDIX

The Sq and Tr Functions

The Sq functions are defined as the sum of unit step functions in the following manner:

$$\begin{aligned} \text{Sq } (\gamma + \gamma_0) &\triangleq U(\gamma) - 2 U(\gamma - 1/2 + \gamma_0) \\ &\quad + 2 U(\gamma - 1 + \gamma_0) - U(\gamma_0 - 1), \\ 0 \leq \gamma_0 &\leq \frac{1}{2}, \quad 0 < \gamma \leq 1 \end{aligned} \tag{A-1}$$

and $U(\gamma)$ is the unit step function. The Sq function shown in Fig. 3a is analytically constructed by this expression. The first unit step function $U(\gamma)$ is zero for γ less than 0 and unity at zero and beyond. The second unit step function is zero for $\gamma < 1/2 - \gamma_0$ and jumps to -2 when $\gamma > 1/2 - \gamma_0$. The sum of the first unit step function and the second unit function add together to give a net value of -1 for $\gamma > 1/2 - \gamma_0$. The third step function is zero for $\gamma < 1 - \gamma_0$; and so in this manner the analytical expression for the Sq is constructed.

The Tr functions are defined as the integral value of the Sq functions; i.e.,

$$\begin{aligned} \text{Tr } (\gamma + \gamma_0) &\triangleq \int_0^\gamma \text{Sq } (\xi + \gamma_0) d\xi \\ &= \gamma U(\gamma) - 2 \left(\gamma - \frac{1}{2} + \gamma_0\right) U\left(\gamma - \frac{1}{2} + \gamma_0\right) \\ &\quad + 2 \left(\gamma - 1 + \gamma_0\right) U\left(\gamma - 1 + \gamma_0\right) - (\gamma - 1) U(\gamma - 1). \end{aligned}$$

The first unit step function in Eq. (A-1) is zero until $\gamma \geq 0$; after which the function increases linearly with γ . Likewise, the integral value of the second unit step function is zero until $\gamma = 1/2 - \gamma_0$; and afterwards increases linearly with γ . The Tr functions shown in Fig. 3b are analytically reproduced by Eq. (A-2).

The value of the product of an Sq and a Tr function integrated over a γ interval of 0 to 1 will be extensively be used in calculating the average power flow in the circuits of interest. Therefore, the following relations are useful:

$$\int_0^1 [Sq(\gamma + \gamma_1) \cdot Tr(\gamma + \gamma_2)] d\gamma = \begin{cases} -2\gamma_1^2 - \gamma_1 + 4\gamma_1\gamma_2 + \gamma_2 - 2\gamma_2^2, & \gamma_1 \leq \gamma_2 \\ 2\gamma_1^2 - \gamma_1 - 4\gamma_1\gamma_2 + \gamma_2 + 2\gamma_2^2, & \gamma_1 \geq \gamma_2 \end{cases} \quad \begin{matrix} (A-3) \\ (A-4) \end{matrix}$$

$$0 \leq \gamma_1 \leq \frac{1}{2}, \quad 0 \leq \gamma_2 \leq \frac{1}{2}$$

The identities, (A-3) and (A-4), are derived by simple term-by-term integration, taking account of the fact the integration has a non-zero contribution only after γ exceeds the larger of the two values of γ_1 and γ_2 .

Note that for $\gamma_1 = \gamma_2$, either one of the above results will give 0.

Two following identities will also be of considerable value:

$$Sq(\gamma - \gamma_0) = -Sq[\gamma + (\frac{1}{2} - \gamma_0)] \quad (A-5)$$

and

$$\text{Tr } (\gamma - \gamma_0) = - \text{Tr } [\gamma + (\frac{1}{2} - \gamma_0)], \quad 0 \leq \gamma_0 \leq \frac{1}{2}. \quad (\text{A-6})$$

These are easily verified by direct substitution. Note that by using the above identities, we can easily derive the operation for lagging functions from the operation for leading functions. For example,

$$\begin{aligned} & \int_0^1 [\text{Sq } (\gamma - \gamma_1) \cdot \text{Tr } (\gamma + \gamma_2)] d\gamma \\ &= - \int_0^1 \left\{ \text{Sq } \left[\gamma + \left(\frac{1}{2} - \gamma_1 \right) \right] \cdot \text{Tr } (\gamma + \gamma_2) \right\} d\gamma \\ &= \begin{cases} - \left[-2 \left(\frac{1}{2} - \gamma_1 \right)^2 - \left(\frac{1}{2} - \gamma_1 \right) + 4 \left(\frac{1}{2} - \gamma_1 \right) \gamma_2 + \gamma_2 - 2 \gamma_2^2 \right], & (\text{A-7}) \\ \left(\frac{1}{2} - \gamma_1 \right) < \gamma_2 \\ - \left[2 \left(\frac{1}{2} - \gamma_1 \right)^2 - \left(\frac{1}{2} - \gamma_1 \right) - 4 \left(\frac{1}{2} - \gamma_1 \right) \gamma_2 + \gamma_2 + 2 \gamma_2^2 \right], & (\text{A-8}) \\ \left(\frac{1}{2} - \gamma_1 \right) > \gamma_2 \\ 0, \quad \left(\frac{1}{2} - \gamma_1 \right) = \gamma_2. & (\text{A-9}) \end{cases} \end{aligned}$$

The Sq and Tr functions may be generalized to periodic functions using an integer summation; i.e.,

$$\begin{aligned}
 \text{Sq } (\gamma' + \gamma_0) &= \sum_{n=-\infty}^{\infty} \text{sq } (\gamma - n + \gamma_0) \\
 &= \sum_{n=-\infty}^{\infty} \left\{ U (\gamma - n) - 2 U (\gamma - n - 1/2 + \gamma_0) \right. \\
 &\quad \left. + 2 U (\gamma - n - 1 + \gamma_0) - U (\gamma - n - 1) \right\}
 \end{aligned} \tag{A-10}$$

and

$$\begin{aligned}
 \text{Tr } (\gamma' + \gamma_0) &= \sum_{n=-\infty}^{\infty} \text{Tr } (\gamma - n + \gamma_0) \\
 &= \sum_{n=-\infty}^{\infty} (\gamma - n + \gamma_0) U (\gamma - n + \gamma_0) \\
 &\quad - 2 (\gamma - n - 1/2 + \gamma_0) U (\gamma - n - 1/2 + \gamma_0) \\
 &\quad + 2 (\gamma - n - 1 + \gamma_0) U (\gamma - n - 1 + \gamma_0) \\
 &\quad - (\gamma - n - 1 + \gamma_0) U (\gamma - n - 1 + \gamma_0)
 \end{aligned} \tag{A-11}$$

where $-\infty < \gamma' < \infty$, $0 < \gamma < 1$ and $\gamma' = n + \gamma$, $n = 0, 1, 2, \dots$.

Clearly, the Sq and Tr functions shown in Fig. 3 are analytically reproduced for any choice of positive or negative integer in Eqs. (A-10) and (A-11).

The period over which the Sq and Tr functions are defined can be generalized from integer to arbitrary values for broader application. The Sq functions are basically independent of the variable. So that,

$$Sq(t + t_0) = Sq(\gamma + \gamma_0), \begin{cases} 0 \leq t \leq T, 0 \leq t_0 \leq \frac{T}{2} \\ 0 \leq \gamma \leq 1, 0 \leq \gamma_0 \leq \frac{1}{2} \end{cases} \quad (A-12)$$

and

$$Tr(t + t_0) = \int_0^t Sq(\sigma + t_0) d\sigma, \begin{cases} 0 \leq t \leq T \\ 0 \leq t_0 \leq \frac{T}{2} \end{cases} \quad (A-13)$$

The variable, t , and the parameter, t_0 , for any arbitrary period of T are related to γ and γ_0 for the unity period by

$$\gamma = t/T \text{ and} \quad (A-14)$$

$$\gamma_0 = t_0/T. \quad (A-15)$$

Substitution of t and t_0 in the integral Eq. (A-13) results in the following

$$Tr(t + t_0) = T \int_0^\gamma Sq(\xi + \gamma_0) d\xi. \quad (A-16)$$

Therefore,

$$\begin{aligned} \text{Tr } (t + t_0) = T [\text{Tr } (\gamma + \gamma_0)] = t u(t) - 2 \left(t - \frac{T}{2} + t_0\right) u \left(t - \frac{T}{2} + t_0\right) \\ + 2 \left(t - T + t_0\right) u \left(t - T + t_0\right) - (t - T) u (t - T) \end{aligned} \quad (\text{A-17})$$

For time base calculations, the Eqs. (A-12), (A-14), (A-15), and (A-17) can be utilized.