MAPPING IN INERTIAL FRAMES

BY

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MAPPING IN INERTIAL FRAMES

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ABSTRACT

World space mapping in inertial frames is used to examine the Lorentz covariance of symmetry operations. It is found that the Galilean invariant concepts of simultaneity (S), parity (P), and time reversal symmetry (T) are not Lorentz covariant concepts for inertial observers. That is, just as the concept of simultaneity has no significance independent of the Lorentz inertial frame, likewise so are the concepts of parity and time reversal. However, the world parity (W) [i.e., the space-time reversal symmetry (P-T)] is a truly Lorentz covariant concept. Indeed, it is shown that only those mapping matrices M that commute with the Lorentz transformation matrix L (i.e., $[M, L] = 0$) are the ones that correspond to manifestly Lorentz covariant operations. This result is in accordance with the spirit of the world space Mach's principle. Since the Lorentz transformation is an orthogonal transformation while the Galilean transformation is not an orthogonal transformation, the formal relativistic space-time mapping theory used here does not have a corresponding non-relativistic counterpart.

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I. INTRODUCTION

In the study of eigenvalue equations, the eigenfunctions of the operator usually exhibit certain symmetry properties if the operator itself is invariant towards certain transformations. For example, in the non-relativistic quantum theory one usually studies the eigenvalue equation¹

$$H\psi(x,y,z,t) = (ih/2\pi) \partial\psi(x,y,z,t)/\partial t = E\psi(x,y,z,t) \quad , \quad (1)$$

where H is the Hamiltonian operator of the system under study. The existence of certain symmetry properties of the wave function ψ follows directly from the invariance of the Hamiltonian operator H towards certain transformations, and this invariance is always associated with a *conservation law* for some physical quantity.^{2,3} For example: (1) the invariance of H with respect to the translational shift of the origin of space coordinates implies the conservation of momentum, i.e., if a coordinate is *cyclic or ignorable* in H , then the associated canonical momentum is a constant of motion, (2) the invariance of H with respect to the time implies the conservation of energy, (3) the invariance of H with respect to spatial inversion implies the conservation of parity, (4) the invariance of H with respect to the angular orientation of the coordinate system used to describe the system implies the conservation of the angular momentum, (5) the invariance of H with respect to the choice of a zero of the electric potential implies the conservation of charge, etc. That is, if the Hermitian operator A representing some dynamical variable does not depend explicitly on the time t (so that $\partial A/\partial t = 0$), then $dA/dt = \partial A/\partial t + (ih/2\pi)^{-1}[A,H] = 0$ if A commutes with H . In this case A is said to be a *constant of motion*. This is usually possible at all times only if H is also a constant of motion, i.e., if H itself does not involve the time explicitly. However, these concepts of the symmetry properties of ψ , the invariance of H for certain operations, and the associated conservation laws are physically meaningful concepts if and only if they

hold true in all inertial frames, i.e., if and only if these concepts are *truly covariant* for all inertial observers, since according to the principle of relativity all inertial frames are equivalent.⁴⁻⁷ In the Newtonian (i.e., non-relativistic) physics we usually demand the covariance with respect to the Galilean transformation in accordance with the *Galilean principle of relativity*. In the Einsteinian (i.e., relativistic) physics we demand the covariance with respect to the Lorentz transformation in agreement with the *Einsteinian principle of relativity*. It is our aim in this paper to examine the Lorentz covariance of these symmetry properties.

We will use the formal theory of *World space mapping* in inertial frames to examine the Lorentz covariance of symmetry operations. It will be seen that the formal relativistic space-time mapping theory used here does not have a corresponding non-relativistic counterpart. The reason for this is the Lorentz transformation is an orthogonal transformation in the world space while the Galilean transformation is not an orthogonal transformation in any space. We will show that the Galilean invariant concepts of simultaneity, parity, and time reversal symmetry are not Lorentz covariant concepts for inertial observers. Indeed, it will be apparent that just as the concept of simultaneity has no significance independent of the Lorentz inertial frame,⁵ likewise so are the concepts of parity and time reversal. However, the *world parity* (i.e., the space-time reversal symmetry) is a *truly Lorentz covariant concept*. Indeed, it will be seen that only those mapping matrices M that *commute* with the Lorentz transformation matrix L (i.e., $[M, L] = 0$) are the ones that correspond to *manifestly Lorentz covariant operations*. This result is in accordance with the spirit of the world space Mach's principle.

II. GALILEAN INVARIANT CONCEPTS

In Newtonian physics the ordinary time t is taken to be an invariant

parameter, the same for all Galilean observers. If we now consider two inertial frames F and F' whose Cartesian axes of coordinates are parallel to each other and the F' frame is moving with a uniform velocity $\mathbf{v} = \beta c \hat{\mathbf{i}}_x$ along the x -direction with respect to the F frame, then the Galilean or the Newtonian transformation equations are^{5,6} :

$$x' = x - vt = x - \beta ct ; y' = y ; z' = z ; t' = t . \quad (2)$$

Here the space is a three-dimensional Euclidean (3D E-) space and t is an invariant scalar parameter. In this 3D E-space and independent scalar time there is absolute sense of simultaneity, absolute sense of parity, absolute sense of time reversal invariance, absolute sense of rigid body constraints, absolute concept of static or "action-at-a-distance" forces, etc. That is, the concepts of simultaneity, parity, time reversal symmetry, rigid body constraints, static or action-at-a-distance forces, etc. are all *Galilean invariant concepts*, the same for all inertial observers.

Let us first examine the familiar concept of simultaneity. Let A be an event with coordinates x_a, y_a, z_a , and t_a , and B be an event with coordinates x_b, y_b, z_b , and t_b in the F frame. For an observer in the F' frame, the corresponding events A' and B' will have the coordinates x'_a, y'_a, z'_a, t'_a and x'_b, y'_b, z'_b, t'_b , respectively. For the F observer if the events A and B are simultaneous, then $t_a = t_b$. By the Galilean transformation Eq.(2), it then follows that $t'_a = t'_b$. Hence, the concept of simultaneity is a Galilean covariant concept. That is, two events that appear simultaneous to one observer will appear simultaneous to all inertial Galilean observers. This, of course, is a well known result of Newtonian physics.

We now examine the concept of intrinsic symmetry of physical systems. For simplicity let us consider a spherical shell of radius r in the F frame. Then at any instant of time t ($= t_a = t_b$), for every point

A on this shell there exists another point B on this shell such that $(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2 = 4r^2$. If the center of the shell is coincident with the origin of the F coordinates at this time t, then the point B is related to the point A by a spatial inversion through the origin or a parity operation. By the Galilean transformation Eq.(2), it then follows that in the F' frame $(x_b' - x_a')^2 + (y_b' - y_a')^2 + (z_b' - z_a')^2 = 4r^2$. That is, a shell that appears spherical to one observer will appear spherical to all Galilean observers. Hence, the concept of intrinsic symmetry of physical objects is a Galilean covariant concept. In a similar way one sees readily that the concepts of rigid body constraints, and static or action-at-a-distance forces are both Galilean invariant concepts.

We now examine the concept of time reversal invariance in classical physics. If we assume that the forces on a particle are time-independent and non-dissipative, then the Newton's equation of motion is $(m d^2x_j/dt^2) = F_j(x,y,z)$, where $j = 1, 2$, and 3 , and $x_1 = x$, $x_2 = y$, and $x_3 = z$. Clearly, if $x_j(t)$ is a possible trajectory, then likewise so is $x_j(-t)$, since $[m d^2x_j/d(-t^2)] = F_j(x,y,z)$. That is, Newtonian mechanics defines no sense of time's direction. In statistical mechanics and kinetic theory, this time reversal invariance of Newton's law manifests itself as the principle of *microscopic* reversibility. This Newton's law is, of course, invariant under the Galilean transformation of Eq.(2). Hence, the concept of time reversal invariance is a Galilean covariant concept.

Thus far we have examined the Galilean covariance of the concepts of simultaneity, parity, time reversal, etc. from a purely classical point of view. We now wish to examine the Galilean covariance of these same concepts from a quantum mechanical point of view. In the non-relativistic quantum theory, the event A has an associated wave function $\psi(x_a, y_a, z_a, t_a)$ and the event B has the corresponding wave function $\psi(x_b, y_b, z_b, t_b)$ in the F frame. Similarly, in the F' frame the

events A' and B' will have the wave functions $\Psi'(x_a', y_a', z_a', t_a')$ and $\Psi'(x_b', y_b', z_b', t_b')$. If the two events A and B are simultaneous in the F frame, then $\Psi(x_b, y_b, z_b, t_b) = S\Psi(x_a, y_a, z_a, t_a) = \pm\Psi(x_a, y_a, z_a, t_b)$, where the simultaneity operator S makes $t_b = t_a = t$. In general, this simultaneity operator S can be built up as a suitable combination of rotation, reflection, and displacement operators. In non-relativistic quantum mechanics the eigenvalues of any symmetry operator are $+1$, and -1 , since the position probability density $\Psi^*\Psi = |\Psi|^2$ should remain the same. Here, $\Psi^* = K\Psi$, where K is the complex conjugation operator. In other words, the square of any symmetry operator has the eigenvalue $+1$. From the Galilean transformation Eq.(2), it then follows that $\Psi(x_b', y_b', z_b', t_b') = S\Psi(x_a', y_a', z_a', t_a') = \pm\Psi(x_a', y_a', z_a', t_b')$. Hence, it is clear that the *concept of simultaneity is a Galilean covariant concept* both in classical and in quantum mechanics.

Suppose now in the F frame the Hamiltonian operator H is invariant under the coordinate inversion through the origin, then the parity operator P commutes with the Hamiltonian, i.e., $[P, H] = 0$. If at some instant of time $t = t_a = t_b$, this operator P maps the event A onto the event B in the F frame, then $\Psi(x_b, y_b, z_b, t_b) = P\Psi(x_a, y_a, z_a, t_a) = \pm\Psi(-x_a, -y_a, -z_a, t_a)$. It may be noted that the parity operator P is a special case of the simultaneity operator S , and can easily be built up from a reflection and a rotation operator.² From the Galilean transformation Eq.(2), it then follows that $\Psi'(x_b', y_b', z_b', t_b') = P\Psi'(x_a', y_a', z_a', t_a') = \pm\Psi'(-x_a', -y_a', -z_a', t_a')$. Thus we see that the *concept of parity is a Galilean covariant concept*, the same for all inertial observers. Since $(dP/dt) = (\partial P/\partial t) + (i\hbar/2\pi)^{-1} [P, H]$ and $(\partial P/\partial t) = 0$, we see that if $[P, H] = 0$, then the parity is conserved for all inertial Galilean observers. Since the non-relativistic Hamiltonian must be a Galilean invariant property of the system only (independent of the particular coordinate system used), we must expect it to be a scalar or a pseudoscalar, invariant under all *proper* Galilean transformations.^{7,8}

The proper transformations are those that can be generated as a continuous unfolding of infinitesimal transformations from the identity transformation. That is, these proper transformations do not include any coordinate reflections or inversions. If H contains only scalars, then $[P, H] = 0$, and parity is conserved universally for all inertial Galilean observers. However, if H contains a pseudoscalar, then $[P, H] \neq 0$, and parity is not conserved. Unlike scalars, pseudoscalars change sign under coordinate inversion or reflection (i.e., for a change from the right-handed to a left-handed coordinate system). A well-known example of a pseudoscalar is the scalar product of a vector and an axial vector, i.e., the scalar triple product of three polar vectors $A \cdot (B \times C)$.

Similarly, if at some spatial point ($x=x_a=x_b, y=y_a=y_b, z=z_a=z_b$), the time reversal operator T maps the event A onto the event B in the F frame, then $\psi(x_b, y_b, z_b, t_b) = T\psi(x_a, y_a, z_a, t_a) = \pm\psi(x_a, y_a, z_a, -t_a)$. From the Galilean transformation Eq.(2), it then follows that $\psi'(x_b', y_b', z_b', t_b') = T\psi'(x_a', y_a', z_a', t_a') = \pm\psi'(x_a', y_a', z_a', -t_a')$. Here, we have assumed that the non-relativistic Hamiltonian H is not an explicit function of time and thus $[T, H] = 0$. Then the time reversal invariance is conserved and the eigenvalues of the time reversal operator $T = \pm 1$ for all inertial Galilean observers. But, the energy operator $E = (i\hbar/2\pi)\partial/\partial t$ in Eq.(1) is not invariant under the action of this pure time reversal operator T . It is, however, invariant under the action of the Wigner time reversal operator $T_w = TK$, where K is the complex conjugation operator.² That is, $T_w\psi(x, y, z, t) = T\psi^*(x, y, z, t) = \pm\psi^*(x, y, z, -t)$. Here again from the Galilean transformation Eq.(2), it then follows that $T_w\psi(x', y', z', t') = T\psi^*(x', y', z', t') = \pm\psi^*(x', y', z', -t')$. Thus we see that the *concept of time reversal symmetry is a Galilean covariant concept*. We emphasize that what we have said so far about simultaneity, parity and time reversal are true only in the Galilean invariant (i.e., non-relativistic) Hamiltonian formalism. However, we shall soon see that simultaneity, parity, and time reversal symmetry are not covariant concepts in the Lorentz covariant Hamiltonian formalism.^{7,8}

Thus far we have examined the symmetry operations of simultaneity, parity, and time reversal as space and time (and not as space-time) mappings in inertial frames. However, we have not been able to formulate this analysis in terms of any formal mapping matrices M in this 3D E-space and time. The reason for this is that the Galilean transformation of Eq.(2) is not an orthogonal transformation in this space. That is,

$$\mathbf{x}' = G\mathbf{x} \quad , \quad (3)$$

where G is an orthogonal matrix (i.e., $\sum_{\mu} G_{\mu\nu} G_{\mu\lambda} = \delta_{\nu\lambda}$) of the Galilean transformation. Here, for example, we could choose the four components of $\mathbf{x}$ as: $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = t$. That is, the 3D E-space of x , y , and z and the absolute time t do not form a space-time continuum in Newtonian physics.

III. WORLD SPACE MAPPING

We now wish to examine the mappings associated with these symmetry operations in the four-dimensional space-time continuum known as the *world space* or the *Minkowski space*. The familiar relativistic Lorentz transformation is an orthogonal transformation in this world space whose coordinates are $x_1 = x$, $x_2 = y$, $x_3 = z$, and $x_4 = ict$. Here the velocity of light c is the same for all Lorentzian observers and $\sum_{\mu} x_{\mu}^2 = \sum_{\mu} (x'_{\mu})^2 \leq 0$, where the equality is for the *null cone* or the *light cone*. Because of the causality principle^{9,10} (which states that the effect should not precede the cause) for particles of positive real mass moving with speeds less than c (i.e., *tardyons* or *bradyons*), we will only consider *time-like four-vectors* in this world space. In the literature,¹¹ the possible existence of *tachyons* (i.e., particles with imaginary mass moving faster than the speed of light c) with *space-like*

four-vectors has been postulated. Here we will only consider the mechanics of the tardyons. The *world time* or the *proper time* τ is an invariant parameter, the same for all inertial Lorentzian observers. The proper time interval $d\tau$ is given by

$$(d\tau)^2 = -c^{-2} \sum (dx_\mu)^2 = (dt)^2 (1 - \beta^2). \quad (4)$$

Here again we consider the two inertial frames F and F' whose axes of coordinates are parallel to each other and the primed system is moving with a uniform velocity $\mathbf{v} = \beta c \mathbf{i}_x$ along the x -direction with respect to the unprimed system. Then the proper Lorentz transformation may be written

$$\mathbf{x}' = L \mathbf{x}, \text{ and } \mathbf{x} = L^{-1} \mathbf{x}', \quad (5)$$

where,

$$L = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \text{ and } L^{-1} = \begin{pmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}. \quad (6)$$

$\gamma = (1 - \beta^2)^{-1/2}$, and $\beta = v/c$. Note that L is an orthogonal matrix and $L^{-1}(\mathbf{v}) = L(-\mathbf{v})$ as it should be, since the unprimed system is moving relative to the primed system with a velocity $-\mathbf{v} = -\beta c \mathbf{i}_x$. The Lorentz transformation of Eq.(5) is $x' = \gamma(x - vt)$, $y' = y$, $z' = z$, $t' = \gamma(t - vx/c^2)$. In the limit $c \rightarrow \infty$, $\gamma \rightarrow 1$ and $(vx/c^2) \rightarrow 0$. Thus it is seen that in the limit of c tending to infinity the Lorentz transformation of Eq.(5) reduces *entirely* to the Galilean transformation of Eq.(2). However, as pointed out earlier in Eq.(3), the limit $c \rightarrow \infty$ of Eq.(5) cannot be written as any orthogonal transformation. That is, the relativistic orthogonal transformation of Eq.(5) in the Minkowskian space-time continuum (i.e.,

four-space) does not reduce to any non-relativistic orthogonal transformation in the 3D E-space and an absolute time. This is the basic reason that the relativistic world space mapping of symmetry operations to be discussed in this section in terms of four-by-four mapping matrices M does not have a formal direct counterpart in the corresponding non-relativistic mapping in the 3D E-space and the absolute time.

An event A in the F frame is given by the tip of the position four-vector $x(a)$ whose Minkowskian coordinates are $x_1(a)$, $x_2(a)$, $x_3(a)$, and $x_4(a) = ict(a)$. Thus, in the F frame the mapping of an event A [i.e., the space-time point $x(a)$] onto an event B [i.e., the space-time point $x(b)$] may be written

$$x(b) = Mx(a) \quad , \quad (7)$$

where M is the mapping matrix with elements $M_{\mu\nu}$ and $\mu, \nu = 1, 2, 3$, and 4. In the F' frame the corresponding mapping of the event A' [i.e., the space-time point $x(a')$] onto the event B' [i.e., the space-time point $x(b')$] is

$$x(b') = M'x(a') \quad , \quad (8)$$

where M' is the corresponding mapping matrix in the F' frame with elements $M'_{\mu\nu}$. But by the Lorentz transformation Eq.(5), $x(a') = Lx(a)$, and $x(b') = Lx(b)$. Hence,

$$x(b') = Lx(b) = LMx(a) = (LML^{-1})Lx(a) = (LML^{-1})x(a') \quad . \quad (9)$$

Thus from Eqs.(8) and (9) we get

$$M' = LML^{-1} \quad . \quad (10)$$

That is, the mapping matrix M' of the F' frame is related to the

corresponding mapping matrix M of the F frame by the usual *similarity or the equivalence* transformation. This result is hardly surprising since, when the space coordinates are subject to an orthogonal transformation, all operators in that space must be subject to a similarity or equivalence transformation.

According to the *principle of relativity* all inertial frames are equivalent. That is, a properly formulated description of a physical phenomenon can contain no reference, either explicit or implicit, to an absolute speed of translational motion of the coordinate frame in which the phenomenon is described. The symmetry operations of simultaneity, parity, time reversal, etc., are indeed descriptions of physical phenomena. Simultaneity of events and the intrinsic symmetry (i.e., intrinsic parity) of physical systems are, of course, well known descriptions of physical phenomena in Newtonian classical physics and mechanics. In quantum mechanics and in quantum electrodynamics (QED) any physically measurable dynamical variable is represented by a Hermitian operator or matrix. If we now inquire into what observable quantities are represented by the parity (P) and the time reversal (T) operators, we see that these quantities have no classical analogue, since reflection and time reversal of a wave function are not classical concepts. We nevertheless find experimentally that the physical consequences of using these operators in quantum mechanics and in QED in the same way other operators are used are actually valid.

Hence, according to the principle of relativity we find that any symmetry operation or mapping is *manifestly Lorentz covariant* if and only if $M' = M$. Then and only then the symmetry operation and the associated mapping M will appear the same (i.e., appear form invariant) for all inertial Lorentz observers, in agreement with the principle of relativity (which states that all inertial frames are equivalent). Thus, from Eq.(10) we get $M = M' = LML^{-1}$. That is,

$$[M, L] = ML - LM = 0 \quad (11)$$

Hence any symmetry operation or mapping matrix M is manifestly Lorentz covariant if and only if it commutes with the Lorentz transformation matrix L . This is the necessary and sufficient condition for the manifestly Lorentz covariance of any physically meaningful symmetry operation and the associated world space mapping. It is somewhat interesting to note that L itself is a Hermitian matrix.

In any relativistic theory we assume that the proper or world time τ is an invariant parameter, that is, the proper time τ is the same for all inertial Lorentz observers. Then the Hamiltonian must, of course, be chosen such that the motion of the particle in the world time interval $d\tau$ can be described by an infinitesimal contact transformation generated by this Hamiltonian. In the Minkowskian geometrical language the Hamiltonian is the generator of the particle (motion with the proper time τ , i.e. the) world line. Thus we see that the Hamiltonian itself is a Hermitian mapping matrix or operator in the world space. Hence, from Eq.(11) it follows that a truly Lorentz covariant Hamiltonian operator (H) of any system must commute with the Lorentz transformation matrix L , that is, $[H, L] = 0$. Hence, for a truly Lorentz covariant description, the Hamiltonian should be a Lorentz-invariant property of the system only, independent of the particular coordinate system used. That is, we must expect the truly Lorentz covariant Hamiltonian to be a world scalar or a world pseudoscalar, invariant under all proper Lorentz transformations. Thus, for example, the simplest choice for a relativistic free-particle Hamiltonian is^{7,8}

$$H = (2m)^{-1} p^2 = (2m)^{-1} \sum_{\lambda} p_{\lambda} p_{\lambda} \quad (12)$$

for spin zero particles, and

$$H = (2m)^{-1} (\gamma \cdot p)^2 = (\sum_{\lambda} \gamma_{\lambda} p_{\lambda}) (\sum_{\mu} \gamma_{\mu} p_{\mu}) \quad (13)$$

for spin 1/2 particles with nonzero mass m . Here, we have chosen γ such that the four components $\gamma_1, \gamma_2, \gamma_3$, and γ_4 are the familiar

four-by-four γ matrices of the Dirac theory,^{8,12} $p = mu$, and $u = dx/d\tau$. These four matrices $\gamma_1, \gamma_2, \gamma_3$, and γ_4 anticommute in pairs and their squares are unity. That is, $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}$. Since the Dirac-Hamiltonian $H^{(D)}$ is the total energy E of the particle which is the fourth component of the energy-momentum four-vector $p = (p_x, p_y, p_z, iE/c)$, the ordinary time t although not a Lorentz invariant parameter, is used to mark the dynamical evolution of the particle (that is, to mark the continuous unfolding of canonical transformations). Hence it is relatively easy to show that $[H^{(D)}, L] \neq 0$, where from Eq.(43.2) of Schiff¹ $H^{(D)} = E = -c\sum_j \alpha_j p_j - \beta mc^2$, $\gamma_4 \gamma_j = i\alpha_j$, $j = 1, 2$, and 3 , and (here only) $\beta = -\gamma_4$. That is, the Dirac-Hamiltonian $H^{(D)}$ cannot be manifestly Lorentz covariant, since it is not a Lorentz invariant world scalar.^{8,7} However, the relativistic invariance of the Dirac eigenvalue equation can be demonstrated¹² by assuming either (1) γ 's transform similar to a four-vector and the wave function $\psi(x,y,z,t)$ remains the same except for the Lorentz transformation of coordinates, or (2) γ 's remain fixed and the wave function ψ undergoes a similarity or equivalence transformation.

In light of Eqs.(10) and (11), let us now examine the Lorentz covariance of the Newtonian concept of simultaneity. We may define a simultaneity mapping matrix S in the F frame as

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 \\ S_{21} & S_{22} & S_{23} & 0 \\ S_{31} & S_{32} & S_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (14)$$

Since $x(b) = Sx(a)$, i.e., $x_\mu(b) = \sum_\nu S_{\mu\nu} x_\nu(a)$, it is clear from Eq.(14) that $x_4(b) = ict(b) = x_4(a) = ict(a)$. That is, $t(b) = t(a)$. Hence Eq.(14) is a simultaneity mapping matrix for the F frame. It should be noted from

Eq.(14) that all Newtonian spatial rotations are also simultaneity mappings, since $S_{ij} \neq 0$ for $i, j = 1, 2$, and 3 . For example, for a rotation about the z -axis through an angle θ , we have $S_{11} = S_{22} = \cos\theta$, $S_{12} = -S_{21} = \sin\theta$, $S_{33} = 1$, $S_{13} = S_{31} = S_{23} = S_{32} = 0$. Since $x(b') = S'x(a')$, and $S' = LSL^{-1}$, i.e., $S'_{\mu\nu} = \sum_{\alpha, \lambda} L_{\mu\alpha} S_{\alpha\lambda} L^{-1}_{\lambda\nu}$, we find that $S' \neq S$ and $x_4(b') \neq x_4(a')$, i.e., $t(b') \neq t(a')$. That is, two events that appear simultaneous to the F observer are not simultaneous to the F' observer. Since $[S, L] \neq 0$, and consequently $S' \neq S$, we see that the simultaneity mapping is not a manifestly Lorentz covariant mapping. Hence, the *Galilean invariant concept* of simultaneity is not a *truly Lorentz covariant concept* for inertial observers. This result is hardly surprising since this is a well-known fact from the early days of the special theory of relativity.

It is also apparent from this simultaneity mapping matrix S in the F frame and the corresponding mapping matrix S' in the F' frame that the Galilean invariant concepts of rigid body constraints, and static or action-at-a-distance forces are not truly Lorentz covariant concepts. Thus we see that the entire field of rigid body dynamics cannot have a truly Lorentz covariant relativistic analog. The two fundamental problems of non-relativistic quantum mechanics and physics are the simple harmonic oscillator and the hydrogen atom. But, both these problems involve static or action-at-a-distance forces and are therefore not directly amenable to a truly Lorentz covariant description. Although the Maxwell's equations and their consequences lend themselves very simply to covariant description, and thus the Coulomb potential in the *proper* frame can be cast into the Lorentz covariant Lienard-Wiechert potentials,⁵ in the literature^{1,3} there does not seem to exist a manifestly Lorentz covariant treatment of the hydrogen atom. This may be also due in part to the fact that the Dirac Hamiltonian itself is not manifestly Lorentz covariant^{8,7} (since it is not a Lorentz invariant world scalar).

Let us now examine the Newtonian concept of parity or the spatial inversion symmetry. We may define a spatial inversion matrix or the parity operator P in the F frame as

$$P = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (15)$$

It may be noted from Eqs.(14) and(15) that the parity operator P is a special case of the simultaneity operator S . Since $x(b) = Px(a)$, it is clear that the matrix P maps the space-time point $[x_1(a), x_2(a), x_3(a), x_4(a)]$ onto a space-time point $[x_1(b) = -x_1(a), x_2(b) = -x_2(a), x_3(b) = -x_3(a), x_4(b) = x_4(a)]$. This is clearly a parity operation in the F frame.

It should be noted that P is a linear, Hermitian operator with a determinant -1 . Orthogonal transformations with a determinant -1 are said to be *improper* rotations in contradistinction to matrices of determinant $+1$ which are *proper* rotations. It may also be noted that a simple spatial reflection operator R_j such that $x_j(b) = R_j x_j(a) = -x_j(a)$ for $j = 1, 2$, or 3 is a special case of the parity operator P . From Eqs.(10) and (15) we get

$$P' = \begin{pmatrix} -\gamma^2(1+\beta^2) & 0 & 0 & i2\beta\gamma^2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ i2\beta\gamma^2 & 0 & 0 & \gamma^2(1+\beta^2) \end{pmatrix} \quad (16)$$

Therefore, in the F' frame the matrix P' maps the space-time point $[x_1(a'), x_2(a'), x_3(a'), x_4(a')]$ onto a space-time point $[x_1(b') = \{-\gamma^2(1+\beta^2)x_1(a') + i2\beta\gamma^2 x_4(a')\}, x_2(b') = -x_2(a'), x_3(b') = -x_3(a'), x_4(b') = \{\gamma^2(1+\beta^2)x_4(a') + i2\beta\gamma^2 x_1(a')\}]$. This is obviously not a spatial

inversion mapping or a parity operation in the F' frame. Hence, a nuclear reaction that appears to conserve parity to an F observer will appear not to conserve parity to the F' observer. Further, it is seen from Eqs.(15) and(16) that $P \neq P'$ and hence $[P,L] \neq 0$. That is, the *Galilean invariant concept* of spatial inversion symmetry or parity is not a *truly Lorentz covariant concept* for inertial observers. It may be noted that when $c \rightarrow \infty$, $\beta \rightarrow 0$, and $P' \rightarrow P$ as it should be. Also, because of the Lorentz contraction, the Newtonian concept of rigid body constraints is not Lorentz covariant. Consequently, the intrinsic parity or the intrinsic spatial symmetry of physical systems are not Lorentz covariant concepts for inertial observers. The only exception is the intrinsic spherical symmetry of a light wave front. That is, the spherical symmetry of a light wave front is a truly Lorentz covariant concept for all inertial observers, since the velocity of light c in free space is the same for all inertial observers.

Let us now examine the Newtonian concept of time reversal symmetry. We may define the time reversal mapping matrix for the F frame as

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (17)$$

It should be noted that T is a linear, Hermitian operator with a determinant -1 . The corresponding mapping matrix T' in the F' frame is

$$T' = \begin{pmatrix} \gamma^2(1+\beta^2) & 0 & 0 & -i2\beta\gamma^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i2\beta\gamma^2 & 0 & 0 & -\gamma^2(1+\beta^2) \end{pmatrix} \quad (18)$$

It is clear that the matrix T maps the space-time point $[x_1(a), x_2(a), x_3(a), x_4(a)]$ onto a space-time point $[x_1(b) = x_1(a), x_2(b) = x_2(a), x_3(b) = x_3(a), x_4(b) = ict(b) = -x_4(a) = -ict(a)]$ in the F frame. However, in the F' frame the corresponding mapping matrix T' maps the space-time point $[x_1(a'), x_2(a'), x_3(a'), x_4(a')]$ onto a space-time point $[x_1(b') = \{\gamma^2(1+\beta^2)x_1(a') - i2\beta\gamma^2x_4(a')\}, x_2(b') = x_2(a'), x_3(b') = x_3(a'), x_4(b') = \{-\gamma^2(1+\beta^2)x_4(a') - i2\beta\gamma^2x_1(a')\}]$. Clearly, this is not a time reversal mapping in the F' frame. Further, it is seen from Eqs.(17) and (18) that $T' \neq T$ and hence $[T, L] \neq 0$. Thus, we see that the *Galilean invariant concept* of time reversal symmetry is not a *truly Lorentz covariant concept* for inertial observers.

It is known in mechanics⁷ that the momentum canonical to the time coordinate t is the total energy E . Thus in quantum mechanics the energy operator $E = (i\hbar/2\pi)\partial/\partial t$. Hence, $TE = -E = (i\hbar/2\pi)\partial/\partial(-t)$. That is, a pure time reversal T corresponds to "negative energy" states. But, in the non-relativistic quantum mechanics the energy cannot be negative. This led Wigner² to suggest that the proper quantum mechanical time reversal operator $T_W = TK$, where K is the complex conjugation operator (that changes $i \rightarrow -i$). Then, $T_W E = TKE = E = (-i\hbar/2\pi)\partial/\partial(-t)$. Since H of Eq.(1) is a Galilean invariant scalar, it is apparent that this Eq.(1) is invariant under this Wigner time reversal operation T_W . However, in the relativistic quantum mechanics Feynman¹² proposed that the negative energy states should be interpreted as states representing particles moving backward in time. He argues that in the classical equation of motion $(m d^2x_\mu/d\tau^2) = \sum_\nu (edx_\nu/d\tau) F_{\mu\nu}$, reversing the direction of the proper time τ amounts to the same as reversing the sign of the charge so that an electron moving backward in time would look like a positron moving forward in time. Here, $F_{\mu\nu}$ is the familiar antisymmetric

electromagnetic field tensor.^{5,12} That is, the time-reversed negative energy states of a particle correspond to the positive energy states of the corresponding antiparticle. Further, it is well known that the Maxwell's equations and their consequences lend themselves very simply to a Lorentz covariant description,⁵ and are invariant with respect to the change in sign of the charge density.² This leads one naturally to consider the charge conjugation on an equal footing with time reversal in quantum electrodynamics. However, this charge conjugation operation does not fall directly within the framework of the theory of world space mapping under study here.

Let us now examine the Minkowskian concept of the *world parity* or *the proper parity* operator W [i.e., the space-time reversal operator (P-T)]. We may define the world parity or the space-time reversal mapping matrix $W = (P-T)$ for the F frame as

$$W = (P-T) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -I \quad (19)$$

It should be noted that W is a linear, Hermitian operator with the determinant $+1$, and is equal to the negative of the identity matrix in this Minkowskian four-space (i.e., the world space or the space-time continuum). It is clear that the matrix W maps the space-time point $[x_1(a), x_2(a), x_3(a), x_4(a)]$ onto the space-time point $[x_1(b) = -x_1(a), x_2(b) = -x_2(a), x_3(b) = -x_3(a), x_4(b) = -x_4(a)]$ in the F frame. The corresponding mapping matrix W' in the F' frame is

$$W' = L W L^{-1} = W = -I \quad (20)$$

Thus the matrix W' maps the space-time point $[x_1(a'), x_2(a'), x_3(a'),$

$x_4(a')$] onto the space-time point $[x_1(b') = -x_1(a'), x_2(b') = -x_2(a'), x_3(b') = -x_3(a'), x_4(b') = -x_4(a')]$ in the F' frame. This is clearly a world parity or the space-time reversal mapping in the F' frame. Since $W' = W$, we see that $[W, L] = 0$. That is, two events that are connected by the space-time reversal operation to an F observer are also connected by the same space-time reversal operation to the F' observer. Thus the *concept of world parity* or the space-time reversal symmetry is indeed a *truly Lorentz covariant concept* for all inertial observers. This result is in accordance with the spirit of the generalized version of the Mach's principle: The laws of physics should not depend on the particular Minkowskian (or world) geometrical coordinate system we happen to choose.

As seen from Eq.(11), any symmetry operator or the world mapping matrix M is manifestly Lorentz covariant if and only if it commutes with the Lorentz transformation matrix L . For example, a null matrix O will clearly satisfy Eq.(11), i.e., $[O, L] = 0$. This truly Lorentz covariant null matrix O maps any space-time point onto the origin of coordinates. Also, the unit or the identity matrix I will clearly satisfy Eq.(11), i.e., $[I, L] = 0$. This truly Lorentz covariant identity matrix I maps any space-time point onto itself. Further, $[\lambda I, L] = 0$, where λ is a number or a Lorentz invariant timelike world scalar such as the Hamiltonians of Eqs.(12) and (13). Indeed, one can generate *the whole set of truly Lorentz covariant symmetry operators or the mapping matrices* by making use of the following commutation relations:

$$\begin{aligned}
 [A, B] &= -[B, A] \quad , \quad [A, \lambda I] = 0 \quad , \\
 [(A+B), C] &= [A, C] + [B, C] \quad , \\
 [AB, C] &= [A, C] B + A [B, C] \quad , \\
 [A, [B, C]] + [B, [C, A]] + [C, [A, B]] &= 0 \quad .
 \end{aligned}
 \tag{21}$$

For example, letting $C = L$ in the last relation of Eq.(21) we find that if $[A, L] = 0$ and $[B, L] = 0$, then $[[A, B], L] = 0$. That is, the commutator bracket of any two truly Lorentz covariant symmetry operators or mappings is itself a truly Lorentz covariant symmetry operator or mapping. This is somewhat analogous to the Poisson's theorem⁷ which states that the Poisson bracket of two constants of the motion is itself a constant of the motion.

IV. CONCLUSIONS

In conclusion we have examined the Lorentz covariance of symmetry operations by using the theory of world space mapping in inertial frames. We have shown that the Galilean invariant concepts of simultaneity, rigid body constraints, static or action-at-a-distance forces, parity, and time reversal symmetry are not Lorentz covariant concepts for inertial observers. In particular, we find that just as the concept of simultaneity has no significance independent of the Lorentz inertial frame, likewise so are the concepts of parity and time reversal. However, the world parity (i.e., the space-time reversal symmetry) is a truly Lorentz covariant concept for all inertial observers. Indeed, it is found that only those mapping matrices M that commute with the Lorentz transformation matrix L are the ones that correspond to manifestly Lorentz covariant operations. This result is in accordance with the spirit of the world space Mach's principle: The laws of physics should not depend on the particular Minkowskian (or world) coordinate system we happen to choose. The formal relativistic mapping theory used here does not have a corresponding non-relativistic analog. The basic reason for this is the Lorentz transformation is an orthogonal transformation in the Minkowskian space while the Galilean transformation is not an orthogonal transformation in the three dimensional Euclidean space and the absolute time.

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