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# GUIDING-CENTER HAMILTONIAN FIGURE-8 PARTICLES IN AXISYMMETRIC FIELD-REVERSED CONFIGURATIONS

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# Guiding-Center Hamiltonian for Figure-8 Particles In Axisymmetric Field-Reversed Configurations

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## ABSTRACT

The guiding-center Hamiltonian K is derived for so-called "figure-8" particles which are present in field-reversed mirror configurations, using a formalism developed previously. For such particles, the gyro-orbit cannot be approximated by a circle, and standard approaches to guiding-center theory are thus totally inapplicable. K manifests this intrinsic difference by a quite different dependence on the gyroaction, and by familiar effects such as mirroring and magnetic-gradient drifts being controlled by the radial derivative of the magnetic field strength B at the point of field-reversal, rather than by B itself, as occurs in standard guiding-center theory.

## I. INTRODUCTION

In standard guiding-center theory, the assumption is made that the parameter $\eta$, the ratio of a particle's gyroexcursion $\rho$ to the magnetic scale length $L_\perp$ normal to the magnetic field $\underset{\sim}{B}$, is a small parameter. This implies that the particle's gyromotion is well approximated by a circular orbit, centered about a particular field line, and characterized by $\underset{\sim}{B}$ in the vicinity of that line. As noted in previous work,[1-3] a valid guiding center theory can be found even for $\eta$ not small, provided only that a separation exists in the time scale of the gyroperiod $\tau_g$ and the slower time scale $\tau_b$ for the variation in magnetic field structure which a particle sees as it moves parallel to the field. That is, one must have $\epsilon << 1$, where $\epsilon \equiv \tau_g/\tau_b$.

For some confinement schemes of practical interest, there are classes of particles for which $\eta$ is not a good small parameter, and yet $\epsilon$ is. Thus the usual expressions for the guiding-center Hamiltonian K and the gyroaction J (the adiabatic invariant of the gyromotion) are invalid, but valid forms for K and J do exist. In Refs. 1 and 3 we have applied a general prescription (described in each of Refs. 1-3) for deriving K, to two specific cases where the standard assumption $\eta \to 0$ is invalid. Here a third case is considered, namely that of the so-called "figure-8" particles in a reversed-field configuration, such as the Berta experiment at Cornell.

In Ref. 1, the problem of particles in a strongly-sheared field (e.g., ions in the Tormac sheath) was considered. There $\eta$ was taken to be less than one, but not negligibly small, and the finite $\eta$ and finite $\epsilon$ corrections to K were found, to second order in both parameters. In Ref. 3, K was obtained for the high-energy ring particles in an unsheared axisymmetric field, such as in the Berta experiment. For these particles one has $\eta \geq 1$, and the physical interpretation of the various terms in K is quite different than that for the $\eta \to 0$ theory. In particular, the gyroaction J no longer corresponds to the full perpendicular kinetic energy $E_\perp$ of a particle as in the $\eta \to 0$ case, but rather describes only that small contribution to $E_\perp$ due to the radial wobble in the gyro-orbit about a constant radial position $R = R_G$.

Though $\eta \gtrsim 1$ for the ring-particle case (and therefore the gyromotion is completely nonlocal), the theory and the expression for K are local, in the sense that only small radial excursions $\delta R$ are made from this "guiding-center" radius $R_G$. Also, the gyro-orbit, while completely unrelated to the field line about which it is centered, is still well approximated by a circle. In the case of figure-8 particles, to be studied here, both of these similarities to the $\eta \to 0$ theory are absent. The gyromotion occurs about the radius $R_0$ of field reversal, so that the character of $\underset{\sim}{B}$ changes fundamentally in the course of a gyroperiod. Because of this, the gyro-orbit has a figure-8 shape (cf. Fig. 2), and so cannot be approximated by a circle.

The paper is organized as follows. In Sec. II the problem is set up and the procedure for deriving K is sketched. The reader is referred to Ref. 3 for a fuller description and justification of the derivation method. The method involves first solving the associated one-dimensional problem which would result if, in addition to the assumed axisymmetry, there were full cylindrical symmetry. A good deal of the physics of the full problem is contained in this associated problem, which is treated in Sec. III. In Sec. IV we make the remaining steps to finding K, relying heavily on the work of Ref. 3. K is given in Eq. (42) of this section, the principal result of this paper. In Sec. V we discuss some of the noteworthy features of the form obtained for K and the concomitant equations of motion, comparing K for figure-8 particles with the form given in Ref. 3 for the higher-energy "ring" particles, and with the form valid for the conventional small-$\eta$ problem.

## II. OVERVIEW

A general description of the procedure for obtaining K from the original particle Hamiltonian H is given in each of Refs. 1-3. Because the magnetic geometry considered here is the same as that in Ref. 3, so are the vector potential $\underset{\sim}{A}$, and the particle Hamiltonian H. Much of the development in Ref. 3 is thus directly applicable here, and we shall accordingly only sketch those portions which are fully dealt with there.

As in Ref. 3, we adopt the right-handed cylindrical coordinate system $(R, z, \phi)$ illustrated in Fig. 1, with R being the distance from the symmetry axis, z the distance along that axis, and $\phi$ the toroidal angle. The magnetic field lies entirely in the (R, z) plane, and so is derivable from the single component $A_\phi \equiv R^{-1}\hat{\phi} \cdot \underset{\sim}{A}$ of the vector potential. The particle Hamiltonian is then given by $(m = c = 1)$

$$H\left(R, p_R;\ \epsilon z, p_z; p_\phi\right) = \frac{1}{2} p_R^2 + U\left(R \,|\, \epsilon z, p_z; p_\phi\right) \quad , \tag{1}$$

where

$$U \equiv \frac{1}{2} p_z^2 + \frac{1}{2} R^{-2} \left[p_\phi - eA_\phi(r,\ \epsilon z)\right]^2 + e\,\Phi(R, \epsilon z) \quad , \tag{2}$$

exactly as in Ref. 3. As done there, we henceforth drop the electrostatic potential $\Phi$, suppress most explicit notation of the constant of motion $p_\phi$, and denote by $\lambda^o$ the pair of variables $(\epsilon z, p_z)$, which primarily describe motion along the field. The superscript "o" on $\lambda$ denotes that $(\epsilon z, p_z)$ are the "original" parallel variables, as opposed to the "mixed" pair $\lambda^m \equiv (\epsilon z, P_z)$, or the "final" pair $\lambda^f \equiv (\epsilon Z, P_z)$, which will be used later.

The magnetic field is $\underset{\sim}{B} = \epsilon \hat{R} R^{-1}\ \partial A_\phi / \partial \epsilon z - \hat{z} R^{-1}\ \partial A_\phi / \partial R$ . (3)

The slow variation of the geometry parallel to $\underset{\sim}{B}$ is manifested by $\epsilon$ multiplying z everywhere it occurs. while the R dependence of H is arbitrary, corresponding to arbitrary $\eta$.

The transformation procedure from H to K has two steps. In the first step, a finite canonical transformation S is made, transforming to action-angle variables $(\Theta^o, J^o)$ of gyration. The resultant Hamiltonian $K^o$ has an $O(\epsilon)$ dependence on $J^o$, and so $J^o$ is not quite the desired adiabatic invariant J, but lies close enough to it that K, J, and gyrophase $\Theta$ can be obtained from $K^o$, $J^o$, and $\Theta^o$ by a sequence T of perturbative canonical transformations. Transformations T comprise the second stage of the transformation procedure.

S is chosen as the transformation which would solve the problem exactly, if $\lambda$ were taken as a constant parameter, as is the case in the limit $\varepsilon \to 0$. It is induced by a generating function $S(R, J^o; \lambda^m)$ of the mixed-variable type (cf. Ref. 4, Sec. 8-1), of the form[3]

$$S(R, J^o; \lambda^m) \equiv zP_z^o + S_1 \quad , \tag{4}$$

$$S_1(R, J^o; \lambda^m) \equiv \int^R dR' \, p_R \left[R', E(J^o, \lambda^m), \lambda^m\right] \quad . \tag{5}$$

The function $E(J^o, \lambda)$ here would equal the particle's true energy if evaluated at $\lambda = \lambda^o$. By evaluating the parameter $\lambda$ at the mixed-variable value $\lambda^m \equiv (\varepsilon z, P_z^o)$, the necessary mixed-variable form for S is achieved, and the canonicity of the transformation is guaranteed.

From Eqs. (4) and (5), one obtains the transformation equations. In particular, one has

$$p_z = \partial S/\partial z = P_z^o + \varepsilon \partial S_1/\partial \varepsilon z \equiv P_z^o + \varepsilon \, \delta p_z (R, J^o, \lambda^m) \quad , \tag{6}$$

and

$$z = \partial S/\partial p_z = Z^o - \partial S_1/\partial p_z \equiv Z^o + \delta z (R, J^o, \lambda^m) \quad . \tag{7}$$

*These transformation equations are used to eliminate the old variables* $\lambda^o$ *in H in* favor of the new variables $\lambda^f$ in $K^o$. Because $\delta z$, $\delta p_z$ themselves are expressed in terms of the mixed variables, this elimination process is iterative, yielding Hamiltonian $K^o$ as an expansion in $\varepsilon$, in which $\delta z$, $\delta p_z$ appear prominantly in the coefficients $K_\ell^o$ of each $\varepsilon^\ell$ [cf. Eqs. (36)].

As in Ref. 3, our first and most difficult task is to set up and solve the "frozen - $\lambda$ " problem, i.e., the one degree-of-freedom problem in which $\lambda$ is taken to be constant in Eq. (1), so that H describes a particle moving in a one-dimensional potential U(R). The solution of this problem is equivalent to the evaluation of $S_1$ in *Eq. (5), and so allows one to obtain* $\delta z$, $\delta p_z$, *and accordingly* $K^o$. Because the desired quantities $\delta z$, $\delta p_z$ are derivatives of $S_1$, as in Ref. 3 it will not be necessary, in the approximation we shall be working, to evaluate $S_1$ itself, but only its derivative $\partial S_1/\partial R = p_R$. In the following section we solve the frozen-$\lambda$ problem appropriate to figure-8 particles, and thereby obtain $K^o$. Given $K^o$, K is readily obtained using Lie perturbative techniques.

## III. THE "FROZEN-$\lambda$" PROBLEM

We consider the one-dimensional problem obtained by taking $\lambda$ to be simply a constant parameter in Eq. (1), as is the case in the limit $\varepsilon \to 0$. When $\varepsilon = 0$, the magnetic geometry is cylindrically symmetric, with only straight field lines, running parallel to the z-axis. In Fig. 2, the trajectory of a typical figure-8 particle in the plane normal to the field lines is drawn. The gyro-orbit is centered about the radial position $R = R_o$ at which the field changes direction. The field reversal causes a change in the sense of the gyro-orbit at $R_o$, resulting in the characteristic figure-8 shape.

We now consider the form of the potential $U(R|\lambda)$ in the vicinity of the radius $R = R_o(\varepsilon z)$ at which $B_z(R, \varepsilon z)$ equals zero, and about which the figure-8 motion occurs. From Eq. (3) one has

$$A_\phi (R, \varepsilon z) = - \int^R dR' R' B_z (R', \varepsilon z) \quad . \tag{8}$$

We take $B_z < 0$ for $R < R_o$ and $B_z > 0$ for $R > R_o$. Then $A_\phi (R)$ has the general form illustrated in Fig. 3; for $R < R_c$, $A_\phi$ increases monotonically toward a maximum $A_\phi (R_o)$ at $R_o$, then decreases monotonically for $R > R_o$.

Expanding $(p_\phi - e A_\phi)$ to second order in $\delta R \equiv R - R_o$ and using Eq. (3) or (8), one has

$$p_\phi - eA_\phi = \delta p_\phi + \frac{1}{2} (\delta R)^2 (R_o \, \partial \Omega_z / \partial R_o) \quad , \tag{9}$$

where $\Omega_z \equiv eB_z$, and $\delta p_\phi \equiv p_\phi - p_{\phi o} \equiv p_\phi - eA_\phi(R_o)$.

In expanding $U(R|\lambda) = \frac{1}{2} R^{-2} (p_\phi - eA_\phi)^2 + \frac{1}{2} p_z^2$ about $R = R_o$, one uses the expansion Eq. (9), and additional terms from the expansion of the denominator $R^{-2}$. The ratio of the logarithmic derivatives of these contributions $\left[\text{i.e., } \partial/\partial R \; \ell n(p_\phi - eA_\phi)^2 \text{ and } \partial/\partial R \; \ell n\, R^{-2}\right]$ is of order $(R_o/\delta R)$, which we take to be much greater than one. We therefore neglect the contribution of derivatives of $R^{-2}$ to the expansion. The potential U thus has the approximate form

$$U(R|\lambda) = \tfrac{1}{2} \left[\delta p_\phi / R_o + \tfrac{1}{2}(\delta R)^2 \, \partial \Omega_z / \partial R_o\right]^2 + \tfrac{1}{2} p_z^2 \quad . \tag{10}$$

For $\delta p_\phi = 0$, $p_\phi - eA_\phi$ is as illustrated in Fig. 4, and U is a simple quartic well $U_o$,

$$U_o(\delta R|\lambda) \equiv (a/4)(\delta R)^4 + \tfrac{1}{2} p_z^2 \quad , \tag{11}$$

where $a \equiv \frac{1}{2} (\partial \Omega_z / \partial R_o)^2$. From Hamilton's equation for $\phi$,

$$\dot{\phi} = R^{-2}(p_\phi - eA_\phi) \quad , \tag{12}$$

one sees that for $\delta p_\phi = 0$, $\dot{\phi}(R_o) = 0$, i.e., a particle is moving exactly radially inward or outward at the point $R_o$ of field reversal. The gyro-orbit thus has half-loops which are somewhat less closed than those in Fig. 2.

Figure 2 is drawn for the case $\delta p_\phi < 0$, for which $\dot{\phi} \propto p_\phi - eA_\phi$ changes sign in the course of a gyroperiod. From Eq. (9), this sign change occurs at $\delta R = \pm \Delta R$, where

$$\Delta R \equiv \left| 2 \, \delta p_\phi \, (R_o \, \partial \Omega_z / \partial R_o)^{-1} \right|^{1/2} \quad . \tag{13}$$

In addition to the quartic portion $U_o$ in U, there is an additional contribution

$$U_1(\delta R|\lambda) \equiv (a/4) \left[(\Delta R)^2 - 2(\delta R)^2\right] (\Delta R)^2 \tag{14}$$

so that U may be written

$$U(R|\lambda) = (a/4) \left[(\delta R)^2 - (\Delta R)^2\right]^2 + \tfrac{1}{2} p_z^2 \quad . \tag{15}$$

Figure 5a shows $p_\phi - eA_\phi$ for this case. The potential U is displayed in Fig. 5b. One sees that the addition $U_1$ to U gives U a double-well form. For figure-8 particles, the energy must be large enough that the particle passes through both wells during a gyroperiod.

For $\delta p_\phi > 0$, $\dot{\phi} > 0$ for all R, and the half-loops of the gyro-orbit are still more open than in the $\delta p_\phi = 0$ case. The potential may be written

$$U(R/\lambda) = (a/4)\left[(\delta R)^2 + (\Delta R)^2\right]^2 + \tfrac{1}{2} p_z^2 \quad , \tag{16}$$

so that $U_1 \equiv U - U_0$ is given by

$$U_1(\delta R/\lambda) = (a/4)\left[(\Delta R)^2 + 2(\delta R)^2\right](\Delta R)^2 \quad . \tag{17}$$

U(R) then has a single minimum, with a well which is quadratic in character for $|\delta R| < \Delta R$, and quartic for $|\delta R| > \Delta R$.

As $\delta p_\phi$ is increased still further, U(R) retains the single-well form, but the approximate form (10) for U becomes invalid. In Ref. 3 it is shown that the minimum in U(R) occurs at that value $R_G$ for which $(\dot{\phi} - \Omega_z)|_{R_G} = 0$. Thus as $p_\phi$, and so $\dot{\phi}$, increase, $\Omega_z$ must also, and $R_G$ moves away from the point $R_0$ about which the particles considered here have their gyro-orbits. This larger $-p_\phi$ case corresponds to the higher energy ring particles, which is the subject of Ref. 3.

For small - $\eta$ particles in a nonvanishing magnetic field[1] or for ring particles,[3] the potential well is dominantly quadratic in character, corresponding to a leading dependence of K on the gyroaction of the form $\Omega_0 J$, with $\Omega_0$ the frequency of oscillation in that quadratic well, about equal to the local gyrofrequency $eB_z$ ($m = c = 1$). Anharmonic corrections to this quadratic potential may then be treated perturbatively, yielding K as an expansion in the small parameter $\eta \equiv \rho/L_\perp$ or $S \equiv r_0/L_\perp$ (where $r_0$ is the amplitude of the excursion in $\delta R$), in addition to the expansion, already noted, in the parameter $\varepsilon$.

For figure-8 particles, however, because of the field reversal, the potential U is dominantly quartic in character. Thus, we shall first solve the "unperturbed" frozen-$\lambda$ problem which has $U_0$ as the potential. We then solve for the motion in the full potential U, treating $U_1$ as a perturbation. From Eq. (14) or (17), the validity of this perturbative treatment requires that

$$\xi \equiv (\Delta R/r_0)^2 << 1 \quad . \tag{18}$$

Thus $\xi$ becomes the appropriate expansion parameter for the frozen-$\lambda$ problem, replacing $\eta$ in Ref. 1 or $\xi$ in Ref. 3. In the remainder of this section we carry out this solution scheme, to $O(\xi)$.

Motion in a Quartic Well

We now solve the "unperturbed" portion of the frozen-$\lambda$ problem, having Hamiltonian

$$h_0(\delta R, p_R|\lambda) \equiv ½\, p_R^2 + U_0(\delta R|\lambda) = ½\, p_R^2 + (a/4)(\delta R)^4 \quad . \tag{19}$$

(The term $½\, p_z^2$, constant in the frozen-$\lambda$ phase space, has been dropped here.) The solution approach here parallels one given by Chirikov.[5] A particle with Hamiltonian $h_0$ attains a maximum momentum $p_0 \equiv (2h_0)^{½}$ and excursion in $\delta R$ $r_0 \equiv (4h_0/a)^{¼}$. Using $d\delta R/dt = p_R$, and $p_R(\delta R, h_0) = \pm[2(h_0 - U_0)]^{½}$, one can write the motion for this unperturbed problem as

$$t = \int_{r_0}^{\delta R} d(\delta R')\, p_R^{-1}(\delta R', h_0) = (r_0/p_0) \int_1^y dw\, (1 - w^4)^{-½} \quad , \tag{20}$$

where $y \equiv \delta R/r_0$. Using the definition[6] of the Jacobi elliptic function cn with modulus $k = 2^{-½}$,

$$u = 2^{½} \int_1^{\mathrm{cn}u} dw (1 - w^4)^{-½} \quad , \tag{21}$$

the solution in Eq. (20) may be written

$$\delta R(t) = r_o \, \mathrm{cn}\, \Omega_1 t \quad , \tag{22}$$

where $\Omega_1 \equiv 2^{\frac{1}{2}} p_o/r_o = 2^{\frac{1}{2}} (h_o a)^{\frac{1}{4}}$. The period $\tau = 2\pi/\Omega_o$ (where $\Omega_o \equiv \dot{\theta}$ is the angular frequency of oscillation in the well) is obtained by taking the integral in Eq. (20) around a complete cycle:

$$\tau = (r_o/p_o) \int dw(1 - w^4)^{-\frac{1}{2}} = 4\kappa/\Omega_1 \quad . \tag{23}$$

Here $\kappa \equiv 2^{\frac{1}{2}} \int_0^1 dw(1 - w^4)^{-\frac{1}{2}} \simeq 1.8541$ is the complete elliptic integral of the first kind, with modulus $k = 2^{-\frac{1}{2}}$ [usually denoted $K(2^{-\frac{1}{2}})$]. From Eq. (23), one has

$$\Omega_o \equiv 2\ /\tau = b\Omega_1 \tag{24}$$

with $b \equiv (\pi/2\kappa\ ) \simeq .8472$.

Now we transform from $(\delta R, p_R)$ to action-angle variables $(\theta, j)$ for this problem. The Hamiltonian in terms of these new variables is denoted by $k_o$, $k_o(j) = h_o(\delta R, p_R)$. Because $\Omega_o = \partial k_o/\partial j$, one has

$$j = \int_0^{k_o} dk'/\Omega_o = (4/3)\, a^{-\frac{1}{4}}\, b^{-1}\, 2^{-\frac{1}{2}}\, k_o^{3/4} \quad , \tag{25}$$

or

$$k_o(j) = d\, j^{4/3}, \quad d \equiv (1/4)\, a^{1/3} (3b)^{4/3} \simeq .87\, a^{1/3} \quad . \tag{26}$$

Writing Eq. (22) in terms of $\theta$, one has

$$\delta R = r(\theta, j|\lambda) \equiv r_o\, \mathrm{cn}(b\,\theta) \tag{27}$$

and since $p_R = \delta\dot{R}$,

$$p_R = p(\theta\ , j|\lambda) \equiv 2^{\frac{1}{2}} p_o\, \mathrm{cn}'(b\theta) \quad , \tag{28}$$

where $\mathrm{cn}'\, u \equiv (\partial/\partial u)\, \mathrm{cn}\, u$. Eqs. (26) - (28) constitute the solution of the unperturbed frozen-$\lambda$ problem. We note the scalings of the various important quantities with gyroaction j:

$$k_o \sim j^{4/3},\ \Omega_c = b\Omega_1 \sim j^{1/3},\ \delta R \sim r_o \sim j^{1/3},\ p_R \sim p_o \sim j^{2/3} \quad . \tag{29}$$

These are to be contrasted with the scalings for the simple-harmonic problem,[3] $k_o \sim j$, $\Omega_o \sim 1$, $\delta R \sim r_o \sim j^{1/2}$, $p_R \sim p_o \sim j^{1/2}$.

Perturbation by $U_1$

We now treat the effect on the solution just obtained of the additional contribution $U_1$ to the potential. We proceed to $O(\xi)$ using Lie methods, wholly analogous to the procedure in Ref. 3. The term in $(\Delta R)^2$ in expressions (14) and (17) is down from the term in $(\delta R)^2$ by $O(\xi)$, and is therefore neglected. We thus consider the problem $k^o(\theta, j) = k_o(j) + \xi k_1^o(\theta, j)$, with

$$\xi k_1^o \equiv U_1(\delta R|\lambda) = (a/2)(\Delta R)^2(\delta R)^2 \tag{30}$$

where the plus (minus) sign holds for $\delta p_\phi > 0$ $(< 0)$.

We apply the operator $\exp \xi \check{G}_1$ to $k^o$, with Lie generator $\xi G_1$ chosen to remove the $\theta$ dependence of the Hamiltonian to $O(\xi^2)$. Because we are working only to $O(\xi)$, the new Hamiltonian $k^1 \equiv \exp(\xi \check{G}_1)k^o$ is an adequate approximation to the fully phase-independent Hamiltonian k for the frozen-$\lambda$ problem.

Just as in Ref. 3, $k_1 = k_1^1$ is the $\theta$-averaged part of $k_1^o$:

$$\xi k_1 \equiv \overline{\xi k_1^o} \approx (a/4)(\Delta R)^2 r_o^2 \quad . \tag{31}$$

Here we have used the fact[5] that $\mathrm{cnb}\theta$ [$= .9550 \cos\theta + (1/23)\cos 3\theta + (1/23)^2 \cos 5\theta + \ldots$] is well approximated by $\cos\theta$. In this same approximation, the generator $\xi G_1$ is given by

$$\xi G_1 = -\Omega_o^{-1} \int^\theta d\theta' \left(\xi k_1^o - \overline{\xi k_1^o}\right) \approx \left(r_o^2/2\,\Omega_1\right)\sin 2b\theta \quad . \tag{32}$$

The solution of the full frozen-$\lambda$ problem to $O(\xi)$ is thus described by Hamiltonian $k = k_o + \xi k_1 + O(\xi^2)$, with $k_o$ and $\xi k_1$ given by Eqs. (26) and (31). In the following section, this solution is used in obtaining the guiding-center Hamiltonian K for the full problem in which $\lambda$ is time-dependent.

## IV. EXPRESSIONS FOR $K^o$, K

From this point, most of the general formulae of Ref. 3 are directly applicable here, if one replaces the expansion point $R_G$ there with $R_o$ here. The specific forms of some of the functions going into these general expressions [e.g., $r(\theta, j|\lambda)$, $p(\theta, j|\lambda)$] are, however, somewhat different, owing to the different nature of the frozen-$\lambda$ problems for the two cases.

It was found in Ref. 3 that neglecting terms of $O(\xi^2)$ and higher, $\delta p_z = -p(\theta, j)(\partial R_G/\partial \varepsilon z)$ and $\delta z = p(\theta, j)(\partial R_G/\partial p_z) = 0$. The same formulae hold here, with the replacement $R_G \rightarrow R_o$:

$$\delta z = 0 \quad , \tag{33}$$

and

$$\delta p_z = -p(\theta, j/\lambda)\,(\partial R_o/\partial \varepsilon z) \quad . \tag{34}$$

Analogous to Eqs. (47) and (48) of Ref. 3, $K^o$ is given by a double expansion in $\varepsilon$ and $\xi$,

$$K^o = K_{00} + \xi K_{01} + \varepsilon K_1^o + \varepsilon^2 K_2^o + O(\varepsilon^3, \varepsilon\xi, \xi^2) \quad , \tag{35}$$

with

$$K_{00} = k_0 + \tfrac{1}{2}(P_z^o)^2 = d(J^o)^{4/3} + \tfrac{1}{2}(P_z^o)^2 \quad , \qquad \xi K_{01} = \xi K_1 \; ,$$

$$K_1^o = \delta p_z P_z^o \; , \text{ and } K_2^o = \tfrac{1}{2}(\delta p_z)^2 \quad . \tag{36}$$

K is now obtained from $K^o$ by applying the Lie operator exp $\varepsilon\check{W}_1$ to $K^o$ to yield Hamiltonian $K^1$, which is $\Theta$-independent to $O(\varepsilon)$, and then applying exp $\varepsilon^2\check{W}_2$ to $K^1$, yielding Hamiltonian $K^2$ which is $\Theta$-independent to $O(\varepsilon^2)$. Since K is desired only to $O(\varepsilon^2)$, $K^2$ is an adequate approximation to K.

As usual, one chooses $K_1 = \overline{K_1^o}$, the gyrophase-averaged part of $K_1^o$. From Eqs. (28) and (36), this vanishes. This choice for $K_1$ determines the Lie generator $W_1$,

$$W_1 = -\Omega_o^{-1}\int^{\Theta} d\Theta' K_1^o = P_z(\partial R_o/\partial \varepsilon Z)\, r(\Theta, J\,|\,\lambda^f) \quad , \tag{37}$$

where the relation $p_R = d\,\delta R/dt = \Omega_o d\,\delta R/\delta\Theta$ has been used.

Proceeding now to remove the $\Theta$ dependence at $O(\varepsilon^2)$, as shown in Ref. 3 one has

$$K_2 = K_2^2 = \overline{K_2^1} = \overline{K_2^o} + \tfrac{1}{2}\,\overline{\check{W}_{1\Theta} K_1^o} \quad , \tag{38}$$

where $\check{W}_{1\Theta} \equiv (\partial W_1/\partial\Theta)\ \partial/\partial J - (\partial W_1/\partial J)\ \partial/\partial\Theta$ is that part of the Lie operator $\check{W}_1$ which acts on the one degree-of-freedom phase space of the frozen-$\lambda$ problem.

From Eqs. (36) and (34),

$$K_1^o = -P_z(\partial R_o/\partial\varepsilon Z)\, p(\Theta, J\,|\,\lambda^f) \quad . \tag{39}$$

Using this and Eq. (37), one has

$$\check{W}_{1\Theta} K_1^o = -\left[P_z(\partial R_o/\partial\varepsilon Z)\right]^2 \left[(\partial r/\partial\Theta)(\partial p/\partial J) - (\partial r/\partial J)(\partial p/\partial\Theta)\right]$$

$$= -\left[P_z(\partial R_o/\partial\varepsilon Z)\right]^2 \quad . \tag{40}$$

Here we have used the fact that the transformation $(\delta R, p_R) \rightarrow (\Theta, J)$ is canonical in the frozen-$\lambda$ phase space, so the Poisson bracket of r with p in this space equals one.

The other term in Eq. (38) is $\overline{K_2^o}$. For this we need to evaluate $\tfrac{1}{2}\overline{p^2}$. One may evaluate this exactly, using a simple application of the virial theorem (cf. Ref. 4, Sec. 3-4) in one dimension. Assuming a Hamiltonian $H = T + U$, with $T \equiv \tfrac{1}{2} P^2$ and $U \equiv \alpha x^{2n}$, the virial theorem implies that $2\overline{T} = n\overline{U}$, or $\overline{T} \equiv \tfrac{1}{2}\overline{p^2} = H\, n/(n+1)$. The overbar here indicates the same time or gyrophase average over one period of oscillation. For the problem in Ref. 3, n = 1 and so $\tfrac{1}{2}\bar{p}^2 = \tfrac{1}{2}(H) = \tfrac{1}{2}\Omega_o J$, the same result as obtained there by direct calculation. For the present case, n = 2, hence $\tfrac{1}{2}\overline{p^2} = (2/3)H$, where H here means $K_{oo}$ in the application we are considering. Using this and Eq. (40) in Eq. (38), one has

$$K_2 = \tfrac{1}{2}(\partial R_o/\partial \varepsilon Z)^2 \left[(4/3)k_o - P_z^2\right] \quad . \tag{41}$$

One notes that both contributions to $K_2$ have been evaluated exactly, by considerations valid for any positive value of n.

From the results of this section, we may write out the guiding-center Hamiltonian K, valid to $O(\varepsilon^2, \xi)$:

$$K = K_{oo} + \xi K_{o1} + \varepsilon^2 K_2 + O(\varepsilon^3, \varepsilon\xi, \xi^2)$$

$$= k_o + \tfrac{1}{2} P_z^2 \pm (a/4)(\Delta R)^2 r_o{}^2 + \tfrac{1}{2}\varepsilon^2 \left[(4/3)k_o - P_z^2\right](\partial R_o/\partial\varepsilon Z)^2 \ , \tag{42}$$

where $k_o = dJ^{4/3}$, $d \approx .87a^{1/3}$, $r_o = (4k_o/a)^{1/4}$, and a and $\Delta R$ are given in Eqs. (11) and (13), respectively.

## V. DISCUSSION

It is instructive to compare and contrast the guiding-center Hamiltonian (42) for figure-8 particles, with the corresponding expression derived in Ref. 3 for ring-particles, here denoted $K^r$:

$$K^r = k_o^r + \tfrac{1}{2} P_z^2 + \tfrac{1}{2} R_G^2 \Omega_z^2 + \tfrac{1}{2}\varepsilon^2(k_o - P_z^2)(\partial R_o/\partial \varepsilon Z)^2 \quad , \tag{43}$$

where $\Omega_z \equiv e B_z(R_G)$, and $k_o^r \equiv \Omega_o J$ is the kinetic energy of radial wobble in the well U(R). The correspondence of the term $\tfrac{1}{2} P_z^2$ and the ponderomotive term $\varepsilon^2 K_2$ in both is clear, and the physical interpretation of these terms here is the same as there.

One might be tempted to identify the terms $k_o$ and $\xi K_{01} \equiv \pm(a/4)(\Delta R)^2 r_o^2$ in Eq. (42) with $k_o^r$ and $\tilde{\mu}\Omega_z \equiv \tfrac{1}{2} R_G^2 \Omega_z^2$, respectively, in Eq. (43), but this is not quite accurate. As shown in Ref. 3, the term $\tilde{\mu}\Omega_z$ represents the bulk of the full perpendicular kinetic energy $E_\perp$ of a ring particle, the contribution $k_o^r$ due to the radial wobble being a small correction. Mathematically, $\tilde{\mu}\Omega_z$ arises as a contribution to the potential U, evaluated at the well bottom $R_G$. It thus corresponds to the term

$\frac{1}{2} R_o^{-2} (\delta p_\phi)^2$ in Eq. (10), which was neglected, being $O(\xi^2)$. Thus, $k_o$ corresponds to $\tilde{\mu}\Omega_z$ in the sense of being the main component of $E_\perp$, but to $k_o^r$ in the sense that, if a particle has negligible $\tilde{\mu}\Omega_z$ (as do figure-8 particles), the radial wobble is all that is left.

The term $\xi K_{01}$ in Eq. (42) represents a small addition to $k_o$ in $E_\perp$. It should be noted that the approximation $\xi \ll 1$ or $(\Delta R)^2 << r_o^2$ made here puts a lower bound on the value of J, and so $E_\perp$, for which expression (42) is valid. In the regime $r_o^2 << (\Delta R)^2$, the radial motion would be governed by a quadratic (rather than quartic) potential U, and $E_\perp$ would emerge in a form linear in J, as in standard $\eta \to o$ guiding-center theories.

The Hamiltonian $K^\eta$ for the $\eta \to o$ theory has the form

$$K^\eta = \Omega_o J + \tfrac{1}{2} g P_z^2 \quad . \tag{44}$$

Here $\Omega_o \equiv eB$, with B evaluated at the guiding-center, and $P_z$ is to be interpreted as the particle momentum along a field line. The metric factor g, whose spatial dependence gives rise to the curvature drift, equals one for an axis-encircling particle. We take g = 1 here, adequate for purposes of the qualitative considerations we are making. Using Hamilton's equations, one sees the mirroring effect of the $\Omega_o J = \mu B$ well:

$$\ddot{Z} = \dot{P}_z = -J\, \partial \Omega_o / \partial Z \quad . \tag{45}$$

Similarly, to $O(\xi)$, K for figure-8 particles is

$$K = dJ^{4/3} + \tfrac{1}{2} P_z^2 \quad . \tag{46}$$

Therefore

$$\ddot{Z} = \dot{P}_z = -J^{4/3}\, \partial d/\partial Z \quad . \tag{47}$$

Since $d \propto a^{1/3} \propto (\partial \Omega_z/\partial R_o)^{2/3}$, one sees that the mirroring effect for these particles is dependent on a different (though related) aspect of the magnetic field from that for particles with approximately circular gyro-orbit.

Summarizing, we have derived the guiding-center Hamiltonian K for figure-8 particles in a field-reversed mirror configuration. Guiding-center theories based on the assumption that the gyro-orbit may be approximated by a circle cannot be applied to such particles, and the scaling of K with gyroaction J manifests this intrinsic difference. The leading dependence of K on J (viz. $K \sim dJ^{4/3}$) arises from the solution of the associated frozen-$\lambda$ problem, in which a particle oscillates in a quartic, rather than quadratic, potential U. This form of U is characteristic of the field reversal, and so the coefficient d accompanying $J^{4/3}$ in K scales with the rapidity $(\partial B_z/\partial R_o)$ of the field reversal, rather than with the strength of B or $B_z$ at some guiding-center position, as in conventional theories. Thus, effects present in conventional theories (e.g., mirroring in a $\mu B$ well, and $\nabla B$ drifts) have counterparts here, but where B is replaced by a power of $\partial B_z/\partial R_o$, and where the scaling with gyroaction is different.

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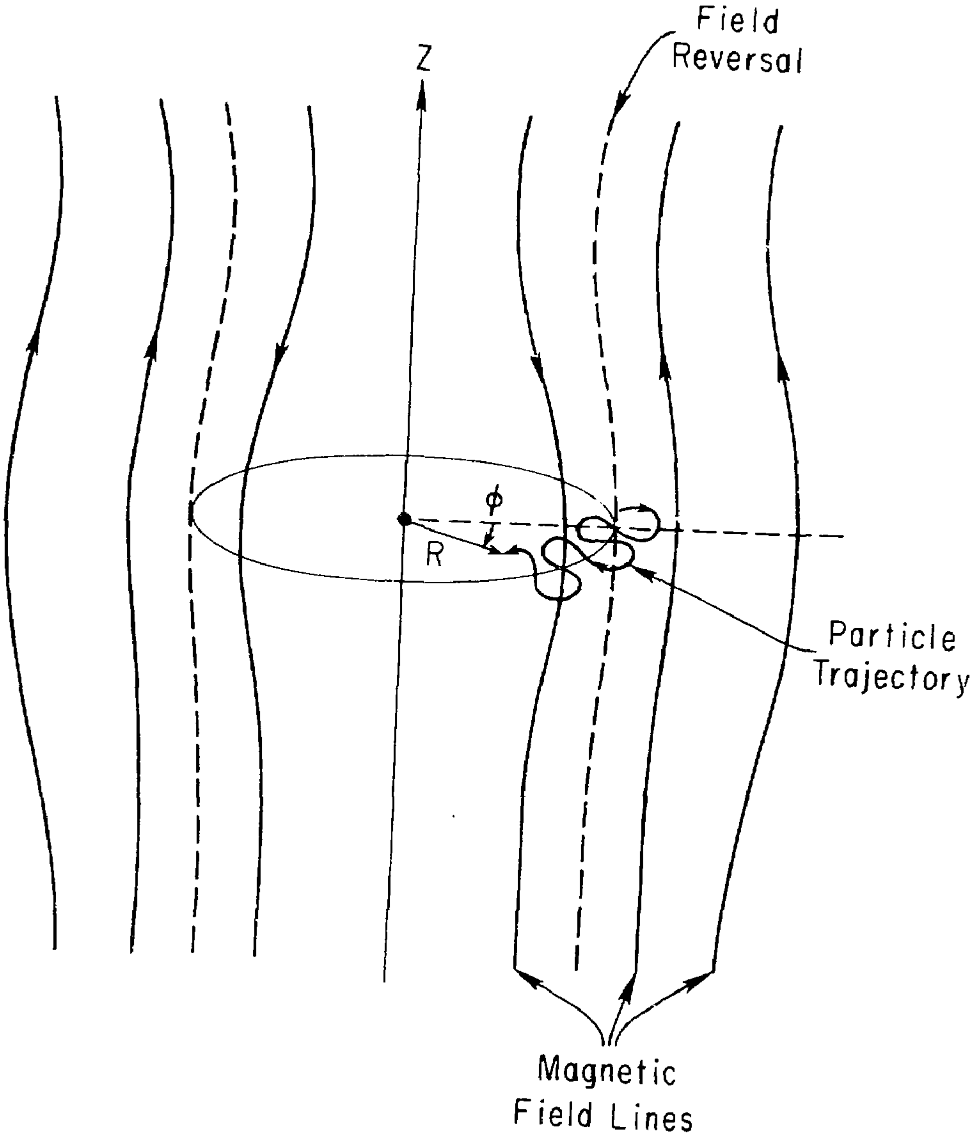


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Fig. 1. Illustration of the configuration considered in the text, showing the cylindrical coordinate (R, z, $\phi$), and picturing a portion of the trajectory for a typical figure-8 particle.

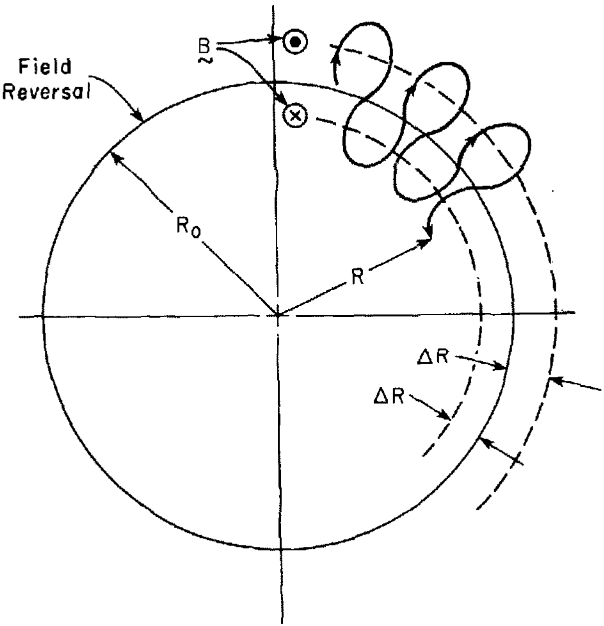


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Fig. 2. Trajectory of a typical figure-8 particle, projected onto the $(R, \phi)$- plane.

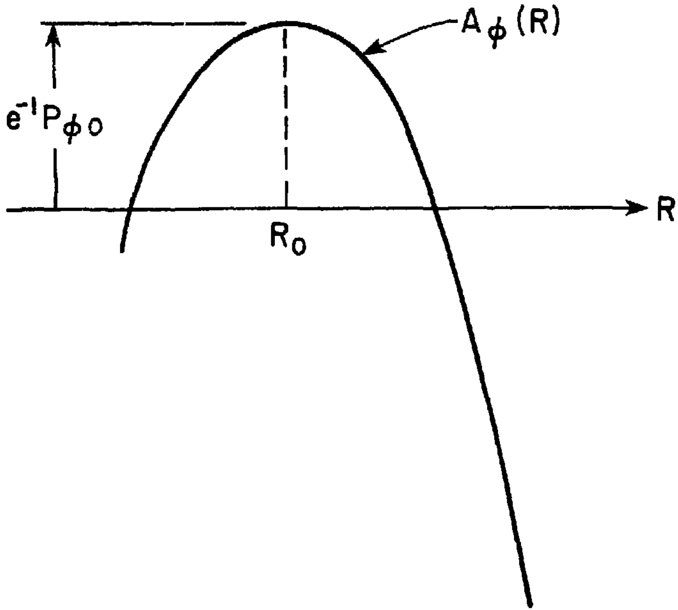


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Fig. 3. Sketch of the vector potential $A_\phi(R)$ in the vicinity of the radius $R_o$ of field reversal.

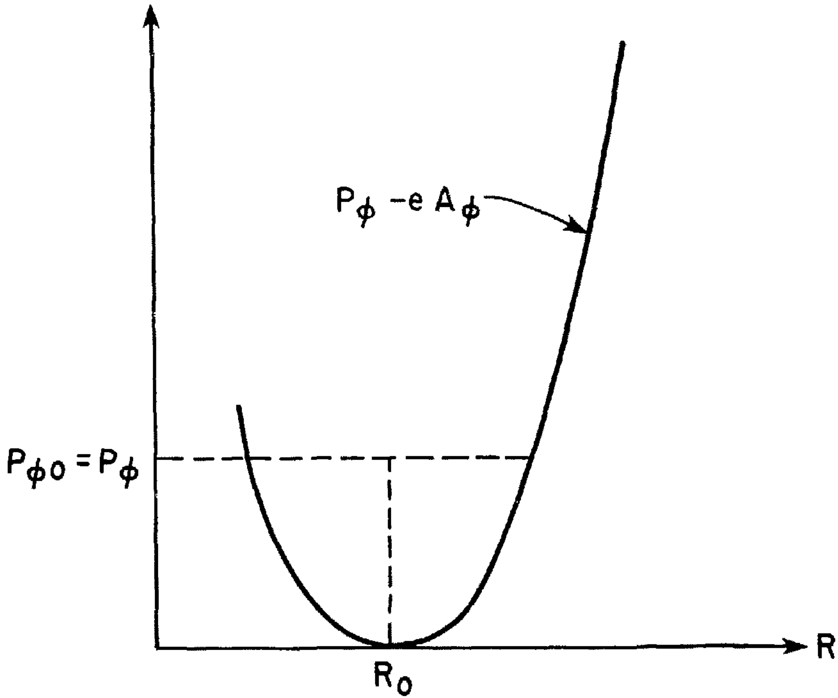


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Fig. 4. The function $p_\phi - eA_\phi(R)$ for $A_\phi$ of Fig. 3, and for $p_\phi = eA_\phi(R_o)$. The corresponding potential $U(R)$ is a simple quartic.

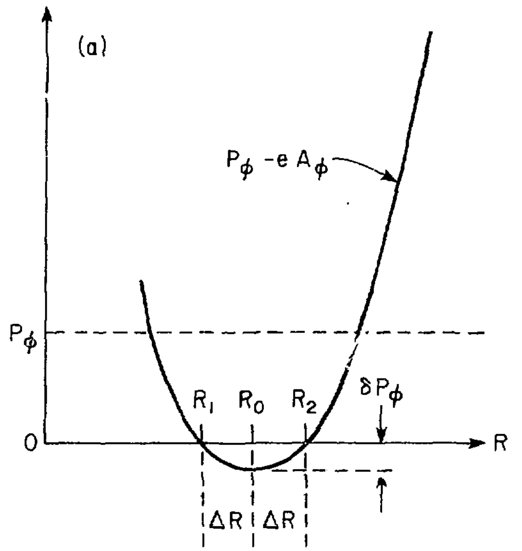


Fig. 5a. The function $p_\phi - eA_\phi(R)$ for $p_\phi < eA_\phi(R_0)$.

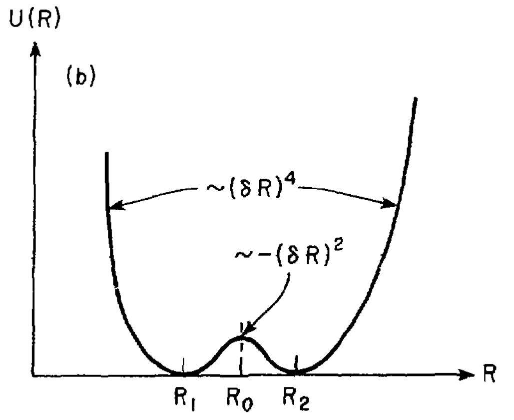


Fig. 5b. The potential U(R) corresponding to Fig. 5a.

Fig. 5. 792390