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Eddy Current Heating of Irregularly Shaped Plates by Slow Ramped Fields

MASTER

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EDDY CURRENT HEATING OF IRREGULARLY SHAPED PLATES
BY SLOW RAMPED FIELDS

Lawrence Dresner

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ABSTRACT

We calculate eddy current heating of thin conducting plates of various shapes by a perpendicular field, assuming that the magnetic field created by the eddy currents is negligible in comparison with the external field. The method is to introduce the stream function of the eddy currents, which is shown to satisfy Poisson's equation, and then employ a pair of complementary variational principles (i.e., a minimum principle and a maximum principle), the extrema of which equal the eddy current heating. Two such complementary principles give not only an estimate of the eddy current heating, but a bound on the error of the estimate as well.

Because the eddy current heating is the extreme value of the functionals in both principles, both can be used straightforwardly to obtain high order estimates of the eddy current heating [as has been done by R. Sikora, J. Purczynski, W. Lipinski, and M. Gramz, IEEE Trans. Magn. MAG-14(5), 383 (September 1978)]. But in addition, the complementary variational principles can be used to prove a variety of interesting and occasionally powerful theorems for estimating eddy current heating. These theorems are of two different kinds. The first kind is the comparison theorems, by which eddy current heating in plates with shapes related by inclusion or Steiner symmetrization may be compared. For comparison theorems to be useful, we must know the eddy current heating exactly for certain shapes. In this paper, exact results are quoted for the circle, ellipse, and equilateral triangle; the series solution is used for the rectangle.

The second kind of theorems is the geometrization theorems, by which the eddy current heating is related to certain geometric quantities dependent on the size and shape of the plate. The most powerful theorem of this class is obtained by application of Pólya and Szegő's method of assigned level lines, in which the level lines of the stream function are taken to be a family of curves geometrically similar to the boundary curve of the plate. This theorem gives a variational estimate of the eddy current heating in all plates having convex shapes and in many (though not all) plates having reentrant shapes. The evaluation of the

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required geometric quantities is extremely simple for polygons, for which results can often be written down by inspection. Another geometrization theorem relates the eddy current heating to the moments of inertia of the plate. Experience has shown that this theorem is less useful than the assigned-level-lines theorem. Attempts to improve it lead to the variational equations of Trefftz.

The plan of the paper is as follows: (1) exhibit the complementary variational principles; (2) quote the exact results for circle, ellipse, equilateral triangle, and rectangle; (3) state and prove the theorems and various corollaries; (4) rework some of Sikora's examples (rectangles, isosceles triangles, sectors of circular annuli) in order to compare his methods with ours; and (5) work some examples for which the methods of this paper are more suitable than Sikora's methods (rhombus, L-shaped plate).

1. INTRODUCTION

Recently, Sikora et al.¹ calculated eddy current heating of thin conducting plates of various shapes by a perpendicular field. They assumed that the magnetic field created by the eddy currents is negligible in comparison with the external field (and we make this assumption here). Their method is to introduce the stream function, which satisfies Poisson's equation, and then to solve Poisson's equation with a variational principle, the maximum of which is the eddy current heating. They give approximate formulas for rectangles, isosceles triangles, and sectors of circular annuli.

This paper starts where Sikora's does, namely, with the Poisson equation obtained by introducing the stream function. But instead of just a maximum principle, we introduce a pair of complementary variational principles,² one of which is a minimum and the other a maximum. Two such complementary principles give not only an estimate of the eddy current heating, but a bound on the error of the estimate as well.

Because the eddy current heating is the extreme value of the functionals in both principles, both can be used straightforwardly to obtain high order estimates of the eddy current heating (Ritz method³ and Kantorovich method⁴). But in addition, the complementary variational principles can be used to prove a variety of interesting and occasionally powerful theorems for estimating eddy current heating. These theorems are of two different kinds. The first kind is the comparison theorems, by which eddy current heating in plates with different but related shapes may be compared. For comparison theorems to be useful, we must know the eddy current heating exactly for certain shapes. In this paper, exact results are quoted for the circle, ellipse, and equilateral triangle; the series solution is used for the rectangle.

The second kind is the geometrization theorems, by which the eddy current heating is related to certain geometric quantities dependent on the size and shape of the plate. The most powerful theorem of this class is obtained by application of Pólya and Szegő's⁵ method of assigned level lines (ALL) in which the level lines of the stream function are taken to be a family of curves geometrically similar to the boundary

curve of the plate. This theorem gives a variational estimate of the eddy current heating in all plates having convex shapes and in many (though not all) plates having reentrant shapes. The evaluation of the required geometric quantities is extremely simple for polygons, for which results can often be written down by inspection. Another geometrization theorem relates the eddy current heating to the moments of inertia of the plate. Experience shows that this theorem is less useful than the ALL theorem. Attempts to improve it lead to the variational equations of Trefftz.⁶

The plan of this paper is as follows: (1) exhibit the complementary variational principles; (2) quote the exact results for circle, ellipse, equilateral triangle, and rectangle; (3) state and prove the theorems and various corollaries; (4) rework some of Sikora's examples in order to compare his methods with ours; and (5) work some examples for which the methods of this paper are more suitable than Sikora's (rhombus, L-shaped plate).

2. BASIC EQUATIONS

Let us choose the axes so that the Z axis is perpendicular to the plate, with the positive Z axis pointing in the direction of the applied external field. Let C denote the boundary curve of the plate and A its interior. The electrodynamic equations we need in this problem are the following:

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} , \quad (1a)$$

$$\nabla \cdot \vec{E} = 0 , \quad (1b)$$

$$P = \frac{1}{A} \iint_A \sigma E^2 dA , \quad (1c)$$

where $\vec{B}$ is the externally applied magnetic field, $\vec{E}$ is the electric field induced by it, σ is the conductivity of the plate, A is the area inside C, and P is the average eddy current dissipation per unit volume of the plate. In Cartesian coordinates, Eq. (1) can be written

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\dot{B} , \quad (2a)$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 , \quad (2b)$$

$$P = \frac{1}{A} \iint_A \sigma (E_x^2 + E_y^2) dx dy , \quad (2c)$$

where $\dot{B}$ is the spatially uniform value of $(\partial \vec{B} / \partial t)_z$.

To solve these equations, we begin by introducing a stream function ψ defined by

$$E_x = -\dot{B} \frac{\partial \psi}{\partial y}, \quad (3a)$$

$$E_y = \dot{B} \frac{\partial \psi}{\partial x}. \quad (3b)$$

These equations satisfy (2b) identically. Equations (2a) and (2c) become

$$\nabla^2 \psi = -1, \quad (4a)$$

$$P = \frac{\sigma \dot{B}^2}{A} \iint_A (\nabla \psi)^2 dA. \quad (4b)$$

The boundary condition that ψ must obey on C expresses the fact that the current density $\sigma \vec{E}$ at C must be parallel to C . Thus, the normal component of $\vec{E}$ vanishes on C . If $\vec{n} = (n_x, n_y)$ is the normal vector to C , then (n_x, n_y) is perpendicular to (E_x, E_y) on C or, what is the same thing, (n_x, n_y) is perpendicular to $(-\partial\psi/\partial y, \partial\psi/\partial x)$. Because $(-\partial\psi/\partial y, \partial\psi/\partial x)$ is perpendicular to $(\partial\psi/\partial x, \partial\psi/\partial y)$, the latter is parallel to (n_x, n_y) . In other words, $\nabla\psi$ is parallel to $\vec{n}$. However, this means that ψ is constant on C ; and because ψ is unique only up to an arbitrary constant, we can choose

$$\psi = 0 \text{ on } C. \quad (5)$$

3. COMPLEMENTARY VARIATIONAL PRINCIPLES

We solve Eqs. (4a) and (5) and obtain estimates of P using a pair of complementary variational principles.² This means we will find two functionals $K[\psi]$ and $J[\psi]$ such that: (1) K is a minimum and J a maximum for the solution of (4a) and (5); and (2) the minimum of K equals the maximum of J , and both are equal to the value of $PA/2\sigma\dot{B}^2$ calculated for the solution of (4a) and (5).

We have

$$K[\phi_2] = \frac{1}{2} \int_A (\nabla \phi_2)^2 dA \geq \frac{PA}{2\sigma\dot{B}^2} \geq \int_A \left[\phi_1 - \frac{1}{2} (\nabla \phi_1)^2 \right] dA \equiv J[\phi_1] , \quad (6a)$$

where

$$\nabla^2 \phi_2 = -1 \text{ but no value of } \phi_2 \text{ is specified on } C , \quad (6b)$$

$$\phi_1 = 0 \text{ on } C \text{ but is otherwise arbitrary} , \quad (6c)$$

and the left- and right-hand terms are the same when $\phi_1 = \phi_2 = \psi$, i.e., $K[\psi] = PA/2\sigma\dot{B}^2 = J[\psi]$. The proof of these assertions is a straightforward but tedious calculation of the first and second variations of J and K and will not be carried out in detail here.

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Fig. 1. A sketch of the equilateral triangle and associated coordinate system.

4. EXACT RESULTS

An elementary calculation shows that for a circle of radius R ,

$$P = \frac{\sigma \dot{B}^2 R^2}{8} = \frac{\sigma \dot{B}^2 A}{8\pi} . \quad (7)$$

As we shall see later on, for all plates of a given area, the circle has the largest average dissipation P . Hence, if we express the average dissipation as a fraction f of the dissipation for a circle of equal area, i.e., if we form the fraction $f = 8\pi P / \sigma \dot{B}^2 A$,

$$f \leq 1 , \quad (8)$$

with equality if and only if the plate is a circle.

The stream function for all ellipses with semimajor axis b and semiminor axis c is

$$\psi = \frac{b^2 c^2}{2(b^2 + c^2)} \left(1 - \frac{x^2}{b^2} - \frac{y^2}{c^2} \right) . \quad (9)$$

Calculating $J[\psi] = K[\psi]$, we find

$$f = 2 \left(\frac{b}{c} + \frac{c}{b} \right)^{-1} \quad (\text{ellipses}) . \quad (10)$$

Consider now the equilateral triangle shown in Fig. 1. The stream function for this triangle is

$$\psi = -\frac{1}{4} (x^2 + y^2) - \frac{1}{4\sqrt{3}} (x^3 - 3xy^2) + \frac{1}{9} , \quad (11a)$$

$$= -\frac{r^2}{4} - \frac{1}{4\sqrt{3}} r^3 \cos 3\theta + \frac{1}{9} . \quad (11b)$$

It is easy to see that $\nabla^2\psi = -1$ from the Cartesian form of ψ , (11a). It is also easy to see from the Cartesian form that $\psi = 0$ on the line $x = 1/\sqrt{3}$ (side QR). From the polar form of ψ , (11b), we see that $\psi(r, \theta + 2\pi/3) = \psi(r, \theta)$; i.e., ψ is invariant to rotation by an angle $2\pi/3$. Thus, $\psi = 0$ on sides RS and SQ of the triangle as well. To evaluate $K[\psi] = J[\psi]$, we first apply Green's theorem to $K[\psi]$ to show it equal to $(1/2) \iint_A \psi dA$. Then, because of the symmetry of ψ , it is only necessary to integrate over triangle OTR, and the integration is most conveniently carried out using the polar form of ψ . We find

$$f = \frac{2\pi\sqrt{3}}{15} = 0.7255 \quad (\text{equilateral triangle}) . \quad (12)$$

The stream function for a rectangle of dimensions $2b \times 2c$ can be expressed as the series

$$\psi = \frac{64}{\pi^4} \sum_{k, \ell=1,3,5,\dots} \frac{\cos\left(\frac{\pi kx}{2b}\right) \cos\left(\frac{\pi \ell y}{2c}\right)}{k\ell \left(\frac{k^2}{b^2} + \frac{\ell^2}{c^2}\right)} , \quad (13)$$

from which it follows that

$$f = \frac{512}{\pi^5} \sum_{k, \ell=1,3,5,\dots} k^{-2} \ell^{-2} [k^2(c/b) + \ell^2(b/c)]^{-1} . \quad (14)$$

Using the following series from Jolley's book,⁷

$$\sum_{n=1,3,5,\dots} (x^2 + n^2)^{-1} = \frac{\pi}{4x} \tanh\left(\frac{\pi x}{2}\right) , \quad (15a)$$

$$\sum_{n=1,3,5,\dots} n^{-4} = \frac{\pi^4}{96} , \quad (15b)$$

one sum can be carried out. Then

$$f = \frac{2}{3} \pi \frac{c}{b} \left[1 - \frac{192}{\pi^5} \frac{c}{b} \sum_{\ell=1,3,5,\dots} \ell^{-5} \tanh \left(\frac{\pi b \ell}{2c} \right) \right], \quad (16)$$

a result already given by Sikora et al.⁸

5. INCLUSION THEOREM

Using the right-hand part of the variational principle (6a), we can prove the following theorem. If a plate A is divided into two plates A_1 and A_2 such that $A = A_1 + A_2$, then $PA > P_1A_1 + P_2A_2$. Proof: Let ψ_1 be the exact solution of (4a) and (5) in A_1 and ψ_2 in A_2 . Then

$$\begin{aligned} \frac{P_1A_1 + P_2A_2}{2\sigma\dot{B}^2} &= \int_{A_1} \left[\psi_1 - \frac{1}{2} (\nabla\psi_1)^2 \right] dA_1 + \int_{A_2} \left[\psi_2 - \frac{1}{2} (\nabla\psi_2)^2 \right] dA_2 \\ &= \int_A \left[\phi_1 - \frac{1}{2} (\nabla\phi_1)^2 \right] dA < \frac{PA}{2\sigma\dot{B}^2}, \quad (17) \end{aligned}$$

where $\phi_1 = \psi_1$ in A_1 and ψ_2 in A_2 .

This theorem says simply that the total eddy current loss PA in a body is greater than the sum of the eddy current losses in its parts, the latter being reckoned separately. In terms of the fraction f , Eq. (17) can be written

$$fA^2 > f_1A_1^2 + f_2A_2^2. \quad (18)$$

This theorem is only of marginal utility in estimating eddy current dissipation.

6. ISOPERIMETRIC THEOREM

The isoperimetric theorem states: Of all plates of a given area A , the circular plate has the largest P . Proof: We use the method of Steiner symmetrization.⁵ When a solid is Steiner-symmetrized, a plane T , called the plane of symmetrization, is selected. The solid is then broken into a bundle of paraxial, infinitesimal, cylindrical elements, all perpendicular to the plane T . These elements are slid parallel to one another until their midpoints all lie in the plane T . Repeated Steiner symmetrization of a plate with respect to an infinitude of planes all containing the normal to the plate will reduce it to a circular plate of the same area. Therefore, any quantity which is decreased (increased) by Steiner symmetrization will be a minimum (maximum) for the circular plate.

We can symmetrize trial functions of the type (6c) by symmetrizing their level surfaces. Thus, if we have a function $\phi_1(x,y)$ which takes the value c on the curve C , we can define a symmetrized function $\phi_1^+(x,y)$ which equals c on the curve C^+ obtained from C by symmetrization.

Now suppose ψ is the exact solution of (4a) and (5) in a cylinder A . Then

$$\frac{PA}{2\sigma B^2} = \int_A \left[\psi - \frac{1}{2} (\nabla\psi)^2 \right] dA . \quad (19)$$

Pólya and Szegő⁵ have proven that integrals $\int_A \psi dA$ remain unchanged under Steiner symmetrization, while integrals $\int_A (\nabla\psi)^2 dA$ decrease. That is to say, they have proven that

$$\int_A \psi dA = \int_{A^+} \psi^+ dA , \quad (20a)$$

$$\int_A (\nabla\psi)^2 dA \geq \int_{A^+} (\nabla\psi^+)^2 dA . \quad (20b)$$

Thus,

$$\frac{PA}{2\sigma\dot{B}^2} \leq \int_{A^+} \left[\psi^+ - \frac{1}{2} (\nabla\psi^+)^2 \right] dA \leq \frac{P_+ A_+}{2\sigma\dot{B}^2}. \quad (21)$$

The first inequality follows from (20a,b), the second from the right-hand side of (6a) since ψ^+ is a trial function of type (6c) in A^+ .

But, since $A_+ = A$, $P_+ \geq P$; i.e., P increases under Steiner symmetrization. Clearly, f also increases under Steiner symmetrization.

The increase of f and P under Steiner symmetrization can be used to prove a number of theorems other than the isoperimetric theorem. First, if a semicircle is symmetrized along its diameter, the result is an ellipse with a ratio of major to minor axes equal to 2. Using (10), we find that for a semicircle, $f < 4/5$. Second, if we symmetrize a triangle repeatedly about its altitudes, we get in the limit an equilateral triangle. Hence, of all triangles of a given area, the equilateral triangle has the largest f . Third, if we symmetrize a rhombus with diagonals $2b$ and $2c$ about an axis perpendicular to a pair of its sides, we get a rectangle, the sides of which are in the ratio $(1/2)(b/c + c/b)$. The value of f for the rectangle, which must exceed that for the rhombus, can be calculated from the series (16). Other theorems can be derived from the increase of f under Steiner symmetrization, but those noted above are the main ones we shall use in this paper.

7. METHOD OF ASSIGNED LEVEL LINES

The theorems proved so far are more or less obvious, and though it is nice to have proofs of them, they do not tell us much we might not already have guessed. Now we turn to some less obvious theorems. If $r = R(\theta)$ is the polar representation of the curve C and the quantity a is defined by

$$a = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{R} \frac{dR}{d\theta} \right)^2 d\theta ,$$

then

$$P \geq \frac{\sigma \dot{B}^2 A}{8\pi(1+a)} , \quad \text{i.e., } f \geq (1+a)^{-1} .$$

Equality holds for circles.

Proof: We use Pólya and Szegő's method of assigned level lines.⁵ We introduce a variable λ , $0 \leq \lambda \leq 1$, which defines the level surfaces of the trial function $\phi_1(\lambda)$ according to the equation $r = \lambda R(\theta)$. To evaluate the functional $J_1[\phi_1(\lambda)]$, it will prove convenient to introduce λ and θ as coordinates rather than r and θ . In λ, θ coordinates the covariant metric tensor is given by the expression for the interval

$$(ds)^2 = (dr)^2 + r^2(d\theta)^2 = (Rd\lambda + \lambda R'd\theta)^2 + \lambda^2 R^2(d\theta)^2 , \quad (22)$$

so that

$$g_{\lambda\lambda} = R^2 , \quad g_{\lambda\theta} = g_{\theta\lambda} = \lambda R R' , \quad g_{\theta\theta} = \lambda^2 (R^2 + R'^2) . \quad (23a)$$

The contravariant metric tensor has the components

$$g^{\lambda\lambda} = \frac{R^2 + R'^2}{R^4} , \quad g^{\lambda\theta} = g^{\theta\lambda} = \frac{-R'}{\lambda R^3} , \quad g^{\theta\theta} = \frac{1}{\lambda^2 R^2} . \quad (23b)$$

Now if $\phi_1 = \phi_1(\lambda)$ and is independent of θ (i.e., surfaces of constant λ are the level surfaces of ϕ_1), then

$$(\nabla\phi)^2 = g^{ij} \frac{\partial\phi_1}{\partial x^i} \frac{\partial\phi_1}{\partial x^j} = g^{\lambda\lambda} \left(\frac{d\phi_1}{d\lambda} \right)^2 = \frac{R^2 + R'^2}{R^4} \left(\frac{d\phi_1}{d\lambda} \right)^2, \quad (24a)$$

and

$$dx dy = \sqrt{g} d\lambda d\theta = \lambda R^2 d\lambda d\theta, \quad (24b)$$

so that

$$J[\phi_1] = \int_0^1 d\lambda \int_0^{2\pi} d\theta \left[\phi_1 - \frac{1}{2} \frac{R^2 + R'^2}{R^4} \left(\frac{d\phi_1}{d\lambda} \right)^2 \right] \lambda R^2. \quad (25)$$

We can carry out the θ integral if we note that $\int_0^{2\pi} R^2 d\theta = 2A$ and use the previous definition of a :

$$J[\phi_1] = \int_0^1 \lambda d\lambda \left[2A\phi_1 - \pi(1+a) \left(\frac{d\phi_1}{d\lambda} \right)^2 \right]. \quad (26)$$

Next we choose $\phi_1(\lambda)$ to maximize $J[\phi_1]$. If we choose $\phi_1 = 0$ at $\lambda = 1$ (i.e., on C), we find that the Euler-Lagrange equation for (26) is

$$A\lambda + \pi(1+a) \frac{d}{d\lambda} \left(\lambda \frac{d\phi_1}{d\lambda} \right) = 0, \quad (27)$$

so that

$$\phi_1 = \frac{A}{4\pi(1+a)} (1 - \lambda^2). \quad (28)$$

Then

$$J[\phi_1] = \frac{A^2}{16\pi(1+a)}, \quad (29)$$

so that

$$P \geq \frac{\sigma_B^2 A}{8\pi(1+a)}, \quad \text{i.e., } f \geq (1+a)^{-1}. \quad (30)$$

If A is a circular plate, $a = 0$ and $f = 1$ when the origin of coordinates is at the center of the plate.

The quantity a is simple to evaluate when the plate has the form of a convex polygon (see Fig. 2). Let O be the center of the polygon and consider the triangle OAB . Let the altitude OH on AB be the line $\theta = 0$, where θ is the polar angle. Then $R(\theta) = h \sec \theta$, and the triangle OAB contributes an amount

$$\frac{1}{2\pi} \int_{-\theta_1}^{\theta_2} \tan^2 \theta \, d\theta = \frac{1}{2\pi} \left(\frac{s_2}{h} + \frac{s_1}{h} \right) - \left(\frac{\theta_2 + \theta_1}{2\pi} \right) = \frac{s}{2\pi h} - \frac{\Delta\theta}{2\pi}, \quad (31)$$

where s is the length of the side AB , h is the length of the altitude OH , and $\Delta\theta$ is the magnitude of the angle AOB . Summing over all the triangles, we find

$$a = \frac{1}{2\pi} \sum_i \frac{s_i}{h_i} - 1, \quad (32a)$$

so that

$$f \geq (1 + a)^{-1} = 2\pi \left(\sum_i \frac{s_i}{h_i} \right)^{-1}, \quad (32b)$$

where s_i is the length of the i th side of the polygon and h_i is the length of the altitude on it from the center O .

The right-hand side of (32b) depends on which interior point of the polygon is chosen as the origin O . The best estimate of f is obtained for that point O which minimizes the sum $\sum_i s_i/h_i$.

The restriction to convex polygons is too confining; the theorem holds whenever a ray from the center O intersects the bounding curve C just once. Reentrant boundary curves are then sometimes admissible, depending on the choice of the center O .

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Fig. 2. A sketch of a convex polygon to aid in evaluating the quantity a that arises in the method of assigned level lines.

8. CONNECTION WITH MOMENTS OF INERTIA

Suppose we choose the origin at the centroid of A and set $I_{xx} = \int_A x^2 dA$, etc. Then

$$P \leq \sigma B^2 A \frac{I_{xx} I_{yy} - I_{xy}^2}{A^2 (I_{xx} + I_{yy})}, \quad \text{i.e., } f \leq \frac{8\pi (I_{xx} I_{yy} - I_{xy}^2)}{A^2 (I_{xx} + I_{yy})}.$$

There is equality for circular plates.

Proof: Set

$$\phi_2 = -\frac{\alpha}{2} x^2 - \frac{1-\alpha}{2} y^2 + \beta xy. \quad (33)$$

Then

$$K[\phi_2] = \frac{\alpha^2}{2} I_{xx} + \frac{(1-\alpha)^2}{2} I_{yy} + \frac{\beta^2}{2} (I_{xx} + I_{yy}) - \beta I_{xy}. \quad (34)$$

K has its minimum when $\alpha = I_{yy}/(I_{xx} + I_{yy})$ and $\beta = I_{xy}/(I_{xx} + I_{yy})$, from which the theorem follows at once. A direct computation shows the equality for circular plates.

Because the numerators and denominators of the fractions expressing P and f involve the determinant and trace of the moment of inertia tensor, respectively, these fractions are invariant to rotation of the coordinate axes.

If the origin of coordinates is chosen at a point other than the centroid, then the moments of inertia in the new coordinates are related to those around the centroid by the parallel axes theorem. A straightforward but tedious calculation shows that the fraction $(I_{xx} I_{yy} - I_{xy}^2)/(I_{xx} + I_{yy})$ has its smallest value when calculated with the origin at the centroid, and this gives the lowest upper bound attainable with the trial function (33).

9. TREFFTZ'S METHOD

Trefftz's method arises out of an attempt to improve the estimate given in the moment of inertia theorem. If we have a function w which satisfies $\nabla^2 w = -1$, and we know several functions u_λ for which $\nabla^2 u_\lambda = 0$, we can use a linear combination as a trial function in K :

$$\phi_2 = w + \sum_{\lambda} a_{\lambda} u_{\lambda} . \quad (35)$$

The coefficients a_{λ} which minimize K solve the simultaneous equations

$$\int_C w \frac{\partial u_{\lambda}}{\partial n} ds + \sum_{\mu} a_{\mu} \int_C u_{\lambda} \frac{\partial u_{\mu}}{\partial n} ds = 0 , \quad (36)$$

where ds is the element of arc length along the boundary curve C and $\partial u_{\mu} / \partial n$ is the derivative of u_{μ} in the direction of the outward normal n to C . The minimum value of K is then

$$K_{\min} = \frac{1}{2} \int_A (\nabla w)^2 dA + \frac{1}{2} \sum_{\lambda} a_{\lambda} \int_C w \frac{\partial u_{\lambda}}{\partial n} ds . \quad (37)$$

Equations (36) and (37) are Trefftz's equations.⁶

10. RECTANGULAR PLATE

Shown in Fig. 3 is the fraction f for rectangles of dimension $2b \times 2c$. The curve marked "series" has been calculated using (16). The curve marked "assigned level lines" is the curve $f = (\pi/2) (b/c + c/b)^{-1}$, while the curve marked "moments of inertia" is the curve $f = (2\pi/3) \times (b/c + c/b)^{-1}$. These two curves are lower and upper bounds, respectively, to f , which must lie in the shaded band. Because the estimates differ by a constant factor of $4/3$, their geometric mean gives an estimate of f which is never in error by more than 15%.

Collatz⁶ has solved the boundary value problem (4a) and (5) for the square by Trefftz's method using the trial functions

$$w = -\frac{1}{4} (x^2 + y^2) , \quad (38a)$$

$$u_1 = \operatorname{Re} (x + iy)^4 , \quad (38b)$$

$$u_2 = \operatorname{Re} (x + iy)^8 . \quad (38c)$$

From his figures we find

$$f = 0.88333 , \quad (39)$$

which differs from the correct result $f = 0.88327$ by only six units in the fifth decimal place. This extremely good result is a consequence of the high symmetry of the shape being considered.

Fig. 3. Various bounds for f for rectangles.

11. TRIANGLES

In order to apply (32b) to triangles most efficiently, we must choose as origin 0 that point which makes theorem $\sum s_i/h_i$ smallest. Suppose we already know the location of this best point 0 in a triangle ABC (see Fig. 4). How can we characterize it? To answer this, first draw a line PQ through 0 parallel to side BC. Let h, k, and l be the altitudes from any point 0' on line PQ to the sides, AB, BC, and CA, of lengths a, b, and c, respectively. Then $cl + ah + bk = 2A$, where A is the area of the triangle. Since k is the same for all points 0' on PQ, $cl + ah$ remains constant on PQ as 0' moves even though l and h individually change. Thus on PQ, $cdl + adh = 0$. Because k is constant on PQ, $c/l + a/h$ attains its minimum on PQ at 0. At 0, therefore, $cdl/l^2 + adh/h^2 = 0$. Thus at 0, $l = h$. This means that 0 lies on the angle bisector of $\angle CAB$. Clearly it must also lie on the other angle bisectors as well so that 0 is the common intersection of the angle bisectors.

It is easy to see that the altitudes h_i from the intersection of the angle bisectors are all equal to the common value $2A/P$, where P is the perimeter of the triangle. It follows then from (32b) that

$$f \geq \frac{4\pi A}{P^2} \quad (\text{any triangle}) . \quad (40)$$

In order to apply the moment of inertia theorem, we must calculate the determinant and trace of the inertia matrix. A tedious but direct calculation gives

$$I_{xx}I_{yy} - I_{xy}^2 = A^4/108 , \quad (41a)$$

$$I_{xx} + I_{yy} = (A/36)(a^2 + b^2 + c^2) , \quad (41b)$$

$$f \leq \frac{8\pi}{3} \frac{A}{a^2 + b^2 + c^2} , \quad (41c)$$

for a triangle with sides a, b, c and area A.

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Fig. 4. A sketch of a triangle showing the constructions used in finding the best point to use as origin.

We can test how good these bounds are for equilateral triangles, for which we know f exactly. We find

$$\frac{\pi}{3\sqrt{3}} = 0.6046 \leq f = \frac{2\pi}{5\sqrt{3}} = 0.7255 \leq \frac{2\pi}{3\sqrt{3}} = 1.2092 . \quad (42)$$

The lower limit, calculated by the ALL theorem, is fairly close to the correct value (17% low), but the upper limit, from the moment of inertia theorem, is much too high (67% high).

Sikora et al.¹ have used the Ritz method with a carefully chosen trial for isosceles triangles with sides $2m$, $\sqrt{1+m^2}$ and $\sqrt{1+m^2}$ (see Fig. 5). Their result, obtained from functional J , reads

$$f \geq \frac{4\pi}{5} \frac{m}{1+3m^2} . \quad (43)$$

From the ALL theorem and the moment of inertia theorem, we have

$$\frac{\pi m}{(m + \sqrt{1+m^2})^2} \leq f \leq \frac{4\pi}{3} \frac{m}{1+3m^2} . \quad (44)$$

Sikora's result is exact for equilateral triangles ($m = 1/\sqrt{3}$), a fact which seems to have escaped their notice.

The bounds (43) and (44) are plotted in Fig. 5 together with the exact value for the equilateral triangle, which, by the isoperimetric theorem, is also an upper bound. The correct value of f must lie in the cross-hatched area. Interestingly, for $m < 0.15$, the ALL theorem gives a better lower bound than Sikora's formula. Near $m = 1/\sqrt{3}$, Sikora's formula should be very accurate.

Fig. 5. Various bounds for f for isosceles triangles.

12. SECTORS OF CIRCULAR ANNULI — PART 1

Sikora et al. have used the method of Kantorovich⁴ to find a lower limit for f of sectors of circular annuli such as shown in Fig. 6.

Their result is

$$f \geq \frac{2\pi}{3\theta_0} \frac{1-k}{1+k} \left[1 - \frac{\tanh(\beta\theta_0)}{\beta\theta_0} \right], \quad (45a)$$

$$\beta = \left[\frac{2(1-k)^2(1-k^2)}{(1-k^2)(1-8k+k^2) - 12k^2 \ln k} \right]^{1/2}, \quad (45b)$$

where θ_0 is the half-angle of the sector and k is the ratio of the inner to the outer radius. When $k = 0$ and $\theta_0 = \pi/2$, the plate is a semicircle. Sikora's formula gives $f \geq 0.7471$ and the symmetrization theorem gives $f \leq 0.8000$, as noted earlier. The geometric mean of these estimates, $f = 0.7731$, has a maximum percentage error of 3.5%. The moment of inertia theorem gives the upper bound $f \leq 0.8738$, which is thus at least 9%, but not more than 17%, high.

In order to apply the ALL theorem to the semicircle most efficiently, we need to find the point for which the lower bound is largest. To do this, we begin by noting that if the plate has a line L of reflection symmetry, the optimum point must lie on L . Proof: We can approximate the plate as closely as we desire by a polygon with the same reflection symmetry. Shown in Fig. 7 are two sides of the polygon that are images of one another. When extended, the two sides intersect at a point C on the symmetry axis L ; L bisects the angle between the two sides.

Suppose that we choose as the origin a point O that lies off the symmetry axis L . It has an image O' on the other side of L for which the $\sum s_i/h_i$ has the same value as at O . The line OO' intersects L at right angles at a point M . We now show that the joint contribution to $\sum s_i/h_i$ from sides AB and $A'B'$ is smaller for point M than for O (or O'). This contribution is $s/h + s/k$. Because $\angle ACM = \angle A'CM = \angle QPT = \angle RPT' = 0$, we see that $h + k = 2\ell \cos \theta$. Thus $dh + dk = 0$ on OO' . The

minimum contribution to $\sum s_i/h_i$ occurs when $sdh/h^2 + sdk/k^2 = 0$. Thus $k = h$, corresponding to point M for the minimum contribution. This argument holds simultaneously for all pairs of sides. The minimum value of $\sum s_i/h_i$, then, must occur for a point on line L, the axis of symmetry. Some slight additional argument, which can easily be supplied by the reader, is necessary in case there are sides that are perpendicular to L.

From this theorem, it follows that the origin for the semicircle should be chosen on the normal bisector of the diametral edge. Its exact location on that line and the corresponding greatest lower bound to f were bound by numerical integration — the origin should be located 0.545 radii from the diametral edge. Then $f \geq 0.7149$, which is between 4.5% and 12% low.

Fig. 6. A sketch of a sector of a circular annulus explaining the nomenclature used.

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Fig. 7. Auxiliary sketch to aid in locating the best origin in a polygon with an axis of reflection symmetry.

13. SECTORS OF CIRCULAR ANNULI — PART 2

Next we turn our attention to very thin circular annuli, i.e., annuli for which $1 - k \ll 1$ (see Fig. 8). We intuitively feel that f for such annuli should be very close to f for rectangles of the "same" dimensions, and what follows next is a rigorous expression of this feeling.

We introduce new coordinates x, y in place of the polar coordinates r, θ :

$$x = R\theta , \quad (46a)$$

$$y = r - R , \quad (46b)$$

where R is the mean radius of the annulus. The coordinates x and y are "almost" Cartesian for their annuli. The components of the covariant metric tensor can be read off the expression for the interval

$$ds^2 = dr^2 + r^2 d\theta^2 = dy^2 + \left(1 + \frac{y}{R}\right)^2 dx^2 , \quad (47a)$$

$$g_{xx} = \left(1 + \frac{y}{R}\right)^2 , \quad g_{yy} = 1 , \quad g_{xy} = 0 . \quad (47b)$$

Then,

$$\sqrt{g} = 1 + \frac{y}{R} , \quad g^{xx} = \left(1 + \frac{y}{R}\right)^{-2} , \quad g^{yy} = 1 , \quad g^{xy} = 0 . \quad (47c)$$

In the x, y coordinate system, then, the functional J becomes

$$J = \int_{-R\theta_0}^{R\theta_0} dx \int_{-\delta}^{+\delta} dy \left[\phi(x, y) \left(1 + \frac{y}{R}\right) - \frac{1}{2} \left(\frac{\partial \phi}{\partial x}\right)^2 \left(1 + \frac{y}{R}\right)^{-1} - \frac{1}{2} \left(\frac{\partial \phi}{\partial y}\right)^2 \left(1 + \frac{y}{R}\right) \right] . \quad (48)$$

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Fig. 8. A sketch of a sector of a thin circular annulus explaining the nomenclature used.

Now let us choose as our trial function the exact stream function $\phi_r(x,y)$ for a rectangle of dimensions $2R\theta_0 \times 2\delta$. The error we commit in J by so doing is $\sim(\delta/R)^2$ because of the variational nature of the functional J (the error in the stream function itself is of order δ/R). Because $\phi(x,y)$ for a rectangle is symmetric in y , it follows from (48) that

$$J = \int_{-R\theta_0}^{R\theta_0} dx \int_{-\delta}^{+\delta} dy \left[\phi_r - \frac{1}{2} \left(\frac{\partial \phi_r}{\partial x} \right)^2 - \frac{1}{2} \left(\frac{\partial \phi_r}{\partial y} \right)^2 \right] - \frac{1}{2} \int_{-R\theta_0}^{R\theta_0} dx \int_{-\delta}^{+\delta} dy \left(\frac{\partial \phi_r}{\partial x} \right)^2 \frac{y^2}{R(R+y)} + O\left(\frac{\delta^2}{R^2}\right), \quad (49a)$$

or

$$J_s = J_r + O\left(\frac{\delta^2}{R^2}\right), \quad (49b)$$

where J_s is the exact value of J for an annular sector and J_r is the exact value of J for a rectangle of dimensions $2R\theta_0 \times 2\delta$. Therefore, if we plot f as a function of δ/R , the slope at $\delta/R = 0$ should be the same as calculated from the "equivalent" rectangle. Comparing with (16), we see that this slope is $2\pi/3\theta_0$. Translated to the variable k , this means that $(df/dk)_{k=1} = -\pi/3\theta_0$.

Shown in Fig. 9 are Sikora's curves (45) of f vs k as well as the exact slopes at $k = 1$. It is easy to see from the figure that Sikora's formula (45) has the correct slope at $k = 1$ [as can be shown rigorously by expanding (45a) in powers of $1 - k$]. Some of Sikora's curves, e.g., that for $\theta_0 = \pi/8$, show maxima. For $\theta_0 = \pi/8$, the maximum comes about because as k changes from 0 to 1, the shape of the sector changes from circular ($k = 0$) to roughly a square ($k \sim 0.5$) and then to a long, thin rectangular strip [$(1 - k) \ll 1$]. From our experience with the symmetrization theorem, we know that equilateral shapes like the square have the highest f .

Fig. 9. Sikora's curves of f for sectors of circular annuli together with the exact slopes at $k = 1$.

14. RHOMBUS

The rhombus is a simple shape not dealt with by Sikora et al. that can be treated by our theorems. Shown in Fig. 10 are three bounding curves for f of the rhombus whose diagonals are $2c$ and $2b$, respectively. The lower bound given by ALL is $f \geq (\pi/2) (b/c + c/b)^{-1}$. The upper bound given by the moment of inertia theorem is $f \leq (2\pi/3) (b/c + c/b)^{-1}$. If we Steiner-symmetrize a rhombus about an axis perpendicular to a pair of sides, we get a rectangle with a ratio of sides of $(1/2) (b/c + c/b)$. The value of f for this rectangle, which is an upper bound to f of the rhombus, can be calculated from Sikora's series (16). The exact value of f must lie in the shaded band.

Fig. 10. Various bounds of f for rhombuses.

15. THE L-SHAPED PLATE

Shown in Fig. 11 is an L-shaped plate which can be dealt with by the methods of this paper. In applying the ALL theorem, we must choose the best origin 0 for the coordinates. It lies on the axis of reflection symmetry PQ. If we choose as origin a point 0 a distance x below and to the left of P, then we find immediately

$$f \geq \pi \left(\frac{1}{a-x} + \frac{a}{1-a+x} + \frac{1-a}{x} \right)^{-1}. \quad (50)$$

Shown in Fig. 12 is a curve determined by maximizing the right-hand side with respect to x by trial and error for each value of a .

We can get an upper limit to f useful near $a = 1$ by using the inclusion theorem, Eq. (18). The L-shaped plate is a square with a smaller square cut out of it. Then,

$$f_{SQ} > f_L [a(2-a)]^2 + f_{SQ} (1-a)^2, \quad (51a)$$

or

$$f_L < \frac{f_{SQ}}{a(2-a)}. \quad (51b)$$

This limit is plotted in Fig. 12 and serves to restrict the region in which f must lie for a near $a = 1$.

We need an upper limit in the region $a \lesssim 0.7$. If we symmetrize the L plate around its axis of reflection symmetry, we get a rectangle with isosceles right triangles capping each end (see Fig. 13). This figure can be handled reasonably well using the moment of inertia method. The resulting estimate, an upper bound, is shown in Fig. 11 — it defines a very wide band when taken with the other bounds.

We can find the exact slope of the curve of f vs a at $a = 1$ just as we did for the annular sectors. In the horizontal half of the L plate we use the stream function of the rectangle ABCD, and in the vertical half, the stream function of the rectangle A'B'C'D' (see Fig. 14).

Owing to the symmetry of the construction, these stream functions join

continuously on the line PQ. Each rectangle has a ratio of sides $a/(2-a)$. To lowest order in a ($a \ll 1$), then, $f = \pi a/3$ so that

$$\left(\frac{df}{da}\right)_{a=0} = \frac{\pi}{3}. \quad (52)$$

This slope is shown in Fig. 12.

Because f must equal f for a square when $a = 1$ and have the slope (52) when $a \ll 1$, we can draw the dashed curve as a practical estimate for the L plate. In drawing the dashed curve, we have been guided by the shape of the ALL bound.

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Fig. 11. A sketch of the L-shaped plate explaining the nomenclature used.

Fig. 12. Various bounds and estimates of f for the L-shaped plate.

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Fig. 13. Steiner symmetrization of the L-shaped plate.

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Fig. 14. A sketch showing the use of the stream function for certain rectangles in constructing a trial function for the L-shaped plate.

16. CONCLUDING REMARKS

The methods presented here, together with those used by Sikora et al., allow us to estimate eddy current heating of thin plates of quite varied geometry. Sikora's methods are capable of high accuracy but work well only for fairly regular shapes. The methods of this paper, on the other hand, sacrifice high accuracy in favor of wide applicability but should suffice for most technical applications.

The theorems presented here have the following two limitations. First, the estimates from the moment of inertia theorem are sometimes (but not always) too high to be of practical use. Second, the ALL theorem fails for many reentrant shapes, e.g., for thin, semicircular annuli. In my view, the second limitation is the more serious of the two, and a way to overcome it would be welcome.

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