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SPECIAL FUNCTIONS AND GROUP THEORY IN THEORETICAL PHYSICS

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Author or Chapter

Title of paper

A widening of the group representation context appears in the book by Miller [10] which is based on the recognition that the special functions defined by separation of variables in second-order linear partial differential equations can be characterized as eigenfunctions defined by commuting elements in the enveloping algebra of the Lie algebra of the symmetry group of the equation. This structure encompasses naturally the factorization method of Schrödinger (see Ref. [11]), the approaches of Tiesdell [12] and Weisner [13] and leads to a uniform approach to hypergeometric special functions as well as nonhypergeometric functions (Mathieu, Lamé, ...). One may discern a trend toward algebraic, as opposed to purely group-

theoretic, methods in the approach to spherical functions in terms of convolution algebras (Helgason [8])).

$$x^j(a)x^j(b) = x^j(ab). \quad (1.1)$$
$$D^j(a) D^j(b) = D^j(ab), \quad (1.2)$$
$$x^j(a) x^{j'}(a) = x^{j+j'}(a), \quad (1.3)$$

Can one generalize relation (1.3) to non-Abelian groups? No really satisfactory answer is known, although there are some fundamental results (for example, Tannaka duality). The Van Kampen-Pontriagin theorem shows that the dual space to the group space of a compact group is discrete; thus we are seeking some kind of product structure on discrete spaces. This structure cannot be that of a group, as the specific examples to be discussed below will show, but the structure does become a group in a well-defined asymptotic limit. The sequence of discrete Abelian groups Z_n as n becomes large can be taken to limit to the Abelian rotation group $R_2(\theta)$ of the plane. Some suitably generalized, but discrete, analog of this for non-Abelian groups would, if it exists!, be of great help to theoretical physics.

Let us, however, proceed straightforwardly, despite the lack of clear guidance. A result for the quantal rotation

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Here the $D^j_{mm}(R)$ are the matrix elements of the representation D^j for the group element R ; the (real, orthonormal) coefficients $C::$ are the Wigner-Clebsch-Gordan coefficients (see, for example, Ref. [14]).

$$\sum_{\substack{m_1 m_1' \\ m_2 m_2'}} C_{m_1 m_2 m_3}^{j_1 j_2 j_3} \cdot C_{m_1' m_2' m_3'}^{j_1 j_2 j_3} D_{m_1 m_1'}^{j_1}(R) D_{m_2 m_2'}^{j_2}(R) = D_{m_3 m_3'}^{j_3}(R). \quad (1.5)$$
$$[D_{11}^{(1)} \otimes D_{22}^{(1)}]^{j_3} = D_{j_3}^{j_3} \quad (1.6)$$

-- Our reason for writing the Wigner product law in this curious way is to motivate two valid identities--each a formal analog of Eq. (1.6)--that involve algebraic elements outside the enveloping algebra of $SU(2)$ whose matrix elements are special functions of generalized hypergeometric type.

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1. *Introduction*

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11 By taking matrix elements in H, one sees that this operator
12 has the WCG coefficients C::: as matrix elements.

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where W_{ij} denotes an $SU(2)$ invariant operator whose matrix elements are the Racah, or 6-j coefficients, and $M'' = M + M'$, $\Delta'' = \Delta + \Delta'$.

product:

where $\textcircled{W}$ denotes the Wigner product (using the 3-j coefficient) and $\textcircled{R}$ denotes the Racah product (using, in effect, the 6-j coefficients).

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Let us note explicitly that the Wigner operators constitute an algebraic structure that goes beyond the concept of the enveloping algebra of the Lie group $SU(2)$, although the enveloping algebra, and the Lie algebra, are contained as subalgebras.

2. THE PROBLEM OF DEFINING WIGNER-CLEBSCH-GORDAN COEFFICIENTS FOR $SU(3)$

$$D^\lambda = \{D^\lambda(g) \mid g \in G\}, \quad (2.1)$$
$$P = \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \geq \lambda_3\}. \quad (2.2)$$
$$[\lambda_1, -\lambda_1, \lambda_2, \lambda_3, 0]$$

$$U (D^{\mu} \otimes D^{\nu}) U^{\dagger} = \sum_{\lambda} \oplus g(\lambda) D^{\lambda}, \quad (2.3)$$

where the right-hand side of this expression denotes a (direct) sum of block diagonal matrices in which a given D^i is repeated a number of times equal to $g(\mu_i)$. We will denote the abstract Clebsch-Gordan series corresponding to the explicit reduction (2.3) of the Kronecker product by

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where $C:::$ is defined in (2.5).

each $\tau = 1, \dots, g_{\mu\nu\lambda}$.

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which is a basis for all linear maps: $H_n \rightarrow H_n$.

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$$\Delta_1 = \alpha_1, \Delta_2 = \alpha_2 + \alpha_3 - \alpha_1, \Delta_3 = \mu_1 + \mu_2 + \mu_3 - \alpha_2 - \alpha_3. \quad (2.13)$$
$$T \begin{pmatrix} \alpha \\ \mu \\ m \end{pmatrix}. \quad (2.14)$$
$$T \begin{pmatrix} \alpha \\ u \\ m \end{pmatrix} : \begin{cases} H_\lambda \rightarrow H_{\lambda+\Delta}, & \text{if } \lambda+\Delta \in \mu \times \lambda \\ \text{or} \\ H_\lambda \rightarrow 0, & \text{if } \lambda+\Delta \notin \mu \times \lambda \end{cases} \quad (2.15)$$
$$T \begin{pmatrix} \lambda \\ m, \lambda \end{pmatrix} |n\rangle = Y_{n', C} \left[\begin{pmatrix} \lambda + \Delta \\ n', \lambda \end{pmatrix} \begin{pmatrix} \lambda \\ m, \lambda \end{pmatrix} \begin{pmatrix} \lambda \\ n \end{pmatrix} \right] |n', \lambda\rangle, \quad (2.16)$$

How many operator patterns $\binom{n}{k}$ are there having given irrep labels $[n_1, n_2, n_3]$ and given shift pattern $[s_1, s_2, s_3]$?

$$M = \begin{cases} (\mu_2 - \mu_3 + 1) - (\delta_1 + \delta_2 + \delta_3) & \text{for } \Delta_1 + \Delta_2 + \Delta_3 = \sum \mu_i, \\ & \mu_1 \geq \Delta_1 \geq \mu_3; \quad i \\ 0 & \text{otherwise,} \end{cases} \quad (2.17)$$

One must prove, of course, that operator patterns--the set of (1) patterns--really work. For this we require properties of the intertwining number function $f_{u,\Delta}$ with values defined on the set P of partitions by

$$I_{\mu, \Delta}(\lambda) = g_{\mu, \lambda, \lambda + \Delta} \quad (2.18)$$

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Here we regard μ and Δ as specified and seek the set of values $\{I_{\mu,\Delta}(\lambda) | \lambda \in P\}$. The numbers in this set are precisely 0, 1, ..., M; the next step is to exhibit in an appropriate graph the "level" subsets of P defined by

$$P_k = \{\lambda \in P | I_{\mu,\Delta}(\lambda) = k\}, \quad k=0, 1, \dots, M. \quad (2.19)$$

The coordinate system most appropriate for graphing the level subsets (2.19) of the intertwining function $I_{\mu,\Delta}$ is the barycentric system shown in Fig. 1 (also called the Möbius plane):

Figure 1. The 3-simplex (equilateral triangle) in R^2 defines the three coordinate axes used for graphing the intertwining function.

The translation properties $I_{\mu+a,\Delta+a} = I_{\mu,\Delta}$ and $I_{\mu,\Delta}(\lambda+a) = I_{\mu,\Delta}(\lambda)$, $a=(a,a,a)$, are important properties used in graphing the intertwining function shown in Fig. 2.

The role of the intertwining function in completing the definition of a canonical tensor operator can be inferred from the special case of $I_{[630],[333]}$ shown in Fig. 3.

Let L_1^1, L_1^2, L_1^3 denote the set of lattice points belonging to the bent solid lines denoted $l=1, l=2, l=3$ in Fig. 3. Also let T_0 denote the lattice points in the equilateral triangle shown in the shaded area, that is, the set of points

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Figure 2. Graph of the intertwining function. The intertwining function I_{Δ} has values as follows on the lattice points in the Möbius plane: value M in the cross-hatched region R , including the Δ at solid boundary line; value k ($k=0, 1, \dots, M$) on the set L_k ; and value 0 in the shaded region R_0 .

Figure 3. Graph of the intertwining function $I[630], [333]$.

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$$\begin{array}{ll} (1, -2, 1) & \bullet \\ (1, -3, 2) & \bullet \quad \bullet (2, -3, 1) \\ (1, -4, 3) & \bullet \quad \bullet (3, -4, 1) \\ & \bullet (2, -4, 2) \end{array}$$
$$T \begin{pmatrix} 6 & 3 & 0 \\ 3 & 0 & 0 \end{pmatrix} : H_\lambda \rightarrow 0 \text{ for all } x \in L_1 \cup L_2 \cup L_3 \cup T_0 \equiv N_1.$$

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$T \begin{pmatrix} 5 & 3 & 1 \\ 3 & 3 & 0 \end{pmatrix} : H_\lambda \rightarrow 0 \text{ for all } x \in L_1 \cup L_2 \cup T_0 \equiv N_2,$

$$T\left(\begin{smallmatrix} 4 & 3 & 2 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right) : H_\lambda \rightarrow 0 \text{ for all } x \in L_{\frac{1}{\lambda}} \cup T_0 \equiv N_3 ,$$

$$T \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 0 \end{pmatrix} : H_\lambda \rightarrow 0 \text{ for all } x \in T_0 \equiv N_\lambda. \quad (2.20)$$

Recall that $(x_1, x_2, x_3) = (\lambda_2 - i_3, \lambda_2 - i_1, \lambda_1 - i_1)$. ~~Observe that~~ ^{Clearly} the characteristic null spaces of these operators are nested (strictly decreasing):

$$N_1 \supset N_2 \supset N_3 \supset N_4. \quad (2.21)$$

The results described above for a special case generalize:
The tensor operators in the multiplicity set

$$\left(\begin{smallmatrix} \Gamma_1 \\ \mu_1 \mu_2 \mu_3 \end{smallmatrix} \right)^{-}, \left(\begin{smallmatrix} \Gamma_2 \\ \mu_1 \mu_2 \mu_3 \end{smallmatrix} \right)^{-}, \dots, \left(\begin{smallmatrix} \Gamma_M \\ \mu_1 \mu_2 \mu_3 \end{smallmatrix} \right)^{-} \quad (2.22)$$

38 corresponding to distinct operator patterns having the same
39 shift pattern $\Delta = [\Delta_1, \Delta_2, \Delta_3]$ are distinguished by the nested
40 characteristic null spaces

$$N_1 \supset N_2 \supset \dots \supset N_{11}, \quad (2.23)$$

each of which can be identified explicitly from the graph of
the intertwining function $I_{\mu, \Delta}$.

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An explicit procedure, based on intrinsic properties of the tensor operators themselves, can also be given for making the assignment of operator patterns to characteristic null spaces, but this result will not be given here, since our main purpose is to indicate the origin of the functions discussed in the sections to follow.

A canonical set of tensor operators (hence, Wigner coefficients) is then a set that is labelled by operator patterns and associated with the characteristic null space as described above. Thus, not only do the operator patterns enumerate all irreducible tensor operators, but they do so in a way that gives exactly a basis for all transformations between any two irrep spaces; equivalently, this solves the Wigner coefficient problem for $SU(3)$ in a structurally meaningful way and eliminates the vague and arbitrary "multiplicity index."

This review, although very incomplete, suffices for our purpose of indicating the origin of the special functions that are the topic of the subsequent discussion: They arise in the explicit construction of the canonical Wigner coefficients discussed in this section.

Remarks: (a) An elegant global construction for the set of all tensor operators in $SU(3)$ has recently been given by Flath [17]. The key to this construction is an embedding of $SU(3)$ in $SO(8)$. A very particular realization of the complex $\mathfrak{so}(3)$ Lie algebra $\mathfrak{g}$ is constructed based on polynomial differential operators in six variables. A unitary subset of $\mathfrak{g}$ belongs to a real form of the D_4 Lie algebra and (it is shown) generates a single unitary irrep V of the group $SO(6,2)$. The algebra of all operators carrying $V \rightarrow V$ is then found to be a simple algebra, A , a quotient algebra of the enveloping algebra of $\mathfrak{so}(3)$.

(b) Can one hope to develop similar results for $SU(n)$ ($n > 3$), or even for all Lie groups? The elegance of Flath's $SU(3)$ result argues otherwise, for the construction admits an S_3 automorphism that accords nicely with the S_3 outer automorphism of the Lie group $SO(8)$. For $SU(4)$ the structure must admit the S_4 group, but there is no simple Lie group with such an automorphism! Something unusual must happen in going from $SU(3)$ to $SU(4)$, and it is not at all clear that the Lie subalgebra analogous to $\mathfrak{g} \in A$ is even finitely generated.

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3. THE FORM OF THE G_q^T FUNCTIONS OCCURRING IN THE DIRECT CALCULATION OF THE CANONICAL WIGNER COEFFICIENTS

The implementation of the characteristic null space properties of the canonical tensor operators leads to the vanishing of certain of the so-called isoscalar factors in the Wigner coefficients. Using these vanishing isoscalar factors, one can then give a complete algorithm for the calculation of all $SU(3)$ Wigner coefficients (see Refs. [18-21]). This procedure is, however, very difficult to implement to obtain complete algebraic expressions for the coefficients. Thus far we have been able to obtain such closed expressions only for a certain $SU(3)$ invariant part of the coefficient--a part that can be identified as essentially the norm of the Wigner operator. It is to these $SU(3)$ invariants that we now direct our attention, giving the very complicated form in which the functions G_q^T first emerge. Henceforth, we use the notation $[p, q, 0] = [p, q, 0]$.

Preliminary to giving the functions G_q^T , it is convenient to introduce a number of notations and definitions that will be used throughout the remainder of Sections 3 and 4:

q is an integer that may assume values $0, 1, 2, \dots$;
 p is an integer that may assume for each q the values $q, q+1, \dots$;
 $\mathbf{q}$ is the 3-tuple of integers (q_1, q_2, q_3) with domain given by $0 = q_1 + p + q, q_2 + q_3 + q_1 = p + q$; (3.1)

$\mathbf{n}$ is the 3-tuple of integers (n_1, n_2, n_3) with domain given by

$$n_1 \in \{0, 1, \dots, q\} \cap \{q - q_1, q - q_1 + 1, \dots, p - q_1\}, \text{ and} \\ n_1 + n_2 + n_3 = q; \quad (3.2)$$

$\mathbf{x} = (x_1, x_2, x_3)$ is an arbitrary point in the Möbius plane;

$(z)_a = z(z+1)\dots(z+a-1)$ for each $a = 0, 1, 2, \dots$ and indeterminate z is Pochhammer's notation for a rising factorial;

$[z]_a = z(z-1)\dots(z-a+1)$ is a falling factorial;

$$\binom{a}{b} = [z]_a / a!;$$

 

$$M = q + 1 - \delta_1 - \delta_2 - \delta_3, \quad \delta_i = \max(0, a - \Delta_i), \text{ each } i = 1, 2, 3.$$
$$D_n^2(\Delta; x) = \prod_{i=1}^3 n_i! (\Delta_i - q + n_i)! \cdot (-1)^{p+q+\Delta_2+1} \prod_{ijk} (x_i + q - \Delta_k - n_j)^{p-\Delta_i-n_j+1} (x_i - n_j)^{q-n_j+1} \cdot \frac{\text{cyclic}}{\prod_{ijk} (x_i + \Delta_j - \Delta_k)(x_i - n_j + n_k)} \text{cyclic}, \quad (3.3)$$

Concerning this definition of $D_n^2(\Delta; x)$, two points need to be noted: The factors in the denominator do divide the numerator, so that the expression is somewhat simpler if we do not carry out the explicit cancellations; for $x_1 \geq 1$ and $x_3 \geq 1$, hence, $x_2 \geq 2$, we have $D_n^2(\Delta; x) \geq 0$, which accounts both for the notation using a square and the presence of the phase factor $(-1)^{p+q+\Delta_2+1}$ [for the identification $(x_1, x_2, x_3) = (\lambda_2 - \lambda_3, \lambda_2 - \lambda_1, \lambda_1 - \lambda_2)$, this is the region in which the irrep labels $(\lambda_1, \lambda_2, \lambda_3)$ obey $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$].

$$\begin{aligned} & (y+1+\frac{1}{3}(x_2-x_3)+\delta_1)_{n_1-\delta_1} (y+1+\frac{1}{3}(x_3-x_1)+\delta_2)_{n_2-\delta_2} \\ & \times (y+1+\frac{1}{3}(x_1-x_2)+\delta_3)_{n_3-\delta_3} = \sum_{s=1}^M y^{M-s} F_n^s(\delta; x). \end{aligned} \quad (3.4)$$
$$V_n^8(\Delta; x) = F_n^8(\delta; x) / D_n(\Delta; x) \text{ for } x_1 \geq 1, x_2 \geq 1, x_3 \leq -2, \quad (3.5)$$
$$[(v^r, v^u)] = \sum_n F_n^r(z; x) F_n^u(z; x) / D_n^2(z; x). \quad (3.6)$$

Concerning this definition of (V^r, V^s) , we note that it is defined even when the conditions on x in Eq.(3.5) are violated, since $D_n^2(\Delta; x)$ is a polynomial--in this case, it is Eq.(3.6) itself that is used for the definition of (V^r, V^s) . The summation over n in Eq.(3.6) is to be carried out over all n in the domain of definition given in Eq.(3.2). It should also be noted that we have suppressed in the notation (V^r, V^s) the fact that this quantity depends on p, q, Δ , and x .

We require still one more definition before introducing the G_q^t functions of interest, namely, the polynomial factors defined by

$$L_s(\Delta; x) = [x_1 x_2 x_3 / (x_1 + \Delta_2 - \Delta_3)(x_2 + \Delta_3 - \Delta_1)(x_3 + \Delta_1 - \Delta_2)] \cdot \prod_{\substack{ijk \\ \text{cyclic}}} (x_i + 1)_{\Delta_j - s + 1} (-x_i + 1)_{\Delta_k - s + 1}, \quad (3.7)$$

each $s = 1, 2, \dots, M$. Again the denominator factor always divides the numerator.

Finally, we define the functions $G_q^r(\Delta; x)$ by

$$G_q^t(\Delta; x) = (\#) \prod_{s=1}^t L_s(\Delta; x) \cdot \det \begin{bmatrix} (v^1, v^1)(v^1, v^1) & \dots & (v^1, v^t) \\ (v^2, v^1)(v^2, v^2) & \dots & (v^2, v^t) \\ \vdots & & \vdots \\ (v^t, v^1)(v^t, v^2) & \dots & (v^t, v^t) \end{bmatrix}, \quad (3.8)$$

each $t = 1, 2, \dots, M$, in which $(\#)$ denotes a quantity that is independent of x_1, x_2, x_3 , that is, depends only on p, q, Δ, t (we will not require the explicit form of $\#$ for this discussion).

The result given by Eq.(3.8) is very complicated, and there would appear to be little hope for much simplification, let alone a proof that G_q^t is, in fact, a polynomial in the x_1, x_2, x_3 with remarkable symmetries and patterns of zeros. Fortunately, the simplest case of the general result, namely,

$$G_q^1(\Delta; x) = (\#) L_1(\Delta; x) (v^1, v^1), \quad (3.9)$$

with

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Chapter 1

$$(V^1, V^1) = \sum_n \frac{1}{D_n^2(\Delta; x)} \quad (3.10)$$

Chapter 2

was studied first (see Refs. [18,19]) and it proved possible to obtain the explicit expression for G_q^1 and find its many nice properties. It was the encouragement coming from these results--and the explicit calculations of Alcaras, et al. [22]--that led us to believe that the complicated expression (3.8) is also a polynomial of a similar sort.

The polynomials $G_q^t(\Delta; x)$ of Eq.(3.8) may be considered as special functions of $SL(3)$. The special functions defined in this way have interesting symmetries, which are most easily grasped by slightly changing the notation: We write $G_q^t(\Delta; x) = G_q^t(A)$, where A denotes the matrix array¹ given by

$$A = \begin{pmatrix} \Delta_1 - t + 1 & \Delta_2 - t + 1 + x_1 & \Delta_3 - t + 1 - x_1 \\ \Delta_2 - t + 1 & \Delta_3 - t + 1 + x_2 & \Delta_1 - t + 1 - x_2 \\ \Delta_3 - t + 1 & \Delta_1 - t + 1 + x_3 & \Delta_2 - t + 1 - x_3 \end{pmatrix}. \quad (3.11)$$

As discussed at the beginning of this section the functions $G_q^t(A)$ characterize the set of $SL(3)$ tensor operators with irrep label $[p, q, 0]$ and shift pattern $(\Delta_1, \Delta_2, \Delta_3)$. The properties of the function $G_q^t(A)$ include the following:

- (a) The function G_q^t is a polynomial in the variables of the matrix array.
- (b) The polynomial is non-negative in the region of the Möbius plane corresponding to $\Delta_1 \geq \Delta_2 \geq \Delta_3$ (lexical region).
- (c) The polynomial is invariant under all determinantal symmetries of the array A (row interchange, column interchange, and transposition).
- (d) The polynomial G_q^t (each $t = 1, 2, \dots, q$) has value zero on a set of lattice points in the lexical region of the Möbius plane, this set of points being exactly that of the weight space diagram for irrep $[q-t, 0, -t+1]$ of $U(3)$, where the multiplicity of a weight space point is the order of the corresponding zero. (The "extremal" functions G_q^1 and G_q^q in the family $\{G_q^t | t = 1, 2, \dots, q\}$ have zeros, respectively, at the lattice points of the triangle and inverted triangle corresponding to the weight space diagrams $[q-1, 0, 0]$ and $[0, 0, -q+1]$.) (These remarkable properties were first noted by Lohe.)

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(e) The polynomials $\{G_q^t\}$ satisfy reduction formulas [see Eqs. (4.6) and (4.7)] such that the subset of polynomials for $t=1,2,\dots,M$ split all multiplicities for tensor operators $[p,q,0]$ with shift pattern $(\Delta_1, \Delta_2, \Delta_3)$; that is, each of the tensor operators of this type, M in number, is uniquely associated with these polynomials (actually to a ratio of successive polynomials).

There is a wealth of special function information contained in the G_q^t functions. Let us illustrate this remark.

Consider first the quantal rotation group $SU(2)$. The map from the Wigner operators, Eq. (1.7), to the invariant denominator function vanishes on the characteristic null space (the null space consisting of entire representations). For $SU(2)$, the space of all irreps, each exactly once, is represented by the positive integers, that is, $2j+1 = p_{11} - p_{21} = x_{12} > 0$. Because of the underlying S_2 symmetry, this becomes the integer points on the real line. The characteristic null spaces are found to be of the form of a "factorial power:" $[x_{12}]_n = x_{12}!/(x_{12}-n)!$. For $SU(2)$, such considerations are so elementary and obvious as to have escaped attention, but they are essential for generalizations. Even for $SU(2)$ they are important: the tensor operators in $SU(2)$ may be constructed entirely from null space considerations (see Ref. [15]).

Let us return now to the tensor operators in $SU(3)$. The space of all irreps, each exactly once, is now represented by lattice points in the N8b1us plane defined by $x_1 = p_{11} - p_{21}$, $x_2 = p_{31} - p_{13}$, $x_3 = p_{13} - p_{31}$, which automatically satisfy the constraint $x_1 + x_2 + x_3 = 0$.

Let us consider tensor operators--transforming as the $SU(3)$ irrep $[p,q,0]$ --that effect the shift $(\Delta_1, \Delta_2, \Delta_3)$. Then for a special class of these operators (the "stretched" operators, $t=1$) the map tensor operator $\rightarrow$ invariant defines the function $G_q^1(X)$. For $q=1$, that is for the operators $[p,1,0]$, we find the explicit result

$$\begin{aligned} G_1^1(\Delta; x) = & -\Delta_1 \Delta_2 (\Delta_1 + x_3) (\Delta_2 - x_3) \\ & - \Delta_2 \Delta_3 (\Delta_2 + x_1) (\Delta_3 - x_1) - \Delta_3 \Delta_1 (\Delta_3 + x_2) (\Delta_1 - x_2) \\ & - \Delta_1 \Delta_2 \Delta_3 (\Delta_1 + \Delta_2 + \Delta_3). \end{aligned} \quad (3.12)$$

We notice that this polynomial defines exactly 6 zeros,

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Footnotes

only one of which $(x_1 = \Delta_3, x_3 = \Delta_2)$ lies in the lexical region. The overall S_3 symmetry yields six 'triangles' of zeros, each 'triangle' having one point.

The G_q^1 functions are defined by two recursion relations, which have the form of contiguous relations in that the parameters and arguments are shifted by integers. If we ignore these shifts in the recursion formulas, then we find that the solution is just $[G_1^1(\Delta; x)]^q$, that is the q^{th} ordinary power of the first solution.

This suggests the actual solution is to be found by some sort of symbolic or umbral power, and this is indeed correct. Let us consider sequences of polynomials of binomial type (see Ref. [19]), that is, polynomials $p_n(x)$ obeying the rule

$$p_n(x+y) = \sum_k \binom{n}{k} p_k(x) p_{n-k}(y), \quad (3.13)$$

and extend this rule to multinomials.

Let us regard the terms in Eq.(3.12) as the terms in a quadriminomial and expand out the q^{th} power, collecting together the powers of $\Delta_1, \Delta_2, \Delta_3$, but leaving the powers in $(\Delta_1 + x_1)$, etc. as well as $(\Delta_1 + \Delta_2 + \Delta_3)$ undisturbed:

$$\begin{aligned} [G_1^1(\Delta; x)]^q &= (-1)^q q! \sum_{(k)} \frac{q-k_1}{1!} \frac{q-k_2}{2!} \frac{q-k_3}{3!} (\Delta_1 + \Delta_2 + \Delta_3)^{k_4} \\ &\cdot (\Delta_2 + x_1)^{k_1} (\Delta_3 - x_1)^{k_2} (\Delta_3 + x_2)^{k_3} (\Delta_1 - x_2)^{k_4} \\ &\cdot (\Delta_1 + x_3)^{k_1} (\Delta_2 - x_3)^{k_2} / (k_1)! (k_2)! (k_3)! (k_4)!, \end{aligned} \quad (3.14)$$

where the sum is over all non-negative integers $(k) = (k_1, k_2, k_3, k_4)$ which add to q , that is, $k_1 + k_2 + k_3 + k_4 = q$. In this result, we now make the replacements

$$\Delta_1^{q-k_1} / (q-k_1)! \rightarrow \binom{\Delta_1}{q-k_1},$$

$$(\Delta_2 + x_1)^{k_1} / k_1! \rightarrow \binom{\Delta_2 + x_1}{k_1}, \text{ etc.,}$$

with the single exception: We replace $(\Delta_1 + \Delta_2 + \Delta_3)^{k_4} / (k_4)!$ by

$$\binom{\Delta_1 + \Delta_2 + \Delta_3 - k_1 - k_2 - k_3}{k_4}$$

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The result is the following polynomial in x_1, x_2, x_3 , which we denote by $G_q^1(\Delta; x)$:

$$G_q^1(\Delta; x) = (-1)^q q! \sum_{(k)} \left(\Delta_1 + \Delta_2 + \Delta_3 - k_1 - k_2 - k_3 \right) \cdot (k_1)!(q-k_1)! \binom{\Delta_1}{q-k_1} \binom{\Delta_2+x_1}{k_1} \binom{\Delta_3-x_1}{k_1} \cdot (k_2)!(q-k_2)! \binom{\Delta_2}{q-k_2} \binom{\Delta_3+x_2}{k_2} \binom{\Delta_1-x_2}{k_2} \cdot (k_3)!(q-k_3)! \binom{\Delta_3}{q-k_3} \binom{\Delta_1+x_3}{k_3} \binom{\Delta_2-x_3}{k_3}. \quad (3.15)$$

minus signs

It can be verified that this explicit result for $G_q^1(x)$ has the properties (a)-(e) mentioned above.

The generalization to all G_q^t is the subject of the next section.

4. CONJECTURED FORM OF THE POLYNOMIAL $G_q^t(\Delta; x)$ AND A VERIFICATION OF SOME OF ITS PROPERTIES

We give below the general polynomial that we conjecture to be the $G_q^t(\Delta; x)$ discussed in the previous sections (for $t=1$, all results given here are known to be true as discussed). We give this polynomial in a somewhat more general form than required for the theory of tensor operators, but only to the extent that the $\xi = (\xi_1, \xi_2, \xi_3)$ variables [see Eq.(4.2)] may be an arbitrary point in R . The relation with the polynomials of interest in the tensor operator problem is given by

$$\xi_i = \Delta_i - t + 1, \quad i = 1, 2, 3. \quad (4.1)$$

With these points in mind, we now define the polynomial $G_q^t(\xi; x)$, each $t = 1, \dots, q$, each $\xi \in R^3$, and each $x \in R$ with $x_1+x_2+x_3=0$ by

$$G_q^t(\xi; x) = \sum_{(k)} \binom{\xi_1}{q-k_1} \binom{\xi_2+x_1}{k_1} \binom{\xi_3-x_1}{k_1} \binom{\xi_2}{q-k_2} \binom{\xi_3+x_2}{k_2} \binom{\xi_1-x_2}{k_2} \binom{\xi_3}{q-k_3} \binom{\xi_1+x_3}{k_3} \binom{\xi_2-x_3}{k_3}.$$

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$$\begin{aligned}
&= (-1)^{t(q-t+1)} \prod_{s=1}^t \frac{(q-s+1)!}{(s-1)!} \sum_{\lambda \mu \nu} h(\lambda \mu \nu) \frac{\prod_{s=1}^t (\xi_1 + \xi_2 + \xi_3 + 2t - q - s) c_s}{M(\tau)} \\
&\quad \cdot F_{q-t+1, \lambda}(\xi_1, \xi_2 + x_1, \xi_3 - x_1) \\
&\quad \cdot F_{q-t+1, \mu}(\xi_2, \xi_3 + x_2, \xi_1 - x_2) \\
&\quad \cdot F_{q-t+1, \nu}(\xi_3, \xi_1 + x_3, \xi_2 - x_3),
\end{aligned} \tag{4.2}$$

where the various symbols in this result have the following definitions:

(a) $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_t]$ denotes an irrep label of $U(t)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_t \geq 0$, each λ_i a non-negative integer; λ may also be regarded as the shape of a Young frame, $Y(\lambda)$. The symbols $\mu, \nu, \dots$ denote irrep labels of the same type as λ .

(b) $h(\lambda \mu \nu)$ denotes the number of times irrep $[q-t+1, \dots, q-t+1]$ ($q-t+1$ repeated t times) is contained in the direct product $\lambda \mu \nu$, and is defined to be zero if $[q-t+1, \dots, q-t+1] \notin \lambda \mu \nu$.

(c) $M(\lambda)$ is the measure of the Young frame $Y(\lambda)$ and has the definition

$$M(\lambda) = \prod_{s=1}^t (\lambda_s - s + 1)! / \prod_{r \leq s} (r - \lambda_s + s - r), \tag{4.3}$$

and $M(\mu), M(\nu), \dots$ have corresponding definitions.

(d) For each non-negative integer k and each Young frame $Y(\lambda)$, the function $F_{k, \lambda}(x, y, z)$ is defined for indeterminates x, y, z by

$$F_{k, \lambda}(x, y, z) = \begin{cases} \frac{1}{M(\lambda)} \prod_{s=1}^t [x+t-s]_{k-\lambda_s} [y+s-1]_{\lambda_s} [z+s-1]_{\lambda_s}, \\ 0, \text{ if any } \lambda_s > k. \end{cases} \tag{4.4}$$

Let us next describe the properties of the polynomials (4.2) to the extent that they have been determined, noting from the outset that we have found no simple recursion relations for these polynomials for general $t(t > 1)$.

Cancel factors
Young diagrams

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[see Eq. (4.1)]:

$$G_q^t \begin{pmatrix} \Delta_1 - t + 1 & \Delta_2 - t + 1 + x_1 & \Delta_3 - t + 1 - x_1 \\ \Delta_2 - t + 1 & \Delta_3 - t + 1 + x_2 & \Delta_1 - t + 1 - x_2 \\ \Delta_3 - t + 1 & \Delta_1 - t + 1 + x_3 & \Delta_2 - t + 1 - x_3 \end{pmatrix}$$

$$= (-1)^{t(q+1-M)} \prod_{s=1}^t \frac{(q+1-s)!}{(M-s)!}$$

(4.6)

$$\prod_{s=1}^t \prod_{ijk \text{ cyclic}} \frac{[\Delta_i + \delta_i - s + 1]_{\delta_j + \delta_k}}{[q - s + 1]_{\delta_i}}$$

$$\prod_{s=1}^t \prod_{ijk \text{ cyclic}} [\Delta_j + x_i - s + 1]_{\delta_i} [\Delta_k - x_i - s + 1]_{\delta_i}$$

$$G_{q'}^{t'} \begin{pmatrix} \Delta'_1 - t' + 1 & \Delta'_2 - t' + 1 + x'_1 & \Delta'_3 - t' + 1 - x'_1 \\ \Delta'_2 - t' + 1 & \Delta'_3 - t' + 1 + x'_2 & \Delta'_1 - t' + 1 - x'_2 \\ \Delta'_3 - t' + 1 & \Delta'_1 - t' + 1 + x'_3 & \Delta'_2 - t' + 1 - x'_3 \end{pmatrix},$$

where

$$q' = M - 1$$

$$x'_i = x_i - \delta_j + \delta_k, \quad ijk \text{ cyclic},$$

$$\Delta'_i = \Delta_i + \delta_i - \delta_j - \delta_k, \quad ijk \text{ cyclic},$$

$$\delta'_i = \max(0, q' - \Delta'_i) = 0, \quad i = 1, 2, 3,$$

$$\Delta'_1 + \Delta'_2 + \Delta'_3 = p + q' \quad (\text{for } \Delta_1 + \Delta_2 + \Delta_3 = p + q).$$

This remarkable formula establishes the important result:
The properties of each $G_q^t(\Delta; x)$ polynomial for $t = 1, 2, \dots, M$ may be obtained through the reduction formula from those in the special class $\{G_{q'}^{t'}(\Delta'; x')\}_{\Delta'_i \geq q', t' = 1, 2, \dots, q'}$.

A second kind of reduction formula for G_q^t can also be proved and is closely related to the fact that one has a conjugation operation defined for tensor operators:

$$G_q^t(\xi; x) = G_q^{q-t+1}(-\xi; -x), \quad t = 1, \dots, q. \quad (4.7)$$

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1. P. H. R. A. M. (1977) 107.

Reduced form: $G_q^t(\xi; x) = G_q^{q-t+1}(-\xi; -x)$

It should be remarked that the proofs of both of the reduction formulas make essential use of the properties of the intertwining number $h(\lambda\mu\nu\rho)$, and this again supports the correctness of that factor in our conjectured form for G_q^t .

An interesting application of Eq.(4.7) occurs in the calculation of the minimal null space tensor operators, where one obtains the following formula (see Refs.[20,21]):

$$G_q^q \begin{pmatrix} 1 & 1+x_1 & 1-x_1 \\ 1 & 1+x_2 & 1-x_2 \\ 1 & 1+x_3 & 1-x_3 \end{pmatrix} = G_q^1 \begin{pmatrix} -1 & -1-x_1 & -1+x_1 \\ -1 & -1-x_2 & -1+x_2 \\ -1 & -1-x_3 & -1+x_3 \end{pmatrix} \quad (4.8)$$

$$= \frac{2(2q+1)!(q+1)!(q+1)!}{x_1 x_2 x_3} \sum_{i=1}^3 \binom{x_i + q + 1}{2q+3}.$$

Let us emphasize again that in establishing the symmetries of the G^t function one is led to a generalization of the Saalschütz identity and to a far-reaching generalization of the Euler hypergeometric function itself. This fact strongly hints that the extension from $SU(3)$ to $SU(n)$ may involve some very interesting special function generalizations.

5. GENERALIZATION TO $SU(n)$

For $SU(3)$ the multiplicity problem is resolved by the canonical labelling induced from the ordering of the characteristic null spaces of tensor operators with the same irrep and shift labels. In Refs. [24,25] this labelling was extended to the $\langle p, q, \dots, q, 0, \dots, 0 \rangle$ tensor operators in $U(n)$ uniquely characterized by maximal null space. To each such tensor operator there was associated an invariant polynomial denoted by

$$\begin{aligned} \mu G_q^{(n)}(X) &\equiv \mu G_q^{(n)}((x_{ij} + \Delta_i)) \\ &\equiv \mu G_q^{(n)}(\Delta_1, \dots, \Delta_n; x_{1,2}, x_{2,3}, \dots, x_{n-1,n}, x_{n,1}), \end{aligned} \quad (5.1)$$

where X is the n by n matrix $[U(n) \text{ array}]$ defined by means of:

Definition 5.2 Let $n \geq 2$. Then

$$(X) \equiv ((x_{ij})) \equiv ((x_{ij} + \Delta_i)), \quad (5.3a)$$

where

$$\Delta_i \in R[x_{rs}], \quad 1 \leq i \leq n, \quad (5.3b)$$

$$x_{ij} \equiv -x_{ji}, \quad i < j, \quad (5.3c)$$

$$x_{ii} \equiv 0, \quad 1 \leq i \leq n, \quad (5.3d)$$

$$x_{ik} \equiv x_{ij} + x_{jk}, \quad 1 \leq i, j, k \leq n. \quad (5.3e)$$

Remark: For $n=3$, we have also denoted above $x_1 = x_{12}$, $x_2 = x_{23}$, and $x_3 = x_{31}$.

The above irrep label $\langle p, q, \dots, q, 0, \dots, 0 \rangle$ consists of one p , p q 's, and $n-p-1$ 0's. Furthermore, q determines p since $\Delta_1 + \dots + \Delta_n = p + nq$ and we are given q , $\Delta_1 + \dots + \Delta_n$, and n .

The problems that give rise to $G_q^{(n)}(X)$ are motivated in more detail and put into a broader mathematical setting in Refs. [14,15,24].

Recently in Refs. [26-28] an alternative method for explicitly writing down $G_q^{(n)}(X)$ in polynomial form has been given. These methods provide a direct connection between the classical theory of symmetric functions as presented in Macdonald [29], and the symmetries satisfied by a family of $U(n)$ invariant polynomials even more general than $G_q^{(n)}(X)$. Reference [26] gives a direct formulation of the invariant polynomials $G_q^{(n)}(x_1, \dots, x_{n-1})$ characterizing $U(n)$ tensor operators $\langle p, q, \dots, q, 0, \dots, 0 \rangle$ in terms of the symmetric functions S_λ .

Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ of length $\leq n$ (the number of nonzero λ_i , denoted by $l(\lambda)$, is called the length of λ), the Schur functions S_λ are defined by

$$S_\lambda(x_1, \dots, x_n) \equiv \frac{\det(x_i^{\lambda_j + n - j})_{1 \leq i, j \leq n}}{\det(x_i^{n-j})_{1 \leq i, j \leq n}}. \quad (5.4)$$

The determinant in the numerator is divisible in $\mathbb{Z}[x_1, \dots, x_n]$ by each of the differences $(x_i - x_j)$, $1 \leq i < j \leq n$, and hence by their product, which is the Vandermonde determinant

$$\prod_{1 \leq i < j \leq n} (x_i - x_j) = \det(x_i^{n-j})_{1 \leq i, j \leq n} \quad (5.5)$$

$$\equiv V_n(x_1, \dots, x_n).$$

Thus, the above quotient of determinants is a symmetric polynomial in $x_1, \dots, x_n$ with coefficients in $\mathbb{Z}$.

After the change of variables

$$\Delta_i = \gamma_i - \delta_i \quad \text{and} \quad x_{i,i+1} = \delta_i - \delta_{i+1}, \quad (5.6)$$

$m_{\mu, q}^{(n)}(\gamma, \delta, x_{i,i+1})$ becomes an integral linear combination of products of Schur functions $S_\lambda(\gamma_i) \cdot S_\mu(\delta_i)$ in the variables $\{\gamma_1, \dots, \gamma_n\}$ and $\{\delta_1, \dots, \delta_n\}$, respectively. That is, it is directly proved that $m_{\mu, q}^{(n)}(\gamma, \delta, x_{i,i+1})$ is a bisymmetric polynomial in the variables $\{\gamma_1, \dots, \gamma_n\}$ and $\{\delta_1, \dots, \delta_n\}$ with integer coefficients. This motivated the study of the yet more general bisymmetric polynomials

$$m_{\mu, q}^{(n)}(\gamma; \delta) \equiv m_{\mu, q}^{(n)}(\gamma_1, \dots, \gamma_n; \delta_1, \dots, \delta_n) \quad (5.7)$$

which are a common generalization of Eq.(2.2b) of Ref. [24] and Eq.(2.17) of Ref. [25] with no numerator parameters. These polynomials are given by

Definition 5.8 Given that $m_{\mu, q}^{(n)}(\gamma; \delta) \equiv 1$, we uniquely determine $m_{\mu, q}^{(n)}(\gamma; \delta)$ by means of

$$m_{\mu, q}^{(n)}(\gamma, \delta) = \sum_{\substack{S \in I_n \\ \|S\| = \mu+1}} (-1)^{\mu+1+\sum(S)} \prod_{\substack{i < j \\ i \in S, j \in S^c}} (\gamma_i - \gamma_j)^{-1} \quad (5.9)$$

$$\cdot \prod_{\substack{i < j \\ i \in S^c, j \in S}} (\gamma_i - \gamma_j)^{-1} \cdot \prod_{i=1}^n \prod_{j=1}^m (\gamma_i - \delta_j) \cdot m_{\mu, q-1}^{(n)}(\gamma_1 - \chi(i \in S), \delta).$$

where $S \subset I_n$ is a $(\mu+1)$ -element subset of $\{1, 2, \dots, n\}$, $\sum(S)$ denotes the sum of the elements in S , and $\chi(A)$ is 1 if statement A is true and 0 otherwise.

Remark. We have $m = j - n$ with $j \geq n$ where j is the number

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of denominator parameters in Eq.(2.17) of Ref. [25]. Only m of these parameters actually appear in Eq. (5.9). The case that is of most interest (in physics) is $m=n$.

At this point it is convenient to establish some notation. Fix u and assume that $m-n \equiv v$ is a constant. Denote the sets of variables $(\gamma_1, \dots, \gamma_{u+1})$, $(\gamma_{u+2}, \dots, \gamma_n)$, $(\delta_1, \dots, \delta_n)$, and $(\delta_1, \dots, \delta_m)$ by A , B , E , and F , respectively. If $Z = m_1 + \dots + m_n$, with m_i monomials, then

$$S_\lambda(Z) \equiv S_\lambda(m_1 + \dots + m_n) \equiv S_\lambda(x_1, \dots, x_n) \Big|_{x_i = m_i}. \quad (5.10a)$$

On the other hand, $S_\lambda(-(Z))$ is defined symbolically by

$$S_\lambda(-(Z)) \equiv S_\lambda(-(m_1 + \dots + m_n)) \equiv (-1)^{|\lambda|} S_{\lambda'}(Z), \quad (5.10b)$$

where $|\lambda| = \lambda_1 + \dots + \lambda_n$ is the sum of the parts of λ and λ' is the conjugate partition to λ . That is, $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_n)$, with $\lambda'_i = \#\{j | \lambda_j \geq i\}$. For example $(5, 2, 1)$ is the conjugate partition of $(3, 2, 1, 1, 1)$.

If $\rho = (\rho_1, \dots, \rho_k)$, where ρ_i are integers then denote by $(X)^\rho$ the monomial $x_1^{\rho_1} \dots x_k^{\rho_k}$ where $X = (x_1, \dots, x_k)$. Let the permutations $w \in S_k$ act on subscripts. For example, $w(1) = x_{w(1)}^{\rho_1} \dots x_{w(k)}^{\rho_k}$. Finally, let λ_k be the partition $(k-1, k-2, \dots, 1)$ and m_k the partition consisting of k parts equal to m .

One of the main results in Ref. [25] is the following fundamental

Theorem 5.11. Let $m_{u,q}^{(n)}(E;F)$ be defined as in Eq.(5.9). We then have

$$m_{u,q}^{(n)}(E;F) = \frac{(-1)^{\binom{u+1}{2}}}{V_n(E)} \sum_{\substack{\lambda = (\lambda_1, \dots, \lambda_{u+1}) \\ \lambda_1 \leq m}} S_\lambda(-F) \cdot \left\{ \sum_{w \in S_n} \epsilon(w) \cdot w[(A)^{m^{u+1} + \delta_{u+1}} - (\lambda_{u+1}, \dots, \lambda_1)(B)^{\delta_{n-(u+1)}}] \right. \\ \left. - m_{u,q-1}^{(n)}(\gamma_1 - 1, \dots, \gamma_{u+1} - 1, B; F) \right\}. \quad (5.12)$$

where, without loss of generality, w acts only on $y_1, \dots, y_{L+1}$ when applied to $\mathbb{M}_{G_q}^{(n)}$.

Starting with (5.12) and making direct use of the new symmetries discovered in Ref. [26], it is shown in Ref. [30] that the bisymmetric polynomials $\mathbb{M}_{G_q}^{(n)}(E;F)$ are an integral linear combination of Schur functions $S_\lambda(E-F)$ in the symbol $E-F$, where $E-F$ denotes the difference of the two sets of variables E and F . Making use of properties of skew Schur functions $S_{\lambda/\rho}$ and $S_\lambda(E-F)$ one puts together an umbral calculus for $\mathbb{M}_{G_q}^{(n)}(E;F)$. That is, working entirely with polynomials, one uniquely determines $\mathbb{M}_{G_q}^{(n)}(E;F)$ from $\mathbb{M}_{G_{q-1}}^{(n)}(E;F)$ and combinatorial rules [such as the Littlewood-Richardson rule (see Ref. [29]) involving Ferrers diagrams, that is, partitions]. The deepest part of this umbral calculus is a summation theorem, involving many different aspects of the theory of Schur functions, which essentially reduces the double sum in Eq. (5.12) to a single term. Although simple to apply, this summation theorem is rather technical. Therefore, we will just give the final formulas for $\mathbb{M}_{G_q}^{(n)}(E;F)$ and $\mathbb{P}_{G_q}^{(n)}(E;F)$, which are a direct result of the umbral calculus and the work in Ref. [26], and state a general theorem from Ref. [30] that illustrates how the structure of $\mathbb{M}_{G_q}^{(n)}(E;F)$ "stabilizes" as v increases while $m-n = v$ remains fixed. square

Before giving these formulas we need some more notation. Let λ, ρ be partitions. The skew Schur function $S_{\lambda/\rho}$ is defined by

$$S_{\lambda/\rho}(E) = \sum_{\nu \subset \lambda} c_{\rho\nu}^\lambda \cdot S_\nu(E), \quad (5.13)$$

where $\nu \subset \lambda$ means $\nu_i \leq \lambda_i$ for all $i \geq 1$, and the integers $c_{\rho\nu}^\lambda$ are the Littlewood-Richardson rule coefficients determined by

$$S_\rho(E) \cdot S_\nu(E) = \sum_{\lambda} c_{\rho\nu}^\lambda S_\lambda(E). \quad (5.14)$$

Note that $S_{\lambda/0} = S_\lambda$, where 0 denotes the zero partition. Also, $c_{\rho\nu}^\lambda = 0$ unless $|\lambda| = |\rho| + |\nu|$, so that $S_{\lambda/\rho}$ is homogeneous of degree $|\lambda| - |\rho|$, and is zero if $|\lambda| < |\rho|$. In fact, $S_{\lambda/\rho} = 0$ unless $\rho \subset \lambda$. The skew Schur function $S_{\lambda/\rho}$ can also be written as the determinant

$$S_{\lambda/\rho}(E) = \det(h_{\lambda_i - \rho_j - i + j}(E))_{1 \leq i, j \leq n}, \quad (5.15)$$

where $n \geq l(\lambda)$, and h_ℓ is the ℓ th homogeneous symmetric function of the variables E .

Given the sets of variables E and F, let E+F denote the set union $(\gamma_1, \dots, \gamma_n, \delta_1, \dots, \delta_m)$. There is then the following classical result

$$S_\lambda(E+F) = \sum_{\rho \in \lambda} S_{\lambda/\rho}(E) S_\rho(F). \quad (5.16)$$

It is immediate from Eqs.(5.10b) and (5.16) that

$$S_\lambda(E-F) = \sum_{\rho \in \lambda} S_{\lambda/\rho}(E) S_\rho(-F) \quad (5.17)$$

$$= \sum_{\rho \in \lambda} (-1)^{|\rho|} S_{\lambda/\rho}(E) \cdot S_{\rho'}(F), \quad (5.18)$$

where ρ' is the conjugate partition to ρ .

The above classical formulas can be found in Chapter I of Macdonald [29]. Eqs.(5.13)-(5.18) are crucial to the analysis in Ref. [30].

We now state a fundamental result of Ref. [30].

Theorem 5.19. Fix $u \geq 0$ and $k = \tau - n$. There are integers $a_{\lambda,q}$, λ a partition, q a non-negative integer such that

$$m_{G_q}^{(n)}(E;F) = \sum_{\substack{|\lambda| \leq (u+1)(u+1+k)q \\ \lambda \in [(u+1+k)q]}} a_{\lambda,q} S_\lambda(E-F). \quad (5.20)$$

If $n \geq (u+1)q$ and $m \geq (u+1+k)q$ then $a_{\lambda,q}$ is independent of n . More generally, if $S_\lambda(E-F) \neq 0$, then $a_{\lambda,q}$ is independent of n .

A similar "stabilization" theorem involving $S_\lambda(E) \cdot S_\rho(F)$ instead of $S_\lambda(E-F)$ was proven in Ref.[26]. Just as in Ref. [26], it follows from Eq. (5.18) that the $n = (u+1)q$ and $m = (u+1+m-n)q$ case of this relation gives the correct formula for $m_{G_q}^{(n+1)}(E;F)$, for all integers λ , when the sets E and F contain $(n+1)$ and $(m+1)$ variables, respectively.

Theorem 1.22 of Ref. [26] expresses $m_{G_q}^{(n)}(E;F)$ as a sum of products of Schur functions $S_\lambda(E) \cdot S_\rho(F)$. Using Eq.(5.18) to rewrite this sum, immediately gives

$$m_{G_q}^{(n)}(E;F) = (-1)^{\binom{u+1}{2}} S_{(u+1+m-n)u+1}(E-F), \quad (5.21)$$

where $(\nu+1+m-n)^{\nu+1}$ denotes the partition consisting of $(\nu+1)$ parts equal to $(\nu+1+m-n)$.

In Ref. [26] it took three pages to write ${}^n_1G_2^{(n)}(E;F)$ as a sum of products of Schur functions. Starting with Eq.(5.21) and making use of the umbral calculus developed in Ref. [30], it is not hard to see that

$$\begin{aligned} {}^n_1G_2^{(n)}(E;F) = & (S_{2^2}(D))^2 + \{-[2S_{4,3}(D) + 2S_{2^3,1}(D) \\ & + 3S_{3,2^2}(D) + 3S_{3^2,1}(D) + S_{4,3,1}(D) + S_{3,2,1^2}(D)] \\ & + [S_{4,2}(D) + S_{2^2,1^2}(D) + 3S_{2^3}(D) + 3S_{3^3}(D) + 3S_{3,2,1}(D)] \\ & - [2S_{3,2}(D) + 2S_{2^2,1}(D)] + S_{2^2}(D)\} , \end{aligned} \quad (5.22)$$

where D denotes the set difference $E-F$.

Further applications of Theorem 5.19 and the umbral calculus, as well as detailed proofs, can be found in Ref.[30].

We conclude roughly where we began: special functions are interesting in theoretical physics because they arise not only from the matrix elements of symmetry group representations, but also from the associated algebras of tensor operators, the symmetry itself still being fundamental.

NOTES

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FOOTNOTE

¹ For $t = 1$, the nine entries in the array A may be written as $p_{jj}^f - p_{k3}^i$ ($j, k = 1, 2, 3$), where $p_{k3}^i = \lambda_k + 3 - k$ and $p_{jj}^f = p_{jj}^i + \Delta_j$. For the generalization to $SU(n)$ discussed in Section 5, it is useful to rearrange the elements in array A into an array X in which the entry appearing in row j and column k is $X_{jk} = p_{jj}^f - p_{k3}^i$. The symmetries of G_q^f may then equally well be expressed as the determinantal symmetries of the array X . [One sees now that X is also invariant under constant shift of $(p_{13}^f, p_{23}^f, p_{33}^f)$ and $(p_{13}^i, p_{23}^i, p_{33}^i)$.]

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