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**Self-Consistent Spectrum
of Electrostatic Drift Wave
Fluctuations Due to Electron
Phase Space Correlations in
a Sheared Magnetic Field**

S. P. Hirshman
P. H. Diamond

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SELF-CONSISTENT SPECTRUM OF ELECTROSTATIC DRIFT
WAVE FLUCTUATIONS DUE TO ELECTRON PHASE SPACE
CORRELATIONS IN A SHEARED MAGNETIC FIELD

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ABSTRACT

The spectrum of electrostatic universal mode fluctuations due to electron phase space correlations (clumps) in a sheared magnetic field is self-consistently calculated. The pair correlation equation for electrons in a sheared field is derived and renormalized, and an approximate solution for the correlation function is obtained. Using the density correlation function as a source term in Poisson's equation, the renormalized dielectric operator is inverted to obtain the turbulent spectrum. Self-consistency is imposed by requiring that the diffusion coefficient derived from the calculated spectrum equal that used to obtain the spectrum originally. The diffusion coefficient obtained in this way is in close agreement with that calculated from one-point turbulence theory. The calculated spectrum is rapidly convergent for large wave numbers and is an even function of the wave numbers, consistent with the quadratic dependence of the relative diffusion coefficient on the orbit separations for small clump dimensions.

I. INTRODUCTION

The nonlinear dynamics of electron orbits in a sheared magnetic field were recently shown to significantly modify drift wave turbulence and stability.¹ In comparison with the uniform magnetic field case, where ions nonlinearly decorrelate at a rate $k_{\perp}^2 D$ (D is the cross-field turbulent diffusion coefficient), electron free streaming along the magnetic field in a sheared field combines with turbulent radial diffusion to produce an enhanced shear decorrelation rate ω_c which typically exceeds the shearless rate for tokamaks. For short wavelength fluctuations ($k_{\perp} \rho_i \geq 1$) thought to be responsible for anomalous electron transport in tokamaks, ω_c can also exceed the mode frequency.

Strong plasma turbulence theory has been used to calculate self-consistently the anomalous diffusion coefficient from considerations of single-mode nonlinear marginal stability. However, the state computed from the one-point theory is not a true steady state, since most modes would be damped in the absence of sources. A turbulent, steady-state spectrum results from the detailed balance of energy emission and absorption and cannot be determined from the one-point theory.

In a turbulent plasma, emission is due in part to phase space structures called "clumps," which are produced by strong mode coupling and incoherent harmonic generation.² These clumps behave like dressed macroparticles, which accounts for their importance in the energy balance of a turbulent plasma. The resonant wave-particle interactions associated with clump formation result in dissipation of mode energy throughout the spectrum, in contrast to an undamped energy cascade, and result in the fine scale rearrangement of the average phase space gradients.

In the present paper, Dupree's clump theory is used to obtain the potential fluctuation spectrum of drift waves for a plasma in a sheared magnetic field. The theory of clump-generated spectra has been previously described for one-dimensional velocity space turbulence²⁻⁴ and low and high frequency three-dimensional turbulence in a uniform magnetic field.^{5,6} Also, several authors have calculated correlation lengths (or K-entropy) for test particles in a sheared magnetic field.^{7,8} In contrast to the test particle problem, the self-consistent theory developed here accounts for the reaction of the clumps back on the electric fields which are producing them, thus allowing the calculation of a self-consistent potential spectrum. A steady state is achieved by a balance of clump emission and energy absorption through turbulently enhanced damping.

The role of phase space granulations in the turbulent steady state can be understood by decomposing the total electron density fluctuation into a coherent part $n_{\vec{k},\omega}^{(c)}$, with the same phase as the potential fluctuation $\phi_{\vec{k},\omega}$, and an incoherent piece $\tilde{n}_{\vec{k},\omega}$ which represents the fine scale phase space clumps. Quasi-neutrality then yields $\langle \phi^2 \rangle_{\vec{k},\omega} = \langle \tilde{n}_{\vec{k},\omega}^2 \rangle / |\epsilon(\vec{k},\omega)|^2$, where ϵ is a dielectric operator in a sheared magnetic field. Thus, the density correlation function is required to compute the spectrum.

In Sec. II, the renormalized equation for the phase space correlation function, whose velocity moment yields the density correlation function, is derived. Section III presents a solution for the equal time correlation function. In Sec. IV, the two-time density correlation function is obtained and used to compute the spectrum in terms of the diffusion coefficient D and mean squared wave numbers characterizing the

clump dimensions. In Sec. V, D and the mean wave numbers, which are defined as averages over the spectrum, are self-consistently determined. In contrast to strong turbulence in a uniform plasma, the fact that $\omega_c \gtrsim k_{\parallel} v_e$ over much of the radial extent of the drift modes and $\omega_c > \omega$ in a sheared field (i.e., the nonlinear broadening dominates the ballistic wave particle resonance) allows the calculation of an explicit value for D , rather than a regeneration criterion. Indeed, D obtained from the two-point theory exceeds only very slightly D obtained from the one-point marginal stability analysis. The excess in D is due to the additional damping required to balance clump emission.

In this paper we assume a low β (vector potential $\vec{A} = 0$) plasma model. Consistent with the fact that the dominant resonant processes affect the electrons, the ions will be treated linearly. Ion nonlinearities could, however, affect the mode structure, and hence the shear damping, and remain to be considered. The presence of many modes and phase space island overlap is used to justify a diffusive treatment of electron orbits and the replacement of spectral sums with integrals.

II. DERIVATION OF TWO-POINT CORRELATION EQUATION FOR ELECTRONS

The drift kinetic equation for electrons with phase space position $(\vec{r}_1, \vec{v}_1, t)$ is

$$\begin{aligned} \frac{\partial f(1)}{\partial t} + v_{\parallel 1} \vec{n} \cdot \nabla f(1) - \frac{c \nabla \Phi(1) \times \vec{n}}{B} \cdot \nabla f(1) \\ + e v_{\parallel 1} \vec{n} \cdot \nabla \Phi(1) \frac{\partial f(1)}{\partial \epsilon_1} = 0, \quad (1) \end{aligned}$$

where $f(1) = f(\vec{r}_1, \vec{v}_1, t)$ and $\vec{n} = \vec{B}/B$. Multiplying Eq. (1) by $f(2)$ and symmetrizing yields

$$\begin{aligned} \left\{ \frac{\partial}{\partial t} + (v_{\parallel 1} \vec{n} \cdot \nabla_1 + v_{\parallel 2} \vec{n} \cdot \nabla_2) - \frac{c}{B} [\nabla \Phi(1) \times \vec{n} \cdot \nabla_1 + \nabla \Phi(2) \times \vec{n} \cdot \nabla_2] \right. \\ \left. + \left[e v_{\parallel 1} \vec{n} \cdot \nabla \Phi(1) \frac{\partial}{\partial \epsilon_1} + e v_{\parallel 2} \vec{n} \cdot \nabla \Phi(2) \frac{\partial}{\partial \epsilon_2} \right] \right\} f(1,2) = 0. \quad (2) \end{aligned}$$

Here $f(1,2) = f(1)f(2)$. Tertiary three-point correlations have been neglected. Equation (2) is recognized as the second member of the BBGKY hierarchy.

In a manner analogous to the renormalization of the one-point Eq. (1), we shall now split Eq. (2) into a quasi-linear equation for the phase averaged part of $f(1,2)$ [denoted by $\langle f(1,2) \rangle$] and the fluctuation $\tilde{f}(1,2) \equiv f(1,2) - \langle f(1,2) \rangle$. Averaging Eq. (2) over the wave phases yields

$$\begin{aligned}
& \left(\frac{\partial}{\partial \tau} + v_{\parallel 1} \vec{n} \cdot \nabla_1 + v_{\parallel 2} \vec{n} \cdot \nabla_2 \right) \langle f(1,2) \rangle \\
&= \frac{c}{B} \left[\left\langle \nabla \Phi(1) \times \vec{n} \cdot \nabla_1 \tilde{f}(1,2) + \nabla \Phi(2) \times \vec{n} \cdot \nabla_2 \tilde{f}(1,2) \right\rangle \right] \\
&- e \left[\left\langle v_{\parallel 1} \vec{n} \cdot \nabla \Phi(1) \frac{\partial}{\partial \epsilon_1} \tilde{f}(1,2) + v_{\parallel 2} \vec{n} \cdot \nabla \Phi(2) \frac{\partial}{\partial \epsilon_2} \tilde{f}(1,2) \right\rangle \right] . \quad (3)
\end{aligned}$$

The first group of terms on the right of Eq. (3) produces spatial diffusion in (1,2) phase space, and is the dominant nonlinear turbulent effect considered here. The last group of terms produces energy diffusion of the pair correlation function and will henceforth be neglected.

To evaluate the right side of Eq. (3), the fluctuating part of $f(1,2)$ is required. An equation for $\tilde{f}(1,2)$ is obtained by subtracting Eq. (3) from Eq. (2) to yield

$$\begin{aligned}
& \left[\frac{\partial}{\partial \tau} + v_{\parallel 1} \vec{n} \cdot \nabla_1 + v_{\parallel 2} \vec{n} \cdot \nabla_2 - \frac{c \nabla \Phi(1) \times \vec{n}}{B} \cdot \nabla_1 \right. \\
& \quad \left. - \frac{c \nabla \Phi(2) \times \vec{n}}{B} \cdot \nabla_2 \right] \tilde{f}(1,2) = L(1,2,t) \tilde{f}(1,2) \\
&= \left[\frac{c}{B} \nabla \Phi(1) \times \vec{n} \cdot \nabla_1 + 1 \leftrightarrow 2 \right] \langle f(1,2) \rangle \\
&- \left[e v_{\parallel 1} \vec{n} \cdot \nabla \Phi(1) \frac{\partial}{\partial \epsilon_1} + 1 \leftrightarrow 2 \right] \langle f(1,2) \rangle . \quad (4)
\end{aligned}$$

The phase averaged component on the left of Eq. (4) is to be omitted. Here, $(1 \leftrightarrow 2)$ indicates the addition of a symmetrizing term obtained by interchanging the coordinates 1 and 2.

It is convenient to introduce the two-point particle propagator, $U(1,2,t,t')$, defined such that

$$LU(1,2,t,t') = 0 , \quad (5a)$$

$$U(1,2,t,t) = 1 . \quad (5b)$$

In terms of U , the solution of Eq. (4) is

$$\begin{aligned} \tilde{f}(1,2,t) = & \int_0^t dt' U(1,2,t,t') \left[\frac{c \nabla \Phi(1,t') \times \vec{n}}{B} \cdot \nabla_1 \right. \\ & \left. - e v_{\parallel 1} \vec{n} \cdot \nabla \Phi(1,t') \frac{\partial}{\partial \epsilon_1} + (1 \leftrightarrow 2) \right] \langle f(1,2,t') \rangle . \quad (6) \end{aligned}$$

Consistent with the physical notion of a "clump" as a long-lasting phase space structure (clump lifetime τ_{CL} exceeds the single-particle decorrelation time τ_c), it is possible to factor the propagator in Eq. (6) into the product of uncorrelated one-point propagators for times less than τ_{CL} ; i.e., $\langle f(1,2,t') \rangle$ changes on τ_{CL} scale, whereas the orbits appearing in Φ diffuse on the τ_c scale. For times greater than τ_{CL} , $U(1,2,t,t')$ is negligible. Thus, Eq. (6) becomes

$$\begin{aligned} \tilde{f}(1,2,t) = \int_0^t dt' \left[U(1,t,t') \left(\frac{c \nabla \Phi(1,t') \times \vec{n}}{B} \cdot \nabla_1 \right. \right. \\ \left. \left. - e v_{\parallel 1} \vec{n} \cdot \nabla \Phi(1,t') \frac{\partial}{\partial \epsilon_1} \right) + (1 \leftrightarrow 2) \right] \langle f(1,2,t) \rangle . \quad (7) \end{aligned}$$

In Eq. (7), the variation of the average correlation function over an orbit diffusion time τ_c has been neglected.

Following Dupree,² the phase-coherent part of the correlation function may be extracted from $\tilde{f}(1,2,t)$ by multiplying Eq. (7) by $\exp(i\beta_{\vec{k}})$ and integrating over $\beta_{\vec{k}}$, the phase of the k^{th} Fourier component of the potential fluctuation. Define

$$\Phi = \sum_{k_y, k_z} \Phi_{\vec{k}}(r,t) \exp[i(k_y y + k_z z)] \quad (8a)$$

where $\Phi_{\vec{k}}(r,t) \propto \exp(-i\beta_{\vec{k}})$. Then

$$\tilde{f}_{\vec{k}}(1,2,t) \equiv \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(1,2,t) \exp(i\beta_{\vec{k}}) d\beta_{\vec{k}} , \quad (8b)$$

where $k_y = m/r$, $y = r\theta$, $k_z = n/R$, and $z = R\phi$. Note that due to nonlinearities in Eq. (4), $\tilde{f}_{\vec{k}}$ generally is not simply a Fourier coefficient.

The resulting expression for $\tilde{f}_{\vec{k}}$ becomes

$$\begin{aligned}
\tilde{f}_{\vec{k}}(1,2,t) \exp(i\beta_{\vec{k}}) = \int_0^t d\tau \left(\langle U(1,t-\tau) \rangle \left\{ \frac{-ik_y c}{B} \Phi_{\vec{k}}(r_1,t) \exp[i(k_y y_1 + k_z z_1)] \frac{\partial}{\partial r_1} + \frac{c}{B} \frac{\partial}{\partial r_1} \Phi_{\vec{k}}(r_1,t) \exp[i(k_y y_1 + k_z z_1)] \frac{\partial}{\partial \theta_1} \right\} \right. \\
\left. + (1 \leftrightarrow 2) \right) \langle f(1,2) \rangle . \quad (9)
\end{aligned}$$

Physically, the averaged propagator $\langle U(1,t) \rangle$ replaces the exact particle orbits with ensemble-averaged (over Gaussian statistics) diffusive trajectories.

Taking the Laplace transform of Eq. (9) yields the coherent two-point response

$$\begin{aligned}
\tilde{f}_{\vec{k},\omega}(1,2) \exp(i\beta_{\vec{k}}) = \left\{ \frac{c}{B} \exp[i(k_y y_1 + k_z z_1)] g_{\vec{k},\omega}(1) \right. \\
\left. \times \left[\frac{-ik_y c}{B} \Phi_{\vec{k},\omega}(r_1) \frac{\partial}{\partial r_1} + \frac{c}{B} \frac{\partial}{\partial r_1} \Phi_{\vec{k},\omega}(r_1) \frac{\partial}{\partial \theta_1} \right] + (1 \leftrightarrow 2) \right\} \langle f(1,2) \rangle , \quad (10)
\end{aligned}$$

where $g_{\vec{k},\omega}(1)$, the Fourier-Laplace transform of the averaged propagator, is the renormalized one-point resonance function for a sheared magnetic field.

Inserting Eq. (10) into Eq. (3) yields a quasi-linear diffusion equation for $\langle f(1,2) \rangle$:

$$\begin{aligned}
& \left[\frac{\partial}{\partial t} + \left(v_{\parallel 1} \vec{n} \cdot \nabla_1 - \frac{\partial}{\partial r_1} D_{r_1, r_1} \frac{\partial}{\partial r_1} - \frac{\partial}{\partial r_1} D_{r_1, r_2} \frac{\partial}{\partial r_2} - \frac{\partial}{\partial y_1} D_{y_1, y_1} \frac{\partial}{\partial y_1} \right. \right. \\
& \quad \left. \left. - \frac{\partial}{\partial y_1} D_{y_1, y_2} \frac{\partial}{\partial y_2} - \frac{\partial}{\partial r_1} D_{r_1, y_1} \frac{\partial}{\partial y_1} - \frac{\partial}{\partial r_1} D_{r_1, y_2} \frac{\partial}{\partial y_2} \right. \right. \\
& \quad \left. \left. - \frac{\partial}{\partial y_1} D_{y_1, r_1} \frac{\partial}{\partial r_1} - \frac{\partial}{\partial y_1} D_{y_1, r_2} \frac{\partial}{\partial r_2} \right) + (1 \leftrightarrow 2) \right] \langle f(1, 2, t) \rangle = 0 . \quad (11)
\end{aligned}$$

The diffusion coefficients are

$$\begin{aligned}
D_{\alpha_i, \beta_j} &= \sum_{\vec{k}, \omega} \frac{c^2}{B^2} [\vec{\alpha}_i \cdot \vec{n} \times \nabla \Phi_{\vec{k}, \omega}(i)] [\vec{\beta}_j \cdot \vec{n} \times \nabla \Phi_{\vec{k}, \omega}^*(j)] g_{\vec{k}, \omega}(j) \\
&\quad \times \exp [ik_y(y_i - y_j) + ik_z(z_i - z_j)] . \quad (12)
\end{aligned}$$

Here, $\vec{\alpha}$ and $\vec{\beta}$ are unit vectors in the $\vec{r}$ - $\vec{y}$ plane. For spectrum densities which are even functions of k_y and k_z , both one-particle cross diffusion coefficients vanish. Furthermore, D_{r_1, y_2} and D_{r_2, y_1} are small for the same reason, when $\Delta r = r_1 - r_2$ and $\Delta y = y_1 - y_2$ are on the order or less than clump dimensions. The evenness of the spectrum can be verified a posteriori. Thus, the diffusion equation for $\langle f(1, 2) \rangle$ reduces to

$$\begin{aligned}
& \left[\frac{\partial}{\partial t} + \left(v_{\parallel 1} \vec{n} \cdot \nabla_1 - \frac{\partial}{\partial r_1} D_{r_1, r_1} \frac{\partial}{\partial r_1} - \frac{\partial}{\partial r_1} D_{r_1, r_2} \frac{\partial}{\partial r_2} - \frac{\partial}{\partial y_1} D_{y_1, y_1} \frac{\partial}{\partial y_1} \right. \right. \\
& \quad \left. \left. - \frac{\partial}{\partial y_1} D_{y_1, y_2} \frac{\partial}{\partial y_2} \right) + (1 \leftrightarrow 2) \right] \langle f(1, 2) \rangle = 0 . \quad (13)
\end{aligned}$$

Making a cluster decomposition of $\langle f(1,2) \rangle$,

$$\langle f(1,2) \rangle = \langle f(1) \rangle \langle f(2) \rangle + \langle \delta f(1,2) \rangle , \quad (14)$$

and noting that $\langle f \rangle$ obeys the single-particle quasi-linear equation, Eq. (13) can be rewritten as an equation for δf :

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + \left(v_{\parallel 1} \vec{n} \cdot \nabla_1 - \frac{\partial}{\partial r_1} D_{r_1, r_1} \frac{\partial}{\partial r_1} - \frac{\partial}{\partial r_1} D_{r_1, r_2} \frac{\partial}{\partial r_2} - \frac{\partial}{\partial y_1} D_{y_1, y_1} \frac{\partial}{\partial y_1} \right. \right. \\ & \quad \left. \left. - \frac{\partial}{\partial y_1} D_{y_1, y_2} \frac{\partial}{\partial y_2} \right) + (1 \leftrightarrow 2) \right] \langle \delta f(1,2) \rangle \\ & = \left(\frac{\partial}{\partial r_1} D_{r_1, r_2} \frac{\partial}{\partial r_2} + \frac{\partial}{\partial r_2} D_{r_2, r_1} \frac{\partial}{\partial r_1} \right) \langle f(1) \rangle \langle f(2) \rangle . \quad (15) \end{aligned}$$

The operator on the left side of Eq. (15) is the renormalized two-point operator. Thus, for drift wave turbulence, the average phase space spatial gradients drive the evolution of the two-point correlation function.

The diffusion coefficients in Eq. (15) may be evaluated from Eq. (12) (discrete sums are now replaced by integrals):

$$D_{r_1, r_1} = \int \frac{d^2 k}{(2\pi)^2} \int \frac{d\omega}{(2\pi)} \frac{c^2 k^2}{B^2} \Phi_{-\vec{k}, -\omega}^{(1)} g_{\vec{k}, \omega}^{(1)} \Phi_{\vec{k}, \omega}^{(1)} , \quad (16a)$$

$$D_{y_1, y_1} = \int \frac{d^2 k}{(2\pi)^2} \int \frac{d\omega}{(2\pi)} \frac{c^2}{B^2} \frac{\partial}{\partial r_1} \Phi_{-\vec{k}, -\omega}^{(1)} g_{\vec{k}, \omega}^{(1)} \frac{\partial}{\partial r_1} \Phi_{\vec{k}, \omega}^{(1)} , \quad (16b)$$

$$D_{r_1, r_2} = \int \frac{d^2 k}{(2\pi)^2} \int \frac{d\omega}{(2\pi)} \frac{c^2 k^2}{B^2} \exp [ik_y(y_1 - y_2) + ik_z(z_1 - z_2)] \\ \times \Phi_{-\vec{k}, -\omega}^{(1)} g_{\vec{k}, \omega}^{(2)} \Phi_{\vec{k}, \omega}^{(2)} \quad (16c)$$

$$D_{y_1, y_2} = \int \frac{d^2 k}{(2\pi)^2} \int \frac{d\omega}{(2\pi)} \frac{c^2}{B^2} \exp [ik_y(y_1 - y_2) + ik_z(z_1 - z_2)] \frac{\partial}{\partial r_1} \Phi_{-\vec{k}, -\omega}^{(1)} \\ \times g_{\vec{k}, \omega}^{(2)} \frac{\partial}{\partial r_2} \Phi_{\vec{k}, \omega}^{(2)} \quad (16d)$$

It is apparent that the strongest spatial variation of $\langle \delta f(1,2) \rangle$ occurs through the relative separations $x_- \equiv (r_2 - r_1)/2$, $y_- \equiv (y_2 - y_1)/2$, and $z_- \equiv (z_2 - z_1)/2$. Note from Eqs. (16c,d) that for $y_- \gtrsim k_{oy}^{-1}$ or $z_- \gtrsim k_{oz}^{-1}$, where $k_{oy}^{-1} \sim 1/m$ and $k_{oz}^{-1} \sim 1/n$ are average spectral wavelengths, that $D_{r_1, r_2} = D_{y_1, y_2} = 0$. Thus, the presence of large poloidal and toroidal wave numbers in the spectrum implies a fine scale structure to the phase space correlations. In contrast, the radial extent of the clump is limited by the radial width of the nonlinear one point resonance function $g_{\vec{k}, \omega}^{(2)}$, appearing in the diffusion coefficients in Eq. (16). Particles separated by a distance Δr_T determined by $k_{||}'(\Delta r_T) v_{||} \tau_c = 1$ will be decorrelated in a time τ_c . Since the clump lifetime exceeds τ_c , the relative radial separation x_- must be less than Δr_T . Otherwise, the clump will be torn apart by free streaming along the magnetic field. Alternatively, since the radial extent of the potentials appearing in the diffusion coefficients is approximately Δr_T , phase mixing reduces D_{r_1, r_2} and D_{y_1, y_2} to zero for $x_- > \Delta r_T$.

In order to emphasize the salient dependence of the correlation function on the relative coordinates, it is convenient to transform to (x_-, y_-, z_-) and $v_{||-} = (v_{||2} - v_{||1})/2$ and $(x_+, y_+, z_+, v_{||+}) \equiv [(r_1 + r_2)/2, (y_1 + y_2)/2, (z_1 + z_2)/2, (v_{||1} + v_{||2})/2]$. Then, neglecting the weak dependence of $\langle \delta f(1,2) \rangle$ in Eq. (15) on the plus coordinates, the two-point relative diffusion equation becomes

$$\left[\frac{\partial}{\partial t} + v_{||+} \frac{x_-}{L_S} \frac{\partial}{\partial y_-} + v_{||-} \left(\frac{r}{Rq} \frac{\partial}{\partial y_-} + \frac{\partial}{\partial z_-} \right) - \frac{\partial}{\partial x_-} D_{x_-} \frac{\partial}{\partial x_-} - \frac{\partial}{\partial y_-} D_{y_-} \frac{\partial}{\partial y_-} \right] \langle \delta f(1,2) \rangle = \left(\frac{\partial}{\partial r_1} D_{r_1, r_2} \frac{\partial}{\partial r_2} + \frac{\partial}{\partial r_2} D_{r_2, r_1} \frac{\partial}{\partial r_1} \right) \langle f(1) \rangle \langle f(2) \rangle, \quad (17)$$

where $L_S^{-1} = (rq'/q^2 R)$ and $x_+ \cong r$ in evaluating L_S and x_+/Rq (but not in the resonance function). Since the plus coordinates correspond to "clump center" coordinates, and noting that clumps resemble macroparticles, the phase space motion of these coordinates is adequately described by the one-point turbulence theory. The relative radial diffusion coefficient D_{x_-} is given by

$$D_{x_-} = \frac{1}{4} (D_{r_1, r_1} + D_{r_2, r_2} - D_{r_2, r_1} - D_{r_1, r_2}). \quad (18a)$$

Because of shear decorrelation, y_- diffusion has a negligible effect in comparison to x_- diffusion and will henceforth be neglected in Eq. (17).

Equation (17) indicates that $\langle \delta f(1,2) \rangle$ evolves by relative free streaming and relative diffusion. The second term, $v_{\parallel+}(x_-/L_S)\partial/\partial y_-$, corresponds to loss of correlation in y by parallel motion at different x . The shearing apart of a clump by this process, together with the turbulent diffusion of x_- (shear decorrelation) is not present in a straight field geometry. The third term, $v_{\parallel-}[(r/Rq)\partial/\partial y_- + \partial/\partial z_-]$, corresponds to loss of correlation by streaming at different velocities along a field line (at the same radial location). This effect is present in shearless systems.

Using the forms for the two-point diffusion coefficients in Eq. (16), the relative diffusion coefficient $D_{x_-} \equiv D_-$ may be expressed as

$$D_- = \frac{1}{4} \int \frac{d^2 k}{(2\pi)^2} \int \frac{d\omega}{(2\pi)} \frac{c^2 k^2 y}{B^2} (I_{1,2} + I_{2,1}) , \quad (18b)$$

where

$$I_{1,2} = \phi_{-\vec{k},-\omega}(r_1) g_{\vec{k},\omega}^+(1) \phi_{\vec{k},\omega}(r_1) - \exp[-2i(k_y y_- + k_z z_-)] \\ \times \phi_{-\vec{k},-\omega}(r_2) g_{\vec{k},\omega}^+(1) \phi_{\vec{k},\omega}(r_1) , \quad (18c)$$

$$I_{2,1} = \phi_{-\vec{k},-\omega}(r_2) g_{\vec{k},\omega}^+(2) \phi_{\vec{k},\omega}(r_2) - \exp[2i(k_y y_- + k_z z_-)] \\ \times \phi_{-\vec{k},-\omega}(r_1) g_{\vec{k},\omega}^+(2) \phi_{\vec{k},\omega}(r_2) . \quad (18d)$$

Note that D_- vanishes for small separations ($x_- \rightarrow 0$, $y_- \rightarrow 0$, $z_- \rightarrow 0$) and reduces to the sum of uncorrelated one-point diffusion coefficients

in the limit of large phase space separations. For small separations on the order of the clump dimensions, D_- may be expanded as follows:

$$D_- = (x_-^2 k_{ox}^2 + y_-^2 k_{oy}^2 + z_-^2 k_{oz}^2) D, \quad (19a)$$

where

$$D = \int \frac{d^2 k}{(2\pi)^2} \int \frac{d\omega}{(2\pi)} \frac{c^2 k^2}{B^2} \Phi_{-\vec{k}, -\omega}(\vec{r}) g_{\vec{k}, \omega}^{(+)} \Phi_{\vec{k}, \omega}(\vec{r}) \quad (19b)$$

is the one-point diffusion coefficient evaluated at the average clump coordinates, and

$$k_{ox}^2 = \frac{1}{2D} \frac{\partial^2}{\partial x_-^2} D_- \Big|_0 \quad (19c)$$

$$k_{oy}^2 = \frac{1}{2D} \frac{\partial^2}{\partial y_-^2} D_- \Big|_0 \quad (19d)$$

$$k_{oz}^2 = \frac{1}{2D} \frac{\partial^2}{\partial z_-^2} D_- \Big|_0 \quad (19e)$$

where $D_-|_0 \equiv D_-(x_- = y_- = z_- = 0)$. In Eq. (19a), terms of first order in x_- , y_- , and z_- vanish explicitly. Terms proportional to $x_- y_-$, $x_- z_-$, and $y_- z_-$ vanish by spectrum symmetry. The quantities k_{ox}^2 , k_{oy}^2 , and k_{oz}^2 are mean square wave numbers characterizing the clump dimensions in the x, y, z directions and will be self-consistently computed once the spectrum has been determined in terms of these quantities. The Taylor expansion

of D_- in Eq. (19a) assumes the finiteness of the quantities k_{ox}^2 , k_{oy}^2 , and k_{oz}^2 , which in turn require that the potential spectrum converge sufficiently rapidly for large wave numbers. That the spectrum has the required convergence properties can be verified a posteriori.

III. INVERSION OF CORRELATION EQUATION

An approximate solution of Eq. (17), neglecting D_{y-} , will now be obtained. Consider the two-point Green's function operator, $G_R(t, \tau)$, which satisfies the equation

$$\left[\frac{\partial}{\partial t} + v_{\parallel} + \frac{x_-}{L_S} \frac{\partial}{\partial y_-} + v_{\parallel} - \left(\frac{r}{Rq} \frac{\partial}{\partial y_-} + \frac{\partial}{\partial z_-} \right) - \frac{\partial}{\partial x_-} D_- \frac{\partial}{\partial x_-} \right] G_R(t, \tau) = 0 \quad (20a)$$

with initial condition

$$G_R(t, t) = \delta(x_- - x'_-) \delta(y_- - y'_-) \delta(z_- - z'_-) \delta(v_{\parallel} - v'_{\parallel}) . \quad (20b)$$

Then, the solution of Eq. (17) is

$$\delta f(1, 2) = 2 \int_0^t \bar{G}_R(t, \tau) \left[D \left(\frac{\partial \langle f \rangle}{\partial r} \right)^2 \right] d\tau \approx 2\tau_{CL} D (\partial \langle f \rangle / \partial r)^2 . \quad (21)$$

Here, $\bar{G}_R \equiv \int dx'_- \int dy'_- \int dz'_- G_R \approx 1$ for $\tau < \tau_{CL}$ and $\bar{G}_R \approx 0$ for $\tau > \tau_{CL}$, where the clump lifetime τ_{CL} must now be computed. This estimate for $\bar{G}_R$ is equivalent to a qualitative integration of Eq. (20a) over turbulent orbits, with τ_{CL} the coherence time for the clump.

A heuristic derivation of the correlation function can be constructed using the fact that, according to the Vlasov equation, phase space density is conserved along particle orbits. The phase space density fluctuation at r is given by $\delta f(r) = \langle f(r - \Delta x) \rangle - \langle f(r) \rangle$. This fluctuation is due to rearrangement of the average phase space density in pieces of size Δx . It

follows that the correlation function $\langle \delta f \delta f \rangle = \langle \delta f(1,2) \rangle$ is given by $\langle \delta f(1,2) \rangle = \langle \Delta x \Delta x \rangle (\partial \langle f \rangle / \partial r)^2$. Now $\langle \Delta x \Delta x \rangle = D \tau_{CL}$, where τ_{CL} (the clump lifetime) is the lifetime of the phase space density chunk of size Δx . Then $\langle \delta f(1,2) \rangle = D \tau_{CL} (\partial \langle f \rangle / \partial r)^2$, which has the form of the correlation function in Eq. (21).

To compute τ_{CL} , a set of ensemble-averaged relative coordinate evolution equations is derived by taking moments of Eq. (20a). For any quantity Q , an ensemble average is defined as $\langle Q[x_-(t), y_-(t), z_-(t), v_{||-}(t)] \rangle \equiv \int Q(x'_-, y'_-, z'_-, v'_{||-}) G_R(t, \tau) dx'_- dy'_- dz'_- dv'_{||-}$. The required second-order moment equations are readily obtained:

$$\frac{\partial}{\partial t} \langle y_-^2(t) \rangle = \frac{2v_{||+}}{L_S} \langle x_-(t) y_-(t) \rangle + \frac{2r}{Rq} \langle v_{||-}(t) y_-(t) \rangle \quad (22a)$$

$$\frac{\partial}{\partial t} \langle x_-^2(t) \rangle = 2 \langle D_- \rangle \quad (22b)$$

$$\frac{\partial}{\partial t} \langle x_-(t) y_-(t) \rangle = \frac{v_{||+}}{L_S} \langle x_-^2(t) \rangle + \frac{r}{Rq} \langle x_-(t) v_{||-}(t) \rangle \quad (22c)$$

$$\frac{\partial}{\partial t} \langle v_{||-}(t) y_-(t) \rangle = \frac{v_{||+}}{L_S} \langle x_-(t) v_{||-}(t) \rangle + \frac{r}{Rq} \langle v_{||-}^2(t) \rangle \quad (22d)$$

$$\frac{\partial}{\partial t} \langle x_-(t) v_{||-}(t) \rangle = 0 \quad (22e)$$

$$\frac{\partial}{\partial t} \langle v_{||-}^2(t) \rangle = 0 \quad (22f)$$

$$\frac{\partial}{\partial t} \langle z_{-}(t) x_{-}(t) \rangle = \langle v_{\parallel -}(t) x_{-}(t) \rangle \quad (22g)$$

$$\frac{\partial}{\partial t} \langle z_{-}(t) v_{\parallel -}(t) \rangle = \langle v_{\parallel -}^2(t) \rangle \quad (22h)$$

$$\frac{\partial}{\partial t} \langle z_{-}^2(t) \rangle = 2 \langle z_{-}(t) v_{\parallel -}(t) \rangle . \quad (22i)$$

Note that Eq. (22a) includes contributions from both relative ballistic free-streaming and shear-induced separation.

Equations (22a-e) can be combined to obtain

$$\frac{\partial^3}{\partial t^3} \langle y_{-}^2(t) \rangle = \frac{4v_{\parallel +}^2}{L_S^2} \langle D_{-} \rangle . \quad (23)$$

Apart from the $v_{\parallel +}$ factor, which accounts for particle motion along the magnetic field, Eq. (23) is identical to the equation describing the exponentiation of two adjacent magnetic field lines, which has been previously obtained by several authors.⁷⁻⁹

Since $\langle D_{-} \rangle$ is an explicit function of $\langle x_{-}^2 \rangle$, $\langle y_{-}^2 \rangle$, and $\langle z_{-}^2 \rangle$ given by Eq. (19), it is possible to solve Eqs. (22b, g-i) and (23) simultaneously. Combining Eqs. (22b) and (23) and integrating once yields

$$\langle x_{-}^2(t) \rangle = \frac{L_S^2}{2v_{\parallel +}} \frac{\partial^2 \langle y_{-}^2(t) \rangle}{\partial t^2} + c , \quad (24)$$

where the integration constant c is determined from initial conditions:

$$c = x_-^2 - \left(x_- + \frac{rL_S}{Rq} \frac{v_{||-}}{v_{||+}} \right)^2. \quad (25)$$

Using this result in Eq. (23) yields

$$\frac{\partial^3}{\partial t^3} \langle y_-^2(t) \rangle - \frac{2}{\tau_{10}} \frac{\partial^2}{\partial t^2} \langle y_-^2(t) \rangle = \frac{12}{\tau_{co}^3} \left[\langle y_-^2(t) \rangle + \frac{k_{oz}^2}{k_{oy}^2} \langle z_-^2(t) \rangle + \frac{k_{ox}^2}{k_{oy}^2} c \right], \quad (26)$$

where $\langle z_-^2(t) \rangle = v_{||-}^2 t^2 + 2z_- v_{||-} t + z_-^2$ is determined from Eqs. (22f-i).

Here, $\tau_{co} = (k_{oy}^2 v_{||+}^2 D / 3L_S^2)^{-1/3}$ and $\tau_{10} = (k_{ox}^2 D)^{-1}$ are the shear and shearless decorrelation times, respectively. The limit where the shear decorrelation frequency $1/\tau_{co}$ exceeds the shearless decorrelation rate $k_{ox}^2 D$ will be considered here, since this is the case of interest for tokamaks.¹ The asymptotic solution to Eq. (26) for times comparable to or greater than τ_{co} is

$$\langle y_-^2(t) \rangle = (3k_{oy}^2)^{-1} I(x_-, y_-, z_-, v_{||-}) \exp(t/\tau_{co}). \quad (27a)$$

Information about the initial phase space positions is contained in the quantity

$$I = [k_{||0} v_{||-} \tau_{co} + k_{oz}^2 z_- / k_{||0} + k_{oy}^2 r Y_- / (Rq k_{||0})]^2 + k_{ux}^2 x_-^2 + \frac{1}{2} k_{uy}^2 y_-^2 + (2k_{oy}^2 k_{oz}^2 / k_{||0}^2) (rz_- / Rq - v_{||-} / ?)^2, \quad (27b)$$

where $k_{\parallel 0}^2 = (2r^2 k_{oy}^2 / R^2 q^2) + k_{oz}^2$, $Y_- = y_- + 2x_- v_{\parallel+} \tau_o / L_S$ and $\tau_o = \tau_{co} / (12)^{1/3}$ is the orbit exponentiation time. Terms of order τ_{co} / τ_{l0} have been neglected in Eq. (27).

The fact that $\tau_{co} / \tau_o \approx 1$ results from the replacement of the *two-particle propagator* in Eq. (6) with the *product of one-particle propagators*. Note that in comparison with results previously obtained for uniform magnetic field systems, the additional terms in Eq. (27) reflect the enhancement of clump destruction due to shear decorrelation.

To determine τ_{CL} , the clump lifetime, recall that k_{ox}^2 , k_{oy}^2 , and k_{oz}^2 characterize the mean square spreads of the clump in the x_- , y_- , and z_- directions, respectively. Thus, $[k_{ox}^2 \langle x_-^2(\tau_{CL}) \rangle + k_{oy}^2 \langle x_-^2(\tau_{CL}) \rangle + k_{oz}^2 \langle z_-^2(\tau_{CL}) \rangle] = 1$ determines τ_{CL} . The term $k_{ox}^2 \langle x_-^2(\tau_{CL}) \rangle$ is of order τ_o / τ_{l0} in comparison to the term $k_{oy}^2 \langle y_-^2(\tau_{CL}) \rangle$. The physical reason for this can be seen by noting from Eqs. (22a,b) that $(\Delta x_-)^2 \sim D \tau_{co}$ (diffusion) and $(\Delta y_-)^2 \sim v_{\parallel+}^2 (\Delta x_-)^2 \tau_{co}^2 / L_S^2$ (shear decorrelation). Using Eq. (27), τ_{CL} is given to within logarithmic accuracy by

$$\tau_{CL} = \tau_o \ln(3/I) . \quad (28)$$

As previously noted, the term driving the correlation function vanishes in Eq. (17) for phase space separations on the order or exceeding clump dimensions (since D_{r_1, r_2} and D_{r_2, r_1} vanish). Thus, the vanishing of the logarithm in Eq. (28) defines an ellipsoid in phase space demarking the extent of the clump.

The correlation function is finally obtained by using Eq. (28) in Eq. (21):

$$\langle \delta f(1,2) \rangle = 2\tau_0 \ln(3/I) D \left(\frac{\partial \langle f \rangle}{\partial r} \right)^2 . \quad (29)$$

The correlation function in Eq. (29) includes contributions from both the coherent and incoherent phase space density fluctuations. In order to isolate the incoherent fluctuation correlation function, the distribution function fluctuation is decomposed as follows:

$$\delta f_{\vec{k}} = f_{\vec{k}}^{(c)} + \tilde{f}_{\vec{k}} . \quad (30a)$$

Here, $f_{\vec{k}}^{(c)}$ is phase coherent with the potential fluctuation $\phi_{\vec{k}}$, and $f_{\vec{k}}^{(c)}$, $\tilde{f}_{\vec{k}}$ are uncorrelated. Thus, the correlation function may be written as

$$\langle \delta f(1,2) \rangle = \langle f^{(c)}(1) f^{(c)}(2) \rangle + \langle \tilde{f}(1) \tilde{f}(2) \rangle . \quad (30b)$$

Noting that the coherent response is given by

$$f^{(c)} = \int \frac{d^2 k}{(2\pi)^2} \int \frac{d\omega}{(2\pi)} \frac{ic}{B_0} \exp [i(k_y y + k_z z)] g_{\vec{k}, \omega} \frac{\partial \langle f \rangle}{\partial r} , \quad (31a)$$

and that $g_{\vec{k}, \omega}^* g_{\vec{k}, \omega} \simeq \tau_c \text{Re } g_{\vec{k}, \omega}$, the coherent correlation function is

$$\langle f^{(c)} f^{(c)} \rangle = \tau_{cu} D (\partial \langle f \rangle / \partial r)^2 . \quad (31b)$$

Note that in the limit of small phase space separation, there is no singular behavior in Eq. (31b). The incoherent correlation function $\langle \tilde{f}(1,2) \rangle = \langle \tilde{f}(1) \tilde{f}(2) \rangle$ is, therefore,

$$\langle \delta \tilde{f}(1,2) \rangle = - \tau_{co} (\ln I) D \left(\frac{\partial \langle f \rangle}{\partial r} \right)^2, \quad (32)$$

where I is defined in Eq. (27b).

IV. SELF-CONSISTENCY AND DETERMINATION OF THE CLUMP-DRIVEN FLUCTUATION SPECTRUM

Equation (32) gives the equal-time phase space correlation function for low frequency electron clump turbulence in a sheared magnetic field. To calculate the self-consistent spectrum, one must obtain the Fourier-Laplace transformed density correlation function from Eq. (32) and treat that as a source in Poisson's equation. Specifically, the quasi-neutrality condition can be written

$$\tilde{n}_e = n_i - n_e^{(c)}, \quad (33a)$$

where $\tilde{n}_e$ and $n_e^{(c)}$ refer to the phase-incoherent and coherent constituents of the electron density fluctuation, respectively. Note that Eq. (33a) is equivalent to

$$(\tilde{n}_e)_{\vec{k},\omega} = \frac{n_o d|e|}{T_e} L_{\vec{k},\omega} \phi_{\vec{k},\omega}, \quad (33b)$$

where

$$L_{\vec{k},\omega} = \left\{ \rho_i^2 \frac{\partial^2}{\partial x^2} - \left[\Lambda - \frac{\mu^2 x^2}{\rho_i^2} + \sigma(x) (\rho_i/x) \right] \right\} \quad (33c)$$

is the turbulently broadened one-point dielectric operator for a sheared magnetic field¹ and $x = x(k_y, k_z)$ is the distance from the mode rational surface where $\vec{k} \cdot \vec{B} = 0$. Here, $\Lambda = [1 + \tau(1 - \Gamma_U) - \Gamma_U \omega_{*e}/\omega] d^{-1}$,

$$\begin{aligned} \mu &= \tau^{-1} (L_n/L_S) (\omega_{*e}/\omega) [\Gamma_0 (\tau + \omega_{*e}/\omega) d^{-1}]^{1/2}, \quad \sigma = \sigma_0 Z[(x_e + ix_c)/x], \\ \sigma_0 &= (\omega/\omega_{*e} - 1) \alpha d^{-1}, \quad \alpha = [(1/2) \tau m_e/m_i]^{1/2} L_S/L_n, \quad \tau = T_e/T_i, \\ x_e &= (\alpha \omega/\omega_{*e}) \rho_i, \quad x_c = (\alpha \omega_c/\omega_{*e}) \rho_i, \quad \text{and } d = (\Gamma_0 - \Gamma_i) (\tau + \omega_{*e}/\omega) + \\ &3i(1 - \omega/\omega_{*e}) (\omega_{*e}/\omega_c) (x_c/\rho_i)^2. \end{aligned}$$

Then, the spectrum is given by

$$\langle \Phi(1) \Phi(2) \rangle_{\vec{k}, \omega} = \left(\frac{T_e}{n_0 e |d|} \right)^2 L_{\vec{k}, \omega}^{-1} (2) L_{-\vec{k}, -\omega}^{-1} (1) \langle \tilde{n}_e(1) \tilde{n}_e(2) \rangle_{\vec{k}, \omega}. \quad (34)$$

The Fourier transform of the two-time density correlation function required in Eq. (34) is given by

$$\begin{aligned} \langle \tilde{n}_e(1) \tilde{n}_e(2) \rangle_{\vec{k}, \omega} &= 2n_0^2 \int 2\pi v_{\perp} dv_{\perp} \int dv_{\parallel} + \int_{-\infty}^{+\infty} \exp(i\omega\tau) d\tau \exp(-ik_y y_-) dy_- \\ &\times \int \exp(-ik_z z_-) dv_{\parallel -} \langle \delta \tilde{f}(1, 2, \tau) \rangle, \quad (35) \end{aligned}$$

where $\tau = \tau_1 - \tau_2$ and $dv_{\parallel 1} dv_{\parallel 2} = 2dv_{\parallel +} dv_{\parallel -}$. The two-time correlation function $\langle \delta \tilde{f}(1, 2, \tau) \rangle$ is obtained from the equal-time correlation function given in Eq. (32) by appropriate application of the one-particle propagator.² The resulting transform of $\langle \delta \tilde{f}(1, 2, \tau) \rangle$ is

$$\begin{aligned} \langle \delta \tilde{f}(1, 2) \rangle_{\vec{k}, \omega} &= [g_{\vec{k}, \omega} (1) + g_{\vec{k}, \omega}^* (2)] \langle \delta \tilde{f}(1, 2) \rangle_{\vec{k}} \\ &= 2 \operatorname{Re} g_{\vec{k}, \omega} (+) \langle \delta \tilde{f}(1, 2) \rangle_{\vec{k}}, \quad (36) \end{aligned}$$

where $\langle \delta \tilde{f}(1, 2) \rangle_{\vec{k}}$ is the Fourier transform of the equal-time correlation function.

Performing the $v_{||-}$ integration in Eq. (35) yields

$$\int dv_{||-} \langle \delta \tilde{f}(1,2) \rangle = \left(\frac{4.5}{k_{||o}} \right) D \left(\frac{\partial \langle f \rangle}{\partial r} \right)^2 [(1 - k^2)^{1/2} - |k| \cos^{-1} |k|] , \quad (37)$$

where $k^2 = k_{ox}^2 x_-^2 + k_{oy}^2 y_-^2/2 + (2k_{oy}^2 k_{oz}^2 / k_{||o}^2) [(r/Rq)z_- - Y_-/2]^2$. Performing the remaining integrals in Eq. (35) yields

$$\begin{aligned} \langle \tilde{n}_e(1) \tilde{n}_e(2) \rangle_{\vec{k}, \omega} &= (9.1) n_o^2 \left(\frac{D}{k_{||o}} \right) \int 2\pi dv_{||-} \int dv_{||+} \left(\frac{\partial \langle f \rangle}{\partial r} \right)^2 \text{Re } g_{\vec{k}, \omega}^{(+)} \\ &\times \int \exp(-ik_y y_-) dy_- \int \exp(-ik_z z_-) dz_- [(1 - k^2)^{1/2} - |k| \cos^{-1} |k|] . \quad (38a) \end{aligned}$$

The symmetry in x_- , y_- , and z_- apparent in the quantity k suggests that Eq. (38a) can be evaluated by first Fourier transforming in x_- and then inverting the result. The details of this procedure are carried out in Appendix A. The result is:

$$\begin{aligned} \langle \tilde{n}_e(1) \tilde{n}_e(2) \rangle_{\vec{k}, \omega} &= (9.1) \left(\frac{n_o^2}{k_{||o}} \right) D \int 2\pi v_{||-} dv_{||-} \int dv_{||+} \left(\frac{\partial \langle f \rangle}{\partial r} \right)^2 \text{Re } g_{\vec{k}, \omega}^{(+)} \\ &\times \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(ik_x x_-) M(k_x, k_y, k_z) dk_x \quad (38b) \end{aligned}$$

where

$$M(k_x, k_y, k_z) = \frac{1}{k_{ox} k_{oy} k_{oz}} \left(\frac{k_{||o}}{\sqrt{2} k_{oy}} \frac{Rq}{r} \right) \left(\frac{\pi^2}{8} \right) \exp [-(h/4)^2] . \quad (38c)$$

The velocity factors $v_{\parallel+}^2$ in the quantity b^2 , which is defined in Appendix A, are to be evaluated at the electron thermal velocity v_{Th} .

Finally, to perform the $v_{\parallel+}$ integral, note that $\partial \langle f \rangle / \partial r = (\sqrt{\pi} v_{Th} L_n)^{-1} \times \exp(-v_{\parallel+}^2 / v_{Th}^2)$. Thus,

$$\begin{aligned} \int 2\pi v_{\perp} dv_{\perp} \int \left(\frac{\partial \langle f(v_{\parallel+}) \rangle}{\partial r} \right)^2 \text{Re } g_{\vec{k}, \omega}^{(+)} dv_{\parallel+} \\ = \frac{1}{\sqrt{\pi} L_n^2 v_{Th}^2 |k_{\parallel}|} \text{Im } Z[(\omega + i\omega_c) / |k_{\parallel}| v_{Th}] \equiv R. \end{aligned} \quad (39a)$$

For $\omega_c > \omega$, which is the case of interest for tokamaks, the right side of Eq. (39a) may be approximated for $x_+ < x_c$ as

$$R \simeq (\sqrt{\pi} v_{Th} \omega_c L_n^2)^{-1}, \quad (39b)$$

and for $x_+ > x_c$ as

$$R \simeq (2 |k_{\parallel}| v_{Th}^2 L_n^2)^{-1}. \quad (39c)$$

Thus, noting $x_c^2 = D/\omega_c$, a uniform approximation for the right side of Eq. (39a) when $\omega_c > \omega$ is

$$R \simeq \left(\frac{L_S}{\sqrt{\pi} L_n^2 v_{Th}^2 |k_y|} \right) (x_c^2 + x_+^2)^{-1/2}, \quad (39d)$$

Eq. (38b) becomes

$$\langle \tilde{n}_e(1) \tilde{n}_e(2) \rangle_{\vec{k}, \omega} = (0.4) \left(\frac{n_o^2 L_S R q}{v_{Th}^2 L_n^2 r} \right) \left[\frac{D}{(x_c^2 + x^2)^{1/2}} \right] \left[\frac{1}{(k_{ox} k_{oy}^2 k_{oz})} \right] \\ \times \frac{1}{|k_y|} \int_{-\infty}^{+\infty} \exp [ik_x x_- - (b/4)^2] dk_x. \quad (40)$$

Using this result, Eq. (34) may be rewritten as

$$\langle \phi(1) \phi(2) \rangle_{\vec{k}, \omega} = \left(\frac{0.4 T_e^2 L_S k q}{e^2 v_{Th}^2 L_n^2 r |d|^2} \right) \frac{D}{(x_c^2 + x_+^2)^{1/2}} \left(\frac{1}{k_{ox} k_{oy}^2 k_{oz}} \right) \\ \times \frac{1}{|k_y|} \int_{-\infty}^{+\infty} \left[L_{\vec{k}, \omega}^{-1} (2) \exp (ik_x x_2/2) L_{-\vec{k}, -\omega}^{-1} (1) \exp (-ik_x x_1/2) \right] \\ \times \exp (-b^2/4) dk_x. \quad (41)$$

The weak dependence on x_+ and the fact that $x_+ < x_c$ is the case of interest, have been invoked in neglecting the effect of L^{-1} on the denominator of the right side of Eq. (41).

It remains to compute the one-point quantity $L_{\vec{k}, \pm\omega}^{-1} \exp (\pm ik_x x/2)$. This is accomplished by constructing a Green's function $G(x, x')$, which satisfies the equation

$$L_{\vec{k}, \omega} G(x, x') = \delta(x - x') \quad (42a)$$

with outgoing wave boundary conditions.¹⁰ In Appendix B, the Green's function is obtained:

$$G(x, x') = \frac{-1}{\rho_i} \left(\frac{i\mu}{\pi} \right)^{1/2} \exp \left[-\frac{i\mu}{2} (x^2 + x'^2) / \rho_i^2 \right] \\ \times \sum_{n=0}^{\infty} \frac{H_n(\sqrt{i\mu} x' / \rho_i) H_n(\sqrt{i\mu} x / \rho_i)}{n! 2^n \epsilon_n(\omega, \vec{k})}, \quad (42b)$$

where H_n are Hermite polynomials and

$$\epsilon_n(\omega, \vec{k}) = \Lambda + (2n + 1)i\mu + i\sigma_0(\rho_i/x_c). \quad (42c)$$

Here the physically relevant limit $x < x_c$ was used to simplify the electron resonance function $\sigma(x)$ appearing in Eq. (33c). Noting the relation

$$\int_{-\infty}^{+\infty} \exp \left(\pm \alpha \sqrt{i\mu} \frac{x'}{\rho_i} - i\mu \frac{x'^2}{2\rho_i^2} \right) H_n \left(\sqrt{i\mu} \frac{x'}{\rho_i} \right) \sqrt{i\mu} \frac{dx'}{\rho_i} \\ = \sqrt{2\pi} (-1)^{n/2} \exp(\alpha^2/2) H_n(\mp i\alpha), \quad (43)$$

it follows that

$$L_{\pm \vec{k}, \pm \omega}^{-1} \exp(\pm i k_x x/2) = \sqrt{2} \exp[\alpha^2/2 - i\mu x^2/(2\rho_i^2)] \\ \times \sum_{n=0}^{\infty} \frac{H_n(\sqrt{i\mu} x/\rho_i) H_n(\mp i\alpha) (i/2)^n}{n! \epsilon_n} \quad (44)$$

where $\alpha = (k_x \rho_i/2)(i/\mu)^{1/2}$. Using the relation

$$\frac{1}{\epsilon_n} = \int_0^\infty d\tau \exp(-\epsilon_n \tau), \quad (45)$$

where $\text{Re } \epsilon_n > 0$ was assumed (corresponding to outgoing waves), Eq. (44)

may be rewritten:

$$L_{\vec{k}, \pm\omega}^{-1} \exp(\pm i k_x x/2) = -\sqrt{2} \exp[\alpha^2/2 - i\mu x^2/(2\rho_1^2)] \int_0^\infty \exp(-\epsilon_0 \tau) \\ \times \sum_{n=0}^{\infty} \frac{H_n(\sqrt{i\mu} x/\rho_1) H_n(\mp i\alpha)}{n!} \left[\frac{i}{2} \exp(-2i\mu\tau) \right]^n d\tau. \quad (46a)$$

Using Mehler's formula,¹¹ the Hermite series may be summed to yield

$$L_{\vec{k}, \pm\omega}^{-1} \exp(\pm i k_x x/2) = -\sqrt{2} \exp[\alpha^2/2 - i\mu x^2/(2\rho_1^2)] \\ \times \int_0^\infty \exp(-\epsilon_0 \tau) s(\tau) d\tau, \quad (46b)$$

where

$$s(\tau) = [1 + \exp(-4i\mu\tau)]^{-1/2} \\ \times \exp \left[\frac{\pm 2\sqrt{i\mu} \alpha x \exp(-2i\mu\tau) + (i\mu x^2 - \alpha^2) \exp(-4i\mu\tau)}{1 + \exp(-4i\mu\tau)} \right], \quad (46c)$$

Note that as τ approaches infinity, $s(\tau)$ approaches one. Therefore, the $n = 0$ mode dominates the asymptotic behavior of the integral in Eq. (46b). Retaining only this dominant contribution finally yields

$$L_{\vec{k}, \pm\omega}^{-1} \exp(\pm i k_x x/2) \approx -\sqrt{2} \exp[\alpha^2/2 - i\mu x^2/(2\rho_1^2)] \epsilon_0^{-1}. \quad (46d)$$

This result is reasonable since ϵ_0 is the dielectric response of the most weakly damped modes in a sheared magnetic field.

Noting the complex conjugacy relations:

$$d_{-\vec{k}, -\omega} = d_{\vec{k}, \omega}^* \quad (47a)$$

$$\mu_{-\vec{k}, -\omega} = -\mu_{\vec{k}, \omega}^* \quad (47b)$$

$$\Lambda_{-\vec{k}, -\omega} = \Lambda_{\vec{k}, \omega}^* \quad (47c)$$

$$(\sigma_0/x_c)_{\vec{k}, \omega} = -(\sigma_0/x_c)_{\vec{k}, \omega}^* \quad (47d)$$

and inserting Eq. (46d) into Eq. (41) yields

$$\begin{aligned} \frac{e^2 \langle \Phi(1) \Phi(2) \rangle_{\vec{k}, \omega}}{T_e^2} &= 0.8 \left(\frac{L_s R_q}{v_{Th}^2 r L_n^2} \right) \left[\frac{D}{(x_c^2 + \bar{x}_+^2)^{1/2}} \right] \left(\frac{k_{ox}^{-1} k_{oy}^{-2} k_{oz}^{-1}}{|d\epsilon_0|_{\vec{k}, \omega}^2 |k_y|} \right) \\ &\times \exp \left[-|\mu_i| (\bar{x}_+^2 + x_-^2) / \rho_i^2 - 2i\mu_r \bar{x}_+ x_- / \rho_i^2 \right] \int_{-\infty}^{+\infty} dk_x \exp [-(b/4)^2 \\ &\quad - |\mu_i| k_x^2 \rho_i^2 / (4|\mu|^2)] \quad (48) \end{aligned}$$

Here $\bar{x}_+ = (x_1 + x_2)/2$, where x_1 and x_2 now refer to the distances from the $\vec{k} \cdot \vec{B} = 0$ surface, and $\mu = \mu_r + i\mu_i$. Performing the k_x integration, the spectrum correlation function is determined to be

$$\frac{e^2 \langle \phi(1) \phi(2) \rangle}{T_e^2} = (3.5) \left(\frac{L_S R q}{v_{Th}^2 r L_n^2} \right) \left[\frac{D}{(x_c^2 + \bar{x}_+^2)^{1/2}} \right] \left(\frac{k_{oy}^{-2} k_{oz}^{-1}}{|d\epsilon_o|_{\vec{k}, \omega}^2 |k_y|} \right) \\ \times \left(|\mu_i| \rho_i^2 k_{ox}^2 / |\mu|^2 + \frac{1}{4} \right)^{-1/2} \exp [-|\mu_i| (\bar{x}_+^2 + x_-^2) / \rho_i^2 - 2i\mu_r \bar{x}_+ x_- / \rho_i^2 - A] , \quad (49a)$$

where

$$A = \left[\left(\frac{k_{oz} v_{Th} r_o}{L_S} \right)^2 \left(\frac{|\mu_i| \rho_i^2}{4|\mu|^2 + \mu_i \rho_i^2 k_{ox}^2} \right) + \frac{k_{||o}^2}{32 k_{oy}^2} \right] \bar{k}_z^2 + \frac{1}{16} \left(\bar{k}_y + \frac{1}{2} \bar{k}_z \frac{k_{oz}}{k_{oy}} \right)^2 . \quad (49b)$$

Here, $\bar{k}_y = k_y / k_{oy}$ and $\bar{k}_z = (k_z / k_{oz}) (Rq/r)$.

Equation (49a) gives an explicit expression for the fluctuation spectrum in terms of the parameters D , k_{ox} , k_{oy} , and k_{oz} (which are independent of the spectral quantities k_y , k_z , k_x , and ω). These parameters will now be determined in terms of fundamental plasma quantities.

V. DETERMINATION OF AVERAGE SPECTRUM PARAMETERS

In this section, the parameters D , k_{ox}^2 , k_{oy}^2 , and k_{oz}^2 which characterize the spectrum given in Eq. (49) will be determined in terms of local equilibrium plasma quantities. Using Eq. (49), it can be easily shown that the defining Eqs. (16) and (19) for these parameters are equivalent to the following spectrum averages (in the limit $\bar{x}_+^2 < x_c^2$):

$$\begin{pmatrix} D \\ k_{ox}^2 \\ k_{oy}^2 \\ k_{oz}^2 \end{pmatrix} = 3.5 \sqrt{2} \left(\frac{\rho_S}{L_r} \right) \left(\frac{m_e}{m_i} \right)^{1/2} \left(\frac{Rq}{r} \right) \int \frac{d^2 k}{(2\pi)^2} \int \frac{d\omega}{2\pi} \begin{pmatrix} D \\ |\mu_i|/\rho_i^2 + 2\mu_r^2 \bar{x}_+^2/\rho_i^4 \\ k_y^2/2 \\ k_z^2/2 \end{pmatrix} \\ \times \frac{\omega_{*e}}{\omega_c^2} \left(\frac{k_y}{k_{oy}} \right)^2 \frac{k_{oz}^{-1}}{|k_y|} \frac{1}{|d\epsilon_o|_{\vec{k}, \omega}^2} \exp [-(|\mu_i| \bar{x}_+^2/\rho_i^2 + A)] . \quad (50)$$

Here, $\rho_S = (T_e/T_i)^{1/2} \rho_i$ and $(1/4 + |\mu_i| \rho_i^2 k_{ox}^2/|\mu|^2)^{-1/2} \cong 1$ was assumed, as may be verified a posteriori. The two contributions to k_{ox}^2 represent nonlinearly enhanced mode broadening ($\propto |\mu_i|$) and a WKB wave number ($\propto \mu_r^2$).

The strongest ω -dependence in Eq. (50) comes from the zero of the real part of the dispersion function ϵ_o , which occurs for

$$\omega \equiv \bar{\omega} = \frac{\Gamma_o \omega_{*e}}{1 + \tau(1 - \Gamma_o)} . \quad (51)$$

Thus, expanding $|d\epsilon_o|^2$ about $\bar{\omega}$ yields

$$|d\epsilon_o|^2 = (\omega - \bar{\omega})^2 \left| \frac{\partial \tilde{\Lambda}}{\partial \omega} \right|^2 + |\tilde{\epsilon}_I|^2, \quad (52)$$

where $\tilde{\Lambda} = \Lambda d$ and $\tilde{\epsilon}_I = \text{Re}(\mu d) + d\sigma_o \rho_i / x_c$. Note that the frequency line width is $\Delta\omega = |\tilde{\epsilon}_I| |\partial \tilde{\Lambda} / \partial \omega|^{-1}$. Therefore, the ω integral in Eq. (50) can be performed, evaluating all other ω -dependent quantities at $\bar{\omega}$, with the result:

$$\begin{pmatrix} D \\ k_{ox}^2 \\ k_{oy}^2 \\ k_{oz}^2 \end{pmatrix} = 0.06 \left(\frac{\rho_s}{L_n} \right) \left(\frac{Rq}{r} \right) \left(\frac{m_e}{m_i} \right)^{1/2} \int \frac{d^2k}{|k_y| k_{oz}} \left(\frac{k_y}{k_{oy}} \right)^2 \exp [-(|v_i| \bar{x}_+^2 / \rho_i^2 + A)]$$

$$\times \frac{1}{\left| \frac{\partial \tilde{\Lambda}}{\partial \omega} \right|_{\bar{\omega}} |\tilde{\epsilon}_I|_{\bar{\omega}}} \begin{pmatrix} D \\ |v_i| / \rho_i^2 + 2u_{\mathbf{r}}^2 \bar{x}_+^2 / \rho_i^4 \\ k_y^2 / 2 \\ k_z^2 / 2 \end{pmatrix} \Big|_{\bar{\omega}}, \quad (53)$$

where $(\partial \tilde{\Lambda} / \partial \omega)_{\bar{\omega}} = (1 + \tau)^2 / (\Gamma_o \omega_{*e})$.

Next we perform the k_z integral. Note that $\bar{x}_+ \equiv r - r_{mn}$ is the radial distance measured from the (m, n) mode rational surface where $m - nq(r_{mn}) = 0$. It is convenient to define a *fixed* rational surface location r_o determined by the relation $m_o - n_o q(r_o) = 0$, where $m_o = k_{oy} r$

and $n_o = k_{oz} R$. With respect to this surface, $\bar{x}_+ = r_o - (r/Rq)^2 (k_{oy}/k_{oz}) \times (L_S/q) [\bar{k}_y - (r/Rq)\bar{k}_z]$, where $\bar{k}_y = k_y/k_{oy}$ and $\bar{k}_z = (k_z/k_{oz})(Rq/r)$.

Neglecting terms of order k_{oz}/k_{oy} and $(\rho_i/r)^2 \tau_{\perp}/\tau_o$ in the exponential term of Eq. (53), the k_z integral yields

$$\begin{pmatrix} D \\ k_{ox}^2 \\ k_{oy}^2 \end{pmatrix} = 0.05 \left(\frac{\rho_S}{L_n} \right) \left(\frac{\rho_i q}{L_S} \right) \left(\frac{Rq}{r} \right)^2 \left(\frac{m_e}{m_i} \right)^{1/2} \int \frac{d\bar{k}_y}{(|\mu_i|^{1/2})|_{\omega}} \frac{\bar{k}_y^2}{|\bar{k}_y|} \times \exp [-(\bar{k}_y^2/4)^2] \frac{\omega_{*e}}{\omega_c^2} \frac{1}{\left(\left| \frac{\partial \tilde{\Lambda}}{\partial \omega} \right| |\tilde{\epsilon}_I| \right) |_{\omega}} \begin{bmatrix} D \\ |\mu_i|/\rho_i^2 + 2\mu_r^2/(|\mu_i|\rho_i^2) \\ k_y^2/2 \end{bmatrix}. \quad (54a)$$

The fourth term in Eq. (53) yields

$$k_{oz}^2 = (r/Rq)^2 k_{oy}^2. \quad (54b)$$

To perform the remaining k_y integral, it should be noted that the $\tilde{\epsilon}_I$ term, which represents a balance between electron growth and nonlinearly enhanced ion shear damping, is the only term in the integrand in Eq. (54a) which exhibits a resonance structure. Indeed, the one-point theory analysis of $\tilde{\epsilon}_I$ indicates that below the maximum value of the diffusion coefficient, $\tilde{\epsilon}_I$ always goes through zero for a certain value of $(k_y \rho_i)^2 \equiv b$. Above this critical value, $\tilde{\epsilon}_I$ remains nonzero. Thus, in order to balance the two sides of Eq. (54), it is necessary that the turbulent diffusion

coefficient be somewhat larger than the maximum value computed from the one-point theory. Thus, it is convenient to expand $|\tilde{\epsilon}_I|$ in the vicinity of the maximum D and associated maximizing poloidal wave number. Introducing the normalized variables $\hat{b} = b/b_o$, $(\omega_c/\omega_{*e}) = b^{-1/6}\omega_o\hat{\omega}_c$ (note that $\hat{\omega}_c$ is independent of b), where

$$b_o = \frac{1}{3} \frac{\bar{\mu}_o (1 + \tau)^{3/2}}{\alpha^2}, \quad (55a)$$

$$\omega_o = \frac{1}{3^{1/6}} [\alpha^{-1/3} \bar{\mu}_o^{-5/6} (1 + \tau)^{-1/4}] , \quad (55b)$$

and $\bar{\mu}_o = \sqrt{\pi}(L_n/\tau L_S)(1 + \tau)^{3/2}$, $\tilde{\epsilon}_I$ can be written

$$\tilde{\epsilon}_I = \bar{\mu}_o (1 + \tau)^{1/2} \left\{ \left[\frac{1}{2} + \left(\frac{1}{4} + \hat{\omega}_c^2 \hat{b}^{5/3} \right)^{1/2} \right]^{1/2} - \hat{\omega}_c^{-1} \hat{b}^{1/6} \right\} . \quad (56)$$

Here $\Gamma_o(b) \approx (2\pi b)^{-1/2}$ was used for $b \geq 1$, which is the regime of interest.

The extremizing values of $\hat{b}$ and $\hat{\omega}_c$ which satisfy $\tilde{\epsilon}_I = 0$ and $\partial\tilde{\epsilon}_I/\partial\hat{b} = 0$ are readily obtained:

$$b_m = 4/3^{3/2} \doteq 0.77 , \quad (57a)$$

$$\omega_{cm} = 3^{1/4}/4^{1/3} \doteq 0.83 . \quad (57b)$$

In the vicinity of b_m and ω_{cm} , we therefore obtain

$$\tilde{\epsilon}_I \cong \bar{\mu}_o (1 + \tau)^{1/2} [1.67(\hat{\omega}_c - \omega_{cm}) + 0.13(\hat{b} - b_m)^2] . \quad (58)$$

Using this result, Eq. (54) becomes:

$$\begin{aligned}
 \begin{pmatrix} D \\ k_{ox}^2 \\ k_{oy}^2 \end{pmatrix} &= 0.01 \left(\frac{\rho_S}{L_n} \right) \left(\frac{\rho_i q}{L_S} \right) \left(\frac{Rq}{r} \right)^2 \left(\frac{m_e}{m_i} \right)^{1/2} \frac{b_o^{1/3} \omega_o^{-2}}{\mu_o^{3/2} (1 + \tau)^{9/4} (k_{oy} \rho_i)^2} \\
 &\times \int_0^{\infty} \hat{b}^{-2/3} \hat{\omega}_c^{-2} \exp \{ -b_o \hat{b} / [16 (k_{oy} \rho_i)^2] \} |1.67 (\hat{\omega}_c - \omega_{cm}) \\
 &+ 0.13 (\hat{b} - b_m)^2|^{-1} \begin{bmatrix} D \\ (|\mu_i| + 2\mu_r^2 / |\mu_i|) / \rho_i^2 \\ b_o b / (2\rho_i^2) \end{bmatrix} d\hat{b} \\
 &= 0.07 \left(\frac{\rho_S}{L_n} \right) \left(\frac{\rho_i q}{L_S} \right) \left(\frac{Rq}{r} \right)^2 \left(\frac{m_e}{m_i} \right)^{1/2} \frac{b_o^{1/3} \omega_o^{-2} \exp [-(0.05) b_o / (k_{oy} \rho_i)^2]}{\mu_o^{3/2} (1 + \tau)^{9/4} (k_{oy} \rho_i)^2 (\hat{\omega}_c - \omega_{cm})^{1/2}} \\
 &\times \begin{bmatrix} D \\ 4.78 \rho_i^{-2} \mu_o b_o (1 + \tau)^{-1/2} \\ 0.39 b_o / \rho_i^2 \end{bmatrix} . \quad (59)
 \end{aligned}$$

It follows from Eq. (59) that

$$k_{ox}^2 \rho_i^2 = 10 \Delta_{PB}^{-1} (L_n / L_S) (1 + \tau)^{5/2} / \tau^3 , \quad (60a)$$

$$k_{oy}^2 \rho_i^2 = 0.47 \Delta_{PB}^{-1} (1 + \tau)^3 / \tau^2 , \quad (60b)$$

where $\Delta_{PB} = (m_e/m_i)(L_S/L_n)^3 \sim 1$ for tokamaks. The normalized diffusion coefficient is determined by

$$\hat{\omega}_c = 0.83 + \epsilon^2, \quad (60c)$$

where $\epsilon = 0.45(\rho_S/L_n)(\rho_i q/L_S)(Rq/r)[\tau(m_e/m_i)L_n/L_S]^{1/6}(1+\tau)^{-7/2} \ll 1$.

Physically, the turbulent damping in this two-point theory must balance both the clump emission as well as the one-point electron growth. Thus, ϵ^2 is a measure of the clump emission rate compared to the linear growth and damping processes near the marginal stability point of the most unstable mode. Thus, through order ϵ^2 , the diffusion coefficient is determined by $\hat{\omega}_c = 0.83$, or

$$D = 15(m_i/m_H) \Delta_{PB}^{3/2} \tau^{5/2} [0.5(1+\tau)]^{-9/2} \rho_S^2 c_S / L_S, \quad (60d)$$

where $c_S = (T_e/m_i)^{1/2}$. This result is in agreement with that obtained from the one-point theory.¹

Finally, the values of k_{oz} , k_{ox} , k_{oy} , and D obtained in Eqs. (54b) and (60) can be used in Eq. (49a) to obtain an explicit expression for the spectrum of electrostatic drift wave turbulence in a sheared magnetic field:

$$\begin{aligned} \frac{e^2 \langle \Phi(1) \Phi(2) \rangle_{\vec{k}, \omega}}{T_e^2} &= \tilde{\epsilon} k_{oy}^{-1} k_{oz}^{-1} \omega_{*o}^{-1} \left(\frac{k_y}{k_{oy}} \right)^{-2/3} \frac{1}{|d\epsilon_o|^2} \\ &\times [1 + (\bar{x}_+/x_c)^2]^{-1/2} \left| \frac{1}{4} + |\mu_i| (k_{ox} \rho_i)^2 / |\mu|^2 \right|^{-1/2} \\ &\times \exp [-|\mu_i| (\bar{x}_+^2 + x_-^2) / \rho_i^2 - 2i\mu_i \bar{x}_+ x_- / \rho_i^2 - A] \quad (61a) \end{aligned}$$

where $\omega_{*0} = \tau(1 + \tau)^{-1} v_i / L_n$ is the characteristic frequency of the short wavelength modes and

$$\tilde{\epsilon} = (400)^{-1} (\rho_i / L_n)^3 (Rq/r) (L_s / L_n) (m_H / m_i)^{1/2} \Delta_{PB}^{3/2} \left(\frac{2\tau}{1 + \tau} \right)^{9/2} (1 + \tau)^{-2} . \quad (61b)$$

Thus, since $\tilde{\epsilon} \ll 1$, it is clear from Eq. (61) that the fluctuation spectrum is peaked around the nonlinear resonances where $|\epsilon_o|_{\vec{k}, \omega}$ approaches zero.

APPENDIX A

FOURIER TRANSFORM OF CORRELATION FUNCTION

Consider the integral

$$M(k_x, k_y, k_z) = \int_{-\infty}^{\infty} dx_- \int_{-\infty}^{\infty} dy_- \int_{-\infty}^{\infty} dz_- \exp [-i(k_x x_- + k_y y_- + k_z z_-)] \\ \times (\sqrt{1 - k^2} - |k| \cos^{-1} |k|) , \quad (A1)$$

which is the Fourier transform of the spatially dependent part of the correlation function in Eq. (38a). Here,

$$k^2 = k_{ox}^2 x_-^2 + \frac{1}{2} k_{oy}^2 y_-^2 + \frac{2k_{oy}^2 k_{oz}^2}{k_{||o}^2} \left(\frac{r}{Rq} z_- + \frac{Y_-}{2} \right)^2 \leq 1 , \quad (A2)$$

where $Y_- = y_- + 2x_- v_{||} \tau_o / L_S$. Defining the spherical coordinate system (k, θ, ϕ) , such that

$$k_{ox} x_- = k \sin \theta \cos \phi , \quad (A3a)$$

$$\frac{1}{\sqrt{2}} k_{oy} y_- = k \sin \theta \sin \phi , \quad (A3b)$$

$$\frac{\sqrt{2} k_{oy} k_{oz}}{k_{||o}} \left(\frac{r}{Rq} z_- - \frac{Y}{2} \right) = k \cos \theta , \quad (A3c)$$

Eq. (A1) becomes:

$$M = (k_{ox} k_{oy} k_{oz})^{-1} \frac{k_{||o}}{\sqrt{2} k_{oy}} \frac{Rq}{r} J(\bar{k}_x, \bar{k}_y, \bar{k}_z) . \quad (A4a)$$

Here, $\bar{k}_x = k_x/k_{ox}$, $\bar{k}_y = k_y/k_{oy}$, $\bar{k}_z = (k_z/k_{oz})(Rq/r)$, and

$$J = 2 \int_0^1 k^2 (\sqrt{1 - k^2} - k \cos^{-1} k) dk \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \\ \times \exp \left\{ -ik \left[\left(\bar{k}_x + \frac{k_{oz} v_{||} + \tau_o \bar{k}_z}{k_{ox} L_S} \right) \sin \theta \cos \phi + \left(\bar{k}_y + \frac{1}{2} \bar{k}_z k_{oz}/k_{oy} \right) \right. \right. \\ \left. \left. \times \sin \theta \sin \phi + \bar{k}_z k_{||o}/(\sqrt{2} k_{oy}) \cos \theta \right] \right\} . \quad (A4b)$$

Performing the ϕ integral yields

$$J = 4\pi \int_0^1 k^2 (\sqrt{1 - k^2} - k \cos^{-1} k) dk \int_0^\pi \sin \theta d\theta \\ \times J_0(k_{||} k \sin \theta) \exp [-ik \bar{k}_z k_{||o} \cos \theta / (\sqrt{2} k_{oy})] , \quad (A5)$$

where $k_{||}^2 = [k_x + (k_{oz}/k_{ox}) \bar{k}_z v_{||} + \tau_o/L_S]^2 + [\bar{k}_y + \bar{k}_z k_{oz}/(2k_{oy})]^2$ and J_0 is a Bessel function. The θ integral is most readily performed by letting $x = \cos \theta$, with the result

$$J = \frac{8\pi}{b} \int_0^1 k^2 (\sqrt{1 - k^2} - k \cos^{-1} k) \left(\frac{\sin bk}{k} \right) dk , \quad (A6)$$

where

$$b^2 = k_{||}^2 + \bar{k}_z^2 k_{||o}^2 / (2k_{oy}^2) . \quad (A7)$$

Finally, the k integral can be performed by repeated integration by parts, i.e., $k \sin bk = d/dk(\sin bk/b^2 - k \cos bk/b)$, etc. The result is

$$J = 8\pi^2[1 - J_0(b) - \frac{1}{2} bJ_1(b)]/b^4, \quad (A8)$$

where $J_n(b)$ are Bessel functions of order n , for $n = 0, 1$. A graph of J is shown in Fig. 1. Note that for very large $b \geq 8$, corresponding to very short wavelengths, J begins to oscillate (with small amplitude) about zero. This spurious oscillation is due to the abrupt cutoff of the correlation function for $k > 1$. This can be traced to the present improper treatment of particle orbits which have diverged significantly so that $D_- \approx D$. Had this orbit divergence and subsequent decorrelation been retained in our analysis, the correlation function would have gone smoothly to zero and the large wave number oscillations in Eq. (A8), which give an unphysical (but *very* small) negative transform of the correlation function, would be absent.

To maintain the physical positiveness of J , the following Gaussian approximation was introduced:

$$J \approx \frac{\pi^2}{8} \exp [-(b/4)^2]. \quad (A9)$$

This approximation is graphed in Fig. 2, and the values of the exact function from Eq. (A8) are indicated at representative points. Its agreement with the exact function over a broad range of b , as well as its evident positive definiteness, make Eq. (A9) a suitable approximation for analytic applications.

APPENDIX B

DERIVATION OF GREEN'S FUNCTION FOR $L_{\vec{k},\omega}^+$

Consider the equation

$$\left[\rho_i^2 \frac{\partial^2}{\partial x^2} - (\Lambda - \mu^2 x^2 / \rho_i^2 + i\sigma_o \rho_i / x_c) \right] G(x, x') \equiv L_{\vec{k},\omega}^+ G$$

$$= \rho_i \delta(x - x') , \quad (B1)$$

where x is measured from the mode rational surface where $\vec{k} \cdot \vec{B} = 0$.

Transforming to $y = (i\mu)^{1/2}(x/\rho_i)$, where $\text{Re}(y^2) > 0$ corresponds to the choice $\text{Im} \mu < 0$, i.e., outward propagation of wave energy, and letting $G = \exp(-y^2/2) \tilde{G}(y, y')$, Eq. (B1) becomes

$$\tilde{G}_{yy} - 2y\tilde{G}_y - (1 + \tilde{\Lambda}/i\mu)G = (i\mu)^{-1/2} \exp(y^2/2)\delta(y - y') , \quad (B2)$$

where $\tilde{\Lambda} = \Lambda + i\sigma_o \rho_i / x_c$. Expanding $\tilde{G} = \sum_{n=0}^{\infty} g_n H_n(y)$ in a complete set of Hermite polynomials and using the equation satisfied by H_n ,

$$(H_n)_{yy} - 2y(H_n)_y + 2nH_n = 0 , \quad (B3)$$

yields upon substitution into Eq. (B2)

$$\sum_{n=0}^{\infty} g_n \left(\frac{\tilde{\Lambda}}{i\mu} + 2n + 1 \right) H_n(y) = -(i\mu)^{-1/2} \exp(y^2/2)\delta(y - y') . \quad (B4)$$

Multiplying Eq. (B4) by $\exp(-y^2) H_n(y)$ and integrating from $y = 0$ to $y = \infty$ yields

$$g_n = \frac{-H_n(y') \exp(-y'^2)}{(i\mu\pi)^{1/2} n! 2^n (\tilde{\Lambda}/i\mu + 2n + 1)} . \quad (B5)$$

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REFERENCES

- ¹S. P. Hirshman and Kim Molvig, Phys. Rev. Lett. 42, 648 (1979);
Kim Molvig and S. P. Hirshman, Phys. Rev. Lett. 43, 582 (1979).
- ²T. H. Dupree, Phys. Fluids 15, 334 (1972).
- ³B. H. Hui and T. H. Dupree, Phys. Fluids 18, 235 (1975).
- ⁴T. H. Dupree, C. E. Wagner, and W. H. Manheimer, Phys. Fluids 18, 1167 (1975).
- ⁵T. H. Dupree, Phys. Fluids 21, 783 (1978).
- ⁶P. H. Diamond, Ph.D. Thesis (MIT, 1979), unpublished.
- ⁷J. A. Krommes, R. G. Kleva, and C. Oberman, Princeton Plasma Physics Laboratory Report PPPL-1389 (1978), to be published.
- ⁸A. B. Rechester, M. N. Rosenbluth, and R. B. White, Phys. Rev. Lett. 42, 1247 (1979).
- ⁹P. H. Diamond, T. H. Dupree, and D. J. Tetreault (submitted for publication to Phys. Rev. Lett.)
- ¹⁰L. D. Pearlstein and H. L. Berk, Phys. Rev. Lett. 23, 220 (1969).
- ¹¹A. Erdélyi, ed., *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 1, p. 159.

FIGURE CAPTIONS

FIG. 1. The function $(8/\pi^2) J(b)$, where $J(b)$ is defined in Eq. (A.8).

FIG. 2. The function $\exp [-(b/4)^2]$, used to approximation $J(b)$.

Fig. 1

Fig. 2

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