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TRANSFORMATION OF HAMILTONIANS TO NEAR
ACTION-ANGLE FORM

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ABSTRACT

A classical Hamiltonian would be solved by a transformation to action-angle variables I, θ in which the Hamiltonian is $\bar{H}(I)$. Generally, such a transformation does not exist, and, at best, the Hamiltonian can be transformed to $\bar{H}(I) + \hat{H}(I, \theta, t)$ with $\hat{H}$ being a sum of Fourier terms that resonate with $\bar{H}$. We give a set of ordinary differential equations in a parameter ϵ that carry out this transformation as the set is integrated from ϵ equal to zero to one. Although the differential equations can be integrated numerically, approximations give classical perturbation theory.

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I. INTRODUCTION

Given a Hamiltonian $H(p,q,t)$, the classical problem of Hamiltonian mechanics is to find a canonical transformation to action-angle variables I, θ so that the Hamiltonian is $\bar{H}(I)$. If such a transformation is found, then the particle trajectories can be trivially evaluated for any initial condition I_0, θ_0 . The trajectory is $I = I_0, \theta = \theta_0 + \omega t$ with $\omega \equiv d\bar{H}/dI$, the rotation number. The transformation equations between the two coordinate sets, $p(I, \theta, t)$ and $q(I, \theta, t)$, give the trajectory in the original canonical coordinate space.¹

Hamiltonians for which action-angle variables exist are called integrable. Few Hamiltonians are integrable, but many Hamiltonians of practical interest are nearly integrable. By nearly integrable we mean that near action-angle variables I, θ exist such that the Hamiltonian is $\bar{H}(I) + \hat{H}(I, \theta, t)$ with $|\hat{H}/\bar{H}| \ll 1$. The function $\hat{H}$ can be assumed to have only resonant Fourier terms (defined below). The resonant Fourier terms give the location and the extent of the so-called islands and stochastic regions of the particle trajectories, which are the topological features that prevent the Hamiltonian from being integrable. This paper gives a method for carrying out the transformation to near action-angle variables by integrating a set of first order differential equations over the finite range zero to one. These equations can be integrated numerically by standard techniques. The method is nonperturbative but does depend on the existence of a continuous deformation or homotopy from the initial canonical variables to the near action-angle variables. Consequently, we will explicitly assume the given Hamiltonian is periodic in q with period 2π . The time dependence is also assumed periodic with the unit of time chosen so that the period is 2π . Although the method is given using the notation of a one degree of freedom Hamiltonian, it can be

extended to Hamiltonians with any number of degrees of freedom by interpreting the canonical variables as vectors.

The reason $\hat{H}$ cannot be made zero is the existence of islands and stochastic regions in the Poincaré plot.¹ A Poincaré plot is constructed by plotting the I, θ location of a trajectory at each periodicity point of time, $t = 0, 2\pi$, etc. Let $\hat{H}$ be given in a cosine series,

$$\hat{H} = \sum_{n,m} \hat{H}_{nm}(I) \cos[nt - m\theta + \lambda_{nm}(I)]. \quad (1)$$

Then there will be an island about each action surface on which $\omega = d\bar{H}/dI$ is a rational number ($M\omega = N$ with N and M integers). The half-width of each island is

$$\Delta_{NM} \approx 2 \left| \frac{\hat{H}_{NM}}{d\omega/dI} \right|^{1/2} \quad (2)$$

with the right-hand side evaluated at the rational surface, $M\omega = N$. If the islands associated with different rational surfaces are sufficiently wide to overlap, then a single trajectory will cover a large fraction of the volume that would have been occupied by the islands. Such trajectories are called stochastic. If the Hamiltonian is an analytic function with no special symmetries, the $\hat{H}_{nm}$ are all finite, but they become exponentially small for large m or n . A crude measure of integrability in a region bounded by the action surface I is given by the sum of the widths of all the islands inside I , $\Delta = 2\sum \Delta_{nm}$. As long as Δ/I is small, the Poincaré plot will consist of nested surfaces over a large fraction of the volume. Since only resonant Fourier terms, $m\omega = n$, directly create islands or stochastic regions, $\hat{H}$ need only contain such terms.

To integrate the differential equations that will be given in Sec. III, one must adopt a precise definition of resonant Fourier terms. The only restriction is that a nonresonant Fourier coefficient must vanish at $m\omega = n$. Here we give two useful definitions. In the first definition a nonresonant Fourier coefficient is defined so that $m\omega \neq n$ for all values of ω in the region of interest. The second definition makes the resonant terms smaller. The resonant $a_{nm}^{(r)}$ and nonresonant parts $a_{nm}^{(nr)}$ of a Fourier coefficient a_{nm} are defined by

$$a_{nm}^{(r)} = \frac{\delta^2}{\xi^2 + \delta^2} a_{nm}, \quad \text{and} \quad a_{nm}^{(nr)} = \frac{\xi^2}{\xi^2 + \delta^2} a_{nm} \quad (3)$$

with $\xi = n - m\omega$. The function $\delta(\xi)$ is arbitrary except $\delta(0)$ is finite.

II. TRANSFORMATIONS

The fundamental concept of the paper is that of a smoothly deformable, canonical transformation between variables I, θ, t and p, q, t . These transformations are characterized by a generating function $s(I, \theta, t, \epsilon)$ which depends on a parameter ϵ . They are defined by the differential equations

$$\frac{\partial I}{\partial \epsilon} \Big|_{p,q} = - \frac{\partial s}{\partial \theta} \quad \text{and} \quad \frac{\partial \theta}{\partial \epsilon} \Big|_{p,q} = \frac{\partial s}{\partial I}. \quad (4)$$

with $I = p$ and $\theta = q$ at $\epsilon = 0$. It is obvious that such transformations are smoothly deformable. The fact that such transformations are canonical follows from the well-known equations of infinitesimal canonical transformations and the fact that the product of two canonical transformations is itself a canonical transformation.²

The Hamiltonian H in I, θ coordinates is determined by the Hamiltonian,

denoted by K , in p, q coordinates. This relation is well-known² when K is independent of ϵ , but the generalization

$$\frac{\partial H}{\partial \epsilon}_{p,q} = \frac{\partial s}{\partial t} + \frac{\partial K}{\partial \epsilon}_{p,q} \quad (5)$$

is obvious. Of course, $H = K$ at $\epsilon = 0$.

The most useful independent variables will be the I, θ variables rather than the p, q variables. To switch independent variables, one uses the identity

$$\frac{\partial f}{\partial \epsilon}_{p,q} = \frac{\partial f}{\partial \epsilon}_{I,\theta} + \frac{\partial I}{\partial \epsilon}_{p,q} \frac{\partial f}{\partial I} + \frac{\partial \theta}{\partial \epsilon}_{p,q} \frac{\partial f}{\partial \theta} . \quad (6)$$

Equation (4) implies that this identity can be written as

$$\frac{\partial f}{\partial \epsilon}_{I,\theta} = \frac{\partial f}{\partial \epsilon}_{p,q} + [f, s] \quad (7)$$

with the Poisson bracket defined by

$$[f, s] \equiv \frac{\partial f}{\partial I} \frac{\partial s}{\partial \theta} - \frac{\partial f}{\partial \theta} \frac{\partial s}{\partial I} . \quad (8)$$

Using this identity, one can easily prove the following relations, which form the basis of the paper,

$$\frac{\partial H}{\partial \epsilon}_{I,\theta} = \frac{\partial K}{\partial \epsilon}_{p,q} + \frac{\partial s}{\partial t} + [H, s], \quad (9)$$

$$\frac{\partial p}{\partial \epsilon}_{I,\theta} = [p, s] \text{ and } \frac{\partial q}{\partial \epsilon}_{I,\theta} = [q, s] . \quad (10)$$

Given an arbitrary Hamiltonian which depends on a parameter, $K(p, q, t, \epsilon)$ these equations determine a large number of mathematically equivalent Hamiltonians $H(I, \theta, t, \epsilon)$ since the generating function $s(I, \theta, t, \epsilon)$ can be chosen in any convenient manner.

III. SOLUTION

Given a Hamiltonian H_g , the problem is to find the transformation to near action-angle variables I, θ . In these variables the Hamiltonian is of the form $\bar{H}(I) + \hat{H}(I, \theta, t)$ with $\hat{H}$ having only resonant Fourier terms. To solve the problem, the Fourier representation of H_g ,

$$H_g = \sum_{n,m} H_{nm}^{(g)}(p) \exp[i(nt - m\theta)], \quad (11)$$

is separated into three parts,

$$H_g = \bar{H}_0(p) + \tilde{H}_0(p, q, t) + \hat{H}_0(p, q, t). \quad (12)$$

The part $\bar{H}_0(p)$ is the $n = 0, m = 0$ Fourier term, and the part $\hat{H}_0$ contains the Fourier terms that resonate with $\bar{H}_0$, $m d\bar{H}_0/dp = n$. The remaining Fourier terms are in $\tilde{H}_0$. A Hamiltonian K , which is dependent on a parameter ϵ , is defined as

$$K(p, q, t, \epsilon) = \bar{H}_0(p) + \epsilon \tilde{H}. \quad (13)$$

The term $\tilde{H}$ is chosen so that $\tilde{H} + \hat{H}$ is independent of ϵ with $\tilde{H} = \tilde{H}_0$ and $\hat{H} = \hat{H}_0$ at $\epsilon = 0$. However, $\tilde{H}$ and $\hat{H}$ individually depend on ϵ with

$$\left(\frac{\partial \tilde{H}}{\partial \varepsilon}\right)_{p,q} = -\sigma. \quad (14)$$

The important features of the Hamiltonian K are that at $\varepsilon = 0$ it is in action-angle form, and at $\varepsilon = 1$ the given Hamiltonian is $H_g = K + \hat{H}$ for any choice of σ . By choosing σ appropriately, K can be made exactly integrable for $0 \leq \varepsilon \leq 1$. By using the freedom of the smoothly deformable, canonical transformations, the Hamiltonian K can be transformed into action-angle form $\tilde{H}(I)$.

To carry out the transformation into action-angle coordinates, the derivative

$$\left(\frac{\partial K}{\partial \varepsilon}\right)_{p,q} = \tilde{H} - \varepsilon \sigma \quad (15)$$

[see Eqs. (13) and (14)] is substituted into Eq. (9) with the Hamiltonian H of the form $\tilde{H}(I, \varepsilon)$. The result is

$$\frac{\partial \tilde{H}(I, \varepsilon)}{\partial \varepsilon} = \frac{\partial s}{\partial t} + \omega(I) \frac{\partial s}{\partial \theta} + \tilde{H} - \varepsilon \sigma \quad (16)$$

with $\omega = \partial \tilde{H} / \partial I$ the rotation number so that

$$[\tilde{H}, s] = \omega \frac{\partial s}{\partial \theta}. \quad (17)$$

In order for the equation for $\partial \tilde{H} / \partial \varepsilon$ to be consistent, σ must be chosen to cancel any resonant terms which may arise in $\tilde{H}(I, \theta, t, \varepsilon)$ for $\varepsilon \neq 0$. The Fourier series for s can be trivially evaluated so that s cancels the nonresonant Fourier terms in $\tilde{H} - \varepsilon \sigma$.

The ϵ derivative of the function $\tilde{H}(I, \theta, t, \epsilon)$ can be determined from the identity, Eq. (7), and Eq. (14). One finds that

$$\frac{\partial \tilde{H}(I, \theta, t, \epsilon)}{\partial \epsilon} = [\tilde{H}, s] - \sigma. \quad (18)$$

Similarly, $\hat{H}$ can be found by integrating

$$\frac{\partial \hat{H}(I, \theta, t, \epsilon)}{\partial \epsilon} = [\hat{H}, s] + \sigma. \quad (19)$$

The nonresonant Fourier terms in σ are to be chosen so that no nonresonant Fourier terms arise in $\hat{H}$. If the set of differential equations, Eqs. (16), (18), and (19), are integrated from $\epsilon = 0$ to $\epsilon = 1$, then the near action-angle Hamiltonian $\tilde{H}(I) + \hat{H}(I, \theta, \phi, t)$ is canonically equivalent to the given Hamiltonian $H_g(p, q, t)$. The canonical transformation equations are obtained by an integration of Eq. (10) for $dp/d\epsilon$ and $dq/d\epsilon$.

IV. EXAMPLE

The differential equations, which evolve canonical coordinates into near action-angle variables, Eqs. (16), (18), and (19) can be integrated using standard numerical techniques. In practical calculations they need not be considered perturbative. However, if $|(H_g - \bar{H}_0)/\bar{H}_0|$ is small, one can perturbatively solve the differential equations with a Taylor expansion in ϵ which closely resembles the expansions of the standard perturbative techniques.^{2,3}

To illustrate the method, consider the example of a harmonic oscillator at the "wrong" frequency. First, note that if a Hamiltonian $H_g(p, q)$ is not periodic in q one can make the canonical transformation

$$p = (2m\omega p)^{1/2} \cos q \text{ and } Q = \left(\frac{p}{2m\omega}\right)^{1/2} \sin q \quad (20)$$

with $m\omega$ an arbitrary constant. The Hamiltonian H_g is then periodic in q . A harmonic oscillator with frequency Ω has the Hamiltonian

$$H_g = \frac{1}{2m} p^2 + \frac{m\Omega^2}{2} Q^2, \quad (21)$$

or transforming to the p, q variables of Eq. (20),

$$H_g = \omega p + \frac{\Omega^2 - \omega^2}{\omega} p \sin^2 q. \quad (22)$$

Obviously, the choice $\omega = \Omega$ puts the Hamiltonian into action-angle form. However, by letting $\Omega = \omega + \omega_1$ with $|\omega_1/\omega| \ll 1$, one can illustrate perturbation theory. The Fourier decomposition of H_g gives

$$\bar{H}_0(I) = \frac{\Omega^2 - \omega^2}{2\omega} I, \quad \tilde{H}_0 = -\frac{\Omega^2 - \omega^2}{2\omega} \cos 2\theta, \text{ and } \hat{H}_0 = 0 \quad (23)$$

with $I = p$ and $\theta = q$ at $\varepsilon = 0$. Equation (16) then implies that the generating function at $\varepsilon = 0$ is

$$S_0 = \frac{\Omega^2 - \omega^2}{\Omega^2 + \omega^2} I \sin 2\theta \quad (24)$$

and Eq. (18) implies

$$\frac{\partial \tilde{H}}{\partial \varepsilon} \bigg|_{\varepsilon=0} = -\frac{\Omega^2 - \omega^2}{\Omega^2 + \omega^2} \frac{\Omega^2 - \omega^2}{\omega} I. \quad (25)$$

Integrating this equation and inserting it into Eq. (16) one finds that

$$\bar{H}(I, \epsilon) = \frac{\Omega^2 + \omega^2}{2\omega} \left[1 - \frac{1}{2} \left(\epsilon \frac{\Omega^2 - \omega^2}{\Omega^2 + \omega^2} \right)^2 + O(\epsilon^4) \right] I. \quad (26)$$

Since each factor of ϵ is multiplied by the small number $(\Omega^2 - \omega^2)/(\Omega^2 + \omega^2)$ one can validly set ϵ to unity to obtain $\bar{H}(I)$, the Hamiltonian in action-angle form. The use of perturbation theory leads to a complicated expression for $\bar{H} = \Omega I$, but one can easily check the convergence term by term. By substituting s_0 into Eq. (10), one obtains the evolution of p and q . Using Eq. (20), one can show convergence to the obvious result that P equals $(2m\omega I)^{1/2} \cos\theta$.

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