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SUBMITTED TO: Proceedings of "AF/DARPA Review of Progress in
Quantitative NDE"

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ELASTIC WAVE SCATTERING CALCULATIONS, THE BORN SERIES, AND THE MATRIX-VARIATIONAL PADÉ APPROXIMANT METHOD

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ABSTRACT

The matrix variational Padé approximant and its generalization to elastic wave scattering are discussed. Predictions of the method for the scattering of a longitudinal plane wave are compared with the exact scattering from spherical voids and inclusions. Its predictions are also compared to those of the first four partial sums of the Born Series for the scattered amplitude. Generally, the fourth partial sum and the variational results compare poorly with the exact results for $ka \lesssim 2$ if the scatterer is strong, but compare well for $ka \lesssim 10$ if the scatterer strength is at best modest. The breakdown of the favorable comparison is traced to the divergence of the Born Series for strong scatterers. It is also demonstrated that by use of the N-point Padé approximant a good comparison with exact results can be obtained for all scatterer strengths.

INTRODUCTION

This is a preliminary report on the application of the Lippmann-Schwinger variational principle to the scattering of elastic waves from voids and inclusions.¹ This particular variational principle has the advantage of being formulated in terms of the scattering matrix and hence directly giving the physical quantities of experimental interest. The trial functions used in this application of the variational principle allow the stationary value of the variational form to be expressed in a compact block-matrix form with the block-matrix elements corresponding to terms in the Born-Neumann series solution to the integral equation of scattering.

Our principal motivation for using the variational approach is to develop a method for treating the scattering of elastic waves from complex-shaped defects when the wavelength of the incident wave is comparable to the size of the defect. Present methods, like the eigenfunction expansion methods,²⁻⁵ are for practical reasons limited to axially-symmetric defects.

Below we summarize some exact results from the integral equation approach to scattering theory. Then we discuss various approximations: partial summations of the Born Series, the matrix Padé approximant of the Born Series, and Lippmann-Schwinger variational approach. Next, to help understand why comparisons of the approximations with exact results for spherical defects fail for strong scatterers, we investigate the singularity structure of the Born Series, find numerous singularities that cause the Born series to diverge, and demonstrate the effectiveness of the N-point Padé approximant, as opposed the standard Padé approximant, for analytically continuing the Born Series into a convergent result.

EXACT RESULTS

The integral equation describing the scattering of an elastic wave from voids and inclusions is

$$u_i(\vec{r}) = u_i^0(\vec{r}) + z \int d\vec{r}' g_{ij}(\vec{r}, \vec{r}') v_{jk}(\vec{r}') u_k(\vec{r}') \quad (1)$$

where $u_i^0(\vec{r})$ is the incident wave, $g_{ij}(\vec{r}, \vec{r}')$ is the Green's function, and $v_{jk}(\vec{r})$ describes the void or inclusion. If the void or inclusion is embedded in an unbounded, isotropic medium, then

$$g_{ij}(\vec{r}, \vec{r}') = \frac{1}{4\pi\rho\omega^2} \left[\beta^2 \delta_{ij} \frac{e^{i\beta R}}{R} + \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left(\frac{e^{i\alpha R}}{R} - \frac{e^{i\beta R}}{R} \right) \right] \quad (2)$$

with

$$R = |\vec{r} - \vec{r}'|$$

and α and β equal to the longitudinal and transverse wavenumbers. We also have that

$$v_{ij}(\vec{r}) = \left[\delta\rho\omega^2\theta(\vec{r})\delta_{ij} + \delta C_{ijkl} \frac{\partial}{\partial x_k} \theta(\vec{r}) \frac{\partial}{\partial x_l} \right] \quad (3)$$

with $\delta\rho$ and δC_{ijkl} equal to the density and stiffness differences between the flaw and host material and with

$$\theta(\vec{r}) = \begin{cases} 1, & \text{for } \vec{r} \text{ inside the defect} \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

The parameter ρ is the density of the host material, and ω is the circular frequency of the incident wave. For later use, we inserted the dimensionless parameter z . For actual defects, $z = 1$.

In the far-field the solution to (1) has the form

$$u_i \sim u_i^0 + A_i \frac{e^{i\alpha R}}{R} + B_i \frac{e^{i\beta R}}{R}$$

with the scattered amplitudes given by

$$A_i = P_i P_j f_j(\vec{\alpha}) \quad (5a)$$

$$B_i = (\delta_{ij} - P_i P_j) f_j(\vec{\beta}) \quad (5b)$$

and the f -vector defined by

$$f_i(\vec{k}) = \frac{zk^2}{4\pi\mu\omega^2} \int d\vec{r} e^{-i\vec{k}\cdot\vec{r}} v_{ij}(\vec{r}) u_j(\vec{r}) \quad (6)$$

where $\vec{k} = k\hat{r}$ and $\hat{r}$ is the direction of scattering.

The quantity of experimental interest is the differential cross-section. If the incident wave is a plane wave of the form

$$u_i^0 = d_i e^{i\vec{k}^0\cdot\vec{r}} \quad (7)$$

then the differential cross-section for the longitudinal component of the scattering is

$$\frac{dP}{d\Omega} = \left| \frac{A}{d} \right|^2 \quad (8)$$

There is an alternate integral equation approach to the scattering that is formulated in terms of the scattered amplitudes.^{7,8} If both sides of (1) were operated on by $z \exp(-i\vec{k}\cdot\vec{r}) v_{ij}(\vec{r})$ and then integrated over $\vec{r}$, it follows from (6) that (1) is now equivalent to

$$f_i(\vec{k}) = \frac{zk^2}{4\pi\mu\omega^2} \left[\int d\vec{r} e^{-i\vec{k}\cdot\vec{r}} v_{ij}(\vec{r}) d_j e^{i\vec{k}^0\cdot\vec{r}} + z \int d\vec{r} \int d\vec{r}' e^{-i\vec{k}\cdot\vec{r}} v_{ij}(\vec{r}) g_{jk}(\vec{r}, \vec{r}') v_{jk}(\vec{r}', \vec{r}) u_k(\vec{r}') \right] \quad (9)$$

so with the definitions

$$f_i(\vec{k}) = \frac{k^2}{4\pi\mu\omega^2} T_{ij}(\vec{k}, \vec{k}^0) d_j \quad (10)$$

$$v_{ij}(\vec{k}, \vec{k}^0) = \int d\vec{r} e^{-i\vec{k}\cdot\vec{r}} v_{ij}(\vec{r}) e^{i\vec{k}^0\cdot\vec{r}} \quad (11)$$

$$g_{ij}(\vec{r}, \vec{r}') = \int d\vec{q} e^{i\vec{q}\cdot\vec{r}} G_{ij}(\vec{q}) e^{-i\vec{q}\cdot\vec{r}'}$$

equation (9) becomes⁸

$$T_{ij}(\vec{k}, \vec{k}^0) = z v_{ij}(\vec{k}, \vec{k}^0) + z \int d\vec{q} v_{ik}(\vec{k}, \vec{q}) G_{kl}(\vec{q}) T_{lj}(\vec{q}, \vec{k}^0) \quad (12)$$

where $z = 1$ for full strength.

The matrix function $T_{ij}(\vec{k}, \vec{k}^0)$ has the following simple relation to the scattered amplitudes⁸

$$A_i = P_i P_j T_{jk}(\vec{\alpha}, \vec{k}^0) d_k / 4\pi(\lambda + 2\mu) \quad (13a)$$

$$B_i = (\delta_{ij} - P_i P_j) T_{jk}(\vec{\beta}, \vec{k}^0) d_k / 4\pi\mu \quad (13b)$$

where λ and μ are the Lamé parameters. The other matrix function $v_{ij}(\vec{k}, \vec{k}^0)$ equals

$$v_{ij}(\vec{k}, \vec{k}^0) = t_{ij}(\vec{k}, \vec{k}^0) S(\vec{k}, \vec{k}^0) \quad (14)$$

where

$$t_{ij}(\vec{k}, \vec{k}^0) = [\delta_{ij} \mu^2 - \delta_{ij} \mu(\vec{k} \cdot \vec{k}^0)] \delta_{ij} - \delta \lambda k_i k_j^0 - \delta \mu k_i^0 k_j \quad (15)$$

and the shape factor

$$S(\vec{k}, \vec{k}^0) = \int d\vec{r} \theta(\vec{r}) e^{i(\vec{k}^0 - \vec{k})\cdot\vec{r}} \quad (16)$$

Several additional definitions and equations will prove useful. First, if we use (9) for incident wave and if we define $U_{ij}(\vec{r})$ by

$$u_i(\vec{r}) = U_{ij}(\vec{r}) d_j \quad (17)$$

then we can rewrite (1) as

$$U_{il}(\vec{r}) = U_{il}^0(\vec{r}) + z \int d\vec{r}' g_{ij}(\vec{r}, \vec{r}') v_{jk}(\vec{r}', \vec{r}) U_{kl}(\vec{r}') \quad (18)$$

where

$$U_{il}^0(\vec{r}) = P_{il}(\vec{k}^0) e^{i\vec{k}^0\cdot\vec{r}} \quad (19)$$

and

$$P_{il}(\vec{k}^0) = k_i^0 k_l^0$$

if the incident wave is longitudinal and

$$P_{il}(\vec{k}^0) = \delta_{il} - k_i^0 k_l^0$$

if the incident wave is transverse.

Also, we note that the specific formula for the scattering matrix is now

$$T_{ik}(\vec{k}, \vec{k}^0) = z \int d\vec{r} e^{-i\vec{k}\cdot\vec{r}} v_{ij}(\vec{r}) U_{jk}(\vec{r}) \quad (20)$$

APPROXIMATIONS

The Born Series - A formal solution to the integral equation for the scattering matrix can be obtained by iterating the equation to produce what is often called the Born-Neumann series. If only the first term in this series is used (i.e., no iterations are performed), then the resulting approximation is called the first Born approximation. The first Born approximation to (12) corresponds to⁹

$$T_{ij}(\vec{k}, \vec{k}^0) \approx z v_{ij}(\vec{k}, \vec{k}^0) \quad (21)$$

If the first two terms of the Born-Neumann series are kept (i.e., one iteration is performed), then the approximation is called the second Born approximation. For (12) the second Born approximation corresponds to

$$T_{ij}(\vec{k}, \vec{k}^0) \approx z T_{ij}^{(1)}(\vec{k}, \vec{k}^0) + z^2 T_{ij}^{(2)}(\vec{k}, \vec{k}^0) \quad (22)$$

where

$$T_{ij}^{(1)}(\vec{k}, \vec{k}^0) = v_{ij}(\vec{k}, \vec{k}^0) \quad (23a)$$

$$T_{ij}^{(2)}(\vec{k}, \vec{k}^0) = \int d\vec{q} t_{ik}(\vec{k}, \vec{q}) S(\vec{k}, \vec{q}) G_{kl}(\vec{q})$$

$$S(\vec{q}, \vec{k}^0) t_{2j}(\vec{q}, \vec{k}^0) \quad (23b)$$

The matrix functions $T_{ij}^{(1)}$ and $T_{ij}^{(2)}$ are called the first and second Born terms.

The [M/N] Matrix Padé Approximation - If $F(z)$ is a matrix function of z and if

$$F(z) = F^{(0)} + F^{(1)}z + F^{(2)}z^2 + \dots \quad (24)$$

then the [M/N] matrix Padé approximation to $F(z)$ is defined by¹⁰

$$F^{[M/N]}(z) = P_M(z) Q_N(z)^{-1} \quad (25a)$$

where

$$F(z) Q_N^{-1}(z) - P_M(z) = O(N \cdot N + 1) \quad (25b)$$

(For compactness, we now denote matrices by underlined, capital Roman letters. Below, all matrices representing physical quantities are 3×3 .)

Specifically, from (18) and (20),

$$\begin{aligned} \underline{T}^{[1/1]}(\vec{k}, \vec{k}^0) &= z \underline{T}^{(1)}(\vec{k}, \vec{k}^0) [\underline{T}^{(1)}(\vec{k}, \vec{k}^0) \\ &- z \underline{T}^{(2)}(\vec{k}, \vec{k}^0)]^{-1} \underline{T}^{(1)}(\vec{k}, \vec{k}^0) \end{aligned} \quad (26)$$

Hence, if the first two terms of the Born series are known, it is a trivial procedure to evaluate the [1/1] matrix Padé approximant. The result may be a better approximation than given by the second Born approximation.

Technically, the definition (21) is of a right-handed matrix Padé approximant. If $P_M(x)$ and $Q_N(x)$ are interchanged, then a left-handed matrix Padé approximant is defined. The different definitions can be shown to give equivalent answers, but the separate numerators and denominators are not necessarily the same.

The Variational Matrix Padé Approximant - First, we identify $\underline{W}(\vec{r})$ as the matrix function satisfying the adjoint equation of (18)

$$\underline{W}(\vec{r}) = \underline{W}^0(\vec{r}) + z \int d\vec{r}' g^*(\vec{r}, \vec{r}') \underline{v}(\vec{r}') \underline{W}(\vec{r}') \quad (27a)$$

where

$$\underline{W}^0(\vec{r}) = \underline{v}(\vec{k}) e^{i\vec{k} \cdot \vec{r}} \quad (27b)$$

Next, we operate on both sides of (18) with $z \underline{W}^*(\vec{r}) \underline{v}(\vec{r})$, integrate over $\vec{r}$, and subtract (20) from the result to find

$$\begin{aligned} \underline{I}(\vec{k}, \vec{k}^0) &= z \int d\vec{r} e^{-i\vec{k} \cdot \vec{r}} \underline{v}(\vec{r}) \underline{U}(\vec{r}) \\ &+ z \int d\vec{r} \underline{W}^*(\vec{r}) \underline{v}(\vec{r}) e^{i\vec{k}^0 \cdot \vec{r}} \\ &- z \int d\vec{r} \underline{W}^*(\vec{r}) \underline{v}(\vec{r}) \left[e^{i\vec{k}^0 \cdot \vec{r}} - z \int d\vec{r}' g(\vec{r}, \vec{r}') \underline{v}(\vec{r}') \underline{U}(\vec{r}') \right] \end{aligned} \quad (28)$$

Clearly, if $\underline{W}(\vec{r})$ and $\underline{U}(\vec{r})$ are exact, the above is an exact expression for the scattering matrix. The expression also has the feature that considered as a functional of $\underline{W}(\vec{r})$ and $\underline{U}(\vec{r})$ it is stationary with respect to independent and arbitrary variations of $\underline{W}(\vec{r})$ and $\underline{U}(\vec{r})$ about their exact values. When used as a variational form, the expression is a generalization to elastic wave scattering of the Lippmann-Schwinger variational principle.^{7,8,11-14}

There are many ways to choose the trial functions $\underline{W}(\vec{r})$ and $\underline{U}(\vec{r})$. We choose the first N terms of the Born Series for equations (18) and (27) as the basis states,

$$\underline{U}(\vec{r}) = \sum_{i=1}^N a_i (g \underline{v})^{i-1} \underline{U}^0 \quad (29a)$$

$$\underline{W}(\vec{r}) = \sum_{i=1}^N b_i (g \underline{v})^{i-1} \underline{W}^0 \quad (29b)$$

When (28) is made stationary with respect to these choices,

$$a_i = b_i$$

and

$$\underline{I}(\vec{k}, \vec{k}^0) = z \begin{bmatrix} \underline{T}^{(1)} \\ \vdots \\ \underline{T}^{(N)} \end{bmatrix}^T \begin{bmatrix} \underline{T}^{(1)} - z \underline{T}^{(2)} & \dots & \underline{T}^{(N)} & - z \underline{T}^{(N+1)} \\ \vdots & \ddots & \vdots & \vdots \\ \underline{T}^{(N)} - z \underline{T}^{(N+1)} & \dots & \underline{T}^{(2N)} & - z \underline{T}^{(2N+1)} \end{bmatrix}^{-1} \begin{bmatrix} \underline{T}^{(1)} \\ \vdots \\ \underline{T}^{(N)} \end{bmatrix} \quad (30a)$$

where

$$\underline{T}^{(i)} = \underline{T}^{(i)}(\vec{k}, \vec{k}^0) \quad , \quad i = 0, \dots, 2N \quad (30b)$$

If $N = 1$, then (30) clearly reduces to (26). In general, (30) is the [N/N] matrix Padé approximant to the N th Born approximation to the scattered amplitude.^{12,13}

(Equation (30) is a block-matrix equation: Matrix multiplication is done by the rule of block-matrix multiplication. The transpose operation is on block elements and not on the individual matrices comprising those elements.)

RESULTS

Comparisons - Our main purpose was to compare the predictions of the matrix, variational Padé approximant with exact calculations for spherical voids and inclusions.¹⁵ For calculations reported here, our trial wave function had $N = 2$; thus, to find the scattered amplitudes by the Padé method, we had to obtain the first, four Born terms. The calculation of these terms involves complicated, multi-dimensional, numerical integrations. For our

"proof-of-concept" analysis, we chose a simpler path and calculated the required terms in the Born series in the following manner: In the exact calculation of the scattering from spherical inclusions, instead of specifying the inclusion by λ' , μ' , and ρ' (the values of an actual defect), we specified the inclusion as a function of z by

$$\begin{aligned}\lambda(z) &= \lambda + z(\lambda' - \lambda) \\ \mu(z) &= \mu + z(\mu' - \mu) \\ \rho(z) &= \rho + z(\rho' - \rho)\end{aligned}$$

When $z = 1$, we are thus calculating the scattering from the actual defect. Since the inclusion is now a function of z , so are the scattered amplitudes. Also, since the Born series for the scattering is simply a power series about $z = 0$, that is,

$$I(z) = zI^{(1)} + z^2I^{(2)} + \dots$$

where

$$I^{(1)} = \left. \frac{dT}{dz} \right|_{z=0}$$

$$I^{(2)} = \left. \frac{1}{2!} \frac{d^2T}{dz^2} \right|_{z=0}$$

then by computing the scattering from the exact equations as a function of z and numerically differentiating the results, we were able to obtain the first four terms of the Born series.

Fig. 1 The matrix Padé approximant (points) compared to the exact forward scattering (solid curves) for three values of z as a function of ka . From top to bottom the values of z are 1, 0.5, and 0.2.

Besides using z as an expansion parameter, we also use it as a measure of the strength of the scatterer. For $0 < z < 1$, the inclusion, while not corresponding to the actual defect, does correspond to a mathematically realizable inclusion weaker than the actual defect. ($z = 0$ corresponds to no defect.) Thus, by varying z , we are able to study the scattering as a function of the "strength of the scatterer."

In Fig. 1 we show one of our comparisons, the $[2/2]$ Padé approximant versus the exact scattering. The incident wave is a longitudinal plane wave; $\lambda' = 0$ so when $z = 1$, the defect is a void; the host material is Si_3N_4 ; and only forward scattering is plotted.

For $z = 0.2$ and 0.5 the agreement between the approximation and the exact result is striking over a large range of ka , but for $z = 1$ the agreement is restricted to $ka < 1$. We also find that the fourth Born approximation gives only slightly less satisfactory results and that the third Born approximation gives only slightly less satisfactory results than the fourth Born approximation. Except for small ka , or small z , the second Born approximation is inadequate. The first Born approximation is identically zero.

From Fig. 1 and calculations at other values of z , we find that the fidelity of the Padé approximant appears correlated to the apparent convergence of the four term Born series. This observation suggests that if the Born series converges for z

Fig. 2 The exact forward scattering as a function of ka for $z = 0.2, 0.5$ and 1 plus the scattering at $z = 0$. The values of z decrease from top to bottom.

near 1, it probably converges very slowly. Figure 2 also suggests that an understanding of the convergence of the Born series would possibly help in understanding the comparisons in Fig. 1.

In Fig. 2, the exact results for $z = 0.2, 0.5$ and 1 are plotted along with the exact results for $z \pm 0.02$. For $z = .2$ and $.5$, from which 0.02 is a relatively large change, the scattering curves appear simply shifted; however, at $z = 1$, from which 0.02 is a relatively small change, radical changes occur.

Singularity Structure - As a function of z , the Born series converges inside a circle whose radius extends from $z = 0$ to the closest singularity of the scattered amplitude in the complex z -plane. For the Born series of an actual defect to converge, this radius must be larger than or at least equal to one. Thus, a study of the convergence properties of the Born series naturally leads to an investigation of the singularity structure of the scattered amplitude.

This structure is easily found from the exact scattering. The exact scattering is given in terms of partial wave amplitudes where each partial wave amplitude is determined from the solution of a linear system of equations that has the following form:

$$\begin{bmatrix} \underline{X}(m) & \underline{Y}(m) \\ \underline{Z}(m) & (1-z)\underline{W}(m) \end{bmatrix} \begin{bmatrix} \underline{A}(m) \\ \underline{B}(m) \end{bmatrix} = \begin{bmatrix} \underline{a}(m) \\ \underline{b}(m) \end{bmatrix}$$

Here, the $\underline{W}(m)$, $\underline{X}(m)$, $\underline{Y}(m)$, and $\underline{Z}(m)$ are 2×2 matrices with $\underline{X}(m)$ and $\underline{Z}(m)$ dependent on α and β (the longitudinal and transverse wave-numbers of the host material), and $\underline{W}(m)$ and $\underline{Y}(m)$ dependent on $\alpha(z)$ and $\beta(z)$. $\underline{A}(m)$ and $\underline{B}(m)$ are 1×2 matrices representing the partial wave coefficients of the scattered and internal waves; $\underline{a}(m)$ and $\underline{b}(m)$ are 1×2 matrices representing the partial wave coefficients of the incident field. The singularities associated with each partial wave are the values of z that make the determinant, $\Omega(z)$, of this system of equations equal to zero. In general, this determinant is a highly nonlinear function of z ; however, when $\lambda' = \mu' = \rho' = 0$, one finds that $\alpha = \alpha(z)$ and $\beta = \beta(z)$. Consequently, the determinant reduces to a quadratic equation

$$\Omega(z) = (1-z)^2 \Lambda + (1-z)\Theta + \Xi$$

where

$$\Lambda = \det \underline{X}(m) \det \underline{W}(m)$$

$$\Xi = \det \underline{Z}(m) \det \underline{Y}(m)$$

$$\Theta = \Omega(0) - \Lambda - \Xi$$

Complex z -plane

Fig. 3 The singularity structure for $ka = 2$.

Complex z -plane

Fig. 4 The singularity structure for $ka = 8$.

This form means that each partial wave contributes only two simple poles to the singularity structure. In principle, since there are an infinite number of partial waves, the singularity structure has an infinite number of simple poles. In practice, only a finite number ($m > 2ka$) of partial waves contribute significantly to the scattering, and poles from the partial waves with $m > 2ka$ are found to tend to fixed points outside the unit circle.

The singularities for $ka = 2$ and 8 are plotted in Figs. 3 and 4.

Only the right-half, complex z -plane is shown since the exceptional singularity found in the left-half plane was always far from the origin and the real axis. In Fig. 3 the radius of convergence is slightly less than 1. At $ka = 1$ the radius was slightly greater than 1. Presumably, the cross over from a convergent to divergent series occurs at $ka \approx 1.8$, that is, where the Padé approximant begins to disagree with the exact result. Figure 4 is typical of the results for $ka \geq 3$. The radius of convergence is around 0.7. From these figures, one sees that the divergence of the Born series involves more than one critical singularity. Generally, there are several singularities inside the unit circle, several just outside, and many relatively close to the positive real axis.

The N-Point Padé Approximant - By examining the singularity structure, we have found that in general the Born series converges in a region of the complex z -plane that does not include the value of z we need. To get the scattering at this value requires an analytic continuation of the Born series. Padé approximants are a standard method for such con-

tinuation, especially when simple poles are present. Because of the large number of simple poles in the present case, we would need a high order approximant to be able to incorporate all the relevant poles. To do this would require finding a large number of terms in the Born series. Unfortunately, this is impractical.

We were able to perform the analytic continuation by using another type of Padé approximant called the N-point Padé approximant.¹⁰ The $[L/M]$ approximant is still defined as the ratio of a polynomial of degree L and a polynomial of degree M , but now the coefficients in these polynomials are fixed by requiring

$$F^{[L/M]}(z_i) = F(z_i) \quad , \quad i = 1, 2, \dots, L+M+1 = N$$

where the z_i are a set of N points in the analytic domain of $F(z)$. If a sequence of approximants converge, then the correct answer is found. We investigated the "stair-case" sequence, which contains

$[0/1], [1/1], [1/2], \text{etc.}$

This sequence was generated by the Thiele Reciprocal Difference Method.¹⁰

We started our investigation by generating the approximants for 20 points spread uniformly over interval $[0, 0.5]$. In Fig. 5 we show a comparison of the approximation with the exact scattering at $z = 0.6, 0.8$ and 1. Thus, we have extended the agreement from 1.8 to nearly 4.

By keeping the same density of points, we were eventually able to extend the agreement to $ka = 10$ by using the interval $[0, 0.9]$. Therefore, the N-point Padé approximant works; however, in this problem, to work it needs values of the scattering quite close to the point at which it is trying to predict the scattering. This difficulty arises

Fig. 5 The N-point Padé approximant compared to the exact forward scattering for three values of z as a function of ka . From top to bottom the values of z are 1, 0.8 and 0.6. Twenty points uniformly spread over the interval $[0, 0.5]$ were used.

Fig. 6 The N-point Padé approximant compared to the exact forward scattering for three values of z as a function of ka . From top to bottom the values of z are -1, -0.8 and -0.6. Twenty points uniformly spread over the interval $[-0.5, 0]$ were used.

because of the clustering of singularities around $z = 1$. If one were to analytically continue towards $z = -1$, the situation would be quite different because of the absence of singularities around $z = -1$. We illustrate this point in Fig. 6 where we show comparisons for $z = -0.6, -0.8$ and -1 for 20 points selected uniformly from the interval $[-0.5, 0]$. The agreement is excellent.

The investigation we report are only relevant to spherical voids. We were not surprised that its Born series diverged because the void does represent a strong perturbation of the host material. We were surprised that the scattering amplitudes have such a rich singularity structure so that the analytic continuation process requires considerable work. Preliminary work on other naturally occurring defects (WC, Fe, etc.) in Si_3N_4 suggest similar difficulties may exist. This work will be reported elsewhere.

ACKNOWLEDGEMENTS

This work was supported by the Materials Sciences Division of the Office of Energy Research of the Department of Energy.

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