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**The Stability of the Mesh-Cornered
Synthetic Method of Diffusion
Acceleration of the Dot-IV
Transport Code**

John Aull

MASTER

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THE STABILITY OF THE MESH-CORNERED SYNTHETIC METHOD OF DIFFUSION
ACCELERATION OF THE DOT-IV TRANSPORT CODE

John Aull

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ABSTRACT

Three methods for accelerating the discrete ordinates equations using variations of the synthetic method are examined. It is found that the use of difference equations written about the mesh corners, rather than a specific form of the equations, brings stability. These methods are compared with three other methods in test problems covering the stability domain of interest.

I. INTRODUCTION

The mono-energetic steady state Boltzmann transport equation in one dimensional slab geometry with isotropic scattering and source is

$$\mu \frac{\partial}{\partial x} \psi(x, \mu) + \sigma(x) \psi(x, \mu) = \sigma_s(x) \phi_0(x) + Q(x) . \quad (1)$$

It can be solved iteratively using

$$\mu_m \frac{\tilde{\psi}_{mi}^k + 1/2 - \tilde{\psi}_{mi}^k - 1/2}{\Delta x} + \sigma \tilde{\psi}_{mi}^k = \sigma_s \phi_{0i}^{k-1} + Q_i \quad (2)$$

and the diamond difference equation,

$$\tilde{\psi}_{mi}^k = (\tilde{\psi}_{mi}^k + 1/2 + \tilde{\psi}_{mi}^k - 1/2)/2 . \quad (3)$$

Without rebalancing this iteration proceeds as follows:

1. $k = 1$.
2. Equations (2) and (3) are solved for $\{\tilde{\psi}_{mi}^k\}$ based on a knowledge of $\{\phi_{0i}^{k-1}\}$.
3. $\tilde{\phi}_{0i}^k = \sum_{m=1}^M w_m \tilde{\psi}_{mi}^k$.
4. $\phi_{0i}^k = \tilde{\phi}_{0i}^k$.
5. If $\max_i \left| \left[\phi_{0i}^k - \phi_{0i}^{k-1} \right] / \phi_{0i}^{k-1} \right| < \varepsilon$ then convergence has been obtained.
6. $K = K + 1$.
7. Go to step (2).

With rebalancing, the equation in step (4) of the above procedure is replaced by a correction equation which gives an improved scalar flux, ϕ_0^k , based on the unimproved scalar flux, $\tilde{\phi}_0^k$. The three rebalance methods discussed in this work are different versions of the synthetic method.

II. CORRECTION EQUATIONS FOR THE THREE METHODS

Reed¹ investigated the stability of the mesh-centered synthetic correction equation:

$$\begin{aligned}
 & - \frac{D}{(\Delta x)^2} (\phi_{0i+1}^k - 2\phi_{0i}^k + \phi_{0i-1}^k) + \sigma_R \phi_i^k \\
 & = - \frac{D}{(\Delta x)^2} (\tilde{\phi}_{0i+1}^k - 2\tilde{\phi}_{0i}^k + \tilde{\phi}_{0i-1}^k) + \sigma_R \tilde{\phi}_{0i}^k + \sigma_s (\tilde{\phi}_{0i}^k - \phi_{0i}^{k-1})
 \end{aligned}
 \tag{4}$$

where $\sigma_R = \sigma - \sigma_s$ and $D = 1/3\sigma$.

Alcouffe² used a different formulation on the right-hand side along with mesh-cornered differencing obtaining,

$$\begin{aligned}
 & - \frac{D}{(\Delta x)^2} (\phi_{0i}^k + 3/2 - 2\phi_{0i}^k + 1/2 + \phi_{0i}^k - 1/2) \\
 & + \sigma_R (\phi_{0i}^k + 3/2 + 2\phi_{0i}^k + 1/2 + \phi_{0i}^k - 1/2)/4 = \frac{1}{2} (Q_i + Q_{i+1}) \\
 & - \left[\frac{\tilde{\phi}_{1i+1}^k - \tilde{\phi}_{1i}^k}{\Delta x} \right] - \frac{D}{(\Delta x)^2} (\phi_{0i}^k + 3/2 - 2\tilde{\phi}_{0i}^k + 1/2 + \tilde{\phi}_{0i}^k - 1/2)
 \end{aligned}
 \tag{5}$$

for the correction equation where the current, $\tilde{\phi}_{1i}$, is defined using

$$\tilde{\phi}_{1i} = \sum_{m=1}^M \mu_m W_m \tilde{\psi}_{mi} .
 \tag{6}$$

This method is called the source correction scheme.

This author³ helped to develop a method that is similar to the Reed formulation except with mesh-cornered differencing:

$$\begin{aligned}
& - \frac{D}{(\Delta t)^2} (\phi_{0i+3/2} - 2\phi_{0i+1/2} + \phi_{0i-1/2}) \\
& + \sigma_R (\phi_{0i+3/2} + 2\phi_{0i+1/2} + \phi_{0i-1/2})/4 \\
& = - \frac{D}{\Delta x^2} (\tilde{\phi}_{0i+3/2} - 2\tilde{\phi}_{0i+1/2} + \tilde{\phi}_{0i-1/2}) \\
& + \sigma_R (\tilde{\phi}_{0i+3/2} + 2\tilde{\phi}_{0i+1/2} + \tilde{\phi}_{0i-1/2})/4 \\
& + \sigma_S \left[(\tilde{\phi}_{0i} - \phi_{0i}^{k-1}) + (\tilde{\phi}_{0i+1} - \phi_{0i+1}^{k-1}) \right] / 2 . \tag{7}
\end{aligned}$$

It was found that this correction equation can be used consistently when weighted difference methods⁴ are used in the transport calculation in place of Eq. (3).

III. STATEMENT OF THE PROBLEM

Reed and Alcouffe have done stability analyses on the synthetic method using, respectively, Eq. (4) and Eq. (5). Reed¹ found that the condition

$$\frac{(2c - 1)\sigma\Delta x}{4} < \frac{d}{\Delta x} \tag{8}$$

(where the scattering ratio, $c = \sigma_s/\sigma$) is sufficient for convergence for the mesh-centered synthetic method. Numerical testing indicated that this is in fact a necessary condition for convergence with Reed's mesh-centered synthetic method.¹ Alcouffe found that a sufficient convergence criterion for his source correction scheme (SCS) using Eq. (5) is $c < 1$.

The mesh-cornered synthetic method is similar to Reed's method except for the mesh-cornered formulation. It is similar to Alcouffe's SCS except that the right-hand side of Eq. (7) differs from the right side of Eq. (5).

The questions to be answered in this work are the following. Is the mesh-cornered synthetic method as stable as the SCS method or is it only as stable as the mesh-centered synthetic method? If it is stable then why is it stable?

IV. STABILITY ANALYSIS OF THE MESH-CORNERED SYNTHETIC METHOD

In this section it is shown that the mesh-cornered synthetic method is as stable as the SCS method.

Reed¹ showed that if Eqs. (2) and (3) are considered in an infinite slab with zero source then they can be written in the form

$$\tilde{\phi}_0^k = M\phi_0^{k-1} \quad (9)$$

where $\tilde{\phi}_0^k$ and ϕ_0^{k-1} are vectors of infinite dimension of the form $[\dots, \phi_{0i-1}, \phi_{0i}, \phi_{0i+1}, \dots]^T$. By following Reed's analysis it is a straightforward calculation to show that Eq. (9) can be written using vectors of the form $[\dots, \phi_{0i-1/2}, \phi_{0i+1/2}, \phi_{0i+3/2}, \dots]$. Reed¹ has shown that the eigenvectors of the matrix M are

$$\phi_N = \left\{ \left[\dots, \cos \frac{(i-1/2)\pi}{N}, \cos \frac{(i+1/2)\pi}{N}, \cos \frac{(i+3/2)\pi}{N}, \dots \right] \right\}$$

for $N = 1, 2, \dots$. The eigenvalues of M are

$$\lambda_N = c \left\langle \frac{\sigma^2(\Delta x)^2/4(\mu_m)^2}{\sigma^2(\Delta x)^2/4(\mu_m)^2 + \tau_N} \right\rangle \quad (10)$$

where the symbol $\langle \rangle$ denotes a summation over all discrete directions,

$$\langle x_n \rangle = \sum_{m=1}^M w_m x_m, \quad (11)$$

and

$$\tau_N = \frac{1 - \cos \frac{\pi}{N}}{1 + \cos \frac{\pi}{N}}. \quad (12)$$

Alcouffe² showed that his correction equation (5) with zero source can be written as

$$\begin{aligned} & - \frac{D}{(\Delta x)^2} \left(\phi_{0i}^k + 3/2 - 2\phi_{0i}^k + 1/2 + \phi_{0i}^k - 1/2 \right) \\ & + \frac{\sigma_R}{4} \left(\phi_{0i}^k + 3/2 + 2\phi_{0i}^k + 1/2 + \phi_{0i}^k - 1/2 \right) \\ & = \frac{2D}{(\Delta x)^2} \left(\tilde{\phi}_{2i}^k + 3/2 - 2\tilde{\phi}_{2i}^k + 1/2 + \tilde{\phi}_{2i}^k - 1/2 \right) \end{aligned}$$

where

$$\tilde{\phi}_{2i}^k + 1/2 = \sum_{m=1}^M \tilde{\psi}_{mi} + 1/2 W_m P_2(u_m), \text{ and}$$

where $P_2(u_m)$ is the second order Legendre polynomial. In the case of an infinite slab this system of equations can then be written as

$$\phi_0^k = A\tilde{\phi}_2^k. \quad (13)$$

The matrix A has the eigenvectors $\{\phi_N\}$ with eigenvalues θ_N where,

$$\theta_N = \frac{2\eta_N}{\eta_N + 1/2(1 - c)\sigma(1 + \cos \pi/N)} \quad (14)$$

and

$$\eta_N = \frac{2d}{\Delta x^2} (1 - \cos \pi/N) . \quad (15)$$

Alcouffe also showed that

$$\tilde{\phi}_2^k = B \phi_0^{k-1}$$

where the matrix B has eigenvalues

$$B_N = c \left\langle \frac{\sigma^2 (\Delta x)^2 P_2(\mu_m) / 4(\mu_m)^2}{\sigma^2 (\Delta x)^2 / 4(\mu_m)^2 + \tau_N} \right\rangle .$$

Alcouffe then found that $B_N \theta_N < C$ for all $N = 1, 2, \dots$

Equation (7) can be obtained from Eq. (5) by substituting a form of the discrete transport equation (2) along with the diamond difference equation (3). Taking the first order discrete moment of Eq. (2) yields

$$\frac{\tilde{\phi}_{1i}^k + 1/2 - \tilde{\phi}_{1i-1}^k}{\Delta x} + \sigma \tilde{\phi}_{0i}^k = \sigma_s \phi_{0i}^{k-1} + Q_i . \quad (16)$$

Combining Eq. (16) with the same relationship from the $i + 1$ cell, dividing by two and using Eq. (3) gives

$$\frac{\tilde{\phi}_{1i+1}^k - \tilde{\phi}_{1i-1}^k}{\Delta x} + \frac{\sigma}{2} (\tilde{\phi}_{0i}^k + \tilde{\phi}_{0i+1}^k) = \frac{\sigma_s}{2} (\phi_{0i}^{k-1} + \phi_{0i+1}^{k-1}) + \frac{1}{2} (Q_i + Q_{i+1}) . \quad (17)$$

Rearranging Eq. (17) yields

$$\begin{aligned} \frac{\tilde{\phi}_{1i+1}^k - \tilde{\phi}_{1i-1}^k}{\Delta x} &= \left(-\frac{\sigma_R}{2} - \frac{\sigma_s}{2} \right) (\tilde{\phi}_{0i}^k + \tilde{\phi}_{0i+1}^k) + \frac{\sigma_s}{2} (\phi_{0i}^{k-1} + \phi_{0i+1}^{k-1}) \\ &\quad + \frac{1}{2} (Q_i + Q_{i+1}) \end{aligned} \quad (18)$$

which gives

$$\begin{aligned} \frac{\tilde{\phi}_{1i+1}^k - \tilde{\phi}_{1i-1}^k}{\Delta x} = & -\frac{\sigma_R}{4} \left(\phi_{0i} - 1/2 + 2\tilde{\phi}_{0i} + 1/2 + \tilde{\phi}_{0i} + 3/2 \right) \\ & - \frac{\sigma_S}{2} \left[\left(\tilde{\phi}_{0i} - \phi_{0i}^{k-1} \right) + \left(\tilde{\phi}_{0i+1} - \phi_{0i+1}^{k-1} \right) \right] + \frac{1}{2} (Q_i + Q_{i+1}) \end{aligned} \quad (19)$$

when Eq. (3) is used. Substituting Eq. (19) into Eq. (5) yields Eq. (7).

Thus the correction equation (7) for the mesh-cornered synthetic method can be written in the form

$$\phi_0^k = A' \tilde{\phi}_2^k$$

for an infinite slab with zero source where $A' = A$. Thus the mesh-cornered synthetic method is stable for $c < 1$.

V. NUMERICAL RESULTS AND CONCLUSIONS

A series of problems was run to determine the stability limits of the three rebalance methods discussed above.

A smoothing method has been invented by Rhoades which is designed to extend the stability limits of the mesh-centered synthetic (CES) method. With this method the rebalance factors, $\{f_i\}_{i=1,\dots,I}$ are defined by

$$f_i' = \phi_{0i}^k / \tilde{\phi}_{0i}^k \quad (20)$$

where the prime indicates a rebalanced, unsmoothed flux gotten by solving Eq. (4). These mesh-centered rebalance factors are averaged to form mesh-cornered rebalance factors using

$$\hat{f}_i + 1/2 = \frac{1}{2} (\hat{f}_i + \hat{f}_{i+1}) . \quad (21)$$

Then the mesh-cornered rebalance factors are averaged to obtain smoothed mesh-centered rebalance factors using

$$f_i = \frac{1}{2} (\hat{f}_i - 1/2 + \hat{f}_i + 1/2) . \quad (22)$$

Finally, these smoothed rebalance factors are used to find the mesh-centered rebalanced, smoothed flux with

$$\phi_{0i}^k = f_i \tilde{\phi}_{0i}^k . \quad (23)$$

The series of problems was run using the CES method both with and without smoothing. Also the problems were run using the DOT⁴ conventional rebalance method both with and without damping.⁵ The mesh-cornered synthetic (MCS) and Alcouffe's SCS were also used.

The series of problems was run in two-dimensional slab geometry, which is illustrated in Fig. 1.

Fig. 1. X-Y Slab Geometry.

All of the problems have constant mesh spacing with $\Delta x = \Delta y$. The total cross section, $\sigma = 2.0 \text{ cm}^{-1}$. A source is distributed throughout all of the mesh intervals. These P-0 transport calculations were done using an S-6 quadrature set with the DOT code. Diamond differencing was used in the transport calculation.

There were two sets in the series of problems; the C set and the DX set. With the C set σ_s is varied which amounts to varying the scattering ratio since σ is kept constant. In the DX set $\Delta x = \Delta y$ is varied by changing the dimension of the problem, R, as seen in Fig. 1. A description of both the C set and the DX set is in Table 1.

Table 1. A description of the problem series

Problem	σ_s	C	R	Δx	$\frac{(2c - 1)\sigma\Delta x}{4}$	$\frac{d}{\Delta x}$
C2	1.2	0.6	8	1	0.1	0.166667
C3	1.4	0.7	8	1	0.2	0.166667
C4	1.6	0.8	8	1	0.3	0.166667
C5	1.8	0.9	8	1	0.4	0.166667
C6	2.0	1.0	8	1	0.5	0.166667
DX1	2.0	1.0	6.66667	0.833333	0.416667	0.2
DX2	2.0	1.0	4.44444	0.555556	0.277778	0.3
DX3	2.0	1.0	3.33333	0.416667	0.208	0.4
DX4	2.0	1.0	2.66667	0.333333	0.166667	0.5
DX5	2.0	1.0	2.22222	0.277778	0.138889	0.6
DX6	2.0	1.0	16.0	2.0	1.0	0.083333

The number of iterations required for convergence for each of the five methods is listed in Table 2.

The most obvious observation to be made is that Alcouffe's SCS gives identical results to the mesh-cornered synthetic (MCS) method. This supports the theoretical finding that there is no difference in the stability criterion between these two methods when diamond differencing is used in the transport calculation.

Using Reed's stability criterion, inequality (8), it is to be expected that the CES method would not converge for problems C3 through C6 and problem DX1. With the exception of problem C3 this prediction holds true. In problem DX6 the mesh width is equal to 4 mean free

Table 2. Results with the six methods

Problem	C	Δx	Iterations to converge					
			DOT conventional rebalance		Mesh-centered synthetic (CES) methods		Mesh-cornered methods	
			With damping	Without damping	With smoothing	Without smoothing	Synthetic (MCS)	Alcouffe's (SCS)
C2	0.6	1.0	6	8	9	44	7	7
C3	0.7	1.0	6	10	13	19	7	7
C4	0.8	1.0	7	14	14	*	8	8
C5	0.9	1.0	8	*	12	*	9	9
C6	1.0	1.0	12	*	16	*	11	11
DX6	1.0	2.0	16	*	23	*	14	14
DX1	1.0	0.833333	11	*	12	*	10	10
DX2	1.0	0.555556	8	16	12	54	9	9
DX3	1.0	0.416667	7	11	11	16	8	8
DX4	1.0	0.333333	7	10	7	10	8	8
DX5	1.0	0.277778	6	10	8	9	8	8

paths. It is seen that the smoothing procedure is quite effective in expanding the stability limits of the CES method, but the procedure is usually inferior to the cornered methods.

The mesh-cornered methods are superior to DOT conventional rebalance without damping in all cases. However when DOT conventional rebalance is damped it is superior to the mesh-cornered methods in all of the cases studied except the case of $c = 1$ and large mesh spacing.

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