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INTERACTION OF SUPERCONDUCTING VORTICES

Lectures presented at the Canadian Mathematical Society
Summer Research Institute on Gauge Theories

Montreal, Canada

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I. Introduction

An Abelian gauge field minimally coupled to a charged matter field in two dimensions constitutes one of the simplest examples of a system with interesting topological properties. The model is physically relevant: it can be used, as in the Ginzburg-Landau theory, to describe cross-sections of superconductors with translational symmetry along some axis [1]; more recently it has been applied to the physics of elementary particles [2].

A topological quantum number q , related to the boundary values of the fields, characterizes all finite energy configurations. In the sector with $|q|=1$ localized vortex-like solutions of the field equations are known to exist [3]. Multi-vortex solutions with $|q| > 1$ exist in special cases [4]. In general, configurations with many vortices are not in equilibrium.

In these two lectures I shall describe results recently obtained on multi-vortex configurations. After a brief review of the model, I shall first illustrate a numerical analysis, performed by variational methods, of the interaction between two vortices [5]. The study, done in collaboration with Laurence Jacobs, shows that two vortices attract or repel each other according to whether a dimensionless coupling constant λ , characterizing the relative strength of the matter self-coupling versus the gauge coupling, takes a value smaller or greater than one. (This agrees with results previously obtained for asymptotic separations of the vortices [6]). For $\lambda = 1$, in particular, the vortices appear in equilibrium at any separation, hinting to the existence of a much wider class of solutions to the field equations. In the second lecture I shall consider in detail the case $\lambda = 1$, illustrating analytical results which demonstrate that for this special value of the coupling constant solutions with any number of vortices at arbitrary positions do indeed exist [5,7,8,9].

II. Interaction Energy of Two Vortices

The model is defined by the following energy functional:

$$E = \int d^2x \left[\frac{1}{2} (\partial_\mu \phi - ieA_\mu \phi)^2 + \frac{1}{2} F_{ij} F^{ij} + c_A (\phi^2 - c_0^2)^2 \right]. \quad (2.1)$$

A_i stands for the two components of an Abelian gauge potential; $F_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{ij} B$, $B = \frac{1}{2} \epsilon_{ij} F^{ij}$; ϕ is a complex matter field and the integration is over the two-dimensional plane. In the application to superconductivity ϕ is the "gap parameter", describing the density of superconducting pairs, B is the magnitude of the magnetic field oriented orthogonally to the plane and E is the free energy per unit length. In the applications to particle physics ϕ represents a Higgs field, since its vacuum value is different from zero, $\phi_{\text{vacuum}} = c_0$, and is coupled to a gauge potential.

Some of the coupling constants can be absorbed into a rescaling of the fields. Defining

$$x^i = \frac{1}{ec_0} \tilde{x}^i, \quad \phi = c_0 \tilde{\phi}, \quad A_i = c_0 \tilde{A}_i$$

the energy functional takes the form

$$E = \frac{c_0}{e} \int d^2x \left[\frac{1}{2} |(\tilde{\partial}_i - i\tilde{A}_i)\tilde{\phi}|^2 + \frac{1}{2} \tilde{F}_{ij} \tilde{F}^{ij} + 1/8 \lambda^2 (|\tilde{\phi}|^2 - 1)^2 \right] \quad (2.2)$$

$$\text{with } \lambda^2 = \frac{8c_4}{e^2}. \quad (2.3)$$

λ , which measures the relative strengths of the self-coupling of the matter field versus the gauge coupling, is a physically relevant parameter.

It shall prove quite convenient to use a complex notation for the coordinates in the plane and the gauge potentials. We define therefore

$$z = \tilde{x}_1 + i\tilde{x}_2, \quad \bar{z} = \tilde{x}_1 - i\tilde{x}_2, \quad (2.4a)$$

$$\partial \equiv \frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial \tilde{x}_1} - i \frac{\partial}{\partial \tilde{x}_2} \right), \quad \bar{\partial} \equiv \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial \tilde{x}_1} + i \frac{\partial}{\partial \tilde{x}_2} \right), \quad (2.4b)$$

$$A = \frac{1}{2}(\tilde{A}_1 - i\tilde{A}_2), \quad \bar{A} = \frac{1}{2}(\tilde{A}_1 + i\tilde{A}_2). \quad (2.4c)$$

In terms of these variables the energy functional is

$$E = \frac{c_0 \pi}{e} \mathcal{E}, \quad (2.5)$$

with

$$\begin{aligned} \mathcal{E} = & \frac{1}{2\pi} \int d^2z \left[|(\partial - iA)\tilde{\phi}|^2 + |(\bar{\partial} - i\bar{A})\tilde{\phi}|^2 + 2|\bar{\partial}A - \partial\bar{A}|^2 + \right. \\ & \left. + 1/8 \lambda^2 (|\tilde{\phi}|^2 - 1)^2 \right] \end{aligned} \quad (2.6)$$

The tilde over ϕ will be omitted from now on.

Finiteness of the energy demands

$$\lim_{|z| \rightarrow \infty} |\phi| = 1 \quad (2.7)$$

and

$$\lim_{|z| \rightarrow \infty} (\partial - iA)\phi = 0. \quad (2.8)$$

From (2.7) we see that ϕ must approach the value $e^{iX(\theta)}$ as $z \rightarrow \infty$ with fixed argument θ . Also, from continuity,

$$X(\theta + 2\pi) = X(\theta) + 2\pi q, \quad (2.9)$$

where q is an integer. Eq. (2.8) implies that, as $z \rightarrow \infty$, A must tend to a pure gauge form,

$$A = -i\partial \ln \phi + O\left(\frac{1}{|z|}\right) = X + O\left(\frac{1}{|z|}\right). \quad (2.10)$$

Gauss' theorem then gives the total magnetic flux through the plane:

$$\begin{aligned} \phi(B) &= -\frac{i}{e} \int dz d\bar{z} (\partial \bar{A} - \bar{\partial} A) = \frac{1}{e} \lim_{|z| \rightarrow \infty} \oint (A dz + \bar{A} d\bar{z}) = \\ &= \frac{1}{e} \oint dX = \frac{2\pi q}{e}. \end{aligned} \quad (2.11)$$

The field configurations with finite energy are therefore divided into classes, labelled by q . Each class contains all those finite-energy field configurations which can be continuously distorted into each other (hence the denomination of topological quantum number for q) and within each class the total magnetic flux is $\frac{2\pi q}{e}$.

Notice that q is also given by

$$q = \frac{-i}{2\pi} \oint_{\gamma} d \ln \phi, \quad (2.12)$$

where γ is a closed contour enclosing all zeroes of ϕ . Distorting γ with continuity the integral can change only when the contour crosses a zero of ϕ . It follows

$q = n_+ - n_-$, n_+ , n_- being the numbers of zeroes of ϕ where ϕ vanishes as

$z \rightarrow z_i^{(+)}$ or $\bar{z} \rightarrow \bar{z}_i^{(-)}$, respectively. We shall say that the field configuration exhibits vortices at the points $z_i^{(+)}$ and anti-vortices at the points $z_i^{(-)}$. Then q is the difference between the number of vortices and the number of anti-vortices.

Fields which make the energy stationary obey the Euler-Lagrange equations

$$(\partial - iA)(\bar{\partial} - i\bar{A})\phi + (\bar{\partial} - i\bar{A})(\partial - iA)\phi - \frac{1}{2}(\phi\bar{\phi} - 1) = 0, \quad (2.13)$$

$$4\partial\bar{\partial}A - 4\partial^2A - i\bar{\phi}\partial\phi + i\phi\bar{\partial}\bar{\phi} - 2A\phi\bar{\phi} = 0. \quad (2.14)$$

These equations are invariant under rotations of the plane. It is therefore possible to satisfy them with a rotationally symmetric Ansatz of the form

$$\begin{aligned} \phi &= e^{iq\theta} f(r), \\ A &= -(qi/2z)a(r), \\ r &= |z|, \quad f(\infty) = a(\infty) = 1. \end{aligned} \quad (2.15)$$

The boundary behavior of the fields characterizes them as having vorticity q . It follows in particular that ϕ must have q zeroes, which, because of the symmetry, must be at the origin. Hence $f(r) = 0(r^q)$ for $r \rightarrow 0$. Regularity of $A(z, \bar{z})$ demands $a(r) = 0(r^2)$ for $r \rightarrow 0$.

Inserting the Ansatz of Eq. (2.15) into the Euler-Lagrange equations, these reduce to

$$\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \frac{q^2(a-1)^2}{r^2} - \frac{1}{2} \lambda^2 f(f^2 - 1) = 0, \quad (2.16)$$

$$\frac{d^2 a}{dr^2} - \frac{1}{r} \frac{da}{dr} - (a-1)f^2 = 0. \quad (2.17)$$

Equations (2.16) and (2.17) cannot be solved analytically, but it is straightforward to verify that the values $f = a = 1$ are approached exponentially as

$$\begin{aligned} f(r) - 1 &= 0(e^{-\lambda r}), \\ a(r) - 1 &= 0(e^{-r}) \end{aligned} \quad (2.18)$$

for $r \rightarrow \infty$. The coupling constant λ thus specifies the relative rate of spatial decay of the matter field versus that of the electromagnetic field. For intermediate values of r the equations must be solved numerically. Alternatively, an Ansatz, compatible with Eqs. (2.15) and (2.18) and containing further variational parameters, may be inserted into the expression for the energy, which is then minimized. Such a computation has been carried out for $q = 1$ and 2 by Jacobs and Rebbi [5] and I shall describe now briefly the method followed and the results.

The variational Ansatz is obtained by further specializing the functions $f(r)$ and $a(r)$ through an expansion

$$f(r) = 1 + e^{-\lambda r} \sum_{n=0}^N (f_n r^n / n!), \quad (2.19)$$

$$a(r) = 1 + e^{-r} \sum_{n=0}^N (a_n r^n / n!), \quad (2.20)$$

which, in the actual computation, has been truncated at $N = 10$. f_0, a_0, a_1 are set

equal to -1 and f_1 is set equal to $-\lambda$ for vorticity two in order to reproduce the correct behavior at the origin of the fields. The remaining coefficients of the expansion are the variational parameters.

Inserting the Ansatz into the expression for $\mathcal{E}$ produces a polynomial function of the f_n and a_n variables, which will be denoted collectively by v_n . The polynomial is of the fourth order and all the coefficients can be evaluated analytically. A convenient numerical procedure to find a minimum consists in approximating the quartic $\mathcal{E}(v_n)$ with a quadric, tangent to it (or more precisely osculating it) at a definite point $v_n^{(0)}$. The minimum $v_n^{(1)}$ of this quadric is found and taken as an approximation to the true minimum of $\mathcal{E}$. The energy surface is then approximated again by a quadric, but now tangent to it at $v_n^{(1)}$ and the whole procedure iterated. The method converges very fast, undoubtedly also because of good convexity properties of the energy functional, depending on the physical nature of the problem. The values we found for the energies of a single vortex and of two superimposed vortices are reproduced in the following table and in Fig. 1.

λ	$E(\lambda, n=1)$	$E(\lambda, n=2)$	$\Delta E = E(n=2) - 2E(n=1)$
0.5	0.75742	1.39129	-0.12355
0.6	0.81305	1.52627	-0.09983
0.7	0.86440	1.65337	-0.07543
0.8	0.91230	1.77407	-0.05053
0.9	0.95736	1.88936	-0.02536
1.0	1.00000	2.00000	0.00000
1.1	1.04053	2.10655	0.02549
1.2	1.07921	2.20945	0.05103
1.3	1.11625	2.30905	0.07655
1.4	1.15181	2.40567	0.10205
1.5	1.18639	2.49953	0.12675

It is apparent from these numerical results that for $\lambda < 1$ a configuration of two superimposed vortices has a lower energy than one of two vortices asymptotically separated, whereas the opposite is true for $\lambda > 1$. It becomes interesting then to follow the behavior of the energy as a function of the separation of the vortices. This cannot be determined from a solution of the Euler-Lagrange equations, which will not be satisfied unless the energy is at a stationary point, but can be obtained from a variational minimization of $\mathcal{E}$ subject to the constraint that ϕ have two zeroes (vortices) at a fixed separation.

A two-vortex Ansatz is constructed as follows. Let us denote by $f^{(q)}(r)$ and $a^{(q)}(r)$ ($q = 1, 2$) the functions determined numerically in the variational search for solutions with rotational symmetry. We set then

$$\phi(z, \bar{z}) = \left\{ [z^2 - \left(\frac{d}{2}\right)^2] / [\bar{z}^2 - \left(\frac{d}{2}\right)^2] \right\}^{\frac{1}{2}} f(z, \bar{z}), \quad (2.21)$$

$$\begin{aligned} f(z, \bar{z}) = & w f^{(1)} \left[\frac{|z-d|}{2} \right] f^{(1)} \left[\frac{|z+d|}{2} \right] + \\ & + (1-w) \frac{|z^2 - (d/2)^2|}{|z|^2} f^{(2)}(|z|) + \delta f(z, \bar{z}), \end{aligned} \quad (2.22)$$

$$A = w \left\{ -\frac{i}{z\bar{z}-d} a^{(1)} \left[\frac{|z-d|}{2} \right] - \frac{i}{2z+d} a^{(1)} \left[\frac{|z+d|}{2} \right] \right\}$$

Fig. 1

$$- (1 - w) \frac{1}{z} a^{(2)}(|z|) + \delta a(z, \bar{z}). \quad (2.23)$$

An explanation of the various terms in this Ansatz follows. The first factor in the r.h.s. of Eq. (2.21) defines the phase of the matter field ϕ . The phase changes by 4π along any circle enclosing the points $z = \pm \frac{d}{2}$ so that $q = 2$. In a small loop around $z = \frac{d}{2}$ or $z = -\frac{d}{2}$ the phase changes by 2π ; these are then the location

of the vortices and f must vanish there. Notice that expressing ϕ in terms of a fixed phase factor times a real function f is equivalent to a choice of gauge (a generalized unitary gauge). The first terms in the r.h.s. of Eqs. (2.22) and (2.23) (excluding δf and δa) represent a zero order Ansatz, in which two single vortex configurations centered at $z = \frac{d}{2}$ and $z = -\frac{d}{2}$ are mixed with a symmetric two vortex

configuration centered at the origin. A few factors are inserted to guarantee the appropriate behavior of the fields asymptotically and at the vortices; w is a mixing parameter, which is in a sense also a variational parameter, but in the actual computation it has been chosen minimizing $\mathcal{E}$ with $\delta f = \delta a = 0$; w approaches zero as $d \rightarrow 0$ whereas it approaches 1 as $\frac{d}{2}$ increases beyond the range of the vortices.

$\delta f(z, \bar{z})$ and $\delta a(z, \bar{z})$ are the variational terms proper. These functions have been expanded as

$$\delta f(z, \bar{z}) = |z^2 - \left(\frac{d}{2}\right)^2| (\cos \lambda |z|)^{-1} \times \\ \times \sum_{n=0}^N \sum_{m=0}^n f_{mn} \frac{(z\bar{z})^n}{2} \left[\left(\frac{z}{\bar{z}}\right)^m + \left(\frac{\bar{z}}{z}\right)^m \right], \quad (2.24)$$

$$\delta a(z, \bar{z}) = \frac{1}{\cosh |z|} [za^I(z, \bar{z}) + \bar{z}a^{II}(z, \bar{z})] \quad (2.25)$$

$$a^{I(II)} = \sum_{n=0}^N \sum_{m=0}^n a_{mn}^{I(II)} \frac{(z\bar{z})^n}{2} \left[\left(\frac{z}{\bar{z}}\right)^m + \left(\frac{\bar{z}}{z}\right)^m \right] \quad (2.26)$$

The coefficients f_{mn} , a_{mn}^I and a_{mn}^{II} are the variational parameters. Eqs. (2.24) + (2.26), apart from a few factors inserted to reproduce the correct zeroes and asymptotic behavior of the fields, represent expansions into a series of terms $r^{2n} \cos 2m\theta$, compatible with the symmetry of the configuration. In the numerical computations N was set equal to 2, giving a total of 18 variational parameters. A few checks were performed with a larger value for N .

As in the case with rotational symmetry, the energy functional $\mathcal{E}$ is a quartic polynomial in the variational parameters. Once the coefficients of this polynomial are evaluated (because of the more complex configuration of the fields some coefficients must now be determined by numerical integrations), the iterative procedure already described for the rotationally symmetric configuration can be employed again very efficiently and the constrained minimum of $\mathcal{E}$ is found.

The results obtained for the energy of two interacting vortices, for three representative values of λ and a few values of the separation d , are presented in the following table and in Figure 2.

Fig. 2

d	$\lambda = 0.7$	$\lambda = 1.0$	$\lambda = 1.3$
0	1.653	2.000	2.309
1	1.653	2.000	2.308
2	1.656	2.000	2.299
3	1.665	2.000	2.276
4	1.680	2.000	2.254
5	1.696	2.000	2.242
6	1.710	2.000	2.236
7	1.718	2.000	2.234
8	1.723	2.000	2.233
∞	1.728	2.000	2.232

The following Figures (3a : d) illustrate the profiles of the energy density, matter field and magnetic field for $\lambda = 1$ and various separations of the vortices.

Clearly, the interaction between vortices is always attractive for $\lambda = 0.7$ and repulsive for $\lambda = 1.3$. The value $\lambda = 1$ also emerges as a critical value, for which the vortices appear to be in equilibrium at any separation. The point $\lambda = 1$ is indeed of particular physical and mathematical significance. For this critical value of the coupling constant it is possible to derive a lower bound on the energy, additive in the topological quantum number q [7]. The bound is saturated when a set of first order non-linear partial differential equations in the fields A_j and ϕ is satisfied; then, of course, also the second order Euler-Lagrange equations are fulfilled. The numerical analysis indicates that the bound can be saturated with two vortices at arbitrary separations. More recent analytical developments have shown that, always for $\lambda = 1$, solutions with any number of vortices at arbitrary locations in the plane do indeed exist. These developments will be the object of my second lecture. To conclude the present one, let me mention that in superconductors the value $\lambda = 1$ separates material exhibiting superconductivity of the first ($\lambda < 1$) or second ($\lambda > 1$) type. In type I superconductors a magnetic field is expelled to the surface, whereas in type II superconductors it penetrates the material, forming an array of tubes of flux. This behavior is easily understood keeping in mind the tendency of vortices to coalesce ($\lambda < 1$) or separate ($\lambda > 1$) and considering the geometry of the system.

III. Multi-vortex Configurations for $\lambda = 1$

When $\lambda = 1$ it is possible to derive a lower bound on the energy functional [7]. The first term in the r.h.s. of Eq. (2.6) is integrated by parts to express it as the sum of the second term in the same equation plus a remainder, involving the strength of the magnetic field and the density of the matter field:

$$\int d^2z d\bar{z} [(\partial - iA)\phi] [(\bar{\partial} - i\bar{A})\bar{\phi}] = \int d^2z d\bar{z} [(\bar{\partial} - i\bar{A})\bar{\phi}] [(\partial - iA)\bar{\phi}] - i \int d^2z d\bar{z} (\partial\bar{A} - \bar{\partial}A)\phi\bar{\phi}. \quad (3.1)$$

Using Eq. (3.1) the energy functional is rewritten as

$$\mathcal{E} = \frac{1}{\pi} \int d^2z d\bar{z} [|(\partial - iA)\phi|^2 + |\bar{\partial}A - \partial\bar{A}|^2 + \frac{1}{16} (|\phi|^2 - 1)^2 - \frac{i}{2} (\partial\bar{A} - \bar{\partial}A)\phi\bar{\phi}]. \quad (3.2)$$

Adding and subtracting the integral

$$-\frac{i}{2\pi} \int d^2z d\bar{z} (\partial\bar{A} - \bar{\partial}A) = q, \quad (3.3)$$

Fig. 3-a

Fig. 3-b

Fig. 3-c

Fig. 3-d

giving the topological quantum number, the last three terms in the r.h.s. of Eq. (3.2) are combined into a perfect square (it is crucial for this that $\lambda = 1$), arriving finally at the equation

$$\mathcal{E} = \frac{1}{\pi} \int d^2z \{ |(\partial - i\bar{A})\phi|^2 +$$

$$[\frac{1}{4}(|\phi|^2 - 1) - i(\partial\bar{A} - \bar{\partial}A)]^2 \} + q. \quad (3.4)$$

It is apparent now that $\mathcal{E}$ is bounded below by q , the bound being saturated if and only if the equations

$$(\partial - i\bar{A})\phi = 0 \quad (3.5)$$

and

$$\partial\bar{A} - \bar{\partial}A + \frac{i}{4}(|\phi|^2 - 1) = 0 \quad (3.6)$$

are satisfied. (If q is negative, an analogous procedure allows to establish the bound $\mathcal{E} \geq -q$.)

In Reference [4], it was shown that Eqs. (3.5) and (3.6) may be solved in a rotationally symmetric configuration of q superimposed vortices. The numerical results obtained in Reference [4] led to the conjecture that solutions with q vortices at arbitrary positions in the plane do exist. In a most recent development this conjecture was demonstrated true [9]. In between, a study of the deformation problem made by E. Weinberg [8] showed that the solutions of Eqs. (3.5), (3.6), if they exist, belong to a $2q$ parameter class (as one would indeed expect, if the whole arbitrariness is in the location of the vortices). This nice analysis proceeds via the derivation of an Atiyah - Singer index theorem and it is worthwhile considering it in some detail.

Eqs. (3.5) and (3.6) admit as trivial infinitesimal deformation any gauge transformation of the fields. To eliminate these a Coulomb gauge condition is imposed. The complete set of equations (including also the complex conjugate of Eq. (3.5)) becomes then

$$(\bar{\partial} - i\bar{A})\phi = 0, \quad (3.7a)$$

$$(\partial + iA)\bar{\phi} = 0, \quad (3.7b)$$

$$\partial\bar{A} - \bar{\partial}A + \frac{i}{4}(\phi\bar{\phi} - 1) = 0, \quad (3.7c)$$

$$\partial\bar{A} + \bar{\partial}A = 0. \quad (3.7d)$$

We assume now that $\phi, \bar{\phi}, A$ and $\bar{A}$ represent the fields of an actual solution to Eqs (3.7a) - (3.7d) and inquire what infinitesimal changes $\delta\phi, \delta\bar{\phi}, \delta A, \delta\bar{A}$ (if any) would maintain the equations satisfied. Putting together the four variations into a single vector ξ , it is immediate to show that the infinitesimal deformations should satisfy the equation

$$\mathcal{D}\xi = 0, \quad (3.8)$$

$$\mathcal{D} = \begin{pmatrix} \bar{\partial} - i\bar{A} & 0 & 0 & -i\phi \\ 0 & \partial + iA & i\bar{\phi} & 0 \\ \frac{i\phi}{4} & \frac{i\phi}{4} & -\bar{\partial} & \partial \\ 0 & 0 & \bar{\partial} & \partial \end{pmatrix} \quad (3.9)$$

The index I of $\mathcal{D}$ is defined as the number of solutions of Eq. (3.8) less the number of solutions of the equation

$$\mathcal{D}^+ \xi = 0, \quad (3.10)$$

where $\mathcal{D}^+$ is the adjoint of $\mathcal{D}$. Since it is rather easy to prove (from positivity) that Eq. (3.10) has no solutions, the index of $\mathcal{D}$ gives also the number of infinitesimal deformations.

All solutions of Eq. (3.8) or (3.9) are also solutions of the equation

$$\mathcal{D}^+ \mathcal{D} \xi = 0 \quad (3.11)$$

or, respectively,

$$\mathcal{D} \mathcal{D}^+ \xi = 0. \quad (3.12)$$

On the other hand the hermitian operators $\mathcal{D}^+ \mathcal{D}$ and $\mathcal{D} \mathcal{D}^+$ have the same non-vanishing eigenvalues, because $\mathcal{D}^+ \mathcal{D} \xi = c \xi$ with $c \neq 0$ implies $\mathcal{D} \mathcal{D}^+ \mathcal{D} \xi = c \mathcal{D} \xi$ with non-vanishing $\mathcal{D} \xi$. The index may therefore be expressed as a trace,

$$I = \text{Tr} \left(\frac{M^2}{\mathcal{D}^+ \mathcal{D} + M^2} - \frac{M^2}{\mathcal{D} \mathcal{D}^+ + M^2} \right), \quad (3.13)$$

taken both in function and matrix space.

M^2 in Eq. (3.13) is a free parameter, which can be assigned any value. It is then convenient to evaluate I from Eq. (3.13) by letting $M \rightarrow \infty$.

Simple algebra gives

$$\mathcal{D}^+ \mathcal{D} = -\partial \bar{\partial} + \mathcal{L} \quad (3.14)$$

$$\frac{M^2}{\partial\bar{\partial} + M^2} = \frac{M^2}{-\partial\bar{\partial} + M^2} - \frac{M^2}{-\partial\bar{\partial} + M^2} \mathcal{L} \frac{M^2}{-\partial\bar{\partial} + M^2} + \dots \quad (3.15)$$

and inserting into the trace, it is possible to verify by diagrammatic techniques that terms of higher order do not contribute to I in the limit $M \rightarrow \infty$. It then follows

$$I = \lim_{M \rightarrow \infty} M^2 \int d^2x \langle x | (-\partial\bar{\partial} + M^2)^{-1} \text{tr}(-\mathcal{L} + \tilde{\mathcal{L}}) (-\partial\bar{\partial} + M^2)^{-1} | x \rangle, \quad (3.16)$$

where the trace has been resolved into a sum over position eigenfunctions and a trace over vector indices only. But

$$\text{tr}(-\mathcal{L} + \tilde{\mathcal{L}}) = -2i(\partial\bar{A} - \bar{\partial}A) \quad (3.17)$$

and introducing the Fourier representation of $(-\partial\bar{\partial} + M^2)^{-1}$ one finally arrives at

$$I = \lim_{M \rightarrow \infty} \int d^2z d\bar{z} \{ -2i(\partial\bar{A} - \bar{\partial}A) \} \times \int d^2k \frac{M^2}{2\pi^2(k^2 + M^2)^2} = 2q. \quad (3.18)$$

To demonstrate that Eqs. (3.7a) ÷ (3.7d) have arbitrary multivortex solutions, it is convenient to solve the gauge condition (3.7d) expressing A and $\bar{A}$ in terms of a real function:

$$A = i\partial\psi, \quad \bar{A} = -i\bar{\partial}\psi. \quad (3.19)$$

Eqs. (3.7a) and (3.7b) reduce then to the statement that

$$f \equiv e^{-\psi} \phi$$

is an analytic function of z . Indeed, substituting Eq. (3.19) into Eq. (3.7a) we find

$$(\bar{\partial} - \bar{\partial}\psi)\phi = e^{\psi}\bar{\partial}(e^{-\psi}\phi) = 0,$$

$$\text{i.e.} \quad \frac{\partial f}{\partial \bar{z}} = 0, \quad f = f(z).$$

Setting

$$\phi = e^{\psi} f \quad (3.20)$$

the remaining equation becomes

$$\partial\bar{\partial}\psi = \frac{1}{8} (e^{2\psi} f\bar{f} - 1) \quad (3.21)$$

and, with the further substitution

$$\psi = X - \frac{1}{2} \ln f - \frac{1}{2} \ln \bar{f}, \quad (3.22)$$

$$\partial\bar{\partial}\chi = \frac{1}{8}(e^{2\chi} - 1)$$

(3.23)

The boundary conditions for χ follow from Eqs. (3.20), (3.22) and the behavior of ϕ . χ must approach 0 at infinity and diverge as $\ln |z - z_i|$ at the vortices where $f(z)$ has a zero. The singular behavior of χ at the vortices is the clue to the existence of multi-vortex solutions. Indeed, because of the allowed divergences at $z = z_i$, Eq. (3.23) should more properly be written as

$$\partial\bar{\partial}\chi = \frac{1}{8}(e^{2\chi} - 1) + \pi \sum_i \delta^{(2)}(z - z_i, \bar{z} - \bar{z}_i). \quad (3.24)$$

Given the presence of sources in the r.h.s. of Eq. (3.24) the existence of solutions for arbitrary locations of the vortices is not surprising. (Think of the Poisson

equation $\partial\bar{\partial}\chi = \pi \sum_i \delta^{(2)}(z - z_i, \bar{z} - \bar{z}_i)$). Of course, the equation is not linear and so to prove the existence of solutions remains non-trivial. To obtain such a proof it is convenient to express χ as sum of a zero order Ansatz with the correct singularities and a finite correction $\delta\chi$. The equation satisfied by $\delta\chi$ is the Euler-Lagrange equation for the minimization of a reduced energy functional $\delta\mathcal{E}$. Given the good convexity properties of $\delta\mathcal{E}$ and the regular behavior of $\delta\chi$, theorems of functional analysis can be used to establish the existence of a unique minimum. The proof has been recently given by Taubes [9].

In summary, for $\lambda = 1$ the properties of a system of interacting vortices have been completely elucidated. The existence of a lower bound on the energy and of a change of variables by which finite physical fields are expressed in terms of a superpotential with allowed singularities plays a fundamental role in the mathematical analysis of the system. It is interesting to notice that also in other field theories, in spaces of higher dimensionality, the same ingredients allow to prove the existence of non-interacting, localized structures. For $\lambda \neq 1$ the system is not amenable to an analytic solution. However, through numerical and asymptotic results a good understanding of the properties of interacting vortices has been achieved.

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