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Properties of the Longitudinal Equilibrium Distribution in a Storage Ring*

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ABSTRACT

General properties of the longitudinal equilibrium distribution of a simple model storage ring are discussed using a mapping algorithm for means and correlations of a Gaussian distribution function. The maps for synchrotron oscillations, stochastic excitation, radiation damping and a general localized perturbation are calculated analytically. The fixed point of the concatenated maps is used to characterize the equilibrium distribution.

INTRODUCTION

In Ref. [1] a method to investigate the behavior of bunched beams under the influence of localized constant wake forces was introduced. In Ref. [2] this method was extended to resonator type wake forces. In this report we will extend the analysis to the most arbitrary form of a localized interaction.

The single particle map for a localized interaction, in which the longitudinal position does not change, is then given by

$$\begin{aligned}x'_1 &= x_1, \\x'_2 &= x_2 + f(x_1, x_2).\end{aligned}\quad (1)$$

Here we introduce the scaled variables $x_1 = \omega_s \tau / \alpha$ and $x_2 = (E - E_0) / E_0$ where ω_s is the synchrotron frequency, α the momentum compaction factor, E_0 the energy of the reference particle, and τ the distance between an electron and the reference electron. Clearly the longitudinal coordinate x_1 does not change.

The next element in our model storage ring describes radiation and damping

$$\begin{aligned}x''_1 &= x'_1, \\x''_2 &= \xi x'_2 + \sqrt{(1 - \xi^2)} \sigma_0 \tilde{P},\end{aligned}\quad (2)$$

with $\xi = e^{-T_0/\tau_E}$, where T_0/τ_E is the ratio of the revolution time and the damping time, σ_0 is the natural energy spread, and $\tilde{P}$ is a Gaussian white noise defined by $\langle \tilde{P} \rangle = 0$, $\langle \tilde{P}^2 \rangle = 1$.

The last element describes synchrotron oscillations and is given by

$$\begin{pmatrix} x'''_1 \\ x'''_2 \end{pmatrix} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} x''_1 \\ x''_2 \end{pmatrix}, \quad (3)$$

where φ is related to the synchrotron tune ν_s by $\varphi = 2\pi\nu_s$.

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The model storage ring is then defined by the succession of mappings

$$x_1, x_2 \xrightarrow{\text{Map 1}} x'_1, x'_2 \xrightarrow{\text{Rad.}} x''_1, x''_2 \xrightarrow{\text{Osc.}} x'''_1, x'''_2. \quad (4)$$

For this model we will construct the corresponding mappings for the statistical quantities bunch center X_1 , average energy X_2 , squared bunch length σ_{11} , squared energy spread σ_{22} , and correlation σ_{12} defined by

$$\begin{aligned}X_i &= \langle x_i \rangle, \\ \sigma_{ij} &= \langle (x_i - X_i)(x_j - X_j) \rangle.\end{aligned}\quad (5)$$

The acute brackets denote averages with respect to the electron distribution function $\psi(x_1, x_2)$. Here we will use a normalized Gaussian

$$\begin{aligned}\psi(x_1, x_2) &= \frac{1}{2\pi\sqrt{\det \sigma}} \\ &\times \exp\left(-\frac{1}{2} \sum_{i,j=1}^2 (\sigma^{-1})_{ij}(x_i - X_i)(x_j - X_j)\right).\end{aligned}\quad (6)$$

In the next section we will first construct the mappings for the statistical quantities X_i and σ_{ij} . In the following section the full one-turn map defined by Eq. (4) is calculated, and its period-1 fixed point which describes the equilibrium configuration is determined. A discussion of the equilibrium concludes this paper.

MAPPINGS FOR THE STATISTICAL QUANTITIES

The maps for the statistical quantities for the radiation and oscillation part are taken from Ref. [1]. We only quote the results here. First, for the radiation part we have

$$\begin{aligned}X''_1 &= X'_1, & X''_2 &= \xi X'_2; \\ \sigma''_{11} &= \sigma'_{11}, & \sigma''_{12} &= \xi \sigma'_{12}; \\ \sigma''_{22} &= \xi^2 \sigma'_{22} + (1 - \xi^2) \sigma_0^2;\end{aligned}\quad (7)$$

then for the oscillator part

$$\begin{aligned}\begin{pmatrix} X'''_1 \\ X'''_2 \end{pmatrix} &= U \begin{pmatrix} X''_1 \\ X''_2 \end{pmatrix}, \\ \begin{pmatrix} \sigma'''_{11} \sigma'''_{12} \\ \sigma'''_{21} \sigma'''_{22} \end{pmatrix} &= U \begin{pmatrix} \sigma''_{11} \sigma''_{12} \\ \sigma''_{21} \sigma''_{22} \end{pmatrix} U^T,\end{aligned}\quad (8)$$

where U is the matrix appearing in the single particle map for the oscillation part, Eq. (3). The calculation for the map through the interaction f is rather tedious and

MASTER

10

the details are reported in Ref. [2]. Here we only quote the results

$$\begin{aligned}
 X'_1 &= X_1, \\
 X'_2 &= X_2 + R_1(X, \sigma), \\
 \sigma'_{11} &= \sigma_{11}, \\
 \sigma'_{12} &= (1 + R_{22}(X, \sigma))\sigma_{12} + R_{21}(X, \sigma)\sigma_{11}, \\
 \sigma'_{22} &= (1 + 2R_{22}(X, \sigma))\sigma_{22} \\
 &\quad + 2R_{21}(X, \sigma)\sigma_{12} + R_3(X, \sigma).
 \end{aligned} \tag{9}$$

The functions R are given in terms of integrals over the interaction function f . They still depend on the quantities (X, σ) before the interaction and are given by

$$\begin{aligned}
 R_1(X, \sigma) &= \frac{1}{\pi} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 e^{-it_1^2 - t_2^2} f(y_1, y_2), \\
 R_{21}(X, \sigma) &= \frac{1}{\pi} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 e^{-it_1^2 - t_2^2} \frac{\partial f}{\partial y_1}(y_1, y_2), \\
 R_{22}(X, \sigma) &= \frac{1}{\pi} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 e^{-it_1^2 - t_2^2} \frac{\partial f}{\partial y_2}(y_1, y_2), \\
 R_3(X, \sigma) &= \frac{1}{\pi} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 e^{-it_1^2 - t_2^2} f^2(y_1, y_2) \\
 &\quad - R_1^2(X, \sigma).
 \end{aligned} \tag{10}$$

The argument y_1 and y_2 of the functions $f(y_1, y_2)$ are related to (X, σ) by

$$\begin{aligned}
 y_1 &= X_1 + u_{11}\sqrt{2\lambda_1}t_1 + u_{21}\sqrt{2\lambda_2}t_2, \\
 y_2 &= X_2 + u_{12}\sqrt{2\lambda_1}t_1 + u_{22}\sqrt{2\lambda_2}t_2,
 \end{aligned} \tag{11}$$

where λ_i and $(u_i)_j$ are the eigenvalues and eigenvectors of σ_{ij} . They are calculated in Ref. [2].

$$\begin{aligned}
 \hat{X} &= \frac{\sigma_{22} - \sigma_{11}}{2\sigma_{12}}, \quad \hat{Y} = \sqrt{1 + \hat{X}^2}; \\
 y_1 &= \hat{X} + \hat{Y}, \quad y_2 = \hat{X} - \hat{Y}; \\
 \lambda_{1,2} &= \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sigma_{12}\hat{Y}; \\
 u_{11} &= \frac{1}{\sqrt{1 + y_1^2}}, \quad u_{12} = \frac{y_1}{\sqrt{1 + y_1^2}}; \\
 u_{21} &= \frac{1}{\sqrt{1 + y_2^2}}, \quad u_{22} = \frac{y_2}{\sqrt{1 + y_2^2}}.
 \end{aligned} \tag{12}$$

The interpretation of the functions R_i in Eq. (9) is straightforward. R_1 is a generalized loss factor, because it describes the energy exchange (it changes the energy-like variable x_2) in the interaction f . R_{22} describes the damping (or anti-damping) effect of the interaction f , because negative (positive) R_{22} decreases (increases) the magnitude of σ_{12} and σ_{22} in Eq. (9). Note from the third line of Eq. (10) that R_{22} is proportional to the derivative of f with respect to the energy-like variable x_2 . Therefore damping or anti-damping can only come from an energy dependence of the interaction f . From Eq. (9) we see that R_{21} drives the cross term σ_{12} and is responsible for correlations between energy and position inside the bunch. Note that R_{21} is proportional to the derivative of f with respect to the position-like variable x_1 . Therefore an interaction f that distinguishes different parts inside a bunch can produce a correlation between position and energy inside the bunch. Finally, R_3 is a noise term. It can be shown that it is positive definite and increases the energy spread σ_{22} ; i.e., it introduces noise.

THE CONCATENATED MAPPING AND ITS FIXED POINT

In order to calculate the one-turn map for the model storage ring defined by Eq. (4) we will follow Ref. [1] and use $\det \sigma$, $\text{Tr } \sigma$, and σ_{11} as the mapped quantities. This "trick" facilitates the algebra, because $\det \sigma$ and $\text{Tr } \sigma$ are invariant under oscillations and the other maps leave σ_{11} invariant. After a considerable amount of algebra we obtain for the one-turn map

$$\begin{aligned}
 X_1''' &= X_1 \cos \varphi + \xi(X_2 + R_1(X, \sigma)) \sin \varphi, \\
 X_2''' &= -X_1 \sin \varphi + \xi(X_2 + R_1(X, \sigma)) \cos \varphi, \\
 \sigma_{11}''' &= \sigma_{11} \cos^2 \varphi \\
 &\quad + 2\xi[(1 + R_{22})\sigma_{12} + R_{21}\sigma_{11}] \sin \varphi \cos \varphi \\
 &\quad + [\xi^2(1 + 2R_{22})\sigma_{22} + 2\xi^2 R_{21}\sigma_{12} + \xi^2 R_3 \\
 &\quad + (1 - \xi^2)\sigma_0^2] \sin^2 \varphi,
 \end{aligned} \tag{13}$$

$$\begin{aligned}
 \det \sigma''' &= \xi^2[(1 + 2R_{22}) \det \sigma + R_3 \sigma_{11} \\
 &\quad - (R_{21}\sigma_{11} + R_{22}\sigma_{12})^2] + (1 - \xi^2) \sigma_0^2 \sigma_{11}.
 \end{aligned}$$

$$\begin{aligned}
 \text{Tr } \sigma''' &= \text{Tr } \sigma - (1 - \xi^2 - 2\xi^2 R_{22})(\sigma_{22} - \sigma_0^2) \\
 &\quad + 2\xi^2 R_{22}\sigma_0^2 + 2\xi^2 R_{21}\sigma_{12} + \xi^2 R_3.
 \end{aligned}$$

These maps can now be used to investigate the time dependence of the model storage ring as was done in Ref. [2] for a resonator type wake. Here we will investigate the equilibrium configuration. To this end we will calculate the period-1 fixed point of the maps given by Eq. (13). Equating the primed and the triple primed quantities we get the

following implicit set of equations for the equilibrium values X_i^∞ and σ_{ij}^∞ , and obtain

$$\begin{aligned} X_1^\infty &= \frac{\xi}{1+\xi} R_1 \cot \frac{\varphi}{2}, \\ X_2^\infty &= -\frac{\xi}{1+\xi} R_1, \\ \sigma_{11}^\infty &= \sigma_{22}^\infty - 2\sigma_{12}^\infty \cot \varphi, \\ \sigma_{12}^\infty &= -\frac{\xi R_{21}}{1+\xi(1+R_{22})} \sigma_{11}^\infty, \\ \sigma_{22}^\infty &= \sigma_0^2 + \xi^2 \frac{R_3 + 2R_{22}\sigma_0^2 + 2R_{21}\sigma_{12}^\infty}{1-\xi^2(1+2R_{22})}. \end{aligned} \quad (14)$$

In the following section we will discuss some of the interesting features of Eq. (14).

DISCUSSION

The equilibrium energy X_2^∞ in the presence of the interaction is changed proportional to the generalized loss factor R_1 and the position of the bunch center X_1^∞ is shifted accordingly. Note that X_1^∞ and X_2^∞ implicitly depend on the equilibrium values X_i^∞ and σ_{ij}^∞ through $R_1 = R_1(X_i^\infty, \sigma_{ij}^\infty)$. The dependence of the loss factor R_1 on the bunch sizes is therefore taken into account in a self-consistent way.

Furthermore note that the dependence of the equilibrium correlation σ_{12}^∞ is proportional to R_{21} . Eq. (10) shows that R_{21} is proportional to the derivative of the interaction f to the position-like variable x_1 . Consequently, for an

interaction f that affects all particles inside the bunch in the same way cannot introduce a correlation. Moreover, if σ_{22}^∞ is zero we obtain from the third of Eqs. 14 that the bunch length σ_{11}^∞ is proportional to the energy spread σ_{22}^∞ . Another way to state this is: if the interaction f only depends on the energy x_2 , the correlation σ_{12}^∞ vanishes and the bunch length is proportional to the energy spread.

An example for an interaction that treats all particles in a bunch in the same way is an amplifier Free Electron Laser (FEL), where a continuous external laser is passed over the transversely undulating electrons. Therefore the amplifier FEL does not produce a correlation between energy and position in the bunch.

On the other hand, the light in an oscillator FEL acquires a pulsed structure due to a mode-locking mechanism produced by the FEL process itself. The light pulses are typically much shorter than the bunch length and therefore affect only part of the electrons. Consequently a correlation between energy and position is generated.

A further example is the wake interaction in which the leading particles in a bunch affect the trailing. Obviously, this will then introduce a correlation σ_{12}^∞ in the bunch.

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REFERENCES

- [1] K. Hirata, Part. Acc. **22**, 57, 1987.
- [2] V. Ziemann, Ph.D. Thesis, Universität Dortmund, available as DELTA Internal Report 90-03, Universität Dortmund, 1990.