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UCID-18303

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Velocity Distribution in a Pinched Beam

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September 17, 1979

Work performed by the UCLLL under the joint auspices of the U.S. Department of Energy under contract No. W-7405-Eng-48 and DARPA (DoD), ARPA Order No. 3718. Monitored by NSWC under contract No. N60921-79-PO-W0035.

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Abstract

An explicit procedure is given for distributing transverse velocities to simulate a self pinched beam of specified density profile. The flat top, Bennett and Gaussian profiles are worked examples.

Introduction

The conditions of equilibrium for a self pinched beam are simple in their stated general form, but they are tricky to implement in a simulation. One wishes to construct from N simulation particles a given density profile $n(r)$ with transverse velocities distributed such that the system does not change as it advances in z , except for $O(\sqrt{N})$ fluctuations. The number of particles placed in an annulus of thickness Δr is, with n normalized to unity,

$$\Delta N = 2\pi r \Delta r n(r) N \quad . \quad (1)$$

These particles are uniformly distributed in the azimuthal angle (ϕ).

The velocity of a particle is

$$\underline{v} = v \left(\hat{e}_x \cos \phi + \hat{e}_y \sin \phi \right) \quad , \quad (2)$$

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with the angle ϕ uniformly distributed in the range $(0, 2\pi)$ to realize isotropy. The tricky part is distributing v . To do this it is convenient to associate a variable α with the particles, which is distributed uniformly in $(0,1)$. Then v is a function of α and r :

$$v = g(r, \alpha) \quad (3)$$

The problem is to determine the functional form of g .

In a real experiment the velocities are not specified except for their mean. At best the rms radius (R) and emittance (E) (transverse phase area) are known approximately. Mechanical equilibrium is achieved by focussing the beam to a neck at injection with radius selected to balance thermal expansion against pinch force(1):

$$0 = \frac{\partial^2 R}{\partial z^2} = \frac{E^2}{R^3} - \frac{\overline{k_\beta^2} r^2}{R} \quad (4)$$

where $k_\beta(r,z)$ is the betatron frequency and the average is taken over the profile $n(r)$. If this condition is satisfied there will be no gross expansion or oscillation. However, there will be some internal adjustments of profile and velocity distribution.

A simulation allows much more detail to be imposed in the initial conditions, and this is usually desirable because phenomena to be studied may be subtle and could be masked by transient adjustments during the available computer time. Thus one wishes to specify a particular density profile of given radius R which satisfies eq. (4), and the distribution of velocities is such that no adjustments occur - this is equilibrium.

Conditions of Equilibrium

We start with the single particle equations of motion in the transverse (x,y) plane. For the i th particle in the set of N , the transverse coordinates are $\mathbf{r}_i(z) = [x_i(z), y_i(z)]$; these satisfy equations of motion

$$\frac{d \mathbf{r}_i}{dz} = \mathbf{v}_i \quad , \quad (5)$$

$$\frac{d \mathbf{v}_i}{dz} = -k_\rho^2 (\mathbf{r}_i, z) \mathbf{r}_i = \frac{q}{\beta \gamma m c^2} \nabla A(\mathbf{r}_i, z) \quad . \quad (6)$$

Here $\mathbf{v}$ is a dimensionless velocity and $A(\mathbf{r}, z)$ is the magnetic potential. Constants q , m , β and γ are respectively the particle charge, mass and relativistic factors.

For the statistical treatment a distribution function in the transverse phase space is defined; in a thin beam segment let

$$f(\mathbf{r}, \mathbf{v}, z) d^2 r d^2 v = \left(\text{fraction of particles} \right) \quad . \quad (7)$$

Then the Vlasov equation derived from eqs. (5) through (7) is

$$\frac{\partial f}{\partial z} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f + \frac{q}{\beta \gamma m c^2} \nabla A \cdot \nabla_{\mathbf{v}} f = 0 \quad , \quad (8)$$

and the density profile is

$$n = \int d^2v f \quad (9)$$

The magnetic potential is determined from Ampere's law:

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial A}{\partial r} = -\frac{4\pi}{c} J_z = -\frac{4\pi I}{c} n(r) \quad (10)$$

where I is total beam current.

In an equilibrium state the z dependence disappears from f and A , i.e. $\partial/\partial z = 0$ except in the particle eqns. (5) and (6). There are then two independent, isolating integrals of the motion:

$$\mathcal{L} = (\underline{r} \times \underline{v}) \cdot \hat{e}_z \quad (\text{angular momentum}), \quad (11)$$

$$\mathcal{E} = \frac{v^2}{2} - \frac{qA(r)}{\beta\gamma mc^2} \quad (\text{energy}) \quad (12)$$

In equilibrium the function $f(\underline{r}, \underline{v})$ is also an isolating integral and is therefore a function of $\mathcal{L}$ and $\mathcal{E}$. Rotating beams and other anisotropic equilibria are not considered here, hence we do not use $\mathcal{L}$ and have simply

$$f(\underline{r}, \underline{v}) = F(\mathcal{E}) \quad (13)$$

This expression, along with eqs. (9), (10) and (12) formally define the system.

Solution for $F(r)$

If $n(r)$ is specified we may determine a unique $F(r)$. To do this it is first convenient to define a dimensionless potential

$$\psi(r) = -\frac{q A(r)}{\beta \gamma m c^2} \quad , \quad (14)$$

so

$$v(r) = \frac{v^2}{2} + \psi(r) \quad . \quad (15)$$

Eq. (10) is formally integrated to obtain $\psi(r)$ from $n(r)$:

$$\psi(r) = \frac{4\pi I}{c} \frac{q}{\beta \gamma m c^2} \int_0^r \frac{dr'}{r'^2} \int_0^{r'} dr'' n(r'') + \text{constant}. \quad (16)$$

The integration constant may be set equal to zero, and the other parameters combine into a well-known dimensionless quantity

$$\frac{q I}{\beta \gamma m c^3} = \frac{I}{I_{\text{Alfven}}} = \frac{1}{k_p^2 r^2} = \frac{v_{\text{Budker}}}{\gamma} \quad . \quad (17)$$

however eq. (16) may be used to define r as an implicit function of ψ :

$$\psi = \frac{1}{k_p^2 r^2} \int_0^{r(\psi)} \frac{dr'}{r'^2} \int_0^{r'} dr'' n(r'') \quad . \quad (18)$$

Hence ψ may be used as a radial coordinate, and the profile depends on ψ as

$$n = n[r(\psi)] \quad (19)$$

An example (flat profile):

$$n(r) = \frac{H(a-r)}{\pi a^2} \quad (20)$$

where H is the step function. Then

$$\begin{aligned} \psi &= \frac{1}{k_\beta^2 r^2} \int_0^r \frac{dr'}{r'} \int_0^{r'} dr'' r'' 4\pi \frac{H(a-r'')}{a^2} \\ &= \frac{1}{k_\beta^2 r^2} \left(\frac{r^2}{a^2} \right) \quad (0 \leq r \leq a), \end{aligned} \quad (21)$$

and

$$r(\psi) = a \sqrt{\frac{\psi}{\left(\frac{1}{k_\beta^2 r^2} \right)}} \quad (22)$$

The profile $n(r)$ is related to F by eq. (9), which may be cast in the form

$$\begin{aligned} n &= \int_0^\infty d^2 v f = \int_0^\infty dv \, 2\pi v \, F \left(\frac{v^2}{2} + \psi \right) \\ &= 2\pi \int_0^\infty d \left(\frac{v^2}{2} \right) F \left(\frac{v^2}{2} + \psi \right) \end{aligned} \quad (23)$$

A change of variable from $\frac{v^2}{2}$ to ψ gives

$$n = 2\pi \int_{\psi}^{\infty} d\epsilon F(\epsilon) . \quad (24)$$

Differentiating with respect to ψ , we have

$$F(\psi) = - \frac{1}{2\pi} \frac{dn}{d\psi} . \quad (25)$$

Example (continued); using eq. (21) the flat profile [eq. (20)] is

$$n = \frac{H\left(\frac{k_{\beta}^2 r^2}{a^2} - \psi\right)}{\pi a^2} . \quad (26)$$

We find from eq. (25)

$$F(\psi) = \frac{1}{2\pi} \frac{\delta\left(\frac{k_{\beta}^2 r^2}{a^2} - \psi\right)}{\pi a^2} , \quad (27)$$

where δ is the delta function. The distribution function is

$$\begin{aligned} f(r, y) = F(r) &= \frac{\delta\left(\frac{k_{\beta}^2 r^2}{a^2} - \psi\right)}{2\pi^2 a^2} \\ &= \frac{\delta\left(\frac{k_{\beta}^2 r^2}{a^2} - \frac{v^2}{2} - \psi(r)\right)}{2\pi^2 a^2} . \end{aligned} \quad (28)$$

Expression (25) has been known at least since 1971⁽²⁾. However, it has only been useful for analytically soluable cases - such as the flat profile, where $r(\psi)$ can be explicitly written. The important case of Gaussian $n(r)$ is not this simple and for this reason has not been used. In the present work the problem is evaded by leaving $r(\psi)$ implicit. It is not necessary to find $F(\epsilon)$ to evaluate $g(r, \alpha)$.

Distribution of Velocities

We return to the determination of g . At a particular r the distribution of v 's is defined by

$$P(r, v) dv = \left(\begin{array}{l} \text{fraction of particles} \\ \text{with } v \text{ in } dv \end{array} \right) . \quad (29)$$

If a group of particles have v in dv , then they have α in $d\alpha$. Since α is uniformly distributed from 0 to 1 we have

$$P(r, v) dv = d\alpha . \quad (30)$$

Integrating both sides of eq. (30) from zero to an indefinite upper limit gives

$$\alpha = \int_0^v dv' P(r, v') . \quad (31)$$

The function $g(r, v)$ is therefore defined by the expression

$$\alpha = \int_0^{g(r, \alpha)} \frac{dv'}{dv'} P(r, v') . \quad (32)$$

It is apparent (from normalization) that P and F are related as

$$2\pi r dr \, n(r) \, P(r, v) dv = 2\pi r dr \, 2\pi v dv \, F, \quad (33)$$

ie,

$$P = \frac{2\pi v}{n} F. \quad (34)$$

Inserting this expression in eq. (32) gives

$$\alpha = \int_0^{g(r, n)} dv' \frac{2\pi v' F\left(\frac{v'^2}{2} + \psi(r)\right)}{n(r)}. \quad (35)$$

To evaluate this integral we split it into two pieces and use $v'^2/2$ as the variable of integration:

$$\begin{aligned} \alpha = \frac{2\pi}{n} & \left[\int_0^g d\left(\frac{v'^2}{2}\right) F\left(\frac{v'^2}{2} + \psi\right) \right. \\ & \left. + \int_{g^2/2}^\infty d\left(\frac{v'^2}{2}\right) F\left(\frac{v'^2}{2} + \psi\right) \right]. \end{aligned} \quad (36)$$

Comparison with eq.(23) indicates that the first integral equals $n/2\pi$.

For the second integral we change the integration variable to ϵ , obtaining

$$\alpha = 1 + \frac{2\pi}{n} \int_{g^2/2}^\infty d\left(\frac{\epsilon^2}{2} + \psi\right) d\epsilon \, F(\epsilon). \quad (37)$$

Note that ϵ is only a dummy variable in this expression. Thus we let

$\epsilon \rightarrow \psi'$ and substitute from eq.(25) to get

$$\alpha = 1 + \frac{2\pi}{n} \int_{\omega} \left(\frac{g^2}{2} + \psi \right) d\psi \cdot F(\psi)$$

$$= 1 + \frac{2\pi}{n} \int_{\omega} \left(\frac{g^2}{2} + \psi \right) d\psi \cdot \frac{d}{d\psi} \left(-\frac{n}{2\pi} \right) \quad (38)$$

$$= 1 - \frac{n}{n} \left[\frac{r(g^2/2 + \psi)}{r(\psi)} \right] \quad (39)$$

It remains to determine g as a function of $r(g^2/2 + \psi)$. Eq. (18) yields

$$\left(\frac{g^2}{2} + \psi \right) = \frac{1}{k_B^2 r^2} \int_0^r \frac{\left(\frac{g^2}{2} + \psi \right)}{\frac{dr'}{r'}} \int_0^{r'} dr'' r'' 4\pi n(r'') \quad (40)$$

and

$$\psi = \frac{1}{k_B^2 r^2} \int_0^{r(\psi)} \frac{dr'}{r'} \int_0^{r'} dr'' 4\pi n(r'') \quad (41)$$

Subtracting eq. (41) from eq. (40) we have

$$\frac{g^2}{2} = \frac{1}{k_B^2 r^2} \int_{r(\psi)}^r \frac{\left(\frac{g^2}{2} + \psi \right)}{\frac{dr'}{r'}} \int_0^{r'} dr'' r'' 4\pi n(r'') \quad (42)$$

Computational Procedure

To use the results of the previous section we first define the variable

$$s = r \left(\frac{g^2}{2} + \psi \right) \quad (43)$$

Then eqs. (39) and (42) become

$$\alpha = 1 - \frac{n(s)}{n(r)} \quad , \quad (44)$$

$$\frac{g^2}{2} = \frac{1}{k_B^2 r^2} \int_r^s \frac{dr'}{r'} \int_0^{r'} dr'' r'' 4\pi n(r''). \quad (45)$$

It is assumed that the inverse function $\hat{n}$ for $n(r)$ is known; this is defined by the relation

$$n[\hat{n}(r)] = r \quad . \quad (46)$$

The computational procedure is as follows:

- (a) For a particular value of r , evaluate $n(r)$ from the profile formula.
- (b) Select s uniformly from $(0,1)$ and evaluate $n(s)$ from eq. (44).
- (c) Evaluate

$$s = \hat{n}(n(s)) \quad . \quad (47)$$

- (d) Using r and s evaluate $(g^2/2)$ from eq. (45). This may require a numerical integration but there will usually be a tabulated function of r and s .

- (e) The simulation particle is given velocity

$$v = \sqrt{2 \left(\frac{g^2}{2} \right)} \quad (48)$$

Worked Examples

(a) Flat profile:

$$n(r) = \frac{H(a-r)}{\pi a^2} \quad (49)$$

For $s > a$, $n(s) = 0$, and for $s < a$, $n(s) = n(r)$. Eq. (44) yields

$$s > a \leftrightarrow \alpha = 1, \quad (50)$$

$$s < a \leftrightarrow \alpha = 0.$$

Hence, except for $\alpha = 0, 1$ we have

$$s = a. \quad (51)$$

Eq. (45) gives

$$\frac{g^2}{2} = \overline{k_\beta^2 r^2} \int_r^a \frac{dr'}{r'} \int_0^{r'} dr'' r'' 4\pi \cdot \frac{1}{\pi a^2} \quad (52)$$

$$= \overline{k_\beta^2 r^2} \left(1 - \frac{r^2}{a^2} \right). \quad (53)$$

For this case g is independent of α . It is instructive to evaluate the mean of g^2 taken over $n(r)$; we have

$$\begin{aligned} \overline{v^2} &= \overline{g^2} = \int_0^a dr 2\pi r \frac{g^2}{\pi a^2} \\ &= \int_0^a dr \frac{2\pi r}{\pi a^2} 2 \overline{k_B^2 r^2} \left(1 - \frac{r^2}{a^2}\right) = \overline{k_B^2 r^2} . \end{aligned} \quad (55)$$

This is the well known Bennett pinch condition^(1,3) and is independent of profile. Emittance is given by the formula

$$E^2 = R^2 \overline{v^2} = R^2 \overline{k_B^2 r^2} , \quad (56)$$

with $R = a/\sqrt{2}$ for the flat profile. The macroscopic equilibrium condition eq. (4) is thus recovered.

(b) Bennett profile:

$$n(r) = \frac{1/\pi a^2}{\left(1 + r^2/a^2\right)^2} . \quad (57)$$

The inverse function is

$$\tilde{n}(x) = a \sqrt{\left(\frac{1}{\pi a^2 x}\right)^{1/2} - 1} , \quad (58)$$

but we will not actually need it in this case. Eq. (45) gives

$$\begin{aligned} \frac{g^2}{2} &= \overline{k_B^2 r^2} \int_r^s \frac{dr'}{r'} \int_0^{r'} dr'' r'' 4\pi \frac{1/\pi a^2}{\left(1 + r''^2/a^2\right)^2} \\ &= \overline{k_B^2 r^2} \ln \left(\frac{1 + s^2/a^2}{1 + r^2/a^2} \right) . \end{aligned} \quad (59)$$

Note that this may be written

$$\frac{q^2}{2} = \overline{k_B^2 r^2} \ln \sqrt{\frac{n(r)}{n(s)}} \quad (60)$$

Eliminating $n(s)$ with eq. (44) yields

$$\frac{q^2}{2} = \overline{k_B^2 r^2} \ln \left(\frac{1}{\sqrt{1-\alpha}} \right) \quad , \quad (61)$$

and

$$v = g \sqrt{-\overline{k_B^2 r^2} \ln (1-\alpha)} \quad . \quad (62)$$

This formula is employed in the simulation codes BEAMSIM⁽⁴⁾ and RINGBEARER⁽⁵⁾. Note that it is not necessary that the rms radius be finite.

(c) Gaussian profile:

$$n(r) = \frac{e^{-r^2/R^2}}{\pi R^2} \quad . \quad (63)$$

The inverse function is

$$\tilde{n}(x) = R \sqrt{-\ln(\pi R^2 x)} \quad . \quad (64)$$

Substituting from eq. (44) we have

$$s = \tilde{n}[n(s)] = R \sqrt{-\ln \left[\pi R^2 (1 - \alpha) n(r) \right]} \\ = \sqrt{r^2 - R^2 \ln(1 - \alpha)} \quad (65)$$

Eq. (45) gives

$$\frac{q^2}{2} = \frac{1}{k_\beta^2 r^2} \int_r^s \frac{dr'}{r'} \int_0^{r'} dr'' r'' 4\pi \frac{e^{-r''^2/R^2}}{\pi R^2} \quad (66)$$

Changing the integration variable to $u = r'^2/R^2$ yields

$$\frac{q^2}{2} = \frac{1}{k_\beta^2 r^2} \int_{r^2/R^2}^{s^2/R^2} \frac{du}{u} (1 - e^{-u}) \quad (67)$$

The "exponential integral" (a tabulated function) appears; it is defined for $x < 0$ as

$$E_i(x) = \int_{-\infty}^x \frac{dz e^z}{z} \quad (68)$$

The form explicit of g is

$$\frac{g^2}{2} = \overline{k_B^2 r^2} \left[\ln \left(1 - \frac{R^2}{r^2} \ln(1 - \alpha) \right) + E_i \left(\frac{-r^2}{R^2} \right) - E_i \left(\frac{-r^2}{R^2} + \ln(1 - \alpha) \right) \right] \quad (69)$$

Mismatch at Injection

So far we have considered only equilibrium injection. However it is also of interest to simulate a beam which is deliberately mismatched in a controlled manner - for example to study sausage oscillations or hose growth. At injection the current (I), emittance (E) and profile shape are held constant, while R and $\partial R/\partial z$ are varied; this avoids confusion of interpretation since the basic parameters of the beam are fixed while the focusing varies.

Several cases are considered:

(a) Equilibrium value of R , but $\partial R/\partial z \neq 0$ at injection (not a neck):

Eq. (2) is replaced with

$$\underline{v} = \left(\frac{\partial R/\partial z}{R} r \right) \left(\hat{e}_x \cos \theta + \hat{e}_y \sin \theta \right) + v \left(\hat{e}_x \cos \phi + \hat{e}_y \sin \phi \right), \quad (70)$$

where $\underline{v}$ is the total transverse velocity given to the particle. The random part v is distributed as before. Note that emittance is unaffected since by definition

$$E^2 = R^2 \overline{|v|^2} = R^2 \left[\overline{|v|^2} - \left(\frac{\partial R}{\partial z} \right)^2 \right] \quad . \quad (71)$$

(b) Mismatched R:

At injection the equilibrium radius is missed by an amount ϵ :

$$R^2 = (1 + \epsilon)^2 \frac{E^2}{k_\beta^2 r^2} \quad . \quad (72)$$

To preserve E we must change the distribution of random velocities. This is easily done by changing the scale factor of eq. (45):

$$\frac{g^2}{2} = \frac{k_\beta^2 r^2}{(1 + \epsilon)^2} \int_r^S \frac{dr'}{r'} \int_0^{r'} dr'' r'' 4\pi n(r'') \quad . \quad (73)$$

Note that $n(r)$ must also vary as R is changed since it depends on R through its functional form. An additional disturbance of the type (a) may be added without further modifications.

(c) Hose disturbance

This is a net sideways displacement of position and/or velocity, which is imposed after the equilibrium is set up. If the beam is shifted an amount (Y) then this is added to the coordinates r_i of each simulation particle. If there is an initial velocity displacement $(\partial Y / \partial z)$ then one adds it to the velocities in eq.(2):

$$\underline{v} = \left(\frac{\partial Y}{\partial z} \hat{e}_x + \frac{\partial Y}{\partial z} \hat{e}_y \right) + v \left(\hat{e}_x \cos \phi + \hat{e}_y \sin \phi \right). \quad (74)$$

Emittance is not effected.

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