STABILITY OF THE FIELD-REVERSED MIRROR

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The stability of a field reversed mirror plasma configuration is studied with an energy principle derived from the Vlasov equation. Because of finite orbit effects, the stability properties of a field-reversed mirror are different from the stability properties of similar magnetohydrodynamic equilibria. The Vlasov energy principle developed here is applied to a computer simulation of an axisymmetric field-reversed mirror state. It has been possible to prove that the $\ell = 0$ modes, called tearing modes, satisfy a sufficient condition for stability. Precessional modes, with $\ell = 1, 2$, are found to be unstable at low growth rate. This suggests possible turbulent behavior (Bohm confinement) in the experimental devices aiming at field reversal. Techniques for suppressing these instabilities are outlined, and the applicability of the Vlasov energy principle to more complicated equilibrium models is shown.

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Chapter 1. Preliminaries

1.1 Nature of the Field-Reversed Mirror

The field-reversed mirror concept is a refinement of the older Astron¹ device for thermonuclear plasma containment. The system consists of an applied magnetic field and a self-contained plasma current of sufficient strength to reverse the direction of the magnetic field inside the device. This current is generated by injecting a beam of particles off-axis. In the Astron, this beam was first a beam of relativistic electrons, and in later experiments, a beam of MeV-energy protons. In the field-reversed mirror concept, the beam is a beam of neutral deuterium or tritium. Charge exchange and impact ionization then create a rotating layer of ions in the device. This is the fundamental difference between the two devices.

A second difference is more subtle, but equally important. In the Astron configuration, the layer of protons is actually a ring, that is, every proton makes a nearly circular orbit in the device, encircling the axis. Thus there is essentially one orbit configuration for all the particles. A small temperature spread exists for the ion species, and this causes the ring to have finite thickness. On the whole, however, the particles are nearly coherent in their precession about the axis.

In the field-reversed mirror, however, the temperature spread is on the order of the injection energy itself. The injection energy is less than the energy required for the particles to lock into one coherent orbit. The spread in energy results in a very thick ring structure. The plasma thus is of order unity aspect ratio (i.e.,

spherical or ellipsoidal with comparable major and minor axes), of several ion gyroradius in physical size. A picture of this state is shown as Figure 1-1. A table of device parameters, showing differentiation between the field-reversed mirror, Astron, and the field-reversed theta pinch⁶ is shown as Table 1-1.

This difference in ion gyroradius changes the physics a great deal in analysis of the properties of the device. The Astron type concept can be analyzed as a one-dimensional equilibrium with integrable ion dynamics. The field-reversed mirror ion kinetic equations have strong coupling with the electromagnetic equations and thus are quite difficult to handle analytically. Perhaps the best method of analysis is to use a computer simulation of the plasma equilibrium. This sort of computer solution has been available for some time in the form of a 2 1/2-dimensional Darwin electromagnetic code with self-consistent ion kinetics. This code has been developed in various stages since 1969 at the Lawrence Livermore Laboratory² and is known as SUPERLAYER. The code has as adjustable parameters the input current, the number of particles and the beam profile. This freedom can make the code fit the needs of analysis of the actual experiment. This code has been a valuable tool in the present research work.

1.2 Field-Reversed Mirror Experiments

No experiment has to date demonstrated the field-reversed mirror equilibrium.³ There are several experiments, however, which form equilibria of a similar nature. As mentioned earlier, the Astron program has aimed at ring-like ion layers. The early Livermore experiments with relativistic electrons failed to form a field-reversed state. However, in later

Figure 1-1. Magnetic field geometry and plasma current in the Field-Reversed Mirror. (From Ref. 3)

	Reversed Field Mirror	Astron	Reversed Field Theta Pinch
Size of System: $\frac{R}{\rho_i}$	3 - 5	1	10
Size of System: $\frac{L}{\rho_i}$	3 - 10	10 - 30	30
Relativistic Factor: γ	~ 1	10 - 100	~ 1
Neutral Density: $\frac{n_0}{n_+}$	0	10	?
Injection Lifetime: $\omega_c \Delta T$	Experiment 10^4 - Reactor ∞	Experiment 10^2 Reactor 10^2	No injection (pulsed magnetic fields)

Table 1-1. Parameters of various field reversal schemes.
(Note: all symbols are defined in Appendix D.)

work carried on at Cornell University by H. Fleischman⁴ and co-workers, field-reversed equilibria were obtained. Later work by Fleischman as well as by J. Golden et. al. at the Naval Research Laboratory⁵ also has shown reversal in ion rings. In the Cornell and NRL experiments, a substantial density of neutral atoms was necessary to maintain the configuration for several gyro periods. The neutral gas was required for charge neutralization in the experiments. However, a field-reversed mirror is proposed to exist without a neutral background. Thus, measured lifetimes from these experiments are not directly applicable to the field-reversed mirror.

Another set of reversal experiments have been done by Linford⁶ and co-workers at Los Alamos. Here a quartz tube theta pinch apparatus was set up. In this configuration, coils are pulsed with current in the azimuthal direction around the tube. The resulting fast rise in the longitudinal magnetic field strength causes compression of the plasma column. In reaction to the pulsed magnetic field, the plasma sets up a radial electric field in the column. This causes rotation of the entire plasma by the formula

$$\vec{v}_p = c \frac{\vec{E} \times \vec{B}}{B^2}$$

where $\vec{v}_p$ is the drift velocity of any charged species, and $\vec{E}$ and $\vec{B}$ are the vector electric and magnetic field strength. The higher temperature of the ions causes an additional contribution to their drift due to the diamagnetic drift. Thus ions create a higher current than the electrons, and this again reverses the field. The rotation rate can be observed by doppler shift of line radiation from carbon which exists in small amounts in the plasma. The rotation rate is larger than the ion thermal transit time and this causes different behavior from a field-reversed mirror.

However, the device exists in this configuration for many sound transit times experimentally, and this has been used as some justification for interest in the field-reversed mirror.

Finally, one should point to the behavior of the Livermore mirror experiment 2X-IIB as a reversal experiment.³ Here the beam current was increased to a point greatly in excess of the predicted onset of field reversal, but the equilibrium configuration sat at marginal reversal, that is, with nearly zero magnetic field at the center of the device. (Although the magnetic field could not be measured directly, it could be inferred from other diagnostics.) At the same time, the RF signal emanating from the plasma was measured, and it was found to increase as the neutral-beam current was increased: a sign that the plasma was dumping its excess free energy in the form of microinstabilities. The 2X-IIB group attributed this increased level of RF fluctuation in the plasma to the DCLC (Drift-Cyclotron-Loss-Cone) mode of instability. The RF spectrum was visibly peaked around the harmonics of the ion cyclotron frequency at midplane, and this is taken as indication of some kind of turbulent-ion mode.

This failure to achieve reversal in 2X-IIB was taken to be a pessimistic sign for field reversal. However, the experiment could be criticized in that the intermediate state between field reversal and normal mirror operation may be unstable to velocity-space ion microinstability, while perhaps the field-reversed state is stable to these modes. Thus, it is argued, if one could create the field-reversed state on a time scale fast compared to the electron-ion kinetic time scale of the turbulent transport process involved in the DCLC turbulence, one could then easily

maintain the reversed state without these effects persisting. It is thus argued that field reversal can only be achieved with fast, pulsed injection, rather than with the rather long injection time scale of the Berkeley-designed neutral beam source presently used on 2X-IIB.

It may be possible to build a fast, pulsed ion-neutral source (IPINS), and development of such sources is underway at Livermore.⁴⁵ Such sources were not available, however, at the close of experiments on 2X-IIB. It thus remains an open question whether or not 2X-IIB-like configurations could be reversed, or if they could, by brute-force methods, if the equilibrium would persist.

Another attempt at reversal in the 2X-IIB machine will be made in June 1978 at Livermore. The new experiment, known as the Beta II experiment, will focus on reversal using a pulsed injection of plasma with a Marshall gun. It is hoped that the injected plasma, which is cold but of high energy density, will heat the hot ion background, causing reversal by diamagnetic effects.

1.3 Methods of Stability Analysis: Survey

In general the stability of all field-reversed configurations remains an open question. L. D. Pearlstein at Livermore has studied the stability of the Linford experimental configuration.⁷ A number of authors have studied the stability properties of the early Astron configuration.⁸⁻¹¹

The general method of analyzing the stability properties of a plasma involves either solving the equations of motion directly, and showing the existence or non-existence of instability by the roots of the equation, or by an energy approach, whereby the energy of a perturbation applied to the system determines the stability properties.

The applications of these techniques to problems pertinent to field reversal are shown in Table 1-2.

The first method, the direct one of solving the equations of motion, is used when the particle motion is quite simple. An example of this is the discussion of instability in Ref. 10, where the entire system was modeled by one particular orbit of extreme simplicity. In this method, the quadratic equation in frequency resulting from the perturbed equation of motion yields roots in complex frequency for the eigenfunctions for the particle's perturbed motion. In cases where there is a positive imaginary part to the frequency, an instability is said to exist. However, extension of this method to more than one particle orbit is difficult. Furthermore, with this method care must be taken to avoid singularities in the perturbed electromagnetic fields arising from the localization of the current to an infinitesimally small volume.

The idea of perturbing a single orbit is equivalent to the displacement of a perfectly rigid fluid with very small cross-section. In this limit Lovelace¹² has analyzed a field-reversed ion ring of small cross-section (the "bicycle tire" limit) to perturbations, with stability properties equivalent to a one-orbit system.

In the second method of stability analysis, that of developing an energy integral for the perturbations, a wider class of stability problems have been considered. The complexity of the energy principle involved depends upon the characteristics endowed to the plasma in setting up the fundamental equations. The simplest and most well-understood stability analysis by energy principle is the magnetohydrodynamic or MHD model.^{13,14} Here the plasma is assumed to have infinite conductivity, to be isotropic,

AUTHOR	YEAR	RESULT
Bernstein, Frieman Kruskal, Kulsrud	1958 (Ref. 13)	Ideal MHD stability Hydrodynamic limit with scalar or tensor pressure: diagonal terms only
Frieman, Rotenberg	1961 (Ref. 16)	MHD stability with rotation. Scalar pressure
Rosenbluth, Simon	1958 (Ref. 18)	Finite larmor radius stabilization of weakly unstable systems. Tensor pressure: off diagonal terms
Rosenbluth, Rostoker	1961 (Ref. 17)	Derivation of MHD energy principle from Vlasov equation
Marx, Furth	1965 (Refs.9,10)	Precessional instability of rings. One-orbit kinetics
Friedberg	1971 (Ref. 25)	Vlasov-fluid model for $m=1, 2$ instability in long systems
Rosenbluth, Sudan	1976 (Ref. 24)	Vlasov stability using free energy analysis. Long, thin ringlike approximation
Pearlstein, Friedberg	1978 Ref. 7	Finite larmor radius stability using long, thin approximation. Precessional instability

Table 1-2. Outline of low-frequency stability studies
for high-beta systems.

and to have zero equilibrium velocity. (This last assumption is the so-called static limit.) This model allows instability through bulk motion of the plasma, producing compressional waves and magnetoacoustic Alfvén waves that propagate throughout the system. A refinement of this model is the addition of a tensor pressure, by the equations of state of Chu, Goldberger, and Low.¹⁵ Here the pressure takes the form of a diagonal tensor with unequal parallel and transverse components. In systems of the type of field-reversed mirrors, one can show that, with either scalar or tensor pressure, there is an instability present in the magnetohydrodynamic limit. In a field-reversed mirror the magnetic field is curved inward against the plasma everywhere, and Ref. 13 shows that systems of this type are always unstable in the MHD limit.

A further refinement of the MHD model involves dropping the requirement of static equilibrium, i.e. $\vec{u} = 0$. Instead the fluid is allowed to have a nonvarying velocity field $\vec{u}_0$. This is called a stationary equilibrium. In this regard, E. A. Frieman and M. Rotenberg¹⁶ have extended the treatment of Ref. 13 to include nonzero velocity field in the equilibrium. The energy principle was modified, and the general variational principle includes non-hermitian operators, which indicates the possibility of instability with complex frequency (which is called over-stability). This is unlike the instability in the static MHD limit, which always occurs at zero real frequency. Furthermore, the addition of a small velocity field to a static MHD equilibrium is always destabilizing, as is shown in Ref. 16. Thus the field-reversed mirror, unstable in the static MHD limit and with a large rotational velocity due to the finite orbits, is also unstable when modeled as a stationary equilibrium MHD system.

However, both MHD theories have requirements that are not met in the field-reversed mirror. The most important of these is the requirement that the ion gyroradius be small compared to the scale length of the device: $\frac{\rho_i}{L} \rightarrow 0$. MHD theory can be considered as the outcome of the analysis of a collisionless plasma in the small $\frac{\rho_i}{L}$ limit. In fact, MHD theory can be derived from the Vlasov equation in this limit.¹⁷ The approximation acquired in MHD theory comes from decoupling the frequencies for acoustic waves in the plasma from the ion cyclotron frequency. In MHD theory,

$$\epsilon \equiv \frac{\rho_i}{L}$$

$$\omega \sim \epsilon \Omega_{ci} \quad (1-1)$$

Here ρ_i is the ion gyroradius and $L = (\nabla B/B)^{-1}$. The quantity ϵ is treated as an expansion parameter in the theory. As $\epsilon \rightarrow 0$, MHD theory is recovered from the Vlasov equation.

As an attempt to include the effects of finite ion gyroradius in stability theory, a new energy principle was developed in which the expansion of the basic equations included terms of the next order in the expansion parameter ϵ .¹⁸ In order to do this, the instability growth rate in the MHD limit must be assumed to be one order in ϵ lower, thus:

$$\omega \sim \epsilon^2 \Omega_{ci} \quad (1-2)$$

This could perhaps come about in a system where the driving force for the MHD instability was weak, perhaps due to low curvature, for example. In this model the energy principle changes from that in MHD theory, and the effect added is that the plasma has a special kind of elasticity and lack of isotropicity due to Larmor radius effects.^{19,20} Hence this

model is called finite Larmor radius MHD, or FLR theory. The name, however, is a poor choice; the assumptions of the theory still include an ordering of the ion Larmor radius to be small compared to the scale length of the device. The FLR theory is just another way of expanding the Vlasov equation in the small gyroradius limit, brought about by low growth rates, not large gyroradii.

Thus the FLR theory is not applicable to the problem of stability in the field-reversed mirror, not only because of the large ion orbits in the device, but also because the MHD theory growth rate is first order in the expansion parameter, not second order. The system cannot be described as "weakly unstable" in the language of FLR theory.

Despite the limitation of FLR theory to small-gyroradius systems, the introduction of the FLR concept into plasma physics has caused speculation that finite gyroradius stabilizing effects are a general property of magnetically confined plasma. It is for this reason that the field-reversed mirror configuration has been suggested as a possible stable plasma state. While the field-reversed mirror can be shown to be unstable in the MHD limit, the large ion orbits in the device have been conjectured to stabilize the MHD modes. This stabilization would be brought about by coupling between the acoustic frequencies in the device and the oscillation frequencies of the ion orbits. This coupling is a different stabilizing mechanism from that employed in FLR theory.

To investigate the stability of systems with finite orbit kinetics by means of an energy principle, recourse to the Vlasov equations is

required. This is because the plasma properties depend on the orbit properties, which cause the system response to be velocity-space dependent and nonlocal. The plasma in the Vlasov hierarchy is considered to be an ensemble of particles traversing six-dimensional phase space instead of being considered as a homogeneous fluid medium as before.

Two methods of stability analysis through the Vlasov equation have emerged as valuable tools in plasma physics. The first method looks at the plasma from the thermodynamic viewpoint. In an appendix to an early paper by I. B. Bernstein on plasma waves,²¹ W. A. Newcomb proved the non-existence of growing waves in a maxwellian plasma. By showing that all perturbations would increase the free energy $U-TS$ of the plasma, Newcomb reasoned that growing waves were not allowed. The viewpoint adopted was to prove the fitness of the thermodynamic energy principle by showing that in a collisionless plasma energy and entropy were constants of motion, and therefore self-excited waves that did not preserve these constants could not exist. For a maxwellian system, all perturbations increase the free energy, and therefore a maxwellian system is manifestly stable. The thermodynamic reasoning was extended by T. K. Fowler²² to find bounds on instability in non-maxwellian situations. The technique was modified and applied to the case of Astron stability by R. C. Davidson and N. A. Krall,²³ in which a shifted maxwellian distribution function was employed. Finally, the thermodynamic energy principle for ion Astron rings was developed by Sudan and Rosenbluth.²⁴ In their study, the role of the thermodynamic reasoning was to allow the system to have arbitrary perturbations δf in the equilibrium. The free energy change

to all perturbations is minimized, under constraint of some normalization of the perturbation. Restrictions of the perturbations admitted to the system then reduced the minimization equations to the equations for low-frequency stability of the system. However, for this method to apply, one must show that the equilibrium is one which leaves the free energy variational to first order, under the constraint of constant angular momentum. This leads to the choice of a shifted maxwellian as the only possible equilibrium. In order to study a different equilibrium, one must construct another invariant functional of the Vlasov equation which yields the equilibrium equations upon the application of the first variational principle. Thus the energy principle is not easily extensible to other equilibrium states, nor can it be strengthened from a sufficient condition for stability to a necessary and sufficient condition.

There is, however, an alternate viewpoint for the construction of a Vlasov stability theory, which is the viewpoint chosen for the present work. That approach is to treat the Vlasov equation as the state equation for the medium coupling perturbed electromagnetic fields to perturbed electromagnetic sources. The plasma is viewed as a collisionless medium capable of dielectric response. The plasma is not susceptible to arbitrary perturbations in the distribution functions that could come from collisions and the like. Instability is defined as the existence of growing electromagnetic waves possessing electromagnetic consistency. The electrodynamic viewpoint results in an energy principle for the energy of an electromagnetic perturbation of the system. There is a reduction of the degree of

freedom for the perturbation from the thermodynamic viewpoint.

In the electrodynamic viewpoint, Vlasov consistency is built into the system, and the energy principle has minimal solutions that obey Maxwell's equations.

In this framework, J. P. Friedberg²⁵ has investigated the stability of high- β systems (β is the ratio of kinetic pressure to magnetic pressure) using simple geometries and a simple maxwellian distribution function. More recently M. N. Rosenbluth and R. N. Sudan²⁶ have shown that the thermodynamic reasoning results in the electrodynamic energy principle when the perturbations are restricted to collisionless electromagnetic perturbations. The formation of necessary and sufficient conditions for these problems comes about under the assumption of simple geometry²⁷ or by symmetry assumptions in the operators involved.²⁸

In the following work, the spectral dielectric response properties of a field-reversed mirror were developed by the application of an energy principle to the computer simulation of a field-reversed mirror equilibrium using SUPERLAYER. Data from the simulation has been used to test the condition for stability based on the energy principle. (In retrospect, a simple analytic model has been derived which fits this data for simple low-frequency modes.) The numerical approach has made it possible to avoid simplifying the geometry by, for example, an infinite length assumption or by a bicycle-tire ring approximation, as has been necessary in previous analytic studies.^{12,24} The computer code has allowed freedom to choose an arbitrary kinetic equilibrium self-consistent in the framework of Maxwell's equations. The development of the computational technique has made possible a re-analysis of the stability features of a field-reversed mirror in light of the gross

features of the SUPERLAYER code output. It has been possible to outline proposed extensions of this method of stability technique to more complicated systems and also to formulate a strengthening of the stability argument. In light of the results, an analysis is done of a field-reversed mirror as a thermonuclear experiment and possible implications on the extension to a thermonuclear reactor. This is discussed in Ch. 7.

Chapter 2. The Energy Principle

2.1 Outline of Method

The analysis of stability done in this thesis is based on an energy principle for the system. The argument used is as follows: the total energy U of a system must be a constant. Stability is defined as the absence of any perturbations to the equilibrium configuration of the system which grow with time. In principle, perturbations to the equilibrium configuration are always present in the form of thermal noise. These fluctuating perturbations will excite a growing coherent mode if there is instability in the system. In a system completely stable to all modes, thermal fluctuations of a sufficiently low level will not trigger such modes. In a field-reversed mirror with $\sim 10^{17}$ particles, the thermal fluctuations are of low enough level to involve linear response theory to describe susceptibility to growing waves.

As the total energy U of the system is a constant, one can construct a set of trial functions T_n which are perturbations to the equilibrium. If any of these trial functions are allowed, then for these trial functions U remains constant. One may have a situation, however, where the system is known to have some other constant of motion which is changed by the existence of the trial function perturbation to the equilibrium. Thus not all trial functions $\{T_n | \delta_n U = 0\}$ that satisfy the condition of zero perturbed energy are allowed. The absence of any growing modes for which $\delta U = 0$ is thus a sufficient condition for stability but is not a necessary condition for stability. (In special cases this method can be strengthened to yield a necessary condition. This is shown in Ch. 6.)

We further restrict the time functions to be those that have the form

$$|T_n(t)| = |T_n(t_0)|e^{\gamma(t-t_0)}. \quad (2-1)$$

Then only exponentially growing modes are used. (This is not a real restriction for the stability study, since roots to $\delta U = 0$ are the eigensolutions to a second order partial differential equation in space and time. The solutions of these equations can be Fourier analyzed. If growing waves exist, they may be decomposed into components exponential in time. Linear response theory invokes the principle of superposition, and thus also the validity of decomposition.)

Practically, only values of γ of some minimum size are important to determine stability. Since real systems are not stationary, one must limit the size of $1/\gamma$ to a value less than the time for other changes in the system configuration. For example, the field-reversed mirror may be limited in duration by the lifetime of the neutral beam. At the present time, injector lifetimes are limited by the temperature rise of the accelerator cathode. Thus in present-day experiments the neutral beam can persist for only a short time ($\ll 1$ sec). However this could not be true for a reactor, if some method of cooling the accelerator cathode was instituted. More importantly, the scale of growth rates must be less than the ion-ion collision time scale, since the stability depends on the ion orbit properties. This is ~ 1 msec. Thus a practical sufficient condition for stability could be defined as

$$\delta U \neq 0 \nabla T_n: \gamma \geq 10^3 \text{sec}^{-1}. \quad (2-2)$$

As a further point, the linear response theory can be used to couple the electromagnetic equations to the kinetic equations. Perturbations

in the electromagnetic field are taken to be exact. The effect of these perturbations on the kinetics is then obtained in a power series with an expansion parameter such as $\frac{\delta B}{B}$, where B is the magnetic field. Then the calculation for δU will become

$$\delta U = \delta^{(1)}U + \delta^{(2)}U + \delta^{(3)}U + \dots \quad (2-3)$$

Here the superscript indicates the power of $\frac{\delta B}{B}$ occurring in the term. The higher-order terms ($n \geq 3$) contain information concerning nonlinear saturation of the system. The second-order term corresponds to linear exponential stability, as will be shown in Ch. 3. The first-order term will vanish identically as a result of the equilibrium conditions, as will also be shown in Ch. 3. The truncation of the power series in $(\frac{\delta B}{B})$ at order 2 is thus justified if nonlinear effects are not important. It remains to show that such an upper bound on $(\frac{\delta B}{B})$ still leaves ample room for the validity of the hierarchy. The lower bound on $(\frac{\delta B}{B})$ is the thermal fluctuation noise that is present in any system with a finite temperature.

For the domain of ion acoustic waves in a magnetized plasma, the level of electromagnetic fluctuations can be derived from the fluctuation-dissipation theorem of H. Nyquist²⁹ using the dielectric tensor for a maxwellian plasma in a magnetic field.³⁰ The appropriate magnitude of the fluctuating magnetic field, for a bandwidth $\Delta\omega = \omega_{ci}$, where ω_{ci} is the ion cyclotron frequency is:

$$\langle \frac{\delta B_{\text{noise}}^2}{8\pi} \rangle \sim \frac{1}{L} \left(\frac{\omega_{pi}}{c} \right)^2 kT_i \quad (2-4)$$

Here ω_{pi} is the ion plasma frequency, kT_i is the thermal ion energy, and

L is the scale length of the device. Numerically for a field-reversed mirror, $\delta B_{\text{RMS}}^{\text{noise}} = 10^{-5}$ gauss.

Thus the dynamic range for the first order expansion of the Vlasov equation for the ion kinetics in a field-reversed mirror is:

$$\delta B_{\text{RMS}}^{\text{noise}} \ll \delta B \ll B_0 \text{ or } 10^{-5} \text{G} \ll \delta B \ll 10^4 \text{G} . \quad (2-5)$$

This gives roughly 9 orders of magnitude (180 dB) for the validity of the approximation. Alternatively, this could be expressed as the valid time period for linear response approximation as 20 growth times ($\Delta t \leq \frac{20}{\gamma}$).

2.2 Maxwell-Vlasov Hierarchy

The following approximations are used to describe a field-reversed mirror.

- (1) The system is axisymmetric. $\frac{\partial}{\partial \phi} = 0$.
- (2) The electrons are cold. $T_e = 0$.
- (3) There are no DC electric fields. $E_0 = 0$.
- (4) There are no neutrals. $n_n = 0$.
- (5) The system is stationary. $\frac{\partial}{\partial t} = 0$.
- (6) There are no ion-ion collisions. $\tau_{ii} \omega_{ci} \rightarrow \infty$.
- (7) There are many particles in the system. $N_0 \rightarrow \infty$.

The hierarchy of equations developed from these assumptions are the Vlasov-Maxwell equations. Extensions on the theory developed here, based on dropping some of the above assumptions, are discussed in Ch. 7. There are two distribution functions, f_e and f_i for the electrons and ions, respectively.

The electrons, however, are very cold thus they do not contribute to the total current in the equilibrium, since by the drift equation

$$\vec{u}_e = c \frac{\vec{E}_0 \times \vec{B}_0}{B^2} + \frac{\vec{\nabla}(n_e T_e) \times \vec{B}}{q_e n_e B^2} = 0, \quad (2-6)$$

by (3) and (2). Thus the electrons are considered a pressureless fluid. In this approximation, the main contribution of the electrons to the electromagnetic response is to short out any electric field along the magnetic field direction. Since they are practically massless, electrons have infinite parallel mobility.

In summary, the ions species is the only one considered in the Vlasov hierarchy and is the rest of this paper $f_i \equiv f$. The equation governing f is the Vlasov equation

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} + \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right) \cdot \frac{\partial}{\partial \vec{v}} \right) f \equiv Df = 0. \quad (2-7)$$

The fields E and B are related back to this equation as the source terms to Maxwell equations:

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 4\pi (q_i \int f_i d^3v + q_e n_e), \\ \vec{\nabla} \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \\ \vec{\nabla} \cdot \vec{B} &= 0, \\ \vec{\nabla} \times \vec{B} &= \frac{4\pi}{c} (\vec{J}_e + q_i \int d^3v f_i \vec{v}). \end{aligned} \quad (2-8)$$

The total energy U of the system is

$$U = \int d^3x \left\{ \frac{E^2}{8\pi} + \frac{B^2}{8\pi} + \int d^3v \frac{m_i v^2}{2} f_i \right\}. \quad (2-9)$$

Thus U is the total stored energy in the electromagnetic field plus the kinetic energy in ions and electrons.

The equation for δU is then obtained by repeated perturbation of the Vlasov equation by writing

$$\begin{aligned}\vec{E} &= \vec{E}_0 + \delta\vec{E} = \delta\vec{E} \quad , \\ \vec{B} &= \vec{B}_0 + \delta\vec{B}\end{aligned}\quad (2-10)$$

where δE and δB are exact. We now obtain the perturbed Vlasov equation for f_1 . We define the Vlasov operator D as

$$D = D_0 + D_1 \quad (2-11)$$

$$\text{where } D_0 = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} + (\vec{E}_0 + \frac{\vec{v}}{c} \times \vec{B}_0) \cdot \frac{\partial}{\partial \vec{v}}$$

$$\text{and } D_1 = (\delta\vec{E} + \frac{\vec{v}}{c} \times \delta\vec{B}) \cdot \frac{\partial}{\partial \vec{v}} .$$

Then, since

$$Df = 0 \quad , \quad (2-12)$$

we have

$$(D_0 + D_1) f = 0 \quad , \quad (2-13)$$

and the equilibrium equation is

$$D_0 f_0 = 0 \quad . \quad (2-14)$$

So we have an equation with a zero leading term. Let δf be written as an expansion in powers of δE and δB :

$$\delta f = \delta^{(1)} f + \delta^{(2)} f + \dots \quad (2-15)$$

Here $\delta^{(1)} f$ is linear in the perturbed fields, $\delta^{(2)} f$ is quadratic, etc.

Then we can solve $Df = 0$ as

$$D_0 f_0 = 0 \quad , \quad (2-16)$$

$$D_1 \delta f + D_0 \delta f + D_1 f_0 = 0 \quad , \quad (2-17)$$

or more explicitly :

$$(D_0 + D_1) (\delta^{(1)}_f + \delta^{(2)}_f + \dots) = - D_1 f_0 \quad (2-18)$$

Separating terms into powers of δE and δB , we have

$$\begin{aligned} D_0 \delta^{(1)}_f &= - D_1 f_0, \\ D_1 \delta^{(1)}_f + D_0 \delta^{(2)}_f &= 0, \\ D_1 \delta^{(N)}_f + D_0 \delta^{(N+1)}_f &= 0. \end{aligned} \quad (2-19)$$

We can integrate the following series by noting that D_0 is the convective time derivative in the seven-dimensional phase space. Thus Eq. (2-19) has a first integral:

$$\delta^{(1)}_f = \int_{-\infty}^t dt' \left. \frac{q}{m} (\delta \vec{E} + \frac{\vec{v}}{c} \times \delta \vec{B}) \right|_{t'} \cdot \frac{\partial f_0}{\partial \vec{v}} \quad (2-20)$$

Here $\vec{x}$ and $\vec{v}$ are now regarded as lagrangian quantities. They are defined implicitly by the equations of motion:

$$\begin{aligned} \frac{d\vec{x}}{dt} &= \vec{v}, \\ \frac{d\vec{v}}{dt} &= \frac{q}{m} (\vec{E} + \frac{\vec{v}}{c} \times \vec{B}). \end{aligned} \quad (2-21)$$

The energy equation can also be expanded in orders of δE and δB as

$$\delta U = \delta^{(1)}_U + \delta^{(2)}_U + \dots, \quad (2-22)$$

where

$$\begin{aligned} \delta^{(1)}_U &= \int d^3x \left(\frac{\vec{B}_0 \cdot \delta \vec{B}}{4\pi} + \frac{\vec{E}_0 \cdot \delta \vec{E}}{4\pi} + \int \frac{mv^2}{2} \delta^{(1)}_{f_i} dv \right), \\ \delta^{(2)}_U &= \int d^3x \left\{ \frac{\delta B^2}{8\pi} + \frac{\delta E^2}{8\pi} + \int \frac{mv^2}{2} \delta^{(2)}_{f_i} dv \right\}, \end{aligned} \quad (2-23)$$

etc.

We can now show that $\delta^{(1)}U$ is a functional of the equilibrium equations. Since $\vec{E}_0 = 0$, we have:

$$\delta^{(1)}U = \int d^3x \left[\frac{\vec{B}_0 \cdot \delta \vec{B}}{4\pi} - \frac{q}{m} \int d^3v \frac{mv^2}{2} \int dt' \left(\delta \vec{E} + \frac{\vec{v}}{c} \times \delta \vec{B} \right) \cdot \frac{\partial f_0}{\partial \vec{v}} \right]. \quad (2-24)$$

We introduce the vector potential $\delta \vec{A}$ such that:

$$\delta \vec{E} = -\frac{1}{c} \frac{\partial \delta \vec{A}}{\partial t}, \quad \delta \vec{B} = \vec{\nabla} \times \delta \vec{A}. \quad (2-25)$$

Then

$$\delta \vec{E} + \frac{\vec{v}}{c} \times \delta \vec{B} = -\frac{1}{c} \left(\frac{\partial \delta \vec{A}}{\partial t} - \vec{v} \times \vec{\nabla} \times \delta \vec{A} \right). \quad (2-26)$$

This expression can be simplified as

$$\delta \vec{E} + \frac{\vec{v}}{c} \times \delta \vec{B} = -\frac{1}{c} \frac{d \delta \vec{A}}{dt} + \frac{1}{c} \vec{\nabla} \delta \vec{A} \cdot \vec{v}. \quad (2-27)$$

Expression (2-24) can be integrated by parts to yield:

$$\delta^{(1)}U = \int d^3x \left\{ \frac{\vec{B}_0 \cdot \delta \vec{B}}{4\pi} - \frac{q}{c} \int d^3v f_0 \vec{v} \cdot \int dt' \frac{\partial \delta \vec{A}}{\partial t} \right\}. \quad (2-28)$$

The time integration then gives

$$\delta^{(1)}U = \int d^3x \left\{ \frac{\vec{B}_0 \cdot \delta \vec{B}}{4\pi} - \frac{1}{c} \delta \vec{A} \cdot \int d^3v q f_0 \vec{v} \right\}. \quad (2-29)$$

As a functional only of $\delta \vec{A}$, this expression becomes:

$$\delta^{(1)}U = \int d^3x \left\{ \frac{\vec{B}_0 \cdot \vec{\nabla} \times \delta \vec{A}}{4\pi} - \frac{1}{c} \delta \vec{A} \cdot \int d^3v q f_0 \vec{v} \right\}. \quad (2-30)$$

Integrating by parts, we next obtain:

$$\delta^{(1)}U = \int \delta \vec{A} \cdot \left(\frac{\vec{\nabla} \times \vec{B}_0}{4\pi} - \frac{q}{c} \int f_0 \vec{v} d^3v \right). \quad (2-30a)$$

We see that this integral is a functional of Eq. (2-8), the equilibrium Maxwell equation. To first order, however, the electrons are not important, since they do not contribute to the equilibrium current.

At the second order, we have

$$\delta^{(2)}U = \int \left[\frac{\delta B^2}{8\pi} + \frac{\delta E^2}{8\pi} - \frac{q}{c} \int \left(\frac{\vec{v}}{c} \times \delta \vec{B} + \delta \vec{E} \right) \cdot \frac{\partial \delta f}{\partial \vec{v}} \frac{mv^2}{2} dt' dv \right] d^3x. \quad (2-31)$$

Using the same integration by parts technique as before, we have

$$\delta^{(2)}U = \int \left[\frac{\delta B^2}{8\pi} + \frac{\delta E^2}{8\pi} + q \int \vec{v} \cdot \int \delta f \left(\frac{\vec{v}}{c} \times \delta \vec{B} + \delta \vec{E} \right) dt' dv \right] d^3x,$$

or

$$\delta^{(2)}U = \int \left[\frac{\delta B^2}{8\pi} + \frac{\delta E^2}{8\pi} + \int q \delta^{(1)} f \vec{v} \cdot \delta \vec{E} dt' d^3v \right] d^3x. \quad (2-32)$$

We could similarly include the electrons by defining

$$\int q \delta^{(1)} f_{\alpha} \vec{v} d^3v \equiv \delta J_{\alpha}, \quad (2-33)$$

so that the energy principle becomes

$$\delta^{(2)}U = \int \left(\frac{\delta B^2}{8\pi} + \frac{\delta E^2}{8\pi} + \int (\delta \vec{J}_i \cdot \delta \vec{E} + \delta J_e \cdot \delta \vec{E}) dt' \right) d^3x. \quad (2-34)$$

This is just Poynting's theorem. Thus, we have shown that the first order perturbation $\delta^{(1)}U$ vanishes as a result of Maxwell's equations, which are the equations establishing the equilibrium. The second-order variation $\delta^{(2)}U$ represents the energy flux out of the system, which of course corresponds to Poynting's theorem. In the Vlasov framework, the distribution functions are described for all space, and thus the boundary condition is unimportant.

However, the system could be truncated at some surface Σ , and then the energy flux out of the system would be described by Poynting's theorem as

$$\frac{\partial U}{\partial t} = \int_{\Sigma} d^2x \left\{ \frac{\vec{E} \times \vec{B}}{4\pi C} \cdot \hat{n} \right\} . \quad (2-35)$$

For a conducting boundary, $\vec{E} \cdot \hat{n} = 0$, and we will adopt this boundary condition as a restriction on the trial functions in Ch. 5.

The neglect of the electron energy discussed earlier can now be easily proved. From eq. (2-34), one can see that the quantity $\delta \vec{J} \cdot \delta \vec{E}$ appears. This can be evaluated using the equation for cold electrons:

$$nq_e \left\{ \delta \vec{J}_e \times \vec{B} + \delta \vec{E} \right\} = 0 . \quad (2-36)$$

Thus, the perturbed electron current can be dotted in this equation:

$$\delta \vec{J}_e \cdot \left\{ \delta \vec{J}_e \times \vec{B} + nq_e \delta \vec{E} \right\} = 0 . \quad (2-37)$$

This result leads directly to:

$$\delta \vec{J}_e \cdot \delta \vec{E} = 0 . \quad (2-38)$$

Thus the cold electron fluid does not exchange energy with the perturbed field. This observation justifies the shorter energy principle of eq. (2-32).

In summary, the stability of the field-reversed mirror, in the collisionless Vlasov-Maxwell hierarchy, can be described by noting the perturbed energy of the perturbed magnetic and electric field. If the system has no zeroes in the perturbed energy, no instability is possible. This is a "topological" argument for stability theory. If the system can be described with the language of response theory, it is also possible to view Poynting's theorem as the variational principle producing as its

Euler-Lagrange equation the wave equation for perturbed electromagnetic waves. This will be discussed in Ch. 6. The evaluation of eq. (2-3) is the subject of Ch. (4) and Ch. (5), and this gives an analysis of the stability of a particular field-reversed mirror equilibrium. In Ch. (3), the implications of the first-order variational principle (eq. (2-30a)) for the field-reversed mirror are discussed; this gives rise to a computational algorithm that produces, in an electromagnetic sense, an "equilibrium."

Chapter 3. Equilibrium

The equilibrium description of a field-reversed mirror must be described by a self-consistent solution to the time-independent Vlasov-Maxwell set of equations. Since only the ions contribute to the total current, the solution does not involve the equations for the electron fluid. In order to maintain charge neutrality, however, one must first consider if the electrons will destroy the equilibrium. In general, the thermodynamic driving force instilled by collisions will eventually cause kinetic equilibrium between the electrons and ions. However, the time scale for this process is roughly the electron-ion equipartition time τ_{ie} derived from binary collision (Spitzer) theory.³¹ This time scale is

$$\tau_{ie} \sim \frac{3.9 \cdot 10^{13}}{n_e \ln \Lambda} T_e^{3/2} \sim 100 \text{ msec} , \quad (3-1)$$

for typical densities and temperatures of a field-reversed mirror. This time scale is very long compared to typical operating times and thus one may more correctly analyze the device as a collisionless plasma. (The ions, as a separate species, have a faster self-collision time. However, the ions as a whole will conserve angular momentum on the shorter time scale and this cannot cancel the total current except by collisions with the electrons.) We can regard the electrons as a maxwellian plasma with $T_e \rightarrow 0$. Then a small electrostatic field can effectively cancel the ion charge density, since

$$n_e \sim n_0 e^{-\frac{q\phi(r)}{kT_e}} . \quad (3-2)$$

Thus a very small value of the electrostatic potential ϕ will enforce charge neutrality.

Solving the Vlasov equation is in general a difficult process. There are, however, special cases which often arise when the symmetries of the problem reduce the number of dimensions of the solution. In an axisymmetric time-independent system there are two canonical invariants for each particle. The first, corresponding to axisymmetry, is the total angular momentum P_θ :

$$P_\theta = mRv_\theta + \frac{q}{c} RA_\theta \quad (3-3)$$

Here θ is the symmetry (azimuthal) direction and R is the cylindrical distance from the axis of symmetry. $\vec{A}$ is the vector potential such that $\vec{\nabla} \times \vec{A} = \vec{B}$.

The second constant, corresponding to time independence, is the hamiltonian H :

$$H = \frac{1}{2} mv^2 + q\phi \quad (3-4)$$

Here $\phi \rightarrow 0$ so that the hamiltonian is just the kinetic energy.

While they do not represent the total manifold of solutions, all functions of H and P_θ are solutions to the Vlasov equation. Proof of this follows from:

$$Df(H, P_\theta) = \frac{\partial f}{\partial H} DH + \frac{\partial f}{\partial P_\theta} DP_\theta = 0 \quad (3-5)$$

Specifically, the so-called "rigid rotor" distribution often arises in an equilibrium Vlasov system. This has the form

$$f_0 = A \exp \left\{ - \frac{(H - \Omega P_\theta)}{T} \right\} \quad (3-6)$$

where Ω and T are constants.

Alternatively one can form a self-consistent Vlasov solution by computer simulation. Here one integrates the equations of motion

$$\frac{d\vec{v}}{dt} = \frac{q}{m} (\vec{E} + \frac{\vec{v}}{c} \times \vec{B}), \quad \frac{d\vec{x}}{dt} = \vec{v} \quad (3-7)$$

by finite-step integration methods. Self-consistency with the electromagnetic field is obtained by forming the magnetic field from the vector potential which is coupled by a second-order equation to the total current of the particles:

$$\Delta^* \psi = \frac{4\pi}{c} J_\theta \text{ where } \psi = RA_\theta.$$

Then the fields $\vec{E}$ and $\vec{B}$ are calculated from

$$\vec{B} = \vec{\nabla} \times \left(\hat{\theta} \frac{\psi}{R} \right), \quad \vec{E}_\theta = \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{\psi}{R} \right). \quad (3-8)$$

These relations form an integration scheme which, if it converges, will represent an equilibrium solution to the Vlasov-Maxwell equations with

$$\Delta^* \psi = \frac{4\pi}{c} \int f v_\theta d^3v = \frac{4\pi}{c} \sum_{i=1}^N w_i v_{\theta i}, \quad (3-9)$$

where w_i is the charge given to the i_{th} particle.

In the Superlayer code, up to $\sim 10^4$ particles can be injected into a system of electromagnetic fields. The fields are contained as a two-dimensional (65 x 129) array. As the particles are injected over many gyroperiods, they are "packed" into different parts of the phase space by the electric field E_θ . Thus there is a broad dependence on energy and momentarily the particles progress self-consistently and are built up in the system by injections. In fact, one finds that, for a broad injection profile, the distribution is a shifted maxwellian rigid-rotor of form

$$f_0 \sim \exp \left[\frac{-(H - \Omega P_\theta)}{T} \right]. \quad (3-10)$$

Although there have been no collisions, the particles form this state of low free energy due to the electromagnetic self-interactions, which operate in a way similar to a collision mechanism.

This collisionless maxwellian randomization process has been reported to happen often in beam injection problems and the rigid-rotor

profile is often seen to develop in computer simulations.³³ The generalization of Liouville's theorem, known as Boltzmann's H theorem, shows that the phase space in an interactive system never decreases in entropy. Thus a beam, even when interacting only through the self-magnetic field, will be driven to optimally match the phase area available.³⁴ Since the system must conserve angular momentum as a whole, the highest entropy is realized with the rigid-rotor profile. Thus the appearance of rigid-rotor equilibria in the SUPERLAYER code runs is not surprising. It is, however, important to point out that a rigid-rotor profile does not have to come about in a collisionless computer simulation, since coupling through the self-magnetic field is not the most complete source of plasma interaction. It also is important to see that the stability technique is not tied to the use of a rigid-rotor profile for the equilibrium. The equilibrium requires only a self-consistent magnetic field. In fact, there are other SUPERLAYER runs showing marked nonrigid-rotor behavior, and these in principle could be used in the stability analysis developed. However the rigid-rotor equilibrium present in SUPERLAYER runs with a broad injection profile would seem to be the state most likely to avoid instability. Also, the other equilibria were similar in gross features (such as rotation rates and reversal factors) and one would not expect the stability properties to be markedly dependent on the rigid or nonrigid character of the equilibrium.

Chapter 4. Rigid-Rotor Energy Principle

For the rigid-rotor distribution, we can derive a simple form of the Poynting theorem that will give a sufficient condition for stability of the field-reversed configuration.

Recalling that

$$\delta^{(2)}U = \int d^3x \left\{ \frac{\delta B^2}{8\pi} + \int_{-\infty}^t \delta J \cdot \delta \vec{E}(t') dt' \right\} , \quad (4-1)$$

we can write

$$\delta J = \int \delta^{(1)} f_q \vec{v} d^3v . \quad (4-2)$$

Then, using the linearized Vlasov equation, we obtain

$$\delta f = - \frac{q}{m} \int dt' \left(\delta \vec{E} + \frac{\vec{v}}{c} \times \delta \vec{B} \right) \cdot \frac{\partial f_0}{\partial \vec{v}} . \quad (4-3)$$

For a rigid rotor $f_0 \sim \exp\left[-\frac{H-\Omega P}{T}\theta\right]$, so that

$$\frac{\partial f_0}{\partial \vec{v}} = - \frac{f_0}{Tm} (\vec{v} - \Omega R \hat{\theta}) . \quad (4-4)$$

Then δf becomes, using $\delta \vec{E} = - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$,

$$\delta f = + \frac{f_0}{T} q \int \vec{v} \cdot \delta \vec{E} dt' - \frac{\Omega f_0}{T} \int R \hat{\theta} \cdot \left(\delta \vec{E} + \frac{\vec{v}}{c} \times \delta \vec{B} \right) dt' . \quad (4-5)$$

Now we introduce the vector potential and complex notation letting

$$\delta \vec{A} = \underline{\delta A} e^{-i\omega t} e^{i\ell\theta} + \underline{\delta A}^* e^{i\omega t} e^{-i\ell\theta} . \quad (4-6)$$

Then, the electric field becomes

$$\delta \vec{E} = \frac{i\omega}{c} \underline{\delta A} e^{i\ell\theta} e^{-i\omega t} + \text{c.c.} \quad (4-7)$$

where c.c. means complex conjugate.

We also define

$$g \equiv \int_{-\infty}^t dt' \vec{v} \cdot \underline{\delta A} e^{i\ell\theta} e^{-i\omega t} . \quad (4-8)$$

Then we use the notion of the convective derivative to calculate

$$\begin{aligned} \delta \vec{E} + \frac{\vec{v}}{c} \times \delta \vec{B} &= -\frac{1}{c} \frac{\partial \delta \vec{A}}{\partial t} + \vec{v} \delta \vec{A} \cdot \vec{v} - \vec{v} \cdot \vec{v} \delta \vec{A} \\ &= -\frac{1}{c} \frac{d\delta \vec{A}}{dt} + \vec{v} \delta \vec{A} \cdot \vec{v} . \end{aligned} \quad (4-9)$$

Note that $R\hat{\theta} \cdot \vec{v} = R \cdot \left(\frac{1}{R} \frac{\partial}{\partial \theta}\right) = \frac{\partial}{\partial \theta} = i\ell$.

Then δf becomes

$$\delta f = \frac{f_0}{T} q \left[i(\omega - \ell\Omega)g + \Omega R \delta A_\theta \right] + c. c. . \quad (4-10)$$

Then we have, noting that $\vec{v} \cdot \delta \vec{E} = \frac{i\omega}{c} \frac{dg}{dt} + c. c. :$

$$\begin{aligned} \int d^3v \int \delta f q \vec{v} \cdot \delta \vec{E} dt' &= \int \delta \vec{J} \cdot \delta \vec{E} dt' = \\ &= \int d^3v \frac{f_0}{T} \int \left(\frac{i\omega}{c} \frac{dg}{dt} + c. c. \right) \frac{q^2}{c} \left(i(\omega - \ell\Omega)g + \Omega R \delta A_\theta \right) dt + c. c. . \end{aligned} \quad (4-11)$$

Cross terms like $\frac{dg}{dt} \cdot g$ and $\frac{dg^*}{dt} \cdot g^*$ are eliminated by the integration over space $\int d\theta$. We are left with, after integration over all space

$$\begin{aligned} &\int \delta \vec{J} \cdot \delta \vec{E} dt' d^3x \\ &= \int d^3x \int d^3v \frac{q^2}{c^2} \frac{f_0}{T} \left\{ \begin{aligned} &(-\omega\omega^* + \omega_0 \ell\Omega) g^*g + \Omega^2 R^2 f_0 \delta A_\theta^* \delta A_\theta \\ &+ i\gamma\omega\Omega \int dt' \left(\frac{g^* dg}{dt} - g \frac{dg^*}{dt} \right) \end{aligned} \right\} . \end{aligned} \quad (4-12)$$

Here we have chosen $\omega = \omega_0 + i\gamma$ where ω_0 and γ are real. The energy principle then becomes

$$\delta^{(2)}_U = \int d^3x \left(\frac{\delta B^2}{4\pi} - \frac{nq^2}{c^2} \left(\frac{m\Omega^2 R^2}{T} \right) \delta A_\theta^* \cdot \delta A_\theta \right)$$

$$+ \frac{q^2}{c^2} \left[\frac{(\omega\omega^* - \omega_0^2 \ell \Omega)}{T} \langle g^* g \rangle + \frac{i\gamma \ell \Omega}{T} \left\langle \int g^* \frac{dg}{dt} - g \frac{dg^*}{dt} \right\rangle \right]. \quad (4-13)$$

Here $\langle \rangle$ stands for $\int f_0 d^3v$. This result for $\delta^{(2)}U$ is particularly simple to evaluate. It is a functional only of the spatially dependent mode $\delta \vec{A}$ and of the orbit integral for each particle g . The velocity space integration is an unweighted ensemble average. This results because of the uniform population of phase space with the shifted Maxwellian velocity profile.

One should draw attention to special cases of this expression. First, for very high frequencies, $\omega_0 \gg \ell \Omega$, $\omega_0 \gg \gamma$ (frequencies large compared to the acoustical times of the system),

$$\delta^{(2)}U \rightarrow \infty \text{ as } \omega_0 \rightarrow \infty.$$

This indicates that there is high frequency stability inherent in the exponential rigid rotor distribution.

Secondly, for axisymmetric modes $\ell = 0$. Then the energy principle is greatly reduced in complexity. It becomes

$$\begin{aligned} \delta^{(2)}U = \int d^3x \left\{ \frac{\delta B^2}{4\pi} - \frac{q^2}{c^2} \frac{m\Omega^2 R^2}{T} n \delta A_\theta^2 \right. \\ \left. + \frac{\omega\omega^*}{T} \langle g^* g \rangle - \frac{q^2}{c^2} \right\}. \end{aligned} \quad (4-14)$$

The third term on the right-hand side is positive definite and is the only term that depends on frequency. Therefore the minimum of $\delta^{(2)}U$ for all modes occurs at zero frequency and zero growth rate. Then a sufficient condition for the stability to axisymmetric ($\ell = 0$) modes is that (multiplying by 4π and using $\omega_p^2 = 4\pi n g^2 m^{-1}$):

$$\int (\delta B^2 - (\frac{\omega_p}{c})^2 \frac{m\Omega^2 R^2}{T} \delta A_\theta^2) d^3x > 0. \quad (4-15)$$

Thirdly, one can deduce the behavior at high mode numbers ($\ell \rightarrow \infty$) by using the asymptotic form of the orbit integral as $\ell \Omega \gg \omega_0$.

Here we have

$$g = \int \vec{v} \cdot \delta \vec{A} \approx \int \vec{v}_0 \cdot \delta \vec{A}_0 e^{i\ell\Omega t} \sim \frac{v_0 \delta A_0}{i\ell\Omega} e^{i\ell\Omega t}, \quad (4-16)$$

so that

$$i\gamma\ell\Omega \int g^* \frac{dg}{dt} - g \frac{dg^*}{dt} = i\gamma\ell\Omega \int \frac{|v_0 \delta A_0|^2}{i\ell\Omega} dt = \gamma \int |v_0 \delta A_0|^2 dt > 0.$$

$$\text{And} \quad (\omega_* - \ell\Omega\omega_0) g^* g \sim \frac{1}{(\ell\Omega)} v_0 \delta A_0 \rightarrow 0. \quad (4-17)$$

Then, we find

$$\delta^{(2)}U(\ell\Omega \rightarrow \infty) > \min \delta^{(2)}U(\ell=0) = \int (\delta B^2 - (\frac{\omega_p}{c})^2 \frac{m\Omega^2 R^2}{T} \delta A_\theta^2) d^3x. \quad (4-18)$$

Thus instability at very high mode numbers is not possible if the equilibrium is stable to all $\ell=0$ modes.

This leaves the question of stability to intermediate mode numbers at frequencies ω_0 on the order of the drift frequency $\ell\Omega$. These modes may or may not present the minimum to $\delta^{(2)}U$, depending on the behavior of the system to $\ell=0$ modes. This is where the real work of this thesis begins. In general the orbit integrals are not tractable because the particle orbits are complicated. Thus the variational Euler-Lagrange equation cannot be solved for the minimizing perturbation because the perturbation accumulates over the time history of the particle. Thus the Euler-Lagrange equation is at best an integro-differential equation over seven dimensions. (In fact finding a solution to the Euler-Lagrange equation is no simpler than solving the perturbed Vlasov equation exactly for all time.)

The energy principle given above is evaluated by two methods in this thesis. The first is a computer calculation of perturbed energy using a set of trial functions and a set of orbit data representing a

field-reversed mirror equilibrium. The second is an analytic method which shows the fundamental physical processes involved in the stability properties of a field-reversed mirror.

The lack of a minimizing equation for the energy principle is the motivation for the treatment of this problem by computer methods. With the computer code it is possible to find the perturbed energy as a function of frequency and growth rate for any trial function $\delta\vec{A}$ that is allowed in the particular electromagnetic gauge chosen. This serves two purposes. First, the computer code has allowed the construction of an optimum trial function for a particular instability mode by heuristic methods. Second, from the frequency-dependent data an analytic model has been constructed which gives insight into the kinetic properties of the device which are important in establishing the stability or instability of the equilibrium.

The method of computer analysis of this energy principle, along with the results obtained for particular equilibrium and for some "optimized" trial functions, is explained in Ch. 5 and in Appendices B and C.

Chapter 5. Stability Numerical Calculations

5.1 Introduction

The stability of a field reversed mirror to low frequency acoustic modes is studied here by the use of the SUPERLAYER equilibrium code with an added stability subroutine. The numerical results for the stability of a particular simulation (referred to as the SUPP58 data set) are given below. Detailed information about the equilibrium is given in Appendix A. The method of calculation for the energy principle terms for the trial functions used is given in Appendices B and C.

5.2 Trial Functions

The energy principle developed in Chapter 4 is evaluated as a functional of the perturbation $\delta\vec{A}$ applied to the system. It is important that the perturbation $\delta\vec{A}$ used in the energy principle be allowed by the assumptions used in the derivation of the energy principle. The two assumptions used in the present development of the energy principle for the field reversed mirror were those of charge neutrality and transverse electromagnetic fields. These two assumptions eliminate electron current effects. This gives rise to two equations to be satisfied for all allowed perturbed fields:

$$\begin{aligned}\vec{\nabla} \cdot \delta\vec{A} &= 0, \\ \vec{B} \cdot \delta\vec{A} &= 0.\end{aligned}\tag{5-1}$$

These two equations can be used to derive a general form for $\delta\vec{A}$. The first equation requires that $\delta\vec{A}$ can be derived from a vector potential, i.e. $\delta\vec{A} = \vec{\nabla} \times \vec{F}$. (The vector $\vec{F}$ is often called the Hertz vector.) The second equation is satisfied if $\vec{F}$ is in the direction of the magnetic field $\vec{B} = B_0 \hat{x}$. Thus a general form for the perturbation $\delta\vec{A}$ can be written as:

$$\delta \vec{A} = \vec{\nabla} \times (\hat{\chi} F(\psi, \chi) e^{i\ell\theta}). \quad (5-2)$$

The remaining question in the restriction of trial functions is the boundary condition on $\delta \vec{A}$. In the field reversed mirror it is reasonable to assume that metal walls will be located close to the edge of the plasma. This requires that all perturbations allowed on the system will have vanishing electric field at the edge of the plasma. In a field reversed mirror, this boundary is outside the closed field region of the device. If the conducting wall is taken to be immediately outside the last closed line, the normal vector $\hat{n}$ of the last closed line satisfies

$$\hat{n} = \hat{\theta} \times \hat{\chi}. \quad (5-3)$$

Here $\hat{\theta}$ is the toroidal unit vector and $\hat{\chi}$ is the direction of the magnetic field. This results in a simple condition on $\delta \vec{A}$:

$$\begin{aligned} \vec{\nabla} \times \delta \vec{A}(\text{wall}) \times \hat{n} &= 0, \\ \text{or } (\delta A_{\psi} \hat{\psi} + \delta A_{\theta} \hat{\theta}) \times \hat{n} &= 0, \\ \therefore \delta A_{\theta} &= 0 \text{ at wall.} \end{aligned} \quad (5-4)$$

Thus equations (5-2) and (5-4) give a complete description of the trial function.

It is important to note that the equations above do not represent the entirety of possible electromagnetic perturbations on a field reversed mirror configuration. They simply represent the type of perturbation believed to be analogous to low-frequency modes of instability that often occur in high-temperature plasmas. For example, the classical Rayleigh-Taylor instability in a plasma⁴¹ involves no charge separation in the device and has electric fields perpendicular to the magnetic field. Thus these modes can be represented by trial functions obeying

Eq. (5.1). Other modes, involving charge separation for example, may exist in the device; these could also be treated by an energy principle but are not considered here.

5.3 Computational Method

Perturbations $\delta\vec{A}$ that are consistent with the above equations are used to calculate the terms appearing in the energy principle. The SUPERLAYER code is used to collect the orbit data necessary to construct the orbit integrals g and g^* for each particle in the system. The code also does a Fourier analysis of this data and presents the total perturbed energy as a function of frequency for the trial functions chosen. Trial functions are constructed in such a way as to minimize the total energy $\delta^{(2)}U$ with respect to a variation within the set of solutions to equation (5-2). Details of the construction of perturbed energies for the $\ell = 0$ case are presented in Appendix B. Details for the calculation in the $\ell = 1, 2$ case are presented in Appendix C.

5.4 Results for the $\ell = 0$ Modes

For the $\ell = 0$ case, a sufficient condition for stability can be written for the energy principle by dropping a positive definite term. This expression is

$$\delta^{(2)}U = \int d^3x \left\{ \delta B^2 - \left(\frac{\omega_p}{c} \right)^2 \frac{m\Omega^2 R^2}{T} \delta A_\theta^2 \right\} . \quad (5-5)$$

This equation can be studied using the diagnostic output of the SUPERLAYER run without the need to calculate orbit integrals. The quantity $\left(\frac{\omega_p}{c} \right)^2$, the inverse square of the collisionless skin depth, is linear in the density. Thus trial functions are minimized in this integral expression of the traditional self-adjoint Ritz form.

Using the parameters of the equilibrium, variations among the set of trial functions $\delta\vec{A}$ were found to produce a minimum in $\delta^{(2)}U$ for

$$\delta A_\theta = R(1-R^2-Z^2) \quad . \quad (5-6)$$

The value of $\delta^{(2)}U$ obtained for this trial function is positive, indicating that all perturbed energies are positive for this case. (The details of this calculation are in Appendix A.) The normalization on R and Z above is

$$R = \left\{ \frac{R_{\text{physical}}}{13.6 \text{ cm}} \right\} \quad . \quad (5-7)$$

This normalization length, 13.6 cm, corresponds to the zero density point in the device. (There is, however, some fluctuation in the density during the simulation. Thus the 13.6 cm is only roughly the cutoff point.)

The above results indicate that a field reversed mirror similar in properties to the SUPERLAYER equilibrium model studied is stable to $\ell = 0$ perturbations. These are called tearing modes,³⁷ and are usually found in equilibria only when the length of the device is large compared to the radius.³⁸

Thus the lack of unstable tearing modes in the simulation of a short, fat equilibrium is not surprising. All such equilibria should be stable to tearing. There is an even stronger reason for the lack of a tearing instability in the system studied. Since the SUPERLAYER code is self-consistent in particle motion and electromagnetic field dynamics for axisymmetric movements, the dynamical code would become unstable numerically if the system were susceptible to tearing instability. Indeed, unstable equilibria have been observed using the SUPERLAYER

code.³⁸ In these simulations, an instability is seen which has a structure like traditional tearing modes. It is likely that the final development of a stable simulation using SUPERLAYER is to shed any unstable parts and form a new equilibrium stable to tearing.

This last comment justifies, in some sense, the appearance of the minimal mode given by eq. (5-6). This perturbation has a structure similar to the equilibrium vector potential for the equilibrium. The lowest energy motion is a simple "breathing" of the equilibrium, that is, a simple expansion of the device. Since the simulation shows a small oscillatory expansion and contraction of the plasma, it is understandable that the perturbation of lowest energy be a breathing mode. The low damping rate observed for this process in the simulation explains the low perturbed energy for this mode as a growing instability. (It is apparent that the mode is stable by its presence as a small damped mode in the simulation code.) The form of this mode is shown in Fig. 5.1.

5.5 Results for the $\ell = 1$ Modes

Nonaxisymmetric modes are not allowed in the equilibrium simulation and thus analysis is required to determine the system's stability or instability to these modes. The form of $\ell = 1$ modes is much like the kink modes in a long system. For long systems, the kink modes are expressed as:

$$\vec{\xi} = (i\ell\xi_{\theta}\hat{\theta} + \xi_R\hat{R})e^{i\ell\theta} e^{ikZ} e^{-i\omega t} . \quad (5-8)$$

Here ξ_{θ} and ξ_R are both real numbers. The vector $\vec{\xi}$ is the displacement vector in MHD theory, and it can be translated into an electromagnetic perturbation by:

$\ell = 0$ (Tearing) Mode

Side View

Top View

$\ell = 1$ (Kinking) Mode

Side View

Top View

$\ell = 2$ (Fluting) Mode

Side View

Top View

Figure 5-1. Deformations accompanying instabilities. (In this figure, the top view is the view along the magnetic axis.)

$$\delta \vec{A} = \vec{\xi} \times \vec{B} \quad (5-9)$$

The derivation of trial functions of the form presented as eq. (5-2), with appropriate boundary condition, is not in itself a trivial task. However, some experimentation yielded a form for the Hertz function χF that results in low magnetic energy and high negative spectral energy:

$$F = \frac{1}{2} R^2 \left(1 - \frac{R^2}{2} - Z^2\right) e^{i\ell\theta} \quad (5-10)$$

The dimensionless distances R and Z are normalized so that $\delta A_\theta = 0$ at $(R^2 + Z^2)^{1/2}_{\text{physical}} = 14$ cm. This function thus fits the boundary condition imposed by the conducting wall. This particular form was also chosen to be similar to the known minimizing perturbations for $\ell = 1$ modes for a long reversed configuration in finite larmor radius MHD theory⁷.

The choice of the perturbation given by eq. (5-10) for detailed calculation was made by examining partial output from the code for a variety of perturbed fields. The sensitivity of the energy integrals to the vector direction of the perturbation and to the spatial distribution of the perturbation led not only to eq. (5-10) as a near-minimum, but also to an understanding of the driving forces for instability in this simulation model.

The calculation of perturbed energies for various frequencies and growth rates for this particular trial function is given in Appendix C. One can show that there is a minimum energy that is negative for this trial function at low growth rates. In fact, for $\gamma = 0.033 \omega_{ci}$, the total energy $\delta^{(2)}U = -1.062$, for a normalization for $\delta \vec{A}$ such that

$$\int d^3x \frac{m\Omega_e^2 R^2}{T} \delta A_\theta^2 \left(\frac{\omega_p}{c}\right)^2 = 0.764 \quad (5-11)$$

(Note that here and in the following development, the perturbed energy is written as a dimensionless quantity.)

This trial function can thus be used to show that the energy principle fails to prove that $\ell = 1$ modes are stable. In fact, it is likely that this failure represents a true instability. The argument for this is shown in Chapter 6. The implications of a positive growth rate in linear stability theory are discussed in Chapter 7.

5.6 Results for the $\ell = 2$ Modes

The $\ell = 2$ mode is a flute mode of the traditional kind affecting plasmas with a magnetic field curved inward against the more dense region³⁹. In spherical geometry, the flute mode takes the form of a flattening of the plasma along a plane containing the axis of symmetry. Since this instability occurs in a system with finite particle drifts at a positive real frequency, this instability is also sometimes called a precessional instability.¹⁰

The perturbed energies for various trial functions for $\ell = 2$ were calculated using SUPERLAYER. Again, however, the optimum trial function was one of the simple kind used for the $\ell = 1$ perturbations of Section (5-4). The perturbation $\delta\vec{A}$ is expressed as:

$$\delta\vec{A} = \vec{\nabla} \times \left(\frac{R^2}{2} \left(1 - \frac{R^2}{2} - Z^2 \right) e^{i2\theta} \hat{x} \right) . \quad (5-12)$$

With this trial function, perturbed energies could be found with minima much lower than the $\ell = 1$ case. For a growth rate of $\gamma = 0.033 \omega_{ci}$ and real frequency $\omega_0 \sim 0.8\ell\Omega$, the total energy is $\delta^{(2)}U = -7.524$, using the normalization of eq. (5-11). The $\ell = 2$ mode is a slightly stronger instability with growth rate $\sim 0.07 \omega_{ci}$. (This growth rate is

obtained by a linear extrapolation of the energy curves as functions of growth rate to yield a curve of zero minimum energy for the extrapolated growth rate.) Assuming that these modes saturate at low amplitude and cause enhanced diffusion, the $\ell = 1$ and $\ell = 2$ modes could make roughly equal contributions to quasilinear transport arising from turbulence induced by instability. This process will be discussed in Chapter 7.

5.7 Comment on Higher ($\ell \geq 3$) Modes

It is likely that higher mode numbers are also unstable in this plasma equilibrium. However, the frequencies involved for these instabilities ($\omega_0 \sim \ell \Omega$, where $\ell \geq 3$) fall outside the range of validity for the kinetics code. The SUPERLAYER code, designed for equilibrium simulation, has a stepsize in time of $\Delta t = 0.196 / \omega_{ci}$. This stepsize, used both for the particle mover and for the numerical computation of the orbit integral, sets an upper limit on the frequency range or bandwidth of the orbit statistics.

The bandwidth $\Delta\omega$ that can be safely accommodated can be derived by treating each iteration as a sampling interval. The bandwidth $\Delta\omega$ contained in the sampled data with sample time Δt corresponding to the stepsize for the Fourier integration is given by³⁵:

$$\Delta\omega\Delta t \leq \frac{1}{2\pi} \quad . \quad (5-13)$$

This is called the Courant condition in numerical analysis.⁴⁰ This condition is satisfied for the $\ell = 1$ and $\ell = 2$ modes. For $\ell = 3$ and higher modes, we can calculate that $\Delta\omega\Delta t \sim 0.04\ell$. This implies that the orbit data deteriorates in quality for these modes.

There are, however, physical arguments for the disappearance of instability for some higher mode number. One argument is that the magnetic energy has a slight ℓ^4 dependence, and this will cause the higher-order modes to have prohibitively high magnetic energy to cause instability. Another argument is that the higher order modes occur at higher frequencies, and thus interactions with the electrons will become important. A small but finite electron temperature will be stabilizing.

This electron stabilization can be demonstrated by examining the dispersion relation for ion acoustic waves in the presence of a warm electron background. Davidson and Ogden³⁶ have examined roots to the ion-acoustic wave equation for $k\rho_i > 1$ and have found damping for these waves when there is a small electron temperature. Since the wavenumber for $\ell \gg 1$ corresponds to this limit, these waves correspond to the modes of instability if they are not damped. Thus while $\ell \gg 1$ modes are intractable by computer simulation, analytic theory shows evidence of stabilization at some intermediate mode number.

Chapter 6. Extensions and Analysis of Computational Method

6.1 Extension to Necessary and Sufficient Condition for Stability

The energy principle used here has been derived under the philosophy that the energy, being a perturbation of a constant of motion for a system, must be zero for an allowed motion (i.e. an instability if $\gamma > 0$ for that motion). It thus manifestly represents a sufficient condition for stability; that is, if there are no zeroes in the energy, there is no instability, but if there are zeroes in the perturbed energy, instability may or may not exist. In what follows, the method for proving the existence of instability is shown.

It is first necessary to define what instability means. An instability exists if a perturbation $\delta\vec{A}$ exists which (1) grows in time as $e^{\gamma t}$, (2) represents a solution to Maxwell's equations, and (3) is self consistent with the Vlasov equation to lowest order by linking the source $\delta\vec{J}$ with the perturbation $\delta\vec{A}$ through the linearized Vlasov equation.

Maxwell's equation, called for in the definition of instability, here means

$$\square \delta\vec{A} = - \frac{4\pi}{c} \delta\vec{J} \quad , \quad (6-1)$$

where $\square = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$. Since we are only interested in modes that are slow compared with the light transit time $\frac{L}{c}$, we drop the displacement-current term $\frac{1}{c^2} \frac{\partial^2}{\partial t^2}$ so that (6-1) becomes Poisson's equation with a source

$$\nabla^2 \delta\vec{A} = - \frac{4\pi}{c} \delta\vec{J} \quad . \quad (6-2)$$

The energy principle used here simply represents an integral of this equation over space and time, when a scalar product is formed with the electric field, i.e.

$$\delta^{(2)}U = \int_{-\infty}^t dt' \int d^3x \delta \vec{E} \cdot \left(\nabla^2 \delta \vec{A} + \frac{4\pi}{c} \delta \vec{J} \right) \quad (6-3)$$

A trivial variational principle shows that if $\delta^{(2)}U$ has no zeroes, then $\nabla^2 \delta \vec{A} \neq -\frac{4\pi}{c} \delta \vec{J}$ for all space and all time; thus there is no instability. This is the variational principle used earlier, now viewed from the standpoint of the equation as a functional of Maxwell's equations. We now proceed to view eq. (6-3) as a variational principle, with the minimization corresponding to the optimization of trial functions, under some normalization, to make $\delta^{(2)}U$ be some minimum. The time dependence can be removed by noticing that

$$\delta^{(2)}U(t) = \delta^{(2)}U(0)e^{\gamma t} \quad (6-4)$$

where γ is the imaginary part of ω , the complex frequency. Clearly ω_0 and γ can be manipulated separately from the spatial variation of the trial eigenfunctions. At fixed growth rate γ and real frequency ω_0 , eq. (6-3) is linearly related to

$$\delta^{(2)}U_t = \int d^3x \delta \vec{E} \cdot \left(\nabla^2 \delta \vec{A} + \frac{4\pi}{c} \delta \vec{J} \right) \quad (6-5)$$

by differentiation with respect to time, and so the variational principle

$$\delta^{(2)}U_t \equiv \frac{\partial}{\partial t} \delta^{(2)}U = 0 \quad (6-6)$$

corresponds to minimizing the (normalized) variational principle

$$Q \equiv \delta^{(2)}U_t = \frac{\int d^3x \delta \vec{E} \cdot \left(\nabla^2 \delta \vec{A} + \frac{4\pi}{c} \delta \vec{J} \right)}{\int d^3x \delta \vec{E} \cdot \delta \vec{E}} \quad (6-7)$$

Notice that the right hand side of the dot product in the numerator integrand of eq. (6-7) can be regarded as an operator acting on $\delta\vec{E}$. Clearly $\delta\vec{A}$ and $\delta\vec{J}$ can be derived from $\delta\vec{E}$ by integrodifferentiation operations and from the equilibrium velocity space. Then Q becomes

$$Q = \frac{\int d^3x \delta\vec{E} \cdot F \delta\vec{E}}{\int d^3x \delta\vec{E} \cdot \delta\vec{E}} . \quad (6-8)$$

For certain cases the operator F appearing in Q can be proved self-adjoint, although for general Vlasov systems it cannot. When F is self-adjoint, the operator has the property

$$\int \vec{A} \cdot F \vec{B} d^3x = \int \vec{B} \cdot F \vec{A} d^3x ; \quad \forall \vec{A}, \forall \vec{B} . \quad (6-9)$$

This makes eq. (6-7) correspond to a simple Ritz variational principle

$$\delta Q = 0 , \quad (6-10)$$

which has the well-known solution (Ref. 8)

$$F \delta\vec{E} = \lambda \delta\vec{E} \quad (6-11)$$

where λ is a real constant; in fact, $\lambda = Q_{\min}$.

The algorithm for finding instability with a self-adjoint operator is thus to find the largest growth rate γ and the optimum frequency ω_0 so that for the entire space of trial functions $\delta\vec{A}$ (or equivalently $\delta\vec{E}$) there is an infimum to Q with eigenvalue zero. For this minimizing trial function $\delta\vec{E}$, eq. (6-11), the Euler-Lagrange equation for this system, becomes

$$F \delta\vec{E} = 0 = \nabla^2 \delta\vec{A} + \frac{4\pi}{c} \delta\vec{J} , \quad (6-12)$$

which is Maxwell's equation eq. (6-2). Thus if the Vlasov operator can be shown to be self-adjoint, the sufficient condition for stability becomes a necessary and sufficient condition for stability. Thus

finding a negative value for $\delta^{(2)}U$ at some frequency and growth rate would guarantee an instability at least that growth rate.

However no general proof of self-adjointness for Vlasov operators exists, and thus modification of the method is required where the operators are manifestly non-self-adjoint. Then what is required is the substitution of an adjoint variational principle for the Ritz variational principle. Here Maxwell's equation eq. (6-2) is dotted not by the electric field $\delta\vec{E}$, but by another fictitious electric field $\delta\vec{E}^+$. This is called the adjoint trial function. A variational principle is generated by minimizing an energy-like quantity exactly as in eq. (6-7), but now with trial functions $\delta\vec{E}$ and $\delta\vec{E}^+$ varied independently. Thus the variational principle $\delta Q = 0$ is represented by the form

$$Q = \frac{\int d^3x \delta\vec{E}^+ \cdot F \delta\vec{E}}{\int d^3x \delta\vec{E}^+ \cdot \delta\vec{E}} \quad (6-13)$$

Here F is the same operator equation as before:

$$F\delta\vec{E} = \nabla^2\delta\vec{A} + \frac{4\pi}{c} \delta\vec{J} \quad (6-14)$$

Minimization of Q with respect to independent variation of $\delta\vec{E}$ and $\delta\vec{E}^+$ in the real space and the adjoint space results in two equations for F :

$$\begin{aligned} F\delta\vec{E} &= \lambda\delta\vec{E} \quad , \\ F^+\delta\vec{E}^+ &= \lambda\delta\vec{E}^+ \quad . \end{aligned} \quad (6-15)$$

Again $\lambda = Q_{\min}$, so $\lambda = 0$ corresponds to allowed instability. Thus again real frequency and growth rate are adjusted so that $Q_{\min} = 0$.

Thus in a non-self-adjoint system, instability can be proved using the same variational principle written in adjoint form. Here, however, the variational principle also yields extraneous information about the eigenvalues of F^+ , the adjoint operator, which is defined by the equation

$$\int \vec{A} \cdot F^+ \vec{B} d^3x = \int (F\vec{A}) \cdot \vec{B} d^3x \quad (6-16)$$

for all $\vec{A}$, $\vec{B}$. This added information may reveal the non-adjoint character of the operators involved and yield some physical insight. However, this method doubles the variational space involved and makes calculation more difficult. In the next section, it is shown that operators which are self-adjoint can be used to approximate the actual Vlasov operator, yielding energy integrals in good agreement with the SUPERLAYER perturbed energy output. This gives evidence that the axisymmetric exact Vlasov operators for field-reversed mirror equilibrium may be nearly self-adjoint, and thus the identification of negative trial energies closely corresponds to the existence of an instability.

6.2 Re-Analysis of Orbit Data

The use of SUPERLAYER for stability calculations is advantageous because it is a self-consistent kinetics code and can give an accurate knowledge of the orbit integrals and thus of the spectral response of the device. However, it may be desirable to interpret the results of the complicated numerical integration and variational method employed by developing a simple analytical model for the general features of the energy-frequency relations discovered in the numerical simulation. A simple model for the orbits involved in field-reversed mirrors will give a clear picture of the spectral properties involved in stability analysis of the device.

The main feature of orbits in axisymmetric field-reversed mirrors is that the particles, on the average, drift in the $\hat{\theta}$ direction. The largest part of the orbit integral (and in the magnetic energy $-(\delta\vec{A} \cdot \nabla^2 \delta\vec{A}))$

comes from V_θ and δA_θ . By ignoring terms in V_Z and V_R (which, as they appear only as time-averaged quantities in the orbit integral, do not contribute to the spectral energy if the R and Z motions are uncoupled in frequency to the mean drift motion $\langle \Omega \rangle$), we can arrive at a simple energy principle. Also, the fluctuations in δA_θ that the particle experiences as it is involved in radial and axial betatron-type motion are uncoupled in frequency and $\delta A_\theta(t)$, the perturbed vector potential as a function of time, can be replaced by $\delta A_{\theta \text{ RMS}}$, a time-averaged vector potential. We then have, for each particle:

$$V_\theta = \Omega R + \tilde{V}_\theta, \quad \langle \tilde{V}_\theta \rangle = 0 \quad (6-17)$$

Then

$$\langle V_\theta \rangle = \Omega R$$

and $e^{i(\omega t - \ell \theta)} \delta A_\theta = \delta A_{\theta \text{ RMS}} + \delta \tilde{A}_\theta$, $\langle \delta \tilde{A}_\theta \rangle = 0$, so that, for each particle

$$g \approx \int_{-\infty}^t \delta A_{\theta \text{ RMS}} e^{-i\omega t} e^{i\ell \theta} \Omega R$$

$$\approx \left[\int_{-\infty}^t e^{i\ell \Omega t} e^{-i\omega t} \right] \delta A_{\theta \text{ RMS}} \Omega R, \quad (6-18)$$

or

$$g = \frac{\Omega R \delta A_\theta}{i(\ell \Omega - \omega)}.$$

If the plasma behaved as a coherent ring, this approximation would be exact. In this case the spectral energy terms would exactly cancel the rotational energy term and stability to all perturbations would be assured. One can see this from

$$g = \int_0^t \vec{v} \cdot \delta \vec{A} = \frac{\Omega R \delta A_\theta}{-i(\omega - \ell \Omega)}$$

Then we have

$$\langle g^* g \rangle = \frac{\langle \Omega^2 R^2 \delta A_\theta^2 \rangle}{\gamma^2 + (\omega_0 - \ell \Omega)^2} \langle i \int g^* \frac{dg}{dt} - g^* \frac{dg}{dt} \rangle = \frac{\omega - \ell \Omega}{\gamma} g^* g$$

so that

$$G(\omega_0) \equiv \langle (\omega \omega^* - \omega_0 \ell \Omega) g^* g + i \gamma \ell \Omega \int g^* \frac{dg}{dt} - g \frac{dg^*}{dt} \rangle = \langle \Omega^2 R^2 \delta A_\theta^2 \rangle \times$$

$$\left[\frac{\gamma^2 + (\omega_0 - \ell \Omega) \omega_0}{\gamma^2 + (\omega_0 - \ell \Omega)^2} - \frac{\ell \Omega (\omega_0 - \ell \Omega)}{\gamma^2 + (\omega_0 - \ell \Omega)^2} \right] = \langle \Omega^2 R^2 \delta A_\theta^2 \rangle \quad (6-19)$$

Then the entire energy principle reads:

$$\begin{aligned} \delta^{(2)} U &= \int d^3 x \left\{ \delta B^2 - \frac{m \Omega^2 R^2}{T} \left(\frac{\omega_p}{c} \right)^2 \delta A_\theta^2 + \frac{m}{T} \left(\frac{\omega_p}{c} \right)^2 G(\omega_0) \right\} \\ &= \int d^3 x \left\{ \delta B^2 - \frac{m \Omega^2 R^2}{T} \left(\frac{\omega_p}{c} \right)^2 \delta A_\theta^2 + \frac{m \Omega^2 R^2}{T} \left(\frac{\omega_p}{c} \right)^2 \delta A_\theta^2 \right\} \\ &= \int d^3 x \delta B^2 \end{aligned} \quad (6-20)$$

Thus stability is assured in the pure ring approximation since $\delta B^2 \geq 0$ for all allowed modes.

However in a field-reversed mirror there is a spread in the drift frequencies Ω which were considered to be the same for each particle in the thin-ring approximation. We shall see that this leads to the negative energy minima found in the simulation on SUPERLAYER.

Now recall that the mean drift frequency Ω depends on the initial conditions of the particle, and that in general there is a spread $\Delta \Omega$ of drift frequencies about the mean frequency Ω_0 which is the constant

appearing in the distribution function

$$f(v) = n_0 e^{-\left(\frac{H - \Omega_0 P_\theta}{T}\right)},$$

so that

$$\begin{aligned} n(\Omega) &\approx e^{-\frac{(\Omega - \Omega_0)^2 m R^2}{2T}} \\ &\equiv e^{-\frac{(\Omega - \Omega_0)^2}{2\Delta^2}}. \end{aligned} \quad (6-21)$$

We can then repeat the same type of analysis as was done above for the case where $\Delta\Omega = 0$. Here the canonical averages of the singular terms in $\frac{1}{\omega - \ell\Omega}$ are not just simple multiplications by the number of particles ($\langle \cdot \rangle_x$). In this case the spectral energy terms become

$$G(\omega_0) = \frac{\gamma^2 + (\omega_0 - \ell\Omega_0)^2 + \ell\Omega_0(\ell\Omega - \ell\Omega_0)}{\gamma^2 + (\omega_0 - \ell\Omega)^2} \delta A_\theta^2 \Omega^2 R^2 \cdot n(\Omega) d\Omega. \quad (6-22)$$

Here $n(\Omega) \approx \exp\left[-\frac{(\Omega - \Omega_0)^2}{2\Delta^2}\right]$. This then becomes a simple integral which can be evaluated numerically. A graph of this function for a $\Delta\Omega$ consistent with the SUPERLAYER data follows as Figure 6-1. One can see that the effect of the finite spread of drift frequencies results in a destabilizing energy contribution which is minimum at $\omega_0 \approx 0.7\ell\Omega$. This is in good agreement with the SUPERLAYER orbit energies. At minimum the perturbed energy is about $-14x \left(\frac{m\Omega^2 R^2}{T}\right) \delta A_\theta^2 \left(\frac{\omega_p}{c}\right)^2 d^3x$ which is slightly greater than the SUPERLAYER value at the same growth rate, $\gamma = 0.033$. This simple model gives a good insight into the fundamental physics involved. The resonance at $\omega_0 \approx \ell\Omega$ becomes broadened by the finite thermal spread of the particle drift velocities. The mathematical form and physics

Figure 6-1. Comparison of SUPERLAYER perturbed energy vs. frequency for $\ell = 2$, $\gamma = 0.033$ with analytic model.

involved in this process are similar to the process of Doppler broadening of a resonance in an atomic or nuclear cross section. In the energy analysis, Doppler broadening has caused a negative energy contribution before the resonant peak ($\omega_0 < \Omega$) and a positive energy contribution after the resonant peak, much like the increased absorption involved in a Doppler-broadened resonance in an energy-dependent neutron flux. In the field-reversed mirror, a "smearing-out" of the ring resonance has caused the perturbed energy to be negative at low growth rates ($\gamma < \Delta\Omega$) for frequencies below the mean drift frequency ($\omega_0 < \Omega$). Thus this "Doppler broadening" is responsible for a precessional instability at low growth rate.

It is also interesting to note that in this approximation the total energy function $\delta^{(2)}U$ is self-adjoint. This can be shown from the expression (6-22). The total energy is

$$\delta^{(2)}U = \int \delta B^2 - \frac{m\Omega^2 R^2}{T} \left(\frac{\omega_p}{c}\right)^2 \delta A_\theta^2 + \frac{m\Omega^2 R^2}{T} \left(\frac{\omega_p}{c}\right)^2 F(\omega_0, \gamma) \delta A_\theta^2, \quad (6-23)$$

which is quadratic in the perturbation $\delta \vec{A}$. Equivalently this operator, in adjoint form, is

$$\delta^{(2)}U = \int d^3x \delta \vec{A}^+ \cdot \left(-\nabla^2 + \frac{\hat{\theta}\hat{\theta}}{\Omega} \left[\frac{m\Omega^2 R^2}{T} \left(\frac{\omega_p}{c}\right)^2 \left[-1 + F(\omega_0, \gamma) \right] \right] \right) \delta \vec{A}. \quad (6-24)$$

As this operator involves only the Laplacian operator ∇^2 and multiplication by constants, Eq. (6-24) is manifestly self-adjoint. Thus minimizing this expression is equivalent to minimizing by the Ritz variational principle, as discussed in Section 6-A. This results in making the energy principle used here become a necessary and sufficient

condition for stability for the simple model. In this regard, it is interesting to note that the reason for this self-adjointness is that the mode structure is only important in the symmetry direction. This also happens in Vlasov low-frequency stability problems in slab geometry, as indicated in Refs. (27,28).

Chapter 7. Conclusions and Suggestion for Future Work

7.1 Analysis of Stability Study

The significance of the instabilities found in the plasma simulation must be determined in deciding whether or not the field-reversed mirror could be a practical fusion device. One needs to relate the existence of instabilities in the domain of small-amplitude, axisymmetric theory to the behavior of the real system. In this regard, two questions are addressed below: (1) if the instabilities of the theory exist in the real device, what saturation mechanisms will result, and (2) what small changes in the equilibrium configuration may improve the stability picture?

The first question involves the process of turbulence and nonlinear effects in plasma. Assuming that some small growth rate for instability exists, the plasma will support the growth of the unstable wave until it becomes of such an amplitude that the nonlinear effects involved (terms in δE^2 and δB^2 and higher order in the perturbed Vlasov equation) produce a saturation phenomenon. A finite amplitude wave can cause transport of particles and energy out of the device. The transport of particles out of the device represents in some sense the reduction of the free energy of the plasma, and a finite transport of particles and energy can balance the source of free energy being released in an unstable equilibrium. An estimate⁴⁴ of the diffusion coefficient created by a saturation of an unstable wave is:

$$D_{\perp} = \frac{\gamma}{k_{\perp}^2} \left(-\frac{\gamma}{\Delta\omega} \right) .$$

Here $K_{\perp}$ is the wavenumber of the instability and $\Delta\omega = \omega - k_{\perp}V_{\text{Drift}}$.

The simulation results cited earlier give a linear growth rate of $\gamma = 0.033\Omega_{ci}$ and $\Delta\omega = \omega_0 - \ell\Omega = 0.3\Omega_{ci}$. For the $\ell = 2$ mode, we estimate $K_{\perp}^2 \approx \frac{15}{R_0^2}$. This gives a value $D_{\perp}$ of

$$D_{\perp} = 2 \times 10^{-4} \Omega_{ci} R_0^2.$$

This value of the perpendicular diffusion coefficient then results in an estimate of the confinement time to the turbulent transport induced by the instability. For a magnetic field strength of 10 KG and a plasma radius of 15 cm, this gives a value of $\frac{R_0^2}{D_{\perp}} \sim 100 \mu\text{sec}$, which is the order of magnitude of the confinement time. This characteristic time is a factor of several hundred times the ion sound transit time across the plasma radius. However, this characteristic time is much shorter than the confinement time due to classical diffusion.

Turbulent behavior may explain the observed lifetime and apparent "stability" of the reversed-field experiments that have been done to date. Linford and co-workers⁶ have produced equilibria with intermediate S factors that have shown lifetimes of several hundred ion sound times. The decay is quite rapid, however, due to the low temperatures involved. While the above analysis was done for a "short, fat" equilibrium, it is possible that precessional instability exists for the longer field-reversed plasmas which Linford has encountered experimentally.

However, Linford's experiment produces very low ion temperatures (~ 100 eV) and thus the classical confinement time and the Bohm confinement time are of the same order of magnitude ($D \sim 10^5 \text{ cm}^2 \text{ sec}^{-1}$). Thus further work at higher temperatures will be required to find the

actual confinement scaling law. As Bohm confinement is usually associated with the presence of an instability, the stability of Linford's experiment cannot be deduced from the observed confinement time.

In the field-reversed ring experiments at Cornell, Fleischmann and co-workers⁴⁷ have reported axisymmetric field-reversed ring lifetimes of several hundred microseconds, rising to several milliseconds for experiments with toroidal fields or quadrupole fields. The longer lifetimes of the Cornell RECE experiment⁴⁷ can be explained by understanding the driving force for the instability outlined above. The effect causing the negative spectral energy is the spread of azimuthal velocities, causing a widening of the resonant absorption spectrum. Thus the driving force for the instability is the thermal spread ΔV_θ , which is proportional to the width of the ring. In the Cornell RECE experiment, the electrons form the ring, and because the ring electrons are highly relativistic, the ring has a tight, dense cross-section with little width in the $\hat{R}$ direction and thus little variation about the mean azimuthal velocity. Although analysis of the stability of this situation is different than that for a field-reversed mirror reactor, the general features of the particle kinetics are similar, and the lower thermal spread may indicate instability at much lower growth rate. This in turn could lead to a saturation at lower amplitude, and thus the enhancement of transport due to turbulent effects could be less serious.

A final word about the possibility of saturation and turbulence could be said in light of the 2XIIB experiment at Livermore. Here the RF noise was monitored as the neutral beam injectors were turned on. In the low- β mode, 2XIIB represents a stable mirror plasma due to the $\ell = 2$ component of the magnetic field from the Yin-Yang type coil. As the

plasma beta increases, the RF noise increases due to drift-cyclotron-loss-cone (DCLC) instability.⁴³ In an attempt to stabilize against this mode, the 2XIIB experiment includes a streaming plasma source which fills the loss-cone and stabilizes against DCLC turbulence. With this source turned on, the RF noise stays low, indicating quiescent operation to a much higher plasma beta. Although beta cannot be measured directly, a good indication of the plasma beta can be determined from interferometer measurements of the plasma density and Thompson scattering measurements of the temperature. At a beta of roughly unity the RF noise begins to rise with the streaming plasma source turned on.

At unity beta the magnetic field lines start to form closed trajectories inside the device. The average curvature of the field lines changes from negative (interchange stable in fluid theory) to positive (interchange unstable in fluid theory). Thus the average particle motion, which is influenced by the field-line curvature, changes in character at roughly unity beta. It is thus conceivable that the increase in RF noise at about $\beta \approx 1$ may correspond to the onset of macroscopic instability which would cause some turbulence and thus a dissipation through the ion-acoustic wave spectrum of the device. A precessional mode of the type outlined above is a likely candidate for the instability in the reversed state. It is also interesting to note that for the lifetime under turbulent conditions outlined above, the characteristic time of the 2XIIB in quiescent conditions (about 500 μ sec) would be reduced by a factor of 10 or more. The 750 Amperes of available beam current will result in marginal reversal (which means $N \sim 2 \cdot 10^{17}$) if the confinement time is degraded to 50 μ sec. If the turbulent mechanism outlined above serves as the degrading influence on the transport, and

it occurs at $\beta \rightarrow 1$, the amount of beam available on the 2XIIB experiment (750A) will prevent the plasma from passing through marginal reversal, $\beta \approx 1$. (Here beta means the maximum local value.) Thus the failure of 2XIIB to obtain full reversal ($\beta \approx 2$) can be viewed either as a failure to provide enough beam to replace the losses due to enhanced transport in the device or as the onset of instability as the device approaches the field-reversed state, or both.

7.2 Suggestions for Further Study

One may ask also whether there is a possibility of stabilizing the field-reversed mirror by changing the equilibrium. To study this, one can review the approximations and assumptions made in the above work, as outlined in Chapter 2:

- (1) Axisymmetry $\frac{\partial}{\partial \phi} = 0$.
- (2) Aspect Ratio $R \sim a$.
- (3) S factor $\frac{R_0}{\rho_i} \sim O(1)$.
- (4) Electrostatically neutral $\phi_0 = 0$.

By violating any one of the above conditions, it may be possible to create a field-reversed state which may not be subject to the instability studied in the present simulation model. Thus the assumptions of this work can be used to understand the starting point for a study of FRM stabilization techniques. The following illustrates the changes in the device physics that may accompany violations of the above restrictions.

Axisymmetry ($\frac{\partial}{\partial \phi} = 0$) has been used to separate the modes by Fourier analysis and also to reduce the variables in the equilibrium by use of the canonical invariant. When axisymmetry is removed, new effects can

be present which may affect the linear stability of field-reversed mirrors. The inclusion of multipole fields in the equilibrium produces changes in the character of the ion orbit, particularly in the drift motion $\langle v_\theta \rangle$, which is so important in the spectral energy to perturbations for $l \neq 0$. It is possible with the inclusion of a multipole field of sufficient strength that the energy terms depending on the drift rate and thermal spread are dominated by other terms representing the absorbed energy of the plasma due to multipole orbit contributions. This may explain why Fleischmann and co-workers⁴⁷ at Cornell see a much longer plasma lifetime with the addition of multipole fields. The additional field in the RECE experiment may be causing a suppression of the instability growth rates and thus an enhancement of the plasma lifetime to turbulent processes. One should note, however, that multipole fields could create open magnetic field lines, thus enhancing plasma losses.

On a similar note, the addition of a toroidal field acts in much the same way as the addition of a multipole field. Here the particle drifts are changed by the $\hat{\theta}$ component of the magnetic field. The addition of a toroidal field, however, has another effect, probably much stronger: the magnetosonic perturbations, which must have a form $\delta \vec{A} = \vec{\xi} \times \vec{B}$, can no longer be aligned along the $\hat{\theta}$ direction if a toroidal field is added. The additional field forces any mode of instability to have an electric field $i\omega \vec{A}$ perpendicular to the magnetic field, and thus at an angle to the $\hat{\theta}$ direction. This reduces the size of terms in $V_\theta \delta A_\theta$ in the energy principle for allowed trial functions $\delta \vec{A}$. This has a correspondence to the stabilization of an equilibrium in the MHD fluid theory: while purely poloidal fields with positive curvature are unstable in MHD

theory, the addition of a toroidal field can stabilize the equilibrium. This field can be added without changing the pressure profile, since $\vec{\nabla}p = \vec{J} \times \vec{B}$ in MHD theory and $\vec{J} = \hat{\theta} J_{\theta}$, so that $\vec{J} \times B_{\theta} \hat{\theta} = 0$.

There is an interesting analogy between a field-reversed mirror and a Tokamak device at this point. In a Tokamak, the azimuthal current produces a poloidal field which, when coupled to the external poloidal field produced by the vertical field coils, makes a field configuration much like a field-reversed mirror. The additional toroidal field, needed for stability, is provided by current flowing through the center of the device. A field-reversed mirror, which has no "natural" hole in the center, does not lend itself to the inclusion of a toroidal field coil. However, the inclusion of a toroidal field in a reversed-field mirror, by a hard-core center carrying current up the center, could possibly provide a "brute-force" stabilization method used successfully in the Tokamak. As is the case in the tokamak, however, this toroidal field lowers the overall beta and significantly increases the reactor cost.

Work has also been done at Princeton on a field-reversed mirror configuration where the current, as well as the magnetic field, has both toroidal and poloidal components. This device is known as Spheromak.⁴⁶ Stabilization of the Spheromak may exist due to the shear of the magnetic field lines. A fluid stability study of this device has been undertaken, however the results to date indicate that MHD instability sets in with very low values of β , about $\beta = 0.05$. The stability properties of this device with finite ion orbits are not known at the present. There may be a fruitful union of finite-particle (Vlasov) effects and the inherent shear of the magnetic field.

The second assumption used in the work preceding has been that the aspect ratio is of order unity, $R \sim a$. This has been used in specifying trial functions $\delta\vec{A}$ which have characteristic variation of R and Z at about the same rate. There remains the possibility that a "stretched" equilibrium, much longer in Z than in R , may have different stability properties. Linford's experiment⁴⁸ at Los Alamos has a longer Z length than its radius by about a factor of 5-7. The apparent existence of equilibria at that aspect ratio suggest stabilization by forcing perturbations to be of the form

$$\delta\vec{A} = \hat{R}\delta A_R + \hat{\theta}\delta A_\theta$$

and thus be less sensitive to particle effects in the $\hat{R}$ direction. However, stretching the equilibrium in the $\hat{Z}$ direction does not reduce energy terms in $V_\theta\delta A_\theta$ unless the equilibrium is also made very thin. A "long, thin" equilibrium may have improved stability properties to precession over a "short, fat" equilibrium. (The width of the pressure profile in Linford's experiments is not accurately known.) However, tearing instability ($\ell = 0$) sets in at some value of L/R , where L is the axial length; typically, tearing instabilities appear in SUPERLAYER equilibria with $L/R \sim 8$.⁴²

We can also investigate stabilizing effects at lower S factors. As the ion gyroradius becomes larger, the device becomes more of a ringlike system. With the particles coherently rotating about the azimuthal axis, thermal spreading is greatly reduced. In short, as the device becomes a ring, the terms in the energy principle dependent on the spread ΔV_θ become less important. Stabilization as $\Delta V_\theta \rightarrow 0$ was outlined in Ch. 6. While it may be possible to develop a stable Astron - like plasma, this device would require a background low temperature ion

component of high density to produce an acceptable thermonuclear power output⁴. However, the addition of this low temperature background complicates the stability analysis²⁴, and many physics and engineering questions for the ion Astron system remain open.

A final assumption made in this work was that the equilibrium plasma was electrostatically neutral, that is, $\phi_0 = 0$. If the electrons are not cold, the nature of the equilibrium will change and the electrostatic field plays an important part of the configuration. Furthermore, warm electrons make a contribution to the perturbed energy when a trial function is applied, since the law $\delta \vec{J}_e \cdot \delta \vec{E} = 0$ no longer holds when $T_e \neq 0$. The primary stabilizing effect is still due, however, to the ion energy terms in $V_\theta \delta A_\theta$; here the effect of the electrostatic field is to change the drift motion of the ions in the azimuthal direction. The ion drift has the additional term

$$\vec{V} = c \frac{\vec{E}_0 \times \vec{B}}{B^2} + \dots ; \vec{E}_0 = \vec{\nabla} \phi_0 ,$$

and this affects the dielectric response of the plasma to allowed perturbations.

It is this effect which places the Linford experiment at LASL outside the realm of the stability analysis done above. A strong radial electric field is set up in the plasma after reversal has started. This has been observed by measuring the rotation rate of heavy ions in the plasma. The electrostatic field is probably an integral part of the confinement structure. In this experiment, it replaces in some sense the stronger magnetic field gradient that would be needed to confine the plasma. This reduced field gradient relates to reduced spread about

mean drift frequency and thus reduced instability growth rates. This partial electrostatic confinement of a field-reversed state could be looked on as a stabilizing mechanism, although the stability of field-reversed plasmas with electrostatic fields remains an open question.

The energy principle methods developed in this thesis could be adapted to these more complicated cases. However, changes would be required both in the theoretical framework of the stability analysis and in the computer simulation of the equilibrium model.

In the case of non-axisymmetry, one loses P_θ as an invariant. This complicates the stability theory enormously. Fourier analysis in the θ direction loses its usefulness when there is no symmetry in this direction. The equilibrium distribution function cannot be rigid-rotor in nature. Furthermore, a 3-D equilibrium code would be required to model the equilibrium. Complete 3-D equilibrium codes are usually plagued with numerical instability.

The aspect ratio dependence could be tested with the present energy principle development. The problem here is the equilibrium code. Simulations of very low aspect ratio are more sensitive to the injection profile used. Also, these "long, thin" numerical simulations tend to be noisy.

Similarly, the S factor dependence represents a greater problem for the equilibrium code than for the stability method. SUPERLAYER and other equilibrium codes can handle greater or smaller S factors than that employed in the present work ($S \sim 1.5 \rightarrow 2.0$). However, higher S factors require better injection statistics and generally are more susceptible to numerical instability. Low S factors present problems

with the decoupling of the ion betatron motion from the drift frequency. Thus bandwidth problems may arise in collecting the orbit integral data.

Two potentially fruitful applications of the methods developed in this work are for equilibria with either toroidal magnetic fields or electrostatic fields. In these cases the axisymmetric energy principle remains unchanged. Equilibrium models may be able to easily incorporate these added fields with little change in the basic algorithm.

Appendix A. Description of Computer Code

The computer code used in the analysis of field-reversed mirror stability performed above was a modified version of SUPERLAYER, an equilibrium 2 1/2-dimensional orbit-kinetics code with self-consistent electromagnetic fields. The code itself, along with the modifications necessary for stability studies, is outlined in this appendix.

SUPERLAYER, as used in this text, is actually SUPERLAYER II,² the second version of an electromagnetic code developed from 1971 to present at Lawrence Livermore Laboratory. The initial configuration of this code enabled calculations of relativistic electron beam equilibria similar to those studied in the Astron experimental program. The early code predicted the failure of the Astron experiment at Livermore to form reversed relativistic electron equilibria using a slow buildup and also predicted the success at Cornell University to form reversed states with a single high-current pulse of relativistic electrons.⁴⁹

SUPERLAYER II is a revised version of the early SUPERLAYER that treats magnetic equilibria with high-current non-relativistic ion beams. For ion beams, as opposed to relativistic electron beams, the ringlike equilibria contain much more transverse motion, i.e., v_R and v_Z are of significant size compared to v_θ . The code allows for ionization of injected neutral beams through a Monte Carlo technique, and by calculation of the particle orbits after injection and ionization obtains a value for J_θ , the current in the azimuthal direction. The code ignores electrostatic effects, space charge, and all electron currents. Further, the currents in the transverse direction, J_R and J_Z , are assumed to be zero. This forces azimuthal symmetry and purely poloidal magnetic

fields. This then leaves the following set of equations to be solved self-consistently:

$$-\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) \right) - \frac{\partial^2 A_\theta}{\partial z^2} = \frac{4\pi}{c} J_\theta, \quad A_r = 0, \quad A_z = 0, \\ E_\theta = -\frac{\partial A_\theta}{\partial t}, \quad E_r = 0, \quad E_z = 0, \quad (A-1)$$

$$\frac{d\vec{v}}{dt} = \frac{q}{m} (\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$$

This equation is solved by transformation into a difference equation in time. This equation is solved by a predictor-corrector method. By using conservation of angular momentum, one can then find J_θ in terms of A_θ at each timestep. By using time-centering, the positions $(R, Z)^{n+1}$ and the velocities $(v_R, v_Z, v_\theta)^{n+1/2}$ determine the electromagnetic quantities $(A_\theta, B_R, B_Z)^{n+1}$. The above dynamical equations become the following finite difference form:

$$\left\{ -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) \right) - \frac{\partial^2 A_\theta}{\partial z^2} + \frac{\omega_{pi}^2}{c^2} A_\theta \right\}^{n+1} = \sum_i K_i^{n+1}. \quad (A-2)$$

Here K_i is a quantity calculated for each particle and depends only on the position of the particle.² It corresponds to the contribution to the total current from the i_{th} particle.

The equation-of-motion solver for this code, called the mover, uses a technique known as Boris's technique⁵⁰ for cylindrical coordinate equations of motion. The technique is a predictor-corrector differential equation solver. The main feature of the elaborate technique is that the algorithm solves for velocities and positions in rectilinear geometry and then uses a geometric transformation to update the coordinate system due to the angular rotation of the particle about the axis of symmetry.

Thus instead of solving the kinetic equation in cylindrical coordinates, the kinetic equation is solved in a coordinate system that is locally rectangular and then provides a coordinate transformation back to cylindrical coordinates at each timestep. This technique has the advantage that the equation of motion in rectangular coordinates has no singularities, whereas the equation of motion in cylindrical coordinates has a singularity at $R = 0$. This technique then provides accurate particle motion calculation, free of numerical instability, when the particle orbits are near the origin.

The modification of the SUPERLAYER II code done here to investigate stability involved adding an additional calculational package which would calculate the orbit integrals required for stability studies as an additional numerical integration to be done during the normal execution of the code. The stability package consisted of two subroutines, STABLE and PLOTSTABL, used to calculate the perturbed energy for a given trial function, which were "hard-wired" into the code by text editing. The overall code, with these modifications, was named EROICA. This code is on file on magnetic tape storage (FILEM) on the MFE computer network, under account number 304LLC.

The two subroutines accept input data from the mover subroutine, called MOTION, and, at each timestep, calculate the quantity $\vec{V} \cdot \delta \vec{A}$ for each particle in the system. Because many real frequencies and growth rates are used for the same mode, the quantity $\vec{V} \cdot \delta \vec{A}$ is calculated separately for each frequency and growth rate. (Typically, 60 real frequencies and 3 growth rates were used.) The two quantities appearing in the energy principle

$$G \equiv g^*g = \left| \int \vec{V} \cdot \delta \vec{A} dt' \right|^2, \quad (A-3)$$

$$AC \equiv \int (g^* \frac{dg}{dt} - g \frac{dg^*}{dt}) dt' = \int \vec{V} \cdot \delta \vec{A}(t) \vec{V} \cdot \delta \vec{A}^*(t+s) dt ds - c.c.,$$

are kept as variables in a frequency-dependent array, and are known as WERNER and ZIEGLER, respectively. The weighted sum of these two terms,

$$\sum_e \left[(\omega \omega^* - \omega_0 \ell \Omega) g_e^* g_e + i \ell \Omega \gamma \int (g_e^* \frac{dg_e}{dt} - g_e \frac{dg_e^*}{dt}) dt' \right], \quad (A-4)$$

then represents the total spectral energy of the plasma. When multiplied by normalizing factors, this equals $\delta^{(2)}U_2$, the second part of the energy principle outlined in the text. By plotting the perturbed energy as a function of real frequency for a given mode and growth rate, one can find the minimum spectral energy and thus calculate easily the total perturbed energy for the given mode. Typical output for these quantities are graphed in Figs. (A.1) and (A.2).

Because the perturbed magnetic energy $\int \delta B^2 d\tau$ depends on the function $\delta \vec{A}$ through a space integration and a spatial derivative operation, numerical data of this kind would be of low numerical quality. Instead, the first part of the perturbed energy is calculated analytically, and the field energy $\int (\delta B^2) d\tau$ is added to the second part of $\delta^{(2)}U_1$:

$$\delta^{(2)}U_1 = \int d^3x \left[(\delta B^2) - \frac{m\Omega^2 R^2}{T} \left(\frac{\omega_p}{c} \right)^2 \delta A_\theta^2 \right]. \quad (A-5)$$

Thus the total energy of the system can be obtained directly by adding the spectral energy terms calculated above to the magnetic energy and the rotational energy

$$\int \frac{m\Omega^2 R^2}{T} \left(\frac{\omega_p}{c} \right)^2 \delta A_\theta^2 d^3x.$$

Figure A.1. Typical output for the spectral energy array WERNER as a function of frequency. This trial function is an $\ell = 2$ mode.

Figure A.2. Typical output for the autocorrelation energy array ZIEGLER as a function of frequency for one growth rate. This trial function is an $\ell = 2$ mode.

For each mode this term is calculated numerically and the analytic result for the magnetic energy is added. Magnetic energy calculations for $\ell = 0$ and $\ell = 1, 2$ modes are given in Appendices B and C, respectively.

EROICA, the modified version of SUPERLAYER with stability calculations, was run with a specific set of input parameters representing a reversed ion equilibria. The specific equilibrium generated is given in Table A-1. This equilibrium corresponds to the equilibrium parameters used in field-reversal studies in the SUPERLAYER program in June 1975. This particular set of input data, plus the specific form of the code at that time, gave the code the specific title SUPP58. Throughout this text the equilibrium parameters used for stability studies are called the SUPP58 data set.

In addition to the physical parameters: injection current, applied fields, and the like, there are a number of numerical parameters, setting matrix sizes, time steps, and so on. These are given in Table A-2. Most of this data is self-explanatory. One point may need further explanation, however. The time sequence for the buildup portion of the code calls for injection for the first 256 timesteps. After that, some time is required for a quiescent state to develop before stability calculations to be started. Thus, the stability code portion is not activated until the 601st timestep. The stability code then runs until the 1201st timestep, at which time the code stops and the stability graphs, along with other diagnostic information, are printed out.

One would expect that the code would run more quiescently during the stability integration cycle by "freezing" the magnetic field. This can be done by a simple "switch" inserted in the code control sequence.

Table A-1. SUPP58 Physics Parameters

<u>Injection Parameters</u>		
Ion Component		100% Deuterium
External Field	B_0	5×10^3 Gauss
Injection Current	I	2.532×10^4 Ampere
Pulse Length of Injection	τ	2.09×10^{-6} Sec
External Field Mirror Ratio	R_m	1.0
Distance Between Injection and Device Midplane	R_0	12.0 cm
Ion Injection Energy	E_0	38.0 keV
Ion Cyclotron Frequency in External Field	Ω_c	$2.39 \times 10^7 \text{ sec}^{-1}$
Ion Gyroradius in External Field	ρ_0	8.0 cm
Particles Lost After Injection		$1/256$ of total ^a
<u>Steady State Parameters (After Injection)</u>		
Average Ion Energy	$\langle E_i \rangle$	31.7 keV
Ion Temperature	T_i	6.9 keV
Average Rotation Rate	Ω	$1.556 \times 10^7 \text{ sec}^{-1}$
Electron Temperature	T_e	0 eV ^b
Peak Density	n_{max}	$9.5 \times 10^{13} \text{ cm}^{-3}$
Peak Beta	β_{max}	3.36
Radius at Peak Density	R_{peak}	10 cm
Axial Plasma Length	L_z	30 cm

Footnotes:

- a.) One simulation particle left the system by a spiralling orbit along the Z axis.
- b.) The electron temperature enters the code only as an adjustable electron drag. This drag was set to zero. There were no electron pressure effects; therefore, this simulation effectively used $T_e = 0$.

Table A-2. SUPP58 Code Parameters

Timestep Size	$0.1963 \Omega_c^{-1}$
Injection Cycle	Every 8 timesteps
Pulse Length for Injection	256 timesteps
Total Code Run-time	1200 timesteps
Stability Integration Cycle	Last 600 timesteps
Injection Position Groups	8 axial x 1 radial
Minimum & Maximum Z Positions for Injection	$-1.0 R_0$ to $+1.0 R_0$
Mesh for B-field Solver	65 axial x 29 radial
Size of Mesh	$5.334 R_0$ axial x $3.111 R_0$ radial
Total "Particles" Injected	256
Field Solver Boundary Condition	Conducting wall at the 29th radial meshpoint

It turns out, however, that the errors due to finite time differencing, combined with the errors due to fast Fourier transformation and gridding of the magnetic field will cause numerically unstable orbit calculations. Thus if the field is "frozen," at the start of the stability cycle, one finds that the particles will slowly walk out of their equilibrium configuration over the remaining 600 timesteps. With the self-consistent field solver left on, however, the numerical errors are "shorted out" by the azimuthal electric field, which represents a sort of magnetoacoustic noise, and thus the equilibrium is maintained with a small amount of noise. (Typically $\delta n/n \sim 5\%$ RMS.) The runs done for the stability analysis here were thus done retaining the self-consistent fields but contain some noise. It should be stressed, however, the electromagnetic perturbations used in the energy principle are never actually applied to the simulation itself, and thus there can be no interaction between the electromagnetic noise and the perturbed electromagnetic field used in the stability calculation. This was possible because the analysis is formulated so that the unperturbed orbit data form the basic input to the stability integrals. Because the equilibrium parameters are left unperturbed in the simulation, there is an advantage in this technique over stability codes where the orbits are perturbed, as the latter technique suffers from numerical noise.

Appendix B. Calculation of Stability to $\ell = 0$ (Tearing) Modes

This appendix is divided into two parts. The first deals with an algorithm for the proof of stability to axisymmetric modes with a conducting wall at the edge of the plasma. This is done by generating a sufficient condition of stability from the general energy principle. The second part involves a study of stability properties of axisymmetric models for more relaxed boundary conditions in light of the observed stability of the simulation code itself and the code's own (more liberal) electromagnetic boundary conditions. An explanation of the observed stability can be explained in terms of the general energy principle, including the orbit effects.

The SUPP58 equilibrium furnishes diagnostic quantities from which one can make a variational calculation to establish the stability of a field-reversed mirror to axisymmetric perturbations. A sufficient condition for stability can be applied by looking at the energy principle for $\ell = 0$ and dropping a positive definite term.

For $\ell = 0$ perturbations, the perturbed second order variation of the total system energy is

$$\delta^{(2)}U = \int d^3x \left\{ \delta B^2 - \left(\frac{\omega_p}{c}\right)^2 \frac{m\Omega^2 R^2}{T} \delta A_\theta^2 + \left(\frac{\omega_p}{c}\right)^2 \omega \omega^* \int g^* g d^3v \right\}. \quad (B1)$$

The sufficient condition for stability that can be generated from this equation follows from dropping the third term, representing the frequency-dependent spectral energy of the perturbation. It is a positive definite term. This then yields the equation

$$\delta^{(2)}U \geq \int d^3x \left\{ \delta B^2 - \frac{m\Omega^2 R^2}{T} \delta A_\theta^2 \left(\frac{\omega_p}{c}\right)^2 \right\}. \quad (B2)$$

Since $\frac{\omega_p^2}{c^2}$ is a function of density only, and since all the parametric quantities Ω , T , R_0 , etc., can be determined from the diagnostics for the SUPP58 equilibrium, one can formulate eq. (B2) as a variational principle in space only. Azimuthal symmetry makes this a two-dimensional problem. By substituting for the magnetic energy term by an integration by parts, one has

$$\delta^{(2)}U \geq \int d^3x \left\{ -\delta A_\theta \left(\nabla^2 - \frac{1}{R^2} \right) \delta A_\theta - \left(\frac{\omega_p}{c} \right)^2 \frac{m\Omega^2 R^2}{T} \delta A_\theta^2 \right\}, \quad (B3)$$

and this is to be minimized with respect to the normalization $\int \delta A_\theta \delta A_\theta d^3x$. To avoid a numerical calculation of the field energy one develops an analytic fit for the density profile that fits the computer data for the SUPP58 input parameters. A good estimate of the density profile, found by fitting a analytic expression to the diagnostic output from SUPP58, is

$$n(R,Z) = 2 n_{\max} R^2 (1-R^2-Z^2). \quad (B4)$$

The quality of this fit is shown in Fig. B1.

For the SUPP58 equilibrium, $n_{\max} = 8.5 \cdot 10^{13} \text{ cm}^{-3}$ and $R_0 = 13.2 \text{ cm}$ is the normalization length for the density cutoff. Using these parameters, one can fit eq. (B4) into dimensionless form by using

$$\delta^{(2)}U \geq \int d^3x \left\{ -\delta A_\theta \left(\nabla^2 - \frac{1}{R^2} \right) \delta A_\theta - 332 R^4 (1-R^2-Z^2) \delta A_\theta^2 \right\}. \quad (B5)$$

Because of the highly nonlinear nature of this equation, the root for the minimizing mode must be found by successive approximation and trial functions. The trial function boundary condition is determined by the presence of a conducting wall applied at the edge of the plasma, $R_0 = 13.2 \text{ cm}$. Further, only modes with finite magnetic energy need be considered: this restricts the behavior of the trial function at

Figure B.1. Analytic fit for SUPP58 density profile for investigation of $\ell = 0$ modes. The dotted curve is the code diagnostic output; the solid, the analytic model.

the origin. Thus we have

$$\delta A_\theta \sim R^\ell, \ell \geq 1 \quad (B6)$$

near the Z-axis $R=0$. The combination of these two conditions, along with the knowledge that the lowest-order mode will have only one maximum (in other words, the number of nodes $n=0$), yield an educated guess for the form of modes of minimal energy:

$$\delta A_\theta \sim R^\ell (1-r^{2m}) \quad (B7)$$

Here ℓ and m are integers. These trial functions were used, and the resultant values for the right-hand-side of eq. (B5) were tabulated for the lowest three nodes of this form. These numbers are given in Table B-1.

The values for the perturbed energy for these $\ell = 0$ trial functions are all positive. Moreover, the difference in energy between these modes, when normalized by dividing by N , the normalization index, is very small. (This can be shown by noting that $\frac{1}{N} \int \delta B^2 d^3x \sim 20-30$ for these modes, whereas $S/N \sim 0.6 \rightarrow 2$ for these modes.) Thus a mode of this sort must be near that required to make S/N stationary and thus be the minimum. This is evidence that the perturbed energy for any $\ell = 0$ mode is positive and thus the system is stable to $\ell = 0$ perturbations.

As indicated in Chapter 5, this is really not very surprising. Since the dynamical code has as degrees of freedom R and Z , if any instability were physically possible it would be present as a numerical instability in the computer simulation itself. That noise can exist in the magnetic field and density without growing in intensity or disrupting the geometry is a sign that instability does not exist purely for want

Table B-1. Perturbed energy calculations for $\ell=0$ trial functions.

δA_θ	$N \equiv \int \delta A_\theta \delta A_\theta$	$S = \text{expression (B.5)}$	$\frac{S}{N}$
$R(1-r^4)$	0.072933	0.18693	2.56303
$R^2(1-r^2)$	0.012313	0.002403	1.9515
$R(1-r^2)$	0.03386	0.021225	0.626

Note here that $r^2 \equiv R^2 + Z^2$. Note also that the last trial function above is essentially the equilibrium vector potential A_θ .

of an appropriate perturbation. In fact, the almost constant amplitude of noise during the computer run would indicate that the system has poles right on the real axis such that it is marginally stable. This is borne out by the rather low values for the minimum perturbed energy in Table B-1. A graph of the noise in the computer run is given in Fig. B.2.

Another facet of the simulation must be taken into account. For the purpose of inverting the electromagnetic Poisson equation to find the self-magnetic field, the SUPERLAYER code employs boundary conditions that the electric field vanish at $R=30$ cm and at $Z = \pm 50$ cm. This represents the presence of a conducting wall at these points, forming a cylindrical shell around the device. This boundary is much farther away from the plasma than the 13.2 cm wall used to evaluate the sufficient condition for stability to tearing modes.

One may ask, then, if tearing modes are stabilized by a wall at a much further distance than the 13.2 cm spherical wall assumed above. Specifically, is the system stable to tearing with the wall at the 30 cm x 50 cm points used in the equilibrium solver? The answer is that the system may be marginally stable to tearing with the wall at 30 cm x 50 cm. However, the sufficient condition for stability outlined above will fail for modes extending out to the larger radius and axial length. This does not mean that the plasma is unstable to tearing, however. Because the equation (B2) is a sufficient condition for stability, it is generally an underestimate of the stability to various perturbations. The system can be proved stable with the condition that a conducting wall is at $R_0=13.2$ cm. However, a stronger condition is needed to prove

Figure B.2. Average radius vs. time for SUPP58 run indicating noise or "breathing" in the device. The RMS amplitude of density fluctuations from this behavior is about five percent.

stability of the system with the more remote wall. This can be done by including the third term in eq. (B1). This positive definite term, when added, could form a more general condition for stability to $\ell=0$ modes.

The magnitude of this term can be approximated using the method outlined in Chapter 6. By assuming that the orbits are mainly ringlike in their properties, the orbit integral to $\ell=0$ perturbations (where $\delta\vec{A} = \hat{\theta} \delta A_\theta$) becomes

$$g \equiv \int_0^t \vec{v} \cdot \delta\vec{A} dt' = \frac{v_\theta \delta A_\theta(t=0)}{-i\omega} \quad (B8)$$

Then the third term in eq. (B1) becomes

$$\begin{aligned} \int d^3x \left[\left(\frac{\omega_p}{c}\right)^2 \omega \omega^* \int g g^* d^3v \right] &\approx \int \left[\left(\frac{\omega_p}{c}\right)^2 \int v_\theta^2 \delta A_\theta^2 d^3v \frac{f_0}{T} \right] d^3x \\ &= \int \left(\frac{\omega_p}{c}\right)^2 \delta A_\theta^2 \left(\frac{m}{T}\right) \left[\langle v_\theta^2 \rangle + \Delta^2 v_\theta \right] d^3x \\ &= \int \left(\frac{\omega_p}{c}\right)^2 \delta A_\theta^2 \left(\frac{m\Omega^2 R^2}{T} + 1\right) d^3x \end{aligned} \quad (B9)$$

Hence the entire perturbed energy, as given by eqn. (B1), becomes

$$\begin{aligned} \delta^{(2)}U &= \int d^3x \left\{ \delta B^2 - \left(\frac{\omega_p}{c}\right)^2 \left(\frac{m\Omega^2 R^2}{T}\right) \delta A_\theta^2 + \left(\frac{\omega_p}{c}\right)^2 \left(\frac{m\Omega^2 R^2}{T}\right) \delta A_\theta^2 \right. \\ &\quad \left. + \left(\frac{\omega_p}{c}\right)^2 \delta A_\theta^2 \right\} \\ &= \int d^3x \left\{ \delta B^2 + \left(\frac{\omega_p}{c}\right)^2 \delta A_\theta^2 \right\} \end{aligned} \quad (B10)$$

This result for the perturbed energy is positive definite for all trial functions δA_θ . Thus, if the orbit integrals for all perturbations yield perturbed energies similar to the ringlike result, then the system would be stable to all $\ell=0$ modes. In fact, however, as the conducting walls are moved away from the plasma, trial functions can be used which, by

virtue of having a node at the peak density of the plasma, cause reduction of the spectral energy terms and failure of the ringlike model. Perhaps the conducting wall position enabling this to happen with the simulation parameters used corresponds to the placement of the boundaries in the dynamical field solver routine in the code (that is, at the 30 x 50 cm points). This could explain the observed quiescent noise which exists without growth or decay in the code. If the plasma forms an equilibrium marginally stable to tearing (as is often the case both in electromagnetic codes and in actual experiments) with the boundary conditions on allowed dynamical modes prescribed in the field solver, the perturbed energy of the lowest mode would be exactly zero. This corresponds to the plasma's accessibility to modes of higher order which would violate the ringlike approximation.

Appendix C. Numerical Calculations for $\ell = 1$ and $\ell = 2$ Modes

C.I Introduction

The SUPPS8 stability code produces data for spectral energy and rotational energy to apply to the energy principles developed in the text. In the following, the method and data for $\ell = 1$ and $\ell = 2$ modes is given. A particular case of trial function is evaluated fully, showing the tendency for instability at low growth rate in this configuration.

Several preliminary considerations must be taken into account. First, the magnetic energy calculation depends on the equilibrium magnetic field. Since the SUPERLAYER code calculates the magnetic field only as a grid quantity, curvature-dependent quantities cannot be derived from data of this type. Hence an analytic model of the approximate magnetic equilibrium is used to derive the magnetic field energy for a mode. Secondly, consideration must be given to the electron physics at the field null. As the version of SUPERLAYER used is not self-consistent in electron physics, some justification must be given to neglecting electron currents. What has been done in the following treatment is to estimate the region where ion magnetic fields may cause nonfluid electron response (the stochastic electron region), and "smoothed" magnetic and electric fields are applied to this region to obtain the energy. For a mode dominated by the bulk kinetic ion movement, the perturbed fields at the field null must be small. Thus the field-null region is treated separately here.

As the code calculates the rotational energy from the particle statistics, this information is derived directly from output data. The

spectral energy is given as a graph plotting energy vs. real frequency, and the minimum spectral energy is then used to calculate the minimum total energy as a function of growth rate. Finally, this information is tabulated in Table (C.1).

C.2 Magnetic Energy Calculation

Since the quantity $\int \delta B^2 d^3x$ appears in the energy principle, it is necessary to evaluate this integral for a given trial function $\delta \vec{A}$. As indicated in Chapter 5, $\delta \vec{A}$ is chosen as

$$\delta \vec{A} = \vec{\nabla} \chi (\hat{\chi} F(\psi, \chi) e^{i\ell\theta}) .$$

For the purposes of calculation, an approximation of modes of this sort is given by

$$\delta \vec{A} = \hat{\psi} \nabla_{\perp} F + \hat{\theta} \frac{i\ell}{R} F , \quad (C1)$$

which is the form actually applied to the particle code. It is most appropriate, then, to calculate $\delta \vec{B}$ from the trial function actually applied in the code, and thus we take

$$\begin{aligned} \delta \vec{B} &= \vec{\nabla} \chi (\hat{\psi} \nabla_{\perp} F + \hat{\theta} \frac{i\ell}{R} F) \\ &= \nabla_{\perp} F (\vec{\nabla} \chi \hat{\psi}) + \hat{\psi} \chi \vec{\nabla} (\nabla_{\perp} F) + (\vec{\nabla} \chi \hat{\theta}) \frac{i\ell}{R} F \\ &\quad + \hat{\theta} \chi \vec{\nabla} \left(\frac{i\ell}{R} F \right) . \end{aligned} \quad (C2)$$

It is clear from equation (C2) that the exact form of $\delta \vec{B}$ depends on the curvilinear unit vectors $\hat{\theta}$ and $\hat{\psi}$. To evaluate $\delta \vec{B}$, one must thus have a knowledge of $\vec{B}_0$, since $\hat{\psi}$ is determined by

$$\hat{\psi} = \frac{\hat{\theta} \times \vec{B}}{|\vec{B}|} . \quad (C3)$$

For the purposes of calculation it is necessary to have a model for the magnetic geometry. The model used here is that the magnetic lines form toruses around the reverse point R_0 . One then has a new coordinate system to evaluate the magnetic energy. This coordinate system is illustrated in Fig. (C.1). For use in the SUPP58 equilibrium, $R_0 = 0.58 \approx \frac{1}{\sqrt{3}}$. A transformation exists between the old coordinate system and the new coordinate system; it is

$$R = R_0 + r \cos \theta \quad , \quad r < R_0 \quad , \quad (C4)$$

$$Z = r \sin \theta \quad .$$

The total volume of the device is taken to be the toroidal region between $0 \leq r \leq R_0$ plus a remaining region with $\cos \theta < 0$, $r \geq 1$, $-R_0 \leq Z \leq R_0$, an hourglass-shaped region near the magnetic axis. The perturbed magnetic energy from this hourglass-shaped region is added to the magnetic energy from the toroidal section. Thus

$$\int \delta B^2 d^3x = \delta W_{\text{Magnetic}}^{(1)} + \delta W_{\text{Magnetic}}^{(2)} + \delta W_{\text{Magnetic}}^{(3)}$$

where (1), (2), and (3) are the toroidal region, the null region, and the hourglass-shaped region, respectively. This is illustrated in Fig. (C.1). The equivalent fluid perturbation $\vec{\xi}$ resulting in this perturbed field is illustrated in Fig. (C.2).

The hourglass-shaped region can be evaluated directly by using an approximate form of the magnetic field. Since the magnetic lines are nearly in the $\hat{z}$ direction near the axis, the evaluation of the magnetic field there is simple since

$$\delta A_\theta = \frac{\partial}{\partial R} F = R(1-R^2-Z^2) \quad .$$

Then we have

$$\delta B_z = \frac{1}{R} \frac{d}{dR} (R \delta A_\theta) \approx 2 \quad ,$$

Figure C.1. Magnetic field geometry for calculation of perturbed magnetic energies for kink and flute modes.

Figure C.2. Equivalent fluid perturbation ξ_r vs. radius at midplane for $\ell = 1, 2$ modes.

for $R \ll 1$. This is the dominant magnetic field near the axis, since the transverse fields are small (~ 0.1). Thus to within 0.1% accuracy $\delta B^2 = \delta B_Z^2$. We then estimate the magnetic energy contribution from region III as

$$\begin{aligned} \delta W_{\text{mag}}^{(3)} &\approx \delta B_Z^2 \cdot \text{Volume} \\ &= 0.46 \end{aligned} \quad (C5)$$

Next the magnetic energy at the null region can be evaluated by using the regularized fields. It is desired to fit a smoothly varying field to the null region that links onto the field in region II. The boundary field at $r = 0.1$, which was chosen as the boundary for the null region, looks like

$$\delta A_\phi(r=0.1) = A_0 \cos\theta, \quad A_0 = 0.19 \quad (C6)$$

To create a smooth profile inside the null region that goes to zero at the null, the field was chosen as

$$\begin{aligned} \delta A_\phi(r < 0.1) &= A_0 (r/r_0) \cos\theta, \\ \delta A_r(r < 0.1) &= \frac{1}{3} (r/r_0)^2 \frac{i\ell}{R_0} \cos\theta. \end{aligned}$$

This field has zero divergence and fits the boundary condition at $r = r_0 \equiv 0.1$. The magnetic energy is then easily calculated for this region of the device, and it gives a value

$$\delta W^{(2)} = 0.24 \quad (C7)$$

Finally, the magnetic energy for the bulk of the plasma region I, is evaluated using the magnetic geometry of Figure C1. Here the complete field structure is very complicated. On the special geometry utilized, the two components of the vector potential $\delta \vec{A}$ are derived from the functions F and $\nabla_\perp F$ written in the new variables. These forms are

$$\begin{aligned}
F &= \left(\frac{1}{3} + r^2 \cos^2 \theta + \frac{2}{\sqrt{3}} r \cos \theta \right) \left(\frac{5}{6} - \frac{r^2}{2} - \frac{1}{\sqrt{3}} r \cos \theta - \frac{r^2}{2} \sin^2 \theta \right), \\
2\nabla_{\perp} F &= \left(2r \cos^2 \theta + \frac{2}{\sqrt{3}} \cos \theta \right) \left(\frac{5}{6} - \frac{r^2}{2} - \frac{1}{\sqrt{3}} r \cos \theta - \frac{r^2}{2} \sin^2 \theta \right) \\
&+ \left(\frac{1}{3} + r^2 \cos^2 \theta + \frac{2}{\sqrt{3}} r \cos \theta \right) \left(r + \frac{\cos \theta}{\sqrt{3}} + r \sin^2 \theta \right). \quad (C8)
\end{aligned}$$

These expressions are differentiated by r and by θ to get the perturbed magnetic fields δB_{θ} , δB_z , and δB_r . These expressions are then squared and integrated over volume to get the total magnetic energy. The resulting expressions are quite complicated; for example, the δB_{θ}^2 expression contains 64 terms. The result of the calculation is that

$$\delta W_{\text{magnetic}}^{(1)} = 1.2980 + 0.0107 \ell^2 + 0.1466 \ell^4 \quad (C9)$$

so that

$$\delta W_{\text{magnetic}}^{\text{total}} = 2.004 + 0.0107 \ell^2 + 0.1466 \ell^4. \quad (C10)$$

While some care was taken in assuring the accuracy of this magnetic energy calculation, the geometrical assumptions involved may limit its accuracy to perhaps thirty percent. The overall result for growth rate is, however, quite insensitive to the magnetic energy, since the magnetic energy is a small part of the total perturbed energy at low growth rates. In fact, a fifty percent decrease in growth rate for the $\ell = 2$ case would require a factor of three increase in the magnetic energy.

C.3 Spectral Energy and Conclusion

As was mentioned in Chapter 6, the algorithm for calculating the perturbed energy arising from the $\delta \vec{J} \cdot \delta \vec{E}$ term in the energy principle

was evaluated by means of calculating the orbit integral for each particle. These integrals were tabulated as functions of the perturbation frequency ω . Sixty real frequencies and three growth rates were used to provide complete information for the plasma low frequency response to perturbations. The orbit data representing the terms in the energy principle were then stored as data arrays WERNER and ZIEGLER in the SUB58 stability code. These quantities are

$$\text{WERNER} = \langle g^*g \rangle [(\omega_0^2 + \gamma^2) - \omega_0 \ell \Omega] ,$$

$$\text{ZIEGLER} = \langle \int_{-\infty}^t (g^* \frac{dg}{dt} - g \frac{dg^*}{dt}) dt' \rangle [\gamma \ell \Omega] .$$

WERNER AND ZIEGLER can be graphed as a function of real frequency for each growth rate. This was used as a program diagnostic. With appropriate conversion constants, these quantities, plus the rotational energy (a fixed number for all frequencies), plus the magnetic energy, give the total energy for the perturbation. A graph of the total energy versus frequency for various growth rates for the $\ell = 2$ case are given in Figure (C.3). The sharp dip around $0.7 \ell \Omega$ can be seen on this graph. Since the minimum energy as a function of growth rate is the interesting parameter, the total energy for the minimizing frequency is also given as Table (C.1). This data then presents the existence of growing modes which are variational in energy with $\delta^{(2)} U_{\min} = 0$ with growth rates of about 0.055 for $\ell = 1$ and 0.07 for $\ell = 2$. Presumably an optimal mode could exist with slightly higher growth rates, although the mode configuration does not have as significant impact on the energy as does the perturbing frequency. Therefore the above analysis, for the particular trial eigenfunction used, gives evidence of growing instability for kink and flute-like modes for a reversed-field equilibrium of the kind studied.

Figure C.3. Graph of perturbed energy vs. real frequency for $l = 2$ trial mode.

Mode	Growth Rate	Perturbed Energy (arbitrary units)			
		Magnetic	Rotational	Spectral	Total
$\ell = 1$ Kink	0.033	2.16	-0.76	-2.46	-1.06
$\ell = 1$ Kink	0.066	2.16	-0.76	-0.80	0.59
$\ell = 1$ Kink	0.100	2.16	-0.76	-0.41	0.98
$\ell = 2$ Flute	0.033	4.44	-0.76	-11.20	-7.52
$\ell = 2$ Flute	0.066	4.44	-0.76	-4.49	-0.81
$\ell = 2$ Flute	0.100	4.44	-0.76	-2.24	1.43

Extrapolated Growth Rate (units of $\frac{\omega}{\omega_{ci}}$):

$\ell = 1$ Kink Mode 0.0546

$\ell = 2$ Flute Mode 0.07874

Table C-1. Perturbed energies and growth rates for specific trial function.

Appendix D. Definitions of Symbols¹

α = species (=i for ions, e for electrons)

$\vec{A}$ = vector potential : $B = \vec{\nabla} \times \vec{A}$

β = ratio of kinetic pressure to magnetic pressure = $8\pi p B_0^{-2}$

$\hat{b}$ = unit vector in direction of $\vec{B}$

$\vec{B}$ = magnetic field

γ = imaginary part of ω

c. c. = complex conjugate

c = speed of light

$\delta^{(n)} \vec{F}$ = nth order perturbation of $\vec{F}$

D = diffusion coefficient , total derivative

D_n = nth order component of Vlasov operator

$\vec{E}$ = electric field

$G(\omega_0, \gamma)$ = spectral energy function

g = orbit integral = $\int \vec{v} \cdot \delta \vec{A} dt'$

$i = (-1)^{1/2}$

$\vec{J}_\alpha$ = current of α th species

θ = azimuthal angle

$\vec{k}$ = wavevector

λ_D = Debye length = $(T/(4\pi n q^2))^{1/2}$

ℓ = azimuthal mode number

m = mass of ion

n = density of particles

$\mathcal{O}$ = order symbol, indicates order of smallness

$\vec{\xi}$ = displacement vector

$\pi = 3.14159\dots$

ρ = charge density

ρ_i = ion gyroradius

r = radius (spherical coordinates)

R = radius (cylindrical coordinates)

R_0 = normalization length for all parameters

F_{RMS} = root-mean-square amplitude of F

S = size factor = $R_0 \rho_i^{-1}$

T = temperature (in units of energy)

$d\tau$ = spatial metric = $dx dy dz$

U = total energy of system

$\vec{v}$ = velocity of particle

ϕ = electrostatic potential: $\vec{E} = -\vec{\nabla}\phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$

χ = coordinate along $\vec{B}$

ψ = flux function = RA_θ

ω = angular frequency for perturbation

ω_0 = real part of ω

Ω = rotation frequency

Ω_{ci} = ion cyclotron frequency = $qB_0 m^{-1} c^{-1}$

Z = coordinate parallel to symmetry axis

$F_\perp$ = component of $\vec{F}$ perpendicular to $\vec{B}$

Footnote:

1. All electromagnetic quantities in this thesis are written in cgs electromagnetic units.

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