STIMULATED THOMSON SCATTERING

BY

ROSS LOREN SPENCER

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To Margo

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LIST OF SYMBOLS

<u>Symbol</u>	<u>Page Where Defined</u>
I_o	30
J	30
K	30
k	19
k_o	19
M	101
p	81
r_e	11
t_o	81
v_o	21
γ	14
β	86
Γ	26
Υ	12
Υ'	10
Δ	102
δE	21
δn	21
δv	21
$\delta \omega$	19
ϵ	81

<u>Symbol</u>	<u>Page Where Defined</u>
n	13
θ	19
θ_0	19
λ	100
ξ	80
σ	99
τ	9
ϕ	19
ϕ_0	19
Ω	24
ω	9
ω_0	9
$\bar{\omega}$	12
ω^2	58
ω'	26

STIMULATED THOMSON SCATTERING

Ross Loren Spencer

Under the supervision of Professor Keith R. Symon

The theory of stimulated Thomson scattering is investigated both quantum mechanically and classically. Two monochromatic electromagnetic waves of like polarization travelling in opposite directions are allowed to interact for a time τ with the electrons in a collisionless plasma. The electromagnetic waves have frequencies well above the plasma frequency, and their difference frequency is allowed to range upward from the plasma frequency. With the difference frequency well above the plasma frequency, the rate at which energy is transferred from one wave to the other is calculated quantum mechanically, classically from a fluid theory, and classically from an independent electron theory. The rate is calculated in both the homogeneously broadened limit, and in the inhomogeneously broadened limit. In

the unsaturated limit, the three calculations agree. In addition, the classical calculations yield the phase shift of the waves due to stimulated Thomson scattering. The fluid theory shows how stimulated Thomson scattering passes into stimulated Raman scattering as the difference frequency approaches the plasma frequency. The independent electron theory shows how the effect saturates due to non-linear Landau damping. In appendices the free electron amplifier experiment is discussed as an example of stimulated Thomson scattering, and an experiment is proposed to detect stimulated Thomson scattering by passing short high energy laser pulses through a discharge.

CHAPTER 1

INTRODUCTION

In the latter part of the nineteenth century and in the early part of the twentieth century, there was a great deal of interest in penetrating rays of various kinds. One of the most interesting examples of such rays were the rays discovered by Roentgen in 1895, the rays now known as x-rays. It was discovered that when these rays passed through matter, secondary radiation was produced. This phenomenon was explained by J. J. Thomson in 1903¹ by proposing that Roentgen rays were short pulses of electromagnetic radiation, and that the secondary rays were produced by the acceleration of "ions" acted upon by the electric field of the primary pulse. In the 1906 edition of Ref. 1 he showed that the total scattered energy per unit volume from the primary pulse is given by the expression $(8\pi/3)(n_e e^4/m^2)E$, E being the energy per unit area of the primary pulse. (Note that the usual factor of c^4 in the denominator is missing because of the units used by Thomson). This process, called Thomson scattering, is the simplest mechanism by which electromagnetic radiation is scattered by matter.

In 1917, Albert Einstein wrote a brilliant paper² on the connection between the Planck radiation law and the quantum mechanics of radiation. He proposed the process of stimulated emission, wherein the rate at which a system makes a radiative transition at frequency ω is proportional to the energy density of radiation evaluated at the frequency ω . He then showed that this assumption, together with the Boltzmann relation relating the relative numbers of states with different energies, implies a radiation law of the Planck form. The argument is very general, and suggests that any radiative transition can be stimulated.

In 1933, Kapitza and Dirac³ considered the connection between Thomson's simple radiation mechanism and Einstein's stimulated emission. They calculated the effect that stimulated Thomson scattering would have on electrons shot through a pattern of standing light waves produced by coherent light reflecting onto itself from a mirror. They proposed that the effect be detected by observing the recoil of the electrons that produce the scattering. Unfortunately, the recoil is so slight that the effect is very hard to detect in this way; in fact, the experiment has never been successfully performed.

More recently, a few papers appeared which concentrated not on the effect of stimulated Thomson

scattering on the electrons, but on its effect on the amplitudes of the electromagnetic waves involved⁴⁻⁹. The usual arrangement proposed allows two counter-propagating electromagnetic waves of different frequencies to impinge on electrons. If an electron absorbs a photon from one of the waves and emits a photon into the propagation direction of the other wave, and if the electron's velocity is just right, the emitted photon will match in frequency and wave vector the photons in the second wave. Such electrons form a system capable of making a radiative transition that may be stimulated by the presence of the wave that receives the scattered photon. The process works, of course, in both directions, causing both stimulated emission and absorption in each wave. However, because of the electron recoil during absorption and emission, the velocity of an electron that causes energy transfer to one of the waves is different from the velocity that causes the inverse process to take place. If there are more electrons with one velocity than the other, there will be net energy transfer from one wave to the other. The possibility of detecting this energy transfer was at first considered to be very small because of the large coherent electromagnetic fields required. Present lasers may be capable of overcoming this difficulty, but the problem was first overcome by

replacing one of the waves by a strong helical DC magnetic field, and by replacing the stationary electrons by a high energy beam of electrons⁶. In the highly relativistic electron frame, the helical magnetic field looks like a coherent electromagnetic wave of very large amplitude. Using this idea, a group at Stanford University built a free electron amplifier¹⁰, and subsequently a free electron laser¹¹, both working on the principle of stimulated Thomson scattering. The experimental results agree favorably with the quantum mechanical theories of the effect by Madey⁶ and Colson⁹.

The announcement that the effect discussed by Kapitza and Dirac in 1933 had at last been observed caused a great stir. Of particular interest to many investigators was a remark that appeared in a paper by Madey, Schwettman and Fairbank¹². They pointed out that in the low-energy photon limit, $\hbar\omega \ll m_e c^2$, the quantum mechanical formula for the gain of a wave due to stimulated Thomson scattering did not contain Planck's constant, and hence might have a classical interpretation. A few years later, Hopf, Meystre, Scully, and Louisell¹³ published the first paper using classical physics to obtain the correct quantum mechanical gain formula obtained by Colson⁹. Classically, the stimulated scattering is produced by a

bunching of the radiating particles along the direction of propagation of the amplified wave. As seen from the electron frame, this bunching is produced by the non-linear ponderomotive force of the two electromagnetic waves on the electrons. It causes the transverse current driven by the two waves to be non-uniform along the propagation direction of the amplified wave. This non-uniformity produces the equivalent of an array of phased antennas; by proper choice of the frequency difference between the two waves (as observed in the electron frame), this phased array can be made to take energy from one wave, and to put it into the other. The paper of Hopf et al.¹³, was accompanied by a flood of papers extending the classical results, or obtaining them in different ways¹⁴⁻²².

At the same time, there was great interest in using devices with periodic magnetic fields and intense high energy electron beams, (several of which pre-dated the Stanford device), to produce stimulated radiation by other mechanisms, like stimulated Raman scattering, stimulated Brillouin scattering, and cyclotron resonance²³⁻³⁸. In these processes, the two counterpropagating electromagnetic waves are not coupled through Thomson scattering by free electrons, but through coupling to a normal mode of the electron fluid or

plasma. The physical arrangement is the same as in stimulated Thomson scattering, but the mechanism is different; in fact, these other mechanisms often give rise to larger amplifications than are possible with stimulated Thomson scattering. There is a large literature on such processes, and much work is being done to exploit them to achieve tunable, coherent light sources in frequency ranges where conventional lasers do not exist²⁹⁻⁵⁴. Because of the similarity between the physical arrangements for observing stimulated Thomson scattering and resonance effects, there has been some confusion about the kind of process being observed in some experiments.

Apart from applications to radiation sources, however, stimulated Thomson scattering is an interesting physical process in its own right, perhaps even with astrophysical consequences⁵⁵⁻⁵⁸. It invites a closer examination of the relationship between stimulated emission in the quantum mechanical sense and stimulated emission in the classical sense, where one usually speaks of three-wave and resonant particle interactions. Some of these considerations have been discussed by others who have studied stimulated Thomson scattering⁵⁹⁻⁶³; it is hoped that this work will contribute to further understanding of stimulated Thomson scattering.

In this thesis, stimulated Thomson scattering will be discussed in all three of the contexts described above. In Chapter 2, the quantum mechanical theory is reviewed. In Chapter 3, the interaction of two counterpropagating electromagnetic waves with a cold neutralized electron fluid is examined, and expressions are derived that show how both stimulated Raman scattering and stimulated Thomson scattering can be viewed as the result of density perturbations driven by the ponderomotive force of the two electromagnetic waves. The gain formula for stimulated Thomson scattering obtained in this way is identical to that obtained quantum mechanically. In Chapter 4, stimulated Thomson scattering is treated as an effect due to the motion of independent electrons, restricting attention to situations where plasma waves and Debye screening effects can be ignored. Both the gain and dispersion are calculated, and the gain formula is again identical to the one obtained quantum mechanically. In addition, since the orbits of the electrons in the ponderomotive potential are treated exactly, the saturation of the gain and the dispersion when the electromagnetic fields are large can be calculated. In the inhomogeneously broadened limit, the unsaturated gain is shown to be analogous to linear Landau damping, while the saturation

mechanism is shown to be analogous to non-linear Landau damping. In Chapter 5, the results are summarized. In Appendix 1, the free electron amplifier experiment is discussed, and formulas are given for transforming quantities in the highly relativistic electron frame to the laboratory frame for comparing gain coefficients. In Appendix 2, formulas for the gain and dispersion functions discussed in Chapter 3 are given. In Appendix 3, a proposed experiment to observe stimulated Thomson scattering is described. Finally, in Appendix 4 are listed useful properties of the Jacobian Elliptic Functions used in Chapter 4.

CHAPTER 2

QUANTUM MECHANICAL THEORY OF STIMULATED THOMSON SCATTERING

In this chapter is presented a quantum mechanical theory of stimulated Thomson scattering following the approach of Madey⁶. Much of the theory presented here is due to L. W. Anderson and J. E. Lawler.

Consider two oppositely directed electromagnetic waves of frequencies ω_0 and ω , and of matching polarization. The wave at frequency ω_0 will be called the pump wave, that at frequency ω the probe wave. Each of the two waves is assumed to be very nearly monochromatic with narrow bandwidths $\Delta\omega_0$ for the pump and $\Delta\omega$ for the probe. At time $t = 0$ they are suddenly turned on everywhere in a plasma of electron density n_e . The interaction of the waves with the electrons is allowed to last for a time τ after which it is terminated.

This rather artificial situation has for its chief virtue simplicity. The more natural physical situation would be to allow two pulses of electromagnetic radiation to pass through each other; in that case different electrons in the plasma would interact with the two waves for different times, depending on where the electrons

were when the two pulses began to overlap. This more complicated situation can be analyzed in terms of the simpler situation considered here if each pulse contains many wavelengths of the radiation, for then when the two pulses overlap in some region, locally it is as if the interaction were simultaneously turned on over many wavelengths. The simple case may then be applied locally, using the interaction time appropriate for that location, to obtain the effect on the pulses.

The stimulated scattering of photons from the pump wave to the probe wave will now be studied. The total rate of Thomson scattering into a mode, including both spontaneous and stimulated photons, is $\gamma'(n+1)$ where γ' is the rate of spontaneous Thomson scattering into the mode, and where n is the number of photons in the mode. The primary mechanism by which photons are lost from the mode is taken to be the inverse process in which the pump wave at ω_0 stimulates the scattering of photons out of the probe wave at ω . Other loss mechanisms are ignored. The spontaneous rate per unit volume and per unit solid angle at which photons are Thomson backscattered from the pump wave is given by the equation

$$R = \frac{n_e r_e^2 I_0}{h \omega_0} \quad (1)$$

where r_e is the classical electron radius. In Eq. (1) use has been made of the fact that the differential Thomson scattering cross section in the backward direction is r_e^2 . The backscattered frequency is $\omega' = \omega_0 - 2\hbar\omega_0/mc^2$ or, approximately, $\omega' \approx \omega_0 - 2\hbar\omega_0^2/mc^2$. Since the stimulated scattering lasts only for a time τ , the scattered photons will have a band width $\Delta\omega' \sim 1/\tau$ about ω' . If $\tau \gg 1/\Delta\omega'$, then the spectrum of the scattered radiation is determined solely by the interruption of the process after a time τ . Call the normalized lineshape function for the scattered radiation g . The quantity Rg is the scattering rate into all modes at the frequency ω' per unit volume per unit solid angle per unit frequency interval. Dividing Rg by $\omega'^2/(2\pi c)^3$, the number of modes per unit volume per unit solid angle per unit frequency interval will give γ' , the rate of Thomson scattering from the pump wave into a mode. Hence, the scattering rate into a mode of the probe wave is given by the expression

$$\begin{aligned}
 \gamma &= \frac{(2\pi c)^3 R}{\omega^2} g \\
 &= \frac{8\pi^3 c^3 n_e r_e^2 I_o}{\hbar \omega^3} g(\omega - (\omega_o - 2\hbar\omega_o^2/mc^2)) \quad (2)
 \end{aligned}$$

The net gain coefficient is gotten by subtracting the rate at which the pump wave stimulates scattering out of the probe wave. Replacing ω_o and ω by $\bar{\omega} = (\omega + \omega_o)/2$ except when their difference is to be taken, and using $\hbar\omega_o \ll mc^2$ yields for the net gain coefficient

$$\begin{aligned}
 \gamma\tau &= \frac{8\pi^3 c^3 n_e r_e^2 I_o \tau}{\hbar \bar{\omega}^3} [g(\omega - \omega_o + 2\hbar\omega_o^2/mc^2) - g(\omega - \omega_o - 2\hbar\omega_o^2/mc^2)] \\
 &= \frac{32\pi^3 n_e r_e^2 I_o c \tau}{m \bar{\omega}} \frac{d}{d\omega} g(\omega - \omega_o) \quad (3)
 \end{aligned}$$

The quantity $I\gamma\tau$ is the total change in intensity of the probe wave due to the stimulated scattering.

In the homogeneously broadened limit, where the electron gas is very cold and the spectral broadening of the Thomson scattering is the result of the finite interaction time of the electrons with the two waves the normalized lineshape, g , is given by the equation^{7,9}

$$g(\omega - \omega_0) = \frac{\tau}{2\pi} \frac{\sin^2[(\omega - \omega_0)\tau/2]}{[(\omega - \omega_0)\tau/2]^2} \quad (4)$$

and the net gain is

$$\gamma\tau = \frac{8\pi^3 n_e r_e^2 I_0 c \tau^3}{m \bar{\omega}} \frac{d}{d\eta} \left(\frac{\sin^2 \eta}{\eta^2} \right) \quad (5)$$

where $\eta = (\omega - \omega_0)\tau/2$.

In an inhomogeneously broadened system, where the spectral broadening of the Thomson scattering is the result of Doppler shifts from the distribution of velocities in the electron gas, and where $f(v_z)$ is a normalized distribution function for electron velocities parallel to the light beam, the net gain is given by the equation

$$\begin{aligned} \gamma\tau &= \frac{32\pi^3 n_e r_e^2 I_0 c \tau}{m \bar{\omega}} \int_{-\infty}^{\infty} \frac{d}{d\omega} g(\omega - \omega_0 + 2\bar{\omega}v_z/c) f(v_z) dv_z \\ &= - \frac{16\pi^3 n_e r_e^2 I_0 c^2 \tau}{m \bar{\omega}^2} \int_{-\infty}^{\infty} dv_z g(\omega - \omega_0 + 2\bar{\omega}v_z/c) \frac{d}{dv_z} f(v_z) \quad (6) \end{aligned}$$

In the limit that inhomogeneous broadening dominates over homogeneous broadening, the net gain at, or near, line center is given by the equation

$$\gamma\tau = - \frac{8\pi^3 n_e r_e^2 I_0 c^3 \tau}{m \bar{\omega}^3} \left. \frac{d}{dv_z} f(v_z) \right|_{v_z = \frac{(\omega_0 - \omega)c}{2\omega_0}} \quad (7)$$

For a Maxwell-Boltzmann distribution function, the net gain is given by the equation

$$\gamma\tau = \frac{8\pi^{\frac{5}{2}} n_e r_e^2 I_0 c^3 \tau}{\bar{\omega}^3 k_B T} y e^{-y^2} \quad (8)$$

where $y = \left(\frac{mc^2}{8k_B T} \right)^{\frac{1}{2}} \frac{\omega_0 - \omega}{\bar{\omega}}$, (9)

T is the temperature of the electron gas, and k_B is Boltzmann's constant. Dreicer has discussed stimulated Thomson scattering⁴. His results can be used to derive Eq. (8) except for a difference of a factor of 2π .

The gain mechanism of the free electron amplifier is essentially stimulated Thomson scattering¹⁰. In the free electron amplifier, a relativistic electron beam is incident on a transverse magnetic field. The field is

produced by currents in a double helix wound on a long pipe so that electrons shot down the pipe see a magnetic field whose direction rotates. When transformed to a frame of reference moving with the electrons, the spatially periodic magnetic field appears as a plane electromagnetic wave; it plays the role of the pump wave. In Appendix 1, it is shown that Eq. (5), when transformed to the highly relativistic electron frame, is the same as the gain formula for the free electron amplifier calculated by others^{9,13,19,21}.

Since Planck's constant does not appear in the gain coefficient of the free electron laser, a classical approach might be expected to give the same result. This was first suggested by Madey et al¹². Since that suggestion, many papers have appeared exploring the classical interpretation of both the free electron laser and stimulated Thomson scattering^{8,15,20,30,42,54,60-62}. In the remainder of this thesis, a classical calculation is presented that yields the gain and dispersion of a plasma in the presence of two counter-propagating electromagnetic waves. The gain formula so obtained agrees, in the unsaturated limit, with that obtained quantum mechanically in this chapter. In addition, the classical calculation shows the relationship between stimulated Thomson scattering and stimulated Raman

scattering, and how the effect saturates by non-linear Landau damping.

CHAPTER 3

FLUID THEORY OF STIMULATED THOMSON SCATTERING

A. Introduction

In this chapter a fluid calculation will be described that yields the same formula for the gain coefficient as that obtained quantum mechanically in Chapter 2. In addition, the close relationship between stimulated Thomson scattering and stimulated Raman scattering is clarified, and the effect of damping in the fluid equations on the gain coefficient is obtained. Besides the gain coefficient, the dispersion due to stimulated Thomson scattering is also obtained.

In spite of being able to take the limit of stimulated Raman scattering, namely $\omega_0 - \omega = \omega_p$, the results obtained here are difficult to compare with the rather large literature on this subject (see for example Refs. 44 and 51). One of the reasons for this difficulty is that the usual approach is to Fourier transform the equations in space and time, whereas in this chapter an initial value problem is solved. This procedure produces transient solutions, and it is these transients that are of interest in connection with

stimulated Thomson scattering. A second reason is that in Raman scattering an incident electromagnetic wave decays into a plasma wave and a scattered electromagnetic wave. Here, two externally produced electromagnetic waves are incident on a plasma and act to induce a plasma wave if the resonance condition is satisfied. Thus, the approach taken here is more in the spirit of optical mixing of electromagnetic waves or of beat heating in plasmas^{30,33,35,37,39,41,43,50,51,64}.

B. A Perturbation Expansion of the Electron Fluid Equations and Maxwell's Equations

It is assumed that two counterpropagating electromagnetic waves of the same polarization are incident on a neutralized electron fluid. At time $t=0$ the two waves are instantaneously turned on, everywhere in space, in the previously undisturbed fluid. Effects due to the sudden turning on are ignored. The development in time of the intensities and relative phases of the two waves are obtained by solving the electron fluid equations and Maxwell's equations. It is assumed that the timescale for the amplitudes and phases of the waves to change is much longer than the period of either of the two waves, and in fact is much longer than

$1/\delta\omega$, where $\delta\omega = \omega_0 - \omega$. The vector potential describing the two waves is written in the form

$$\vec{A}_T = \left[\frac{c}{\omega_0} E_0 (1 + \theta_0(t)) \sin(k_0 z - \omega_0 t + \phi_0(t)) + \frac{c}{\omega} E (1 + \theta(t)) \sin(kz + \omega t + \phi(t)) \right] \hat{x} \quad (10)$$

where θ and θ_0 represent the change in the wave amplitudes and where ϕ and ϕ_0 represent the change in the phases of the waves. Note that $\vec{A}_T$ satisfies the Coulomb gauge condition, $\vec{\nabla} \cdot \vec{A}_T = 0$. At time $t=0$, θ , ϕ , θ_0 , and ϕ_0 are all zero. These quantities are assumed to satisfy the following ordering.

$$\dot{\theta}, \dot{\phi}, \dot{\theta}_0, \dot{\phi}_0 \ll \delta\omega \quad (11)$$

The electron fluid is taken to be described by the equation of motion,

$$\frac{\partial \vec{v}}{\partial t} + \vec{\nabla} \cdot \vec{v} \vec{v} = -\frac{e}{m} \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right) - \frac{1}{mn} \vec{\nabla} p \quad (12)$$

the equation of continuity,

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n \vec{v}) = 0 \quad (13)$$

an equation of state,

$$\left(\frac{p}{p_0}\right) = \left(\frac{n}{n_0}\right)^{\gamma^*} \quad (14)$$

and Maxwell's equations,

$$\vec{\nabla} \cdot \vec{E} = -4\pi e (n - n_0) \quad (15)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (16)$$

$$\vec{\nabla} \times \vec{B} = -\frac{4\pi}{c} e n \vec{v} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \quad (17)$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad (18)$$

The quantity $\vec{v}$ is the fluid velocity, p is the pressure, n is the electron fluid density, m is the electron mass, and $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields. The variable γ^* represents the adiabatic exponent and n_0 is the density of the neutralizing positive charge. There is a wealth of information contained in these equations. For the purposes of this calculation, it is sufficient to make a perturbation expansion of the equations. To lowest order, the fluid is taken to respond only to the electric fields of the two external electromagnetic waves; in the next order, the motion of the fluid along

the z-direction, the direction of propagation of the waves, is considered. Hence, the following perturbation scheme is adopted.

$$\vec{V} = \vec{V}_0 + \delta v \hat{z} \quad (19)$$

$$n = n_0 + \delta n \quad (20)$$

$$\vec{E} = \vec{E}_T + \delta E \hat{z} \quad (21)$$

$$\vec{B} = \vec{B}_T \quad (22)$$

The quantity $\vec{V}_0$ is the fluid velocity driven by the external electric fields while δv , δn , and δE describe the motion of the fluid in the z-direction and the longitudinal electric field produced by that motion.

Equations (19)-(22) are substituted into the fluid equations and Maxwell's equations, Eqs.(15)-(18), and the system of equations is linearized to obtain the set of equations

$$\frac{\partial \vec{v}_0}{\partial t} = -\frac{e}{m} \vec{E}_T \quad (23)$$

$$\frac{\partial \delta v}{\partial t} = -\frac{e}{mc} (\vec{v}_0 \times \vec{E}_T)_z - \frac{e}{m} \delta E - \frac{\gamma^* k_B T}{mn_0} \frac{\partial \delta n}{\partial z} \quad (24)$$

$$\frac{\partial \delta E}{\partial z} = -4\pi e \delta n \quad (25)$$

$$\frac{\partial^2}{\partial z^2} \vec{A}_T - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A}_T - \frac{\omega_p^2}{\omega^2} \vec{A}_T = \frac{4\pi}{c} e \delta n \vec{v}_0 \quad (26)$$

$$(27)$$

This procedure is similar to that followed in Refs. 38 and 64. In obtaining these equations it was assumed that the fluid and the fields are homogeneous in the x- and y-directions so that all derivatives with respect to x and y vanish.

C. The Motion of the Electron Fluid Driven by the Ponderomotive Force for Weak Plasma Response

With the ordering of Eq. (11), this set of equations may be solved by iteration: find the perturbed fluid quantities assuming that θ , θ_0 , ϕ , and ϕ_0 are negligible,

then substitute the perturbed fluid quantities into the wave equation, Eq. (27), to determine θ , θ_0 , ϕ , and ϕ_0 . Equation (23) is solved first to obtain

$$\vec{v}_0 = \frac{e}{mc} (\vec{A}_T - \vec{A}_T|_{t=0}) \quad (28)$$

Note that the requirement that the fluid be initially undisturbed has been applied. This expression for $\vec{v}_0$ is substituted into the right hand side of Eq. (24). The term containing $\vec{v}_0$ in Eq. (24) acts as an external driving force in the longitudinal equation of motion, and is simply the familiar ponderomotive force together with a term due to the initial conditions:

$$-\frac{e}{mc} \vec{v}_0 \times \vec{B}_T = -\frac{e^2}{2m^2 c^2} \vec{\nabla} A_T^2 + \frac{e^2}{m^2 c^2} \vec{A}_T|_{t=0} \times (\vec{\nabla} \times \vec{A}_T) \quad (29)$$

The ponderomotive force is the force that produces stimulated Thomson scattering, and must be treated with care. The importance of the ponderomotive force may be seen by expanding A_T^2 in the following way.

$$\begin{aligned}
A_T^2 = & E_0^2 \frac{c^2}{\omega_0^2} \sin^2(k_0 z - \omega_0 t) + E^2 \frac{c^2}{\omega^2} \sin^2(kz + \omega t) \\
& + \frac{E_0 E c^2}{\omega \omega_0} [\cos((k_0 - k)z - (\omega_0 + \omega)t) - \cos((k_0 + k)z - \delta \omega t)] \quad (30)
\end{aligned}$$

The first three terms in this expression are high phase velocity terms and can have little effect on the electron motion, but the fourth term has phase velocity much less than the speed of light; it can influence the electron fluid more strongly than the other terms. Assuming that the high phase velocity terms in Eq. (30), as well as similar terms arising from the initial value term in Eq. (29) are negligible, the ponderomotive force term in the fluid equation of motion may be written in the form

$$\frac{\vec{F}_p}{m} = -\frac{e}{mc} (\vec{v}_0 \times \vec{B}_T) \approx -\frac{1}{2} \frac{\Omega^2 c}{\omega} \sin((k_0 + k)z - \delta \omega t) \hat{z} \quad (31)$$

$$\text{where} \quad \Omega^2 = \frac{(k_0 + k) e^2 E E_0}{m^2 c \omega} \approx \frac{2 E_0 E e^2}{m^2 c^2}$$

and where $\omega = (\omega + \omega_0)/2$. Substituting this term into Eq. (24) and Fourier transforming Eqs. (24)-(26) in z with respect to the Fourier transform variable k' yields the following set of equations.

$$\frac{\partial}{\partial t} \tilde{\delta v} = \frac{\tilde{F}_p}{m} - \frac{e}{m} \tilde{\delta E} - \frac{i k' \gamma^* k_B T}{m n_0} \tilde{\delta n} \quad (32)$$

$$\frac{\partial}{\partial t} \tilde{\delta n} + i k' n_0 \tilde{\delta v} = 0 \quad (33)$$

$$i k' \tilde{\delta E} = -4\pi e \tilde{\delta n} \quad (34)$$

Combining these equations yields the single equation for $\tilde{\delta n}$,

$$\frac{d^2}{dt^2} \tilde{\delta n} + \left(\omega_p^2 + \frac{\gamma^* k_B T}{m} k'^2 \right) \tilde{\delta n} = - \frac{i k' n_0}{m} \tilde{F}_p \quad (35)$$

The left hand side by itself would simply yield the familiar Bohm-Gross dispersion relation for plasma waves. The right hand side represents the effect of the low frequency part of the non-linear ponderomotive force. Mathematically, the equation is simply that describing the driven harmonic oscillator.

Equation (35) does not include any damping effects. Since plasma waves may be damped either collisionally or

by Landau damping, a phenomenological damping term is added to Eq. (35) to obtain the equation

$$\frac{d^2}{dt^2} \tilde{n} + \Gamma \frac{d}{dt} \tilde{n} + \omega'^2 \tilde{n} = \frac{k' n_0 \Omega^2 c}{4 \bar{\omega}} [\delta(k' - k_0 - k) e^{-i\delta\omega t} - \delta(k' + k_0 + k) e^{i\delta\omega t}] \quad (36)$$

$$\text{where } \omega'^2 = \omega_p^2 + \frac{\gamma^* k_B T}{m} (k_0 + k)^2$$

Equation (36) is simply solved, but working with its solutions is very messy. However, the technique used to find the time dependent electromagnetic field amplitudes and phases can be illustrated in a very simple case which is interesting in its own right. The more complicated results obtained by working with Eq. (36) are given in Appendix 2 and in the figures. The simple case is to assume that Γ and ω' are both zero. In this case Eq. (36) can be immediately integrated to obtain

$$\tilde{n} = - \frac{k' n_0^2 c}{4\delta\omega^2 \bar{\omega}} [\delta(k' - k_0 - k) e^{-i\delta\omega t} - \delta(k' + k_0 + k) e^{i\delta\omega t}] + Ct + D \quad (37)$$

where C and D are constants. That the fluid be initially undisturbed imposes the conditions

$$\hat{\delta n} \Big|_{t=0} = 0 \quad (38)$$

$$\frac{d}{dt} \hat{\delta n} \Big|_{t=0} = 0 \quad (39)$$

which determine C and D. Equation (37) is now inverse-Fourier transformed in k' to obtain the desired solution.

$$\begin{aligned} \delta n = - \frac{n_0 \Omega^2}{\delta \omega^2} [& \cos((k_0 + k)z - \delta \omega t) \\ & - \delta \omega t \sin(k_0 + k)z - \cos(k_0 + k)z] \end{aligned} \quad (40)$$

D. The Gain and Dispersion of the Waves

Having solved for the fluid motion, the electromagnetic fields may now be determined. Substituting Eq. (10) into the wave equation, Eq. (27), and applying the ordering of Eq. (11) gives for Eq. (27)

$$\begin{aligned}
& \left[\left(-\frac{k_0^2 c}{\omega_0} + \frac{\omega_0}{c} - \frac{\omega_p^2}{c\omega_0} \right) - \frac{2\dot{\phi}_0}{c} \right] E_0 \sin(k_0 z - \omega_0 t + \phi_0) \\
& + \frac{2E_0}{c} \dot{\theta}_0 \cos(k_0 z - \omega_0 t + \phi_0) \\
& + \left[\left(-\frac{k^2 c}{\omega} + \frac{\omega}{c} - \frac{\omega_p^2}{c\omega} \right) + \frac{2\dot{\phi}}{c} \right] E \sin(kz + \omega t + \phi) \\
& - \frac{2E}{c} \dot{\theta} \cos(kz + \omega t + \phi) = \frac{4\pi}{c} e \delta n v_{ox}
\end{aligned} \tag{41}$$

Expressions for θ , ϕ , θ_0 , and ϕ_0 may be obtained by multiplying Eq. (41) by an appropriate factor and integrating over time and space. The time integration is assumed to be over a time, t , long compared to the wave period of either wave, but short compared to the time $1/\delta\omega$. The integration in z is taken to be over a region long compared with the length $1/(k-k_0)$. The procedure is illustrated by multiplying Eq. (41) by $\cos(kz) \cos(\omega t)$ and integrating as described above. Equation (41) becomes

$$-\frac{E}{2c} \dot{\theta} = \frac{4\pi}{c} e \frac{1}{Lt} \int_{-L/2}^{L/2} dz \int_0^t dt' \delta n v_{ox} \cos kz \cos \omega t' \tag{42}$$

Multiplying by $\cos(kz)\sin(\omega t)$ and integrating yields

$$\frac{1}{4} \left(-\frac{k^2 c^2}{\omega} + \frac{\omega}{c} - \frac{\omega_p^2}{c\omega} \right) E + \frac{E}{2c} \dot{\phi} =$$

$$\frac{4\pi}{c} e \frac{1}{L^2} \int_{-L/2}^{L/2} dz \int_0^t dt' \delta n v_{ox} \cos kz \sin \omega t' \quad (43)$$

Carrying out the operations on the right hand sides of Eqs. (42) and (43), and equating terms of like order, yields the familiar dispersion relation for electromagnetic waves in a plasma and differential equations for θ and ϕ .

$$\omega^2 = \omega_p^2 + k^2 c^2 \quad (44)$$

$$\dot{\theta} = \frac{4\pi^2 n_o r_e^2 I_o c}{m \omega} P(t) \quad (45)$$

$$\dot{\phi} = - \frac{4\pi^2 n_o r_e^2 I_o c}{m \omega} Q(t) \quad (46)$$

where

$$P = \frac{4}{\delta\omega^2} (\sin\delta\omega t - \delta\omega t \cos\delta\omega t) \quad (47)$$

$$Q = \frac{4}{\delta\omega^2} (\cos\delta\omega t - 1 + \delta\omega t \sin\delta\omega t) \quad (48)$$

$$I_o = \frac{c}{8\pi} E_o^2 \quad (49)$$

The differential equations are easily integrated over the time of interaction, τ , to obtain

$$\theta = \frac{4\pi^2 n_o r_e^2 I_o c}{m \omega} J \quad (50)$$

$$\phi = - \frac{4\pi^2 n_o r_e^2 I_o c}{m \omega} K \quad (51)$$

where

$$J = \tau^3 \frac{d}{d\eta} \left(\frac{\sin^2 \eta}{\eta} \right) \quad (52)$$

$$K = \tau^3 \frac{d}{d\eta} \left(\frac{1}{\eta} - \frac{\sin 2\eta}{2\eta^2} \right) \quad (53)$$

and where $\eta = -\delta\omega\tau/2$.

The functions J and K given above are displayed in Figures 1 and 2.

A similar calculation shows that

Figure 1

The Gain Curve for Unsaturated
Stimulated Thomson Scattering

The function $-J(\eta) = -\tau^3 \frac{d}{d\eta} \left(\frac{\sin^2 \eta}{\eta^2} \right)$ is displayed for
 $\tau = 1$.

FIGURE 1

Figure 2

The Dispersion Curve for Unsaturated
Stimulated Thomson Scattering

The function $K(\eta) = \tau^3 \frac{d}{d\eta} \left(\frac{1}{\eta} - \frac{\sin 2\eta}{2\eta^2} \right)$ is displayed for
 $\tau = 1$.

FIGURE 2

$$E_0^2 \theta_0 = - E^2 \theta \quad (54)$$

$$E_0^2 \phi_0 = - E^2 \phi \quad (55)$$

Equation (54) shows that energy is conserved between the two waves; to the order of approximation taken here, the fluid acts only to transfer energy from one wave to the other. This is not strictly true, of course, for the fluid obviously gains energy in the process described above. The amount of energy gained by the fluid is of order $\delta\omega/\bar{\omega}$ compared with the energy transferred between the two waves. Terms of this order have been consistently neglected in this calculation.

E. Comparison with the Quantum Mechanical Result;

The Kramers-Kronig Relations

To make the connection between Eq. (50) and the quantum mechanical result, note that $Y\tau$ given in Eq. (5) represents the fractional gain in intensity due to stimulated Thomson scattering acting for a time τ . Since $I \propto E^2(1 + 2\theta)$,

$$\gamma\tau = 2\theta = \frac{8\pi^2 n_o r_e^2 I_o c \tau^3}{m \omega} \frac{d}{d\eta} \left(\frac{\sin^2 \eta}{\eta^2} \right) \quad (56)$$

in agreement with the quantum mechanical result.

The time derivatives of θ and ϕ may be given an interpretation as real and imaginary parts of a non-linear index of refraction. Since $\dot{\theta}$ and $\dot{\phi}$ are small and vary slowly in time compared to ωt , and since $(1 + \theta) \approx \exp(\theta) = \exp(\int_0^t \dot{\theta} dt)$, the vector potential of the probe wave can be expressed as follows.

$$\vec{A} = \frac{c}{\omega} \vec{E} e^{i(kz + \int_0^t \omega^* dt)} \quad (57)$$

where $\omega^* = \omega + \dot{\phi} - i\dot{\theta}$

The index of refraction may now be written in the form

$$n^2 = \frac{k^2 c^2}{\omega^{*2}} \approx 1 - \frac{\omega_p^2}{\omega^2} - \frac{2\dot{\phi}}{\omega} + \frac{2i\dot{\theta}}{\omega} \quad (58)$$

It is interesting to note that $\dot{\theta}$ and $\dot{\phi}$, interpreted in this way, are connected by the Kramers-Kronig relations with respect to ω . The Kramers-Kronig relations apply to linear, time independent systems, and hence might not be expected to apply to the situation

considered here. The question of linearity is answered by noting that although the interaction of the two waves is non-linear, it is non-linear in a special way. The interaction is characterized by the quantity $\Omega^2 \propto EE_0$; this is a non-linear term, but is linear separately in the unperturbed amplitudes E and E_0 . Another way of saying the same thing is to say that the medium seen by the probe wave consists of the electrons plus the pump wave. Viewed in this way, the interaction of the probe wave with the "medium" is linear. The problem that the medium is time dependent is solved by showing that the same results are obtained from a different, albeit contrived, medium which is time independent. Note that Equation (36) is linear in the density of the medium, n_0 , and in the density perturbation δn . Suppose now that instead of one medium of density n_0 , there are many such media of smaller densities in the presence of the two waves. At some time, one of the media has all of its density perturbations smoothed out. It is then allowed to respond for a time τ , at which time it is once again made smooth and allowed to respond again for a time τ . The same thing happens over and over for all of the media, but the smoothing times for the various media are randomly scattered over the time interval τ . At the end of each time interval, the medium which is smoothed out

will have produced a change in the probe wave amplitude. From Eq. (57), the change can be represented, for small ϕ , by

$$\Delta \vec{E} = \vec{E} e^{i(kz + \omega t)} (\theta + i\phi) \quad (59)$$

Thus for this system, θ and ϕ are simply the real and imaginary parts of the linear response function for the probe wave. They are to be thought of as functions of ω with ω_0 and τ as parameters. When viewed in this way, it is clear that θ and ϕ should be connected by the Kramers-Kronig relations. Since these relations are linear relations between analytic functions, and since τ is simply a parameter, $\hat{\theta}$ and $\hat{\phi}$ should also be connected by the Kramers-Kronig relations.

F. The Gain and Dispersion for General Plasma Response

The effect of including nonzero ω' and Γ will now be considered. Equation (36) is easy, if messy, to solve. The solution for δn is combined with $\vec{v}_0$ to obtain the non-linear correction to the transverse current, and hence to obtain θ , ϕ , θ_0 , and ϕ_0 from the procedure outlined in Eqs. (42)-(53). Carrying out this program

yields more complicated expressions for J and K . These expressions, together with the solution of Eq. (36) for δn are given in Appendix 2.

Two cases will now be considered. In case (i) the interaction time τ is held fixed, as are Γ and ω' , while $\delta\omega$ is allowed to vary. In case (ii), the difference frequency, $\delta\omega$, Γ , and ω' are held fixed while τ is allowed to vary. Case (i) is obtained in arrangements like the free electron amplifier experiment where the difference frequency is varied by changing the electron beam energy while the other parameters are held fixed. In case (ii) the time development of the amplitudes and phases of the electromagnetic waves is considered, a natural thing to look at given the way the problem has been solved.

Figures 3-8 show the functions J and K for case (i) with $\tau = 1$ and with $0 \leq \delta\omega\tau \leq 12$. Note that in all cases the maximum gain occurs for $\omega'\tau \ll 1$ and $\Gamma\tau \ll 1$, i.e. for simple stimulated Thomson scattering. Note also that there is very little difference between the cases of critical damping and overdamping of the plasma wave. If τ has a value other than $\tau = 1$, the following identity allows the desired value of J or K to be found in terms of the case $\tau = 1$.

Figure 3

The Gain Curve for Underdamping and τ Fixed

The function J is displayed vs. $\delta\omega\tau$ for $\tau = 1$, for $\Gamma \ll \omega'$, and for 3 values of ω' :

_____	$\Gamma = .0001$	$\omega' = .001$
_____	$\Gamma = .001$	$\omega' = 2$
_____	$\Gamma = .001$	$\omega' = 4$

Figure 4

The Gain Curve for Critical Damping and τ Fixed

The function J is displayed vs. $\delta\omega\tau$ for $\tau = 1$, for $\omega' = \frac{1}{2}\Gamma$, and for 3 values of Γ :

————— $\Gamma = .019 \quad \omega' = .01$
 — — — — $\Gamma = 1.999 \quad \omega' = 1$
 — — — — $\Gamma = 3.999 \quad \omega' = 2$

Figure 5

The Gain Curve for Overdamping and τ Fixed

The function J is displayed vs. $\delta\omega\tau$ for $\tau = 1$, for $\omega' \ll \Gamma$, and for 3 values of Γ :

————	$\Gamma = .002$	$\omega' = .0001$
———	$\Gamma = 2$	$\omega' = .01$
———	$\Gamma = 4$	$\omega' = .01$

FIGURE 5

Figure 6

The Dispersion Curve for Underdamping and τ Fixed

The function K is displayed vs. $\delta\omega\tau$ for $\tau = 1$, for $\Gamma \ll \omega'$, and for 3 values of ω' :

————	$\Gamma = .001$	$\omega' = .01$
— — —	$\Gamma = .001$	$\omega' = 2$
—— —	$\Gamma = .001$	$\omega' = 4$

FIGURE 6

Figure 7

The Dispersion Curve for Critical Damping and τ Fixed

The function K is displayed vs. $\delta\omega\pi$ for $\tau = 1$, for $\omega' = \frac{1}{2}\Gamma$, and for 3 values of Γ :

————	$\Gamma = .19$	$\omega' = .1$
— — — —	$\Gamma = 1.99$	$\omega' \approx 1$
—— — —	$\Gamma = 3.99$	$\omega' \approx 2$

FIGURE 7

Figure 8

The Dispersion Curve for Overdamping and τ Fixed

The function K is displayed vs. $\delta\omega\tau$ for $\tau = 1$, for $\omega' \ll \Gamma$, and for 3 values of Γ :

————	$\Gamma = .1$	$\omega' = .01$
— — —	$\Gamma = 2$	$\omega' = .1$
—— —	$\Gamma = 4$	$\omega' = .1$

FIGURE 8

Figure 9

The Gain Curve for Underdamping and $\delta\omega$ Fixed

The function J is displayed vs. $\delta\omega\tau$ for $\delta\omega = 1$, for $\Gamma \ll \omega'$, and for 4 values of ω' :

————	Γ	$= .001$	ω'	$= .01$
———	Γ	$= .01$	ω'	$= .2$
—— —	Γ	$= .01$	ω'	$= .6$
—— — —	Γ	$= .01$	ω'	$= .99$

FIGURE 9

Figure 10

The Gain Curve for Critical Damping and $\delta\omega$ Fixed

The function J is displayed vs. $\delta\omega\pi$ for $\delta\omega = 1$, for $\omega' = \frac{1}{2}\Gamma$, and for 3 values of Γ :

————	$\Gamma = .0195$	$\omega' = .01$
— — — —	$\Gamma = .495$	$\omega' = .25$
—— — —	$\Gamma = 1.495$	$\omega' = .75$

FIGURE 10

Figure 11

The Gain Curve for Overdamping and $\delta\omega$ Fixed

The function J is displayed vs. $\delta\omega\tau$ for $\delta\omega = 1$, for $\omega' \ll \Gamma$, and for 3 values of Γ :

————	$\Gamma = .02$	$\omega' = .001$
— — —	$\Gamma = .5$	$\omega' = .01$
—— ———	$\Gamma = 1.5$	$\omega' = .01$

FIGURE 11

Figure 12

The Gain Curve at Resonance With the Plasma Wave

The function J is displayed vs. $\delta\omega\tau$ for $\delta\omega = 1$, for $\tilde{\omega} = \sqrt{\omega^2 - \frac{1}{4}\Gamma^2} \approx 1$, and for 3 values of Γ :

————	Γ	$= .01$	$\omega' = .99$
———	Γ	$= .5$	$\omega' = 1.03$
———	Γ	$= 1.5$	$\omega' = 1.24$

FIGURE 12

Figure 13

The Dispersion Curve for Underdamping and $\delta\omega$ Fixed

The function K is displayed vs. $\delta\omega\tau$ for $\delta\omega = 1$, for $\Gamma \ll \omega'$, and for 4 values of ω' :

————	Γ	$= .001$	$\omega' = .01$
— — —	Γ	$= .01$	$\omega' = .2$
—— —	Γ	$= .01$	$\omega' = .6$
— — — —	Γ	$= .00001$	$\omega' = .999$

FIGURE 13

Figure 14

The Dispersion Curve for Critical Damping
and $\delta\omega$ Fixed

The function K is displayed vs. $\delta\omega\tau$ for $\delta\omega = 1$, for
 $\omega' = \frac{1}{2}\Gamma$, and for 3 values of Γ :

_____	Γ	$= .0195$	$\omega' = .01$
-----	Γ	$= .495$	$\omega' = .25$
-----	Γ	$= 1.495$	$\omega' = .75$

FIGURE 14

Figure 15

The Dispersion Curve for Overdamping and $\delta\omega$ Fixed

The function K is displayed vs. $\delta\omega\tau$ for $\delta\omega = 1$, for $\omega' \ll \Gamma$, and for 3 values of Γ :

————	$\Gamma = .02$	$\omega' = .001$
- - - -	$\Gamma = .5$	$\omega' = .01$
————	$\Gamma = 1.5$	$\omega' = .01$

FIGURE 15

Figure 16

The Dispersion Curve at Resonance With the Plasma Wave

The function K is displayed vs. $\delta\omega\tau$ for $\delta\omega = 1$, for
 $\tilde{\omega} = \sqrt{\omega'^2 - \frac{1}{4}\Gamma^2} \approx 1$, and for 3 values of Γ :

————	$\Gamma = .00001$	$\omega' = .999$
-----	$\Gamma = .5$	$\omega' = 1.03$
-----	$\Gamma = 1.5$	$\omega' = 1.249$

FIGURE 16

$$\tau^3 L(\delta\omega\tau, \omega'\tau, \Gamma\tau, 1) = L(\delta\omega, \omega', \Gamma, \tau) \quad (60)$$

where L stands for either J or K .

Figures 9-16 show the functions J and K for case (ii). Of special interest is the change from oscillatory to monotonic behavior as either Γ becomes appreciable, or as resonance with the plasma wave, $\delta\omega = \tilde{\omega} = \sqrt{\omega'^2 - \frac{1}{4}\Gamma^2}$ is approached. The undamped resonant case (see Figs. 9, 12, 13, and 16) is of special interest since it corresponds to stimulated Raman scattering. For this case, i.e. $\Gamma = 0$ and $\delta\omega = \omega'$, the expressions in Appendix II simplify to

$$J = \frac{\tau^2}{\omega'} - \frac{\sin^2 \omega'\tau}{\omega'^3} \quad (61)$$

$$K = \frac{\tau^2}{\omega'^2} - \frac{\sin 2\omega'\tau}{2\omega'^3} \quad (62)$$

The above expression for K agrees, except for a numerical factor, with an estimate for the frequency shift of the probe wave at resonance given by Schmidt⁴². At resonance, the solution for δn given in Appendix II can also be used to obtain a simple expression for the longitudinal electric field, δE .

$$\delta E = \frac{\pi e c n_0}{\omega} \frac{\Omega^2}{\omega'^2} \left[\omega' \tau \cos((k_0 + k)z - \omega' \tau) \right. \\ \left. + \cos(k_0 + k)z \sin \omega' \tau \right] \quad (63)$$

The part of the electric field that grows linearly with time agrees with that obtained by Rosenbluth and Liu³⁷. Hence, the quadratically increasing transfer of energy from the pump wave to the probe wave is accompanied by a linear increase in the amplitude of the resonant plasma wave. Schmidt⁴² also finds the plasma wave to have a linearly growing amplitude. As discussed in Ref. 37, the plasma wave will cease to grow when the particle displacements from equilibrium in the fluid grow to be on the order of the wavelength of the electromagnetic waves. When this large amplitude regime is reached, the plasma wave "breaks" and its energy is rapidly converted to thermal energy through turbulence⁶⁵. Since the particle displacement, δx , and the electrostatic field are related by the expression³⁷

$$\delta E = -4\pi e n_0 \delta x \quad (64)$$

for δx small enough that particle crossing does not occur, setting $\delta x = 1/k$ gives for the time, τ_0 , when wave

breaking will limit the energy transfer between the electromagnetic waves,

$$\tau_c \approx \frac{\omega'}{\Omega^2} \quad (65)$$

Figures 9-16 will not apply for values of τ exceeding this limit. Note that Eq. (60) can be used to find J and K for cases where $\delta\omega \neq 1$.

G. An Approximation to the Inhomogeneously Broadened Limit

It is possible to use these results in the cold fluid limit to obtain an approximation to the gain in the inhomogeneously broadened limit. The kinetic theory of plasma waves predicts that such waves are damped by Landau damping. Indeed, the situation taken here requires the plasma oscillations to have wavelengths in the optical range; for such short wavelengths, the plasma waves are heavily damped. To find the natural frequency corresponding to this short wavelength it is necessary to use the full plasma dispersion function. Once the values of Γ and ω' corresponding to the wavelength $\frac{\lambda}{2\pi} = 1/(k+k_0)$ are found, the correct kinetic treatment of stimulated Thomson scattering can be approximated by using these

values in Eq. (36) and approximating $f(v_2)$ by a superposition of cold fluids with different velocities. In this approximation, the inhomogeneously broadened limit is obtained by folding the general gain curve, Eq. (50), against $f(v_2)$ as was done in Chapter 2. In the limit that $f(v_2)$ is much wider than J , the inhomogeneous gain coefficient is simply proportional to $\int_{-\infty}^{\infty} \eta J(\eta) d\eta$. For ω' and Γ very small, this integral has the value π^3 . This integral may also be calculated in the case where ω' and Γ are appreciable by using the more complicated expressions given in Appendix 2. A contour integration shows that the value of the integral for arbitrary Γ , ω' , and τ is still π^3 . Hence, in this approximation, Eq. (7) still gives the gain coefficient in the inhomogeneously broadened limit. An inspection of the figures, however, shows that J becomes wider as ω' and Γ become appreciable; it then becomes necessary for the distribution function to be wider to obtain the inhomogeneously broadened limit.

CHAPTER 4

INDEPENDENT ELECTRON THEORY
OF STIMULATED THOMSON SCATTERING

A. Gain and Dispersion in the Plasma

In this section sufficient formalism is introduced to determine the gain and dispersion of a plasma, due to stimulated Thomson scattering, in terms of the motion of the electrons in the plasma. As in Chapter 3, two oppositely directed electromagnetic waves, of angular frequencies ω_0 and ω , propagating in a plasma are studied. The difference is that here the electric fields produced by the electron motion will be neglected; the motion of the electrons only under the influence of the low frequency ponderomotive force will be examined. Making this simplification allows the saturation of the gain coefficient for stimulated Thomson scattering by non-linear Landau damping to be obtained. The approach taken here is similar to that used by Johnston^{60,61}, Colson¹⁹, and Cocke⁶². A particularly elegant approach was taken by Bambini and Renieri⁶³ who used the total hamiltonian, including both fields and particles, to calculate the energy transfer between the two waves.

As in Chapter 3, it is convenient to work in the Coulomb gauge, where $\vec{\nabla} \cdot \vec{A} = 0$, so that the vector potential satisfies the inhomogeneous wave equation,

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c} \vec{J} + \frac{1}{c} \vec{\nabla} \frac{\partial \phi}{\partial t} \quad (66)$$

The current density is given by

$$\vec{J} = -e \int d^3r_0 \int d^3v_0 \vec{v}'(\vec{r}_0, \vec{v}_0, t) \delta(\vec{r} - \vec{r}') F(\vec{r}_0, \vec{v}_0) \quad (67)$$

The primed coordinates, $\vec{r}'$ and $\vec{v}'$, represent the instantaneous position and velocity at time t of an electron that at time $t=0$ was at position $\vec{r}_0$ with velocity $\vec{v}_0$. The electron distribution at time $t=0$ is $F(\vec{r}_0, \vec{v}_0)$, normalized so that its integral over all velocities and over the plasma volume gives the number of electrons in the plasma. The vector potential is assumed to be of the form given in Eq. (10).

The procedure used in Eqs. (41)-(42) of Chapter 3 is also used here to obtain the following equation for $\hat{\phi}$.

$$\dot{\theta} = \frac{4\pi\omega}{EVtck} \int_0^t dt^* \int_V d^3\vec{r} \cos(kz+\omega t^*) \hat{x} \cdot \vec{J} \quad (68)$$

Equation (67) is now used to convert the right hand side of Eq. (68) into a time integral along particle orbits, averaged over initial positions and velocities. Since ω and ω_0 are assumed to be well above the plasma frequency, so that $\frac{\omega}{ck} \approx 1$, Eq. (68) can be written in the form

$$\dot{\theta} = - \frac{4\pi e}{VEt} \int_V d^3\vec{r}_0 \int_V d^3\vec{v}_0 \int_0^t dt^* \cos(kz'+\omega t^*) v_x' F(\vec{r}_0, \vec{v}_0) \quad (69)$$

Eq. (68) can be given a simple physical interpretation by writing it in the form

$$(2\dot{\theta}) \frac{E^2}{8\pi} = \frac{1}{Vt} \int_V d^3\vec{r} \int_0^t dt^* \vec{E} \cdot \vec{J} \quad (70)$$

where $\vec{E} = -E \cos(kz+\omega t) \hat{x}$. The left hand side of Eq. (70) is the product of $2\dot{\theta}$ and the energy density, U , of the probe wave. The right hand side of Eq. (70) is the average of $\vec{E} \cdot \vec{J}$ for the probe wave. Hence, Eq. (70) can be written in the form

$$\frac{d}{dt}(U) = \frac{1}{Vt} \int d^3r_0 \int_0^t dt^* E \cdot J \quad (71)$$

as expected.

The same procedure used in Chapter 3 is also used here to obtain an equation for $\frac{1}{2}(\omega - k^2 c^2/\omega) + \dot{\phi}$.

$$\begin{aligned} \frac{1}{2}(\omega - \frac{k^2 c^2}{\omega}) + \dot{\phi} \\ = - \frac{4}{VEt} \int d^3\vec{r} \int_0^t dt^* \sin(kz + \omega t) \hat{x} \cdot \vec{J} \end{aligned} \quad (72)$$

Eq. (67) is now used to convert the right hand side of Eq. (72) into a time integral along particle orbits, averaged over initial positions and velocities.

$$\begin{aligned} \frac{1}{2}(\omega - \frac{k^2 c^2}{\omega}) + \dot{\phi} \\ = \frac{4\pi e}{VEt} \int d^3\vec{r}_0 \int d^3\vec{v}_0 \int_0^t dt^* \sin(kz' + \omega t^*) v_x' F(\vec{r}_0, \vec{v}_0) \end{aligned} \quad (73)$$

Useful expressions for $\dot{\theta}$ and $\dot{\phi}$ can be obtained if the details of the electron motion are known; this is the subject of the next section.

Henceforth the prime notation is dropped on quantities to be evaluated at the position of the

electron. It will be understood that integrals of the type in Eq. (69) are to be taken along electron orbits.

B. Particle Orbits

In this section the orbits of plasma electrons under the influence of the two electromagnetic waves are calculated. It is assumed that θ , ϕ , θ_0 , and ϕ_0 are so small that the wave fields may be taken to be constant over the plasma volume. The scalar potential, ϕ , is neglected in solving for the particle orbits. Comparison of the electrostatic term and the ponderomotive term in Eq. (35) shows that electrostatic effects are negligible if $\omega_p \ll \delta\omega$.

The Lagrangian for an electron with these approximations is

$$L = \frac{1}{2}mv^2 - \frac{e}{c} \vec{v} \cdot \vec{A} = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) - \frac{v_x e}{c} \left[\frac{E_0 c}{\omega_0} \sin(k_0 z - \omega_0 t) + \frac{Ec}{\omega} \sin(kz + \omega t) \right] \quad (74)$$

Since x and y are ignorable coordinates, there are two constants of the motion,

$$p_x = mv_x - \frac{e}{c} \left[\frac{E_0 c}{\omega_0} \sin(k_0 z - \omega_0 t) + \frac{Ec}{\omega} \sin(kz + \omega t) \right] \quad (75)$$

and

$$p_y = m v_y \quad (76)$$

Lagrange's equation for z yields

$$\begin{aligned} m \ddot{v}_z + \frac{e}{c} \left[\frac{E_0 c k_0}{\omega_0} \cos(k_0 z - \omega_0 t) \right. \\ \left. + \frac{Eck}{\omega} \cos(kz + \omega t) \right] v_x = 0 \end{aligned} \quad (77)$$

Combining Eqs. (75) and (77) yields

$$\begin{aligned} m \ddot{v}_z = & - \frac{ep_x}{mc} [E_0 \cos(k_0 z - \omega_0 t) + E \cos(kz + \omega t)] \\ & - \frac{e^2}{mc} \left[\frac{E_0^2}{2\omega_0} \sin(2k_0 z - 2\omega_0 t) + \frac{E^2}{2\omega} \sin(2kz + 2\omega t) \right. \\ & + \frac{E_0 E}{\omega} \cos(k_0 z - \omega_0 t) \sin(kz + \omega t) \\ & \left. + \frac{E_0 E}{\omega_0} \sin(k_0 z - \omega_0 t) \cos(kz + \omega t) \right] \end{aligned} \quad (78)$$

The first four terms on the right hand side of Eq. (78) represent forces that oscillate at optical frequencies.

Assuming electron temperatures of a few kilovolts, or less, and laser intensities of about 10^{13} W/cm², these terms contribute z-displacements far less than the wavelength of either electromagnetic wave. The first four terms are therefore neglected. As in Chapter 3, the last two terms on the right hand side of Eq. (78) are combined to yield a low frequency ponderomotive force,

$$F_p = - \frac{e^2 E_0 E}{m c \bar{\omega}} \sin((k_0 + k)z - \delta \omega t) \quad (79)$$

where $\bar{\omega} = \frac{1}{2}(\omega_0 + \omega)$ and $\delta\omega = \omega_0 - \omega$. This term is the classical recoil force that causes stimulated Thomson scattering; the oscillation of a particle produced by the electric field of one wave combines with the magnetic field of the other wave to produce a force in the z-direction. This low frequency force can contribute displacements comparable to the wavelengths of the two electromagnetic waves for $\delta\omega \leq 10^{-3} \bar{\omega}$. This low frequency approximation to Eq. (78) yields

$$\frac{d^2 z}{dt^2} = - \frac{e^2 E_0 E}{m^2 c \bar{\omega}} \sin((k_0 + k)z - \delta \omega t) \quad (80)$$

The physical significance of this equation can be seen by making a Galilean transformation to a frame of reference where the two waves are Doppler shifted so that they have the same frequency:

$$\tilde{z} = z - \left(\frac{\delta\omega}{k_0 + k} \right) t \quad (81)$$

In the $\tilde{z}$ coordinate system, Eq. (80) becomes

$$\frac{d^2}{dt^2} \tilde{z} = - \frac{e^2 E_0 E}{m^2 c \bar{\omega}} \sin(k_0 + k) \tilde{z} \quad (82)$$

Equation (82) shows that in this frame the particles move in a time independent potential. Particles with small $\tilde{z}$ velocities are trapped in the minima of this potential and execute harmonic motion. Particles with $\tilde{z}$ velocities on the order of $\tilde{v}_2 = \sqrt{4e^2 E_0 E / (m^2 c \bar{\omega} (k_0 + k))}$ are "barely trapped" in the potential minima and tend to "stick" near the peaks of the potential. Particles with $\tilde{z}$ velocities well above $\tilde{v}_2$ are not trapped; the potential only causes a slight ripple in their $\tilde{z}$ velocities.

It is convenient to make the change of variable

$$\xi = (k_0 + k)\tilde{z} = (k_0 + k)z - \delta\omega t \quad (83)$$

so that Eq. (80) becomes the simple pendulum equation

$$\frac{d^2}{dt^2} \xi = -\Omega^2 \sin \xi \quad (84)$$

$$\text{where } \Omega^2 = \frac{(k_0 + k)e^2 E_0 E}{m^2 c \bar{\omega}} \approx \frac{2E_0 E e^2}{m^2 c^2}$$

The 'swinging,' 'sticking,' and rotating solutions of the simple pendulum equation correspond, respectively, to trapped, "barely trapped," and untrapped particles in the reference frame described by Eq. (81). The solutions of the simple pendulum equation are given in terms of Jacobian Elliptic Functions. These functions arise naturally when discussing particle motion in sinusoidal waves, and have been used to estimate the non-linear Landau damping of electrostatic waves^{66,67}. Milne-Thomson's notation for these functions is used⁶⁸. To describe the solutions, two quantities, ϵ and p , are defined by the equations

$$\epsilon^2 = \dot{\xi}^2 + 4 \Omega^2 \sin^2 \frac{\xi}{2} \quad (85)$$

and

$$p = \left(\frac{\epsilon}{2\Omega} \right)^2 \quad (86)$$

Note that ϵ is a constant of the motion, analogous to the energy of a pendulum. The two types of solutions of Eq. (84) are now described.

(1) Swinging Solutions

These solutions are characterized by the condition $0 \leq p \leq 1$, and satisfy the equation

$$\sin \frac{\xi}{2} = \frac{\epsilon}{2\Omega} \operatorname{sn}[\Omega(t-t_0) | p] \quad (87)$$

From Eq. (87) it follows that

$$\dot{\xi} = \epsilon \operatorname{cn}[\Omega(t-t_0) | p] \quad (88)$$

The constants ϵ and t_0 are determined by the initial values of ξ and $\dot{\xi}$ as follows:

$$\sin \frac{\xi_0}{2} = \frac{\epsilon}{2\Omega} \operatorname{sn}[-\Omega t_0 | p] \quad (89)$$

$$\dot{\xi}_0 = \epsilon \operatorname{cn}[-\Omega t_0 | p] \quad (90)$$

(2) Rotating Solutions

These solutions are characterized by the condition $p \geq 1$, and satisfy

$$\sin \frac{\xi}{2} = \operatorname{sn}[\epsilon(t-t_0)/2 | p^{-1}] \quad (91)$$

and

$$\dot{\xi} = \epsilon \operatorname{dn}[\epsilon(t-t_0)/2 | p^{-1}] \quad (92)$$

The constants ϵ and t_0 are determined by the equations

$$\sin \frac{\xi_0}{2} = \operatorname{sn}[-\epsilon t_0/2|p^{-1}] \quad (93)$$

$$\dot{\xi}_0 = \epsilon \operatorname{dn}[-\epsilon t_0/2|p^{-1}] \quad (94)$$

Note that since $\operatorname{dn}(x|p^{-1})$ is always positive, Eq. (94) requires that ϵ have the same sign as $\dot{\xi}_0$.

The integrals on the right hand side of Eqs. (69) and (73) can now be evaluated by using v_x obtained from Eq. (75) and the simple pendulum solutions for the z -motion.

C. The Gain, θ , in the Homogeneously Broadened Limit

In this section the classical expression for the gain of a plasma due to stimulated Thomson scattering (Eq. (69)) is evaluated. A simple formula is obtained for it in the homogeneously broadened limit. The distribution function, $F(\vec{r}_0, \vec{v}_0)$, is assumed to be spatially homogeneous and factorable in velocity space,

making it useful to define a reduced distribution function by the equation

$$f(v_{z0}) = \frac{1}{n_e} \int_{-\infty}^{\infty} dv_{x0} \int_{-\infty}^{\infty} dv_{y0} F(\vec{r}_0, \vec{v}_0) \quad (95)$$

where $\int_{-\infty}^{\infty} f(v_{z0}) dv_{z0} = 1$.

The expression for v_x' obtained from Eq. (75) is substituted into Eq. (69). In the resulting expression, all terms with frequencies on the order of $\bar{\omega}$ nearly time average to zero; they are neglected. If a cylindrical plasma volume of area A in the x - y plane and length L in the z -direction is used, Eq. (69) becomes

$$\begin{aligned} \dot{\theta} = & - \frac{2\pi e^2 n_e E_0}{E m \omega_0 L} \int_{-\infty}^{\infty} dv_{z0} \int_{-L/2}^{L/2} dz_0 f(v_{z0}) \chi \\ & \sin((k_0 + k)z - \delta\omega t) \end{aligned} \quad (96)$$

The variables are now changed from z' , z_0 , and v_{z0} to ξ , ξ_0 , and ξ_0' according to the equations

$$\xi = (k_0 + k)z - \delta\omega t \quad (97)$$

$$\xi_0 = (k_0 + k)z \quad (98)$$

and

$$\dot{\xi}_0 = (k_0 + k) v_{z0} - \delta\omega \quad (99)$$

Using Eq. (84) to perform the time integration yields

$$\theta = \frac{2\pi e^2 n_e E_0}{L E m \omega_0 (k_0 + k)^2 \Omega^2} \chi \quad (100)$$

$$\int_{-\infty}^{\infty} d\xi_0 \int_{-(k_0+k)L/2}^{(k_0+k)L/2} d\xi_0 [\dot{\xi}(\tau) - \dot{\xi}_0] f\left(\frac{\xi_0 + \delta\omega}{k_0 + k}\right)$$

The integrand is periodic in ξ_0 with period 2π ; ignoring end effects, Eq. (100) becomes

$$\theta = \frac{e^2 n_e E_0}{E m \omega_0 (k_0 + k) \Omega^2} \int_{-\infty}^{\infty} d\xi_0 \int_{-\pi}^{\pi} d\xi_0 [\dot{\xi}(\tau) - \dot{\xi}_0] f\left(\frac{\xi_0 + \delta\omega}{k_0 + k}\right) \quad (101)$$

This is the general expression for θ . A similar calculation yields θ_0 . The relationship between θ and θ_0 was discussed in Chapter 3 (see Eq. (54)).

In the homogeneously broadened limit, Doppler broadening is unimportant; this limit is obtained by setting

$$f\left(\frac{\xi_0 + \delta\omega}{k_0 + k}\right) = (k_0 + k) \delta(\xi_0 + \delta\omega) \quad (102)$$

Using this distribution function, Eq. (101) becomes

$$\theta = \frac{4\pi^2 n_e r_e^2 I_0 c}{m \omega_0} J \quad (103)$$

where

$$J(\delta\omega\tau, \Omega\tau) = \frac{4}{\pi\Omega^4} \int_{-\pi}^{\pi} d\xi_0 [\xi(\tau) + \delta\omega] \quad (104)$$

and

$$I_0 = \frac{c}{8\pi} E_0^2 \quad (105)$$

For general values of $\delta\omega\tau$ and $\Omega\tau$, J is difficult to evaluate analytically. In the limit that $\beta = \left(\frac{2\Omega}{\delta\omega}\right)^2 \ll 1$ and $\Omega\tau \leq 1$, however, it is possible to obtain an approximate expression for J by expanding the integrand of Eq. (104) in powers of β .

Note that in this limit, only rotating solutions are used, so that Eq. (104) can be written as

$$J = \frac{4}{\pi \Omega^4} \int_{-\pi}^{\pi} d\xi_0 \left[\epsilon \operatorname{dn} \left(\epsilon(t-t_0)/2 \mid \left(\frac{2\Omega}{\epsilon}\right)^2 \right) + \delta\omega \right] \quad (106)$$

where

$$\epsilon = - \left[\delta\omega^2 + 4\Omega^2 \sin^2 \frac{\xi_0}{2} \right]^{\frac{1}{2}} \quad (107)$$

and

$$\sin \frac{\xi_0}{2} = \operatorname{sn} \left[-\frac{\epsilon t_0}{2} \mid \left(\frac{2\Omega}{\epsilon}\right)^2 \right] \quad (108)$$

The expansion of the integrand of Eq. (106) is tedious, but straightforward; Useful identities are given in Appendix 4. To second order in β , J is given by the equation

$$J = \tau^3 \frac{d}{d\eta} \left(\frac{\sin^2 \eta}{\eta^2} \right) \quad (109)$$

in agreement with Eq. (52). As stated, the expansion for J is valid when $\Omega\tau \leq 1$ and when $\beta \ll 1$. This implies that $\eta \gg \Omega\tau$. Equation (109) is also valid

for values of $\eta \leq \Omega\tau$ with the added restriction that $\Omega\tau \ll 1$. This follows from an expansion of the integrand of Eq. (104) in powers of $\Omega\tau$ with $\delta\omega \leq 2\Omega$. Hence, in the unsaturated limit ($\Omega\tau \ll 1$) the homogeneously broadened gain coefficient is given by the equation

$$\gamma\tau = 2\theta = \frac{8\pi^2 n_e r_e^2 I_0 c \tau^3}{m \omega_0} \frac{d}{d\eta} \left(\frac{\sin^2 \eta}{\eta^2} \right) \quad (110)$$

in exact agreement with the quantum mechanical expression, Eq. (5).

When $\Omega\tau$ is no longer small, the gain saturates. The gain saturates because of the phase mixing of the trapped and nearly trapped electrons as in the non-linear Landau damping of plasma waves. To calculate the saturation behavior, the integral in Eq. (105) is performed numerically, using Eqs. (85) through (94). The homogeneously broadened gain saturates more easily near $\eta = 0$ than in the wings of the lineshape. Figure 17 shows J as a function of $\delta\omega\tau$ for selected values of $\Omega\tau$ and for $\tau = 1$. Figure 18 shows J as a function of $\Omega\tau$ for selected values of $\delta\omega\tau$ and for $\tau = 1$.

Figure 17

The Saturation of the Homogeneously Broadened
Gain Curve vs. $\delta\omega\tau$

The function J is displayed vs. $\delta\omega\tau$ for $\tau = 1$ and for 3
values of $\Omega\tau$.

FIGURE 17

Figure 18

The Saturation of the Homogeneously Broadened
Gain Curve vs. $\Omega\tau$

The function J is displayed vs. $\Omega\tau$ for $\tau = 1$ and for 4
values of $\delta\omega\tau$.

FIGURE 18

D. The Phase Shift, ϕ , in the Homogeneously Broadened Limit

In this section ϕ is calculated using the same assumptions about the distribution function and the plasma volume given in Section C. The expression for v'_x obtained from Eq. (75) is substituted into Eq. (73). In the resulting expression, the high frequency terms are time averaged, the variables z' , z_0 , and v_{x0} are changed to ξ , ξ_0 , and $\dot{\xi}_0$, and the equation is integrated over the interaction time, τ . Equating terms of like order yields Eq. (44) again and an expression for ϕ .

$$\phi = - \frac{e^2 n_e E_0}{m \omega_0 E(k_0 + k)} \int_{-\infty}^{\infty} d\dot{\xi}_0 \int_{-\pi}^{\pi} d\xi_0 \cos \xi f\left(\frac{\xi_0 + \delta\omega}{k_0 + k}\right) \quad (111)$$

This is the general expression for ϕ .

In the homogeneously broadened limit, Eq. (102) is used to obtain

$$\phi = - \frac{4\pi^2 n_e r_e^2 I_0 c}{m \omega_0} K \quad (112)$$

where

$$K(\delta\omega\tau, \Omega\tau) = \frac{4}{\pi\Omega^2} \int_0^\tau dt \int_{-\pi}^\pi d\xi_0 \cos \xi \quad (113)$$

In the limit that $\beta \ll 1$ and $\Omega\tau \leq 1$, K is given by

$$K = \frac{4}{\pi\Omega^4} \int_0^\tau dt \int_{-\pi}^\pi d\xi_0 [1 - 2 \operatorname{sn}^2(\epsilon(t-t_0)/2 | (\frac{2\Omega}{\epsilon})^2)] \quad (114)$$

To first order in β ,

$$K = \tau^3 \frac{d}{d\eta} \left(\frac{1}{\eta} - \frac{\sin 2\eta}{2\eta^2} \right) \quad (115)$$

for $\eta \gg \Omega\tau$. This expression is also valid for $\eta \leq \Omega\tau$ if $\Omega\tau \ll 1$. Hence, in the unsaturated limit ($\Omega\tau \ll 1$)

$$\phi = - \frac{4\pi^2 n_e r_e^2 I_0 c \tau^3}{m \omega_0} \frac{d}{d\eta} \left(\frac{1}{\eta} - \frac{\sin 2\eta}{2\eta^2} \right) \quad (116)$$

Equation (58) may be used to interpret ϕ as a non-linear part of the index of refraction.

If β is no longer small, ϕ must be calculated numerically. Figure 19 shows K as a function of $\delta\omega\tau$

Figure 19

The Saturation of the Homogeneously Broadened
Dispersion Curve vs. $\delta\omega\tau$

The function K is displayed vs. $\delta\omega\tau$ for $\tau = 1$ and for 3
values of $\Omega\tau$.

FIGURE 19

Figure 20

The Saturation of the Homogeneously Broadened
Dispersion Curve vs. $\Omega\tau$

The function K is displayed vs. $\Omega\tau$ for $\tau = 1$ and for 4
values of $\delta\omega\tau$.

FIGURE 20

for selected values of $\Omega\tau$ and for $\tau = 1$. Figure 20 shows K as a function of $\Omega\tau$ for selected values of $\delta\omega\tau$ and for $\tau = 1$. Note that like the homogeneous gain, the homogeneous dispersion saturates more easily near $\eta = 0$ than in the wings of the lineshape.

E. The Gain and the Phase Shift in the Inhomogeneously Broadened Limit

When the electrons in a plasma have a distribution of velocities, the gain can be obtained as a simple convolution of the homogeneous gain formula with the electron distribution function. Equation (101), Eq. (103), and the expression for the distribution function,

$$f\left(\frac{\xi_0 + \delta\omega}{k_0 + k}\right) = \int_{-\infty}^{\infty} d\sigma \delta(\sigma - \xi_0 - \delta\omega) f\left(\frac{\sigma}{k_0 + k}\right) \quad (117)$$

can be combined to yield the equation

$$\theta = \frac{4\pi^2 r_e^2 n_e I_0 c}{m \omega_0 (k_0 + k)} \int_{-\infty}^{\infty} d\sigma f\left(\frac{\sigma}{k_0 + k}\right) J((\delta\omega - \sigma)\tau, \Omega\tau) \quad (118)$$

In the unsaturated limit, Eq. (109) gives the expression for J ; substituting it into Eq. (118) and integrating by parts yields

$$\theta = \frac{16\pi^2 r_e^2 n_e I_0 c \tau}{m \omega_0 (k_0 + k)^2} \int_{-\infty}^{\infty} d\lambda f' \left[\frac{1}{k_0 + k} (\delta\omega + 2\lambda/\tau) \right] \frac{\sin^2 \lambda}{\lambda^2} \quad (119)$$

where $\lambda = (\sigma - \delta\omega) \tau/2$. In the inhomogeneous limit, the width, in λ , of the distribution function is much larger than the width of the homogeneous gain curve. This limit is described by the condition

$$\frac{\overline{v_z}}{c} \gg \frac{1}{\omega\tau} \quad (120)$$

where $\overline{v_z}$ is the width, in velocity, of the distribution function $f(v_z)$. In this limit, the slope of the electron distribution function is nearly constant over the width of the homogeneous gain curve. Setting $f'[(\delta\omega + 2\lambda/\tau)/(k_0 + k)] = f'[\frac{\delta\omega}{k_0 + k}]$ in the integrand of Eq. (119) and using the result that $\int_{-\infty}^{\infty} \sin^2 \lambda / \lambda^2 = \pi$, yields for the inhomogeneously broadened gain in the unsaturated limit

$$\gamma\tau = 2\theta = - \frac{8\pi^3 r_e^2 n_e I_0 c^3 \tau}{m \omega_0^3} \left. \frac{df}{dz} \right|_{v_z = \frac{(\omega_0 - \omega)c}{2\omega_0}} \quad (121)$$

in agreement with the quantum mechanical result, Eq. (7).

If the gain is allowed to saturate, then the homogeneous gain curve is no longer given by Eq. (109). The saturated homogeneous gain curve is flattened near the center, but is still an odd function of $\delta\omega\tau$, as shown in Figure 18. A function M is defined by the equation

$$M(\lambda, \Omega\tau) = \int_{-\infty}^{\lambda} J(-2\lambda', \Omega\tau) d\lambda', \quad (122)$$

Equation (118) is integrated by parts to obtain an equation for the saturated inhomogeneous gain coefficient:

$$\gamma\tau = - \frac{32\pi^2 r_e^2 n_e I_0 c \tau}{m \omega_0 (k_0 + k)^2} \int_{-\infty}^{\infty} d\lambda f'[(\delta\omega + 2\lambda/\tau)/(k_0 + k)] M \quad (123)$$

If the distribution function is much wider than M , Eq. (123) may be written as

$$\gamma\tau = - \frac{8\pi^3 r_e^2 n_e I_0 c^3 \tau}{m \omega_0^3} \Delta(\Omega\tau) \left. \frac{df}{dz} \right|_{v_z = \frac{(\omega_0 - \omega)c}{2\omega_0}} \quad (124)$$

where

$$\Delta(\Omega\tau) = \int_{-\infty}^{\infty} M(\lambda, \Omega\tau) d\lambda = \int_{-\infty}^{\infty} \lambda J(-2\lambda, \Omega\tau) d\lambda \quad (125)$$

The quantity $\Delta(\Omega\tau)$ is shown in Figure 21.

The expression for ϕ , Eq. (111), can also be expressed, in the inhomogeneously broadened limit, as a convolution of the homogeneous expression, Eq. (112), with the distribution function:

Figure 21

The Saturation of the Inhomogeneously Broadened Gain

The function A is displayed vs. $\Omega\tau$.

FIGURE 21

$$\phi = - \frac{4\pi^2 r_e^2 n_e I_0 c}{m\omega_0(k_0+k)} \int_{-\infty}^{\infty} f\left(\frac{\sigma}{k_0+k}\right) K((\delta\omega-\sigma)\tau, \Omega\tau) d\sigma \quad (126)$$

In the unsaturated limit, Eq. (109) is used to obtain the equation

$$\phi = \frac{16\pi^2 r_e^2 n_e I_0 c \tau^2}{m\omega_0(k_0+k)} \frac{d}{d(\delta\omega)} \int_{-\infty}^{\infty} f\left(\frac{\sigma}{k_0+k}\right) \chi \left[\frac{1}{(\sigma-\delta\omega)\tau} - \frac{\sin(\sigma-\delta\omega)\tau}{(\sigma-\delta\omega)^2 \tau^2} \right] d\sigma \quad (127)$$

In the inhomogeneous limit, the distribution function is nearly constant, in σ , over the center of K . Since this curve is odd, the center of K makes only a small contribution to ϕ . There is, however, a significant contribution from the wings of K ; in fact, since the integral of $1/(\sigma - \delta\omega)\tau$ diverges at large σ , it samples the entire width of the distribution function. It is convenient to replace the fairly complicated function K by a simpler function that also contributes nothing for σ near $\delta\omega$, and that has the same behavior for $|\sigma| \gg \delta\omega$. Such a function is $P \frac{1}{(\sigma-\delta\omega)\tau}$, where P stands for principal value. In this approximation, Eq. (127) becomes

$$\phi = \frac{8\pi^2 r_e^2 n_e I_0 c^2 \tau}{m \omega^2} \frac{d}{d(\delta\omega)} \left[P \int_{-\infty}^{\infty} d\sigma \frac{f\left(\frac{\sigma}{k_0 + k}\right)}{\sigma - \delta\omega} \right] \quad (128)$$

If the distribution function is Maxwellian,

$$f(v_z) = \sqrt{\frac{m}{2\pi k_B T}} e^{-mv_z^2/2k_B T} \quad (129)$$

then Eq. (128) becomes

$$\phi = \frac{4\pi^2 r_e^2 n_e I_0 c^3 \tau}{k_B T \omega^3} Z'(y) \quad (130)$$

where y is given by Eq. (9) and where

$$Z'(y) = \frac{d}{dy} \left[\operatorname{Re} \left(\frac{1}{\sqrt{\pi}} P \int_{-\infty}^{\infty} dx \frac{e^{-x^2}}{x-y} \right) \right] \quad (131)$$

is the derivative of the real part of the plasma dispersion function for a real argument⁶⁹. For the case of a Maxwellian distribution function, the ratio of the correct expression, Eq.(127), to the approximate expression, Eq. (130), can, for $y = 0$, be shown to be $1 - \operatorname{erf}(N)/N$, where $N = (2k_B T/mc^2)^{1/2} \omega \tau$. Hence, for

$N \geq 20$, Eq. (130) is good to 10% or better at line center.

If ϕ is allowed to saturate, then K changes near $\delta\omega = 0$, but remains nearly the same as in the unsaturated limit for $\delta\omega \gg 1/\tau$. Figure 19 shows that for $\delta\omega \gtrsim \Omega$, the saturated K and the unsaturated K are nearly the same. From Eq. (126), saturation in the inhomogeneously broadened phase shift should only become important when $2\Omega \gtrsim (k_0 + k)v_{th}$, or when $\Omega \gtrsim v_{th}\omega/c$. The inhomogeneous gain, on the other hand, saturates when $\Omega\tau \sim 1$. Since $v_{th}\omega/c \gg 1$ in most cases, the inhomogeneous gain will usually saturate much more easily than the inhomogeneously broadened phase shift.

It is interesting to note that the way the folding together of the distribution function with the homogeneously broadened lineshape, $\frac{d}{d\eta}(\sin^2\eta/\eta^2)$, gives rise to the first derivative of the distribution function in the inhomogeneously broadened limit, Eq. (108), is analogous to the way the first derivative of the distribution function appears in the Landau damping formula. In the physical derivation of Landau damping given by Dawson⁷⁰, the same function, $\frac{d}{dx}(\sin^2x/x^2)$, appears in exactly the same way. Indeed, the unsaturated gain due to stimulated Thomson scattering arises from a process analogous to linear Landau damping while the

saturation mechanism discussed in Sections C-E of this chapter is analogous to non-linear Landau damping.

CHAPTER 5

SUMMARY

The process of stimulated Thomson scattering can thus be thought of in three ways. Firstly, it can be viewed quantum mechanically as the result of photons obeying Bose-Einstein statistics. Secondly, it can be viewed as the result of the transient response of an electron fluid to the ponderomotive beat force between two counterpropagating electromagnetic waves when the frequency difference between the two waves is far above the plasma frequency. As the difference frequency is allowed to approach the plasma frequency, stimulated Thomson scattering passes continuously into stimulated Raman scattering. Thirdly, it can be viewed as the result of a resonant particle interaction. Thought of in this way, the decrease in the gain coefficient as $\delta\omega$ is increased to values much greater than one is due to the phase velocity of the ponderomotive beat wave becoming so large that the electrons are no longer close to resonance. Pursuing the resonant particle picture leads to the observation of non-linear Landau damping of the effect. This damping arises because different electrons have different phases with respect to the beat wave when

it is suddenly turned on. Each different phase position corresponds to a different period of motion, so that as time goes on the electrons phase mix in space and destroy the coherence that produces stimulated Thomson scattering.

The free electron amplifier and free electron laser experiments come the closest to providing an experimental test of the theory presented in this thesis, but are not entirely satisfactory. The electron density and velocity distribution are not very well known since they are difficult to measure in high energy accelerators. It would be desirable to observe stimulated Thomson scattering from a plasma to which standard diagnostic techniques for temperature and density can be applied. A proposal for such an experiment has been made by L. W. Anderson and J. E. Lawler. A description of the proposed experiment is found in Appendix 3. A possible result of such an experiment could be the development of stimulated Thomson scattering as a plasma diagnostic technique.

Finally, it would be desirable to extend the work discussed in this thesis by doing a full kinetic treatment of stimulated Thomson scattering, perhaps following the treatment of stimulated Raman scattering given by Schmidt⁴². Such a calculation should be able to

predict how the inclusion of plasma wave effects will affect the simple pendulum solutions to the electron motion given in Chapter 4. It should also be able to unify the approach of Chapter 3 with that of Chapter 4 by treating the plasma wave kinetically. In particular, the effect of linear Landau damping of the plasma wave should enter naturally rather than having to be treated phenomenologically through the damping constant used in Chapter 3. It might even be possible to include collisions, thus making it possible for the finite duration time of the interaction to be provided in a time independent way, as discussed in connection with the Kramers-Kronig relations in Chapter 3. Since the collision time is not a definite time, however, but rather the mean of a distribution of times, the oscillations in the functions θ and ϕ should be washed out because of their having to be averaged over many different times τ .

APPENDIX 1

THE FREE ELECTRON AMPLIFIER

The free electron amplifier is an interesting example of the stimulated Thomson scattering process. Madey⁶ presented a quantum mechanical theory of this effect, and Elias, Fairbank, Madey, Schwettman, and Smith¹⁰ performed the first experiment demonstrating the effect. Deacon, Elias, Madey, Ramian, Schwettman, and Smith¹¹ subsequently built a free electron laser. Hopf, Meystre, Scully, and Louisell¹³ showed that the free electron amplifier could be explained using a classical theory. Their classical theory of the free electron amplifier agrees with the quantum mechanical theory. Ichimaru and Iwamoto²¹, Colson¹⁹, Dawson, Kwan and Lin^{15,20}, Sprangle, Smith, and Granatstein⁵⁴, Cocke⁶², and Kroll and McMullin²² have also presented classical calculations of the performance of free electron amplifiers and lasers. Hopf, Meystre, Scully, and Louisell¹⁷ have discussed some aspects of the saturation of the free electron amplifier.

In this appendix, the free electron amplifier experiment is analyzed briefly in terms of the theory of stimulated Thomson scattering. In the free electron

amplifier, highly relativistic electrons are shot down the axis of a helical magnetic field in the presence of an electromagnetic wave propagating in the direction of the electron velocity. The electromagnetic wave was observed to increase, or decrease, in intensity as the electron energy was varied. It is useful to view this experiment from the rest frame of the electrons. In this frame, the transverse part of the helical field is approximately a left-circularly polarized electromagnetic wave; the electromagnetic wave is Doppler shifted to a lower frequency. In the electron frame, the pump wave pulse looks very short, while that of the probe wave looks very long. When a short pulse passes through a long one, all of the electrons in a given region suddenly experience the interaction of the two waves for a time equal to the duration of the short pulse, and each part of the long probe pulse will have its intensity changed by the amount corresponding to this time. Thus, except for the difference in polarization, the free electron amplifier corresponds to the physical situation discussed in this thesis. Assuming that the electron beam is mono-energetic, Eq. (103), properly Lorentz transformed and modified for circularly polarized light, should give the gain of the free electron amplifier.

It is a simple matter to carry out the calculations in Chapters 3 and 4 for circularly polarized light to obtain the following results. If the two oppositely directed waves have the same circular polarization, the gain coefficient is zero. If they are of opposite circular polarization, the gain coefficient is twice as large as for plane polarized light. In the free electron amplifier, these effects are observed¹⁰. In addition, the saturation parameter, $\Omega\tau$, appearing in the function J , is replaced by $\sqrt{2} \Omega\tau$, for waves of opposite circular polarization.

The quantities in Eq. (103) are related to quantities in the free electron amplifier by the Lorentz transformation. With the assumption that $v \approx c$, these quantities are the following.

(i) If L is the length of the helical field and γ is the Lorentz factor for the electrons, then the coherence time, τ , is given by the equation

$$\tau = \frac{L}{\gamma v} \approx \frac{L}{\gamma c} \quad (1.1)$$

(ii) If ω^* is the probe wave frequency in the laboratory frame, then the probe wave frequency in the electron frame is given by the equation

$$\omega = \sqrt{\frac{1-v/c}{1+v/c}} \quad \omega^* = \frac{\omega^*}{2\gamma} \quad (1.2)$$

(iii) If λ_q is the period length of the helical field, then the pump wave frequency in the electron frame is given by the equation

$$\omega_o = \frac{2\pi\gamma v}{\lambda_q} = \frac{2\pi\gamma c}{\lambda_q} \quad (1.3)$$

(iv) If B is the magnitude of the transverse part of the helical magnetic field in the laboratory, then the intensity of each linearly polarized component of the circularly polarized probe wave is given by the equation

$$I_o = \frac{c}{8\pi} \gamma^2 B^2 \quad (1.4)$$

(v) If I^* is the intensity of the circularly polarized probe wave in the laboratory, then the saturation parameter, $\sqrt{2} \Omega \tau$, is given by the equation

$$\sqrt{Z} \Omega \tau = \left[\frac{16\pi r_e^2 B^2 I^*}{m^2 c} \right]^{1/2} \frac{L}{\gamma c} \quad (1.5)$$

(vi) If n^* is the electron density in the laboratory, then the electron density in the electron frame is given by the equation

$$n = \frac{n^*}{\gamma} \quad (1.6)$$

(vii) If $q = 2\pi/\lambda_q$, then the homogeneous line shape variable, n , is given by the equation

$$n = -\frac{\delta\omega\tau}{Z} = \frac{L}{Z} \left(\frac{\omega^*}{2\gamma^2 c} - q \right) \quad (1.7)$$

The gain per pass of the amplifier in the electron frame can be calculated by noting that in the electron frame, two pulses appear to be passing through each other. The pump wave pulse is much shorter than the probe wave pulse. Each part of the probe wave pulse sees the interaction last for a time $\tau/2$, but during that interaction time it encounters electrons with interaction times ranging from 0 to τ , linearly distributed in time.

Hence, the total gain of each part of the probe wave pulse, ignoring end effects, is given by the equation

$$G = 2 \int_0^{\tau/2} [2\hat{\theta}(2t)] dt = 2 \theta(\tau) \quad (1.8)$$

Since this is just the ratio of the final intensity to the initial intensity, Eq. (1.8) also gives the gain per pass in the laboratory. Hence, the small signal gain coefficient per unit length of the magnet in the laboratory, α^* , is given by the equation

$$\alpha^* = \frac{G}{L} = \frac{r_e^2 n^* B^2 \lambda_g L^2 F}{2 mc^2 \gamma^3} \left(\frac{J}{r^3} \right) \quad (1.9)$$

where F is a filling factor that gives the fraction of the electromagnetic wave cross-section that interacts with the electron beam (assuming that the wave cross-section is larger than the beam cross section). In the unsaturated limit, Eq. (109) may be used to write Eq. (1.9) as

$$\alpha^* = \left(\frac{3}{2} n^* \alpha_T \right) \left(\frac{B^2}{8\pi mc^2} \right) \left(\frac{\lambda_q L^2}{\gamma^3} F \right) \frac{d}{dn} \left(\frac{\sin^2 \eta}{\eta^2} \right) \quad (1.10)$$

in agreement with the results obtained by others^{9,13,19,21}.

In evaluating Eq. (1.10) for the free electron amplifier, care must be taken in evaluating γ ; it is reduced from its value when the electrons are outside of the helical field since the electrons spiral down the helix. The factor γ is given by the equation

$$\gamma = \frac{E}{mc^2 [1 + (\lambda_q^2 r_e B^2)/(4\pi^2 mc^2)]^{1/2}} \quad (1.11)$$

where E is the electron energy.

When the parameters of the free electron amplifier experiment are used to calculate the peak gain per pass, G , from Eq. (1.10), it is found to be 5.4%. The experiment measured a peak gain per pass of 7%. When the same parameters are used to calculate the saturation parameter, $\sqrt{2} \Omega \tau$, it is found to be 2.7. Figures 17 and 21 show that this value is just past threshold for saturation; the theory presented in this thesis predicts that for $I^* = 1.4 \times 10^5 \text{ W/cm}^2$, the highest power level

attained in the free electron amplifier experiment, the gain is near saturation. No saturation was observed in the experiment. I believe that any significant increase in the value of the laser intensity in the free electron amplifier above the maximum value used will produce saturation effects in the gain. Others have discussed other mechanisms by which the free electron amplifier may saturate^{8,16,19,21}.

APPENDIX 2

THE PERTURBED DENSITY AND THE FUNCTIONS
J AND K FOR GENERAL Γ AND ω'

In this appendix the general solution of Eq. (36) and the resulting expressions for θ and ϕ are given. When Γ and ω' are non-zero, the solution of Eq. (36) subject to the initial conditions given in Eqs. (38) and (39) is

$$\begin{aligned} \delta n = & \frac{-n_0 \Omega^2}{[(\delta\omega^2 - \tilde{\omega}^2 - \frac{1}{4}\Gamma^2)^2 + \Gamma^2\delta\omega^2]} \times \\ & \{ (\delta\omega^2 - \tilde{\omega}^2 - \frac{1}{4}\Gamma^2) \cos((k_0+k)z - \delta\omega t) + \Gamma\delta\omega \sin((k_0+k)z - \delta\omega t) \\ & + e^{-\Gamma t/2} [- (\delta\omega^2 - \tilde{\omega}^2 - \frac{1}{4}\Gamma^2) \cos(k_0+k)z \cos\tilde{\omega} t \\ & - \frac{\delta\omega}{\tilde{\omega}} (\delta\omega^2 - \tilde{\omega}^2 + \frac{1}{4}\Gamma^2) \sin(k_0+k)z \sin\tilde{\omega} t \\ & - \Gamma\delta\omega \sin(k_0+k)z \cos\tilde{\omega} t \\ & + \frac{\Gamma}{2\tilde{\omega}} (\delta\omega^2 + \tilde{\omega}^2 + \frac{1}{4}\Gamma^2) \cos(k_0+k)z \sin\tilde{\omega} t] \} \end{aligned} \quad (2.1)$$

where $\tilde{\omega} = \sqrt{\omega'^2 - \frac{1}{4}\Gamma^2}$

To find $\dot{\theta}$ and $\dot{\phi}$, it is necessary to time and space average the non-linear current, $\frac{4\pi}{c} e\delta n \vec{v}_0$, against either

$\cos(kz) \cos(\omega t)$ or $\cos(kz) \sin(\omega t)$ as was done in the simple case discussed in Chapter 3. If the symbol $\langle \rangle$ is defined by

$$\langle f \rangle = \frac{1}{Lt} \int_{-L/2}^{L/2} dz \int_0^t dt' f \quad (2.2)$$

where it is assumed that $1/\omega \ll t \ll 1/\delta\omega$ and that $L \gg 1/(k-k_0)$, then $\dot{\theta}$ and $\dot{\phi}$ are given by the equations

$$-\frac{E}{2c} \dot{\theta} = \frac{4\pi e}{c} \langle \delta n v_{ox} \cos kz \cos \omega t' \rangle \quad (2.3)$$

$$\frac{E}{2c} \dot{\phi} = \frac{4\pi e}{c} \langle \delta n v_{ox} \cos kz \sin \omega t' \rangle \quad (2.4)$$

Carrying out the operation on the right hand sides of both equations allows them to be written in the same form as Eqs. (45) and (46):

$$\dot{\theta} = \frac{4\pi^2 n_o r_e^2 I_o c}{m \omega} P \quad (2.5)$$

$$\dot{\phi} = - \frac{4\pi^2 n_o r_e^2 I_o c}{m \omega} Q \quad (2.6)$$

where P and Q are now given by the expressions

$$P = \frac{4}{[(\delta\omega^2 - \omega^2 - \frac{1}{4}\Gamma^2)^2 + \Gamma^2\delta\omega^2]} \{ \Gamma\delta\omega + e^{-\Gamma t/2} [(\delta\omega^2 - \omega^2 - \frac{1}{4}\Gamma^2) \cos\omega t \sin\delta\omega t - \frac{\delta\omega}{\omega} (\delta\omega^2 - \omega^2 + \frac{1}{4}\Gamma^2) \sin\omega t \cos\delta\omega t - \Gamma\delta\omega \cos\omega t \cos\delta\omega t - \frac{\Gamma}{2\omega} (\delta\omega^2 + \omega^2 + \frac{1}{4}\Gamma^2) \sin\omega t \sin\delta\omega t] \} \quad (2.7)$$

$$Q = \frac{4}{[(\delta\omega^2 - \omega^2 - \frac{1}{4}\Gamma^2)^2 + \Gamma^2\delta\omega^2]} \{ -(\delta\omega^2 - \omega^2 - \frac{1}{4}\Gamma^2) + e^{-\Gamma t/2} [(\delta\omega^2 - \omega^2 - \frac{1}{4}\Gamma^2) \cos\omega t \cos\delta\omega t + \frac{\delta\omega}{\omega} (\delta\omega^2 - \omega^2 + \frac{1}{4}\Gamma^2) \sin\omega t \sin\delta\omega t + \Gamma\delta\omega \cos\omega t \sin\delta\omega t - \frac{\Gamma}{2\omega} (\delta\omega^2 + \omega^2 + \frac{1}{4}\Gamma^2) \sin\omega t \cos\delta\omega t] \} \quad (2.8)$$

The functions P and Q can be integrated to obtain the generalizations of the functions J and K defined in

the simple case of Chapter 3 by Eqs. (52) and (53). Two separate cases must be considered, (i) underdamping of the plasma wave where $\omega' \geq \frac{1}{2}\Gamma$, and (ii) overdamping where $\omega' < \frac{1}{2}\Gamma$. For case (i) integration of Eqs. (2.7) and (2.8) yields

$$J = \frac{4}{[(\delta\omega^2 - \tilde{\omega}^2 - \frac{1}{4}\Gamma^2)^2 + \Gamma^2\delta\omega^2]} \{ \Gamma\delta\omega\tau + e^{-\Gamma\tau/2} [A_1 \sin(\delta\omega - \tilde{\omega})\tau + A_2 \cos(\delta\omega - \tilde{\omega})\tau + A_3 \sin(\delta\omega + \tilde{\omega})\tau + A_4 \cos(\delta\omega + \tilde{\omega})\tau] - (A_2 + A_4) \} \quad (2.9)$$

$$K = \frac{4}{[(\delta\omega^2 - \tilde{\omega}^2 - \frac{1}{4}\Gamma^2)^2 + \Gamma^2\delta\omega^2]} \{ -(\delta\omega^2 - \tilde{\omega}^2 - \frac{1}{4}\Gamma^2)\tau + e^{-\Gamma\tau/2} [-A_2 \sin(\delta\omega - \tilde{\omega})\tau + A_1 \cos(\delta\omega - \tilde{\omega})\tau - A_4 \sin(\delta\omega + \tilde{\omega})\tau + A_3 \cos(\delta\omega + \tilde{\omega})\tau] - (A_1 + A_3) \} \quad (2.10)$$

where

$$A_1 = -\frac{\Gamma}{2\omega} (\delta\omega + \tilde{\omega}) \frac{[(\delta\omega + \tilde{\omega})^2 + \frac{1}{4}\Gamma^2]}{[(\delta\omega - \tilde{\omega})^2 + \frac{1}{4}\Gamma^2]} \quad (2.11)$$

$$A_2 = -\frac{1}{2\omega} \frac{[(\delta\omega + \tilde{\omega})^2 + \frac{1}{4}\Gamma^2]}{[(\delta\omega - \tilde{\omega})^2 + \frac{1}{4}\Gamma^2]} \frac{[(\delta\omega - \tilde{\omega})^2 - \frac{1}{4}\Gamma^2]}{[(\delta\omega + \tilde{\omega})^2 + \frac{1}{4}\Gamma^2]} \quad (2.12)$$

$$A_3 = A_1(-\tilde{\omega}) = \frac{\Gamma}{2\omega} (\delta\omega + \tilde{\omega}) \frac{[(\delta\omega - \tilde{\omega})^2 + \frac{1}{4}\Gamma^2]}{[(\delta\omega + \tilde{\omega})^2 + \frac{1}{4}\Gamma^2]} \quad (2.13)$$

$$A_4 = A_2(-\tilde{\omega}) = \frac{1}{2} \frac{[(\delta\omega - \tilde{\omega})^2 + \frac{1}{4}\Gamma^2]}{[(\delta\omega + \tilde{\omega})^2 + \frac{1}{4}\Gamma^2]} \frac{[(\delta\omega + \tilde{\omega})^2 - \frac{1}{4}\Gamma^2]}{[(\delta\omega - \tilde{\omega})^2 + \frac{1}{4}\Gamma^2]} \quad (2.14)$$

For case (ii), J and K are given by the expressions

$$\begin{aligned}
 J = & \frac{4}{\left[(\delta\omega^2 + \sigma^2 - \frac{1}{4}\Gamma^2)^2 + \Gamma^2\delta\omega^2 \right]} \left\{ \Gamma\delta\omega\tau + \right. \\
 & e^{-\Gamma\tau/2} \left[B_1 \sin\delta\omega\tau \sinh\sigma\tau - B_2 \sin\delta\omega\tau \cosh\sigma\tau \right. \\
 & \left. \left. + B_3 \cos\delta\omega\tau \sinh\sigma\tau + B_4 \cos\delta\omega\tau \cosh\sigma\tau \right] - B_4 \right\} \quad (2.15)
 \end{aligned}$$

$$\begin{aligned}
 K = & \frac{4}{\left[(\delta\omega^2 + \sigma^2 - \frac{1}{4}\Gamma^2)^2 + \Gamma^2\delta\omega^2 \right]} \left\{ -(\delta\omega^2 + \sigma^2 - \frac{1}{4}\Gamma^2) \right. \\
 & + e^{-\Gamma\tau/2} \left[-B_3 \sin\delta\omega\tau \sinh\sigma\tau - B_4 \sin\delta\omega\tau \cosh\sigma\tau \right. \\
 & \left. \left. + B_1 \cos\delta\omega\tau \sinh\sigma\tau + B_2 \cos\delta\omega\tau \cosh\sigma\tau \right] - B_2 \right\} \quad (2.16)
 \end{aligned}$$

where $\sigma = \sqrt{\frac{1}{4}\Gamma^2 - \omega'^2}$

$$B_1 = \frac{-1}{\sigma} \frac{(\delta\omega^2 - \sigma^2 + \frac{1}{8}\Gamma^2) [(\delta\omega^2 - \sigma^2 + \frac{1}{8}\Gamma^2)^2 + 4\delta\omega^2\sigma^2] + 4\Gamma^2\delta\omega^2\sigma^2}{[(\delta\omega^2 - \sigma^2 + \frac{1}{8}\Gamma^2)^2 + 4\sigma^2\delta\omega^2]} \quad (2.17)$$

$$B_2 = \Gamma \frac{[(\delta\omega^2 - \sigma^2 + \frac{1}{8}\Gamma^2)^2 - 4\delta\omega^2(\delta\omega^2 + \frac{1}{8}\Gamma^2)]}{[(\delta\omega^2 - \sigma^2 + \frac{1}{8}\Gamma^2)^2 + 4\sigma^2\delta\omega^2]} \quad (2.18)$$

$$B_3 = \frac{\Gamma\delta\omega}{\sigma} \frac{[(\delta\omega^2 - \sigma^2 + \frac{1}{8}\Gamma^2)^2 - 4\sigma^2(\sigma^2 - \frac{1}{8}\Gamma^2)]}{[(\delta\omega^2 - \sigma^2 + \frac{1}{8}\Gamma^2)^2 + 4\sigma^2\delta\omega^2]} \quad (2.19)$$

$$B_4 = -2\delta\omega \frac{[(\delta\omega^2 - \sigma^2 + \frac{1}{8}\Gamma^2)^2 + 4\sigma^2\delta\omega^2 - \frac{1}{8}\Gamma^4]}{[(\delta\omega^2 - \sigma^2 + \frac{1}{8}\Gamma^2)^2 + 4\sigma^2\delta\omega^2]} \quad (2.20)$$

These are the expressions used to generate Figures 3-16.

APPENDIX 3

PROPOSED EXPERIMENTAL DETECTION OF
STIMULATED THOMSON SCATTERING

The following experimental proposal, and Figure 22 which illustrates it, are due to L. W. Anderson and J. E. Lawler. In order to evaluate the possibility of detecting stimulated Thomson scattering, reasonable values for certain experimental parameters must be assumed. The plasma is assumed to be produced by a pinch discharge, and to have an electron density of $n_e = 10^{15} \text{ cm}^{-3}$ and an electron temperature of $k_B T_e = 10 \text{ eV}$. These parameters are reasonable for a pinch discharge⁷¹. It is assumed that the pump laser has a pulse duration of 3 nsec and an energy per pulse of 1.0 J at a wavelength of 694.3 nm. The pump laser has a waist diameter at focus of 0.01 cm and hence a Rayleigh range of $Z_R = \frac{\pi w_0^2}{\lambda} = 4.5 \text{ cm}$. The pump laser intensity at the focus is $I_0 = 2.1 \times 10^{12} \text{ W/cm}^2$.

If the plasma is completely ionized (i.e. the neutral gas pressure is negligible) then the pump laser will heat the plasma primarily by inverse bremsstrahlung. The percentage increase in the electron temperature due to heating by the focussed laser beam is on the order of .

5%, a tolerable perturbation to the plasma⁷². A broadband dye laser centered at wavelength 694.3 nm is used as the probe laser. The dye laser has an energy per pulse of 10^{-2} J and a pulse duration of 3 nsec. The dye laser has the same waist and Rayleigh range as the pump laser. Plasmas and lasers with these properties are available.

The peak gain coefficient is $\alpha = 2.8 \times 10^{-4} \text{ cm}^{-1}$ at a wavelength of 700.4 nm with the assumed experimental parameters. Note that $v_{th} \omega/c$ is more than 5 times as large as the plasma frequency. The single pass gain product for the stimulated Thomson scattering is $\alpha L = \alpha Z_p = 1.3 \times 10^{-3}$. Although the gain product is small, it is possible to detect by the use of polarization enhancement as introduced by Wieman and Hänsch⁷³. Polarization enhancement may provide a three order of magnitude increase in signal to background. In this situation the light transmitted by the blocking polarizer is increased by 130% above the light transmitted without stimulated Thomson scattering. With existing plasmas and lasers it should be possible to detect stimulated Thomson scattering.

The probe beam can be analyzed with a low resolution grating monochromometer and detected with a multielement detector array. The measurement of the gain for

$\lambda > 694.3 \text{ nm}$ or absorption for $\lambda < 694.3 \text{ nm}$ as a function of λ will yield both n_e and T_e for the plasma. If a 30 element array with 33% quantum efficiency were used, then each of the 30 elements would detect about 3.5×10^7 dye laser photons. This number assumes the dye laser has an energy of 10^{-2} J per pulse and assumes an extinction ratio of 10^{-7} for the crossed polarizers. If the signal to noise ratio for the detector array were limited by photon statistics, then an intensity modulation of 1.7×10^{-4} would just be detectable. This corresponds to the gain from a plasma with $n_e = 1.3 \times 10^{11} \text{ cm}^{-3}$ and $T_e = 10 \text{ eV}$. Stimulated Thomson scattering may be useful as a diagnostic for laboratory plasmas and arcs. One additional advantage of using stimulated Thomson scattering as a diagnostic is that spontaneous emission from the plasma would not be a major source of noise. A small gain or absorption is detected with a near diffraction limited probe laser beam. The probe laser beam is spatially filtered after traversing the plasma. Since all the stimulated photons are emitted into the same small solid angle as the probe beam, only the spontaneous emission into this small solid angle acts as a background noise source.

Figure 22

A Schematic Diagram of a Proposed Experiment
for Detecting Stimulated Thomson Scattering

FIGURE 22

APPENDIX 4

PROPERTIES OF JACOBIAN ELLIPTIC FUNCTIONS

The properties of the Jacobian Elliptic Functions are well described by L. M. Milne-Thomson in Ref. 68. A few of the most useful properties of the functions used in Chapter 4 are listed in this appendix. The parameter, p , is suppressed in the notation used, i.e. $\text{sn}(x, p)$ is written simply as $\text{sn}(x)$.

$$\text{sn}(-x) = -\text{sn}(x) \quad (4.1)$$

$$\text{cn}(-x) = \text{cn}(x) \quad (4.2)$$

$$\text{dn}(-x) = \text{dn}(x) \quad (4.3)$$

$$\text{sn}^2 x + \text{cn}^2 x = 1 \quad (4.4)$$

$$\text{dn}^2 x = 1 - p \text{sn}^2 x \quad (4.5)$$

$$\text{sn}(x \pm y) = \frac{\text{sn}(x)\text{cn}(y)\text{dn}(y) \pm \text{sn}(y)\text{cn}(x)\text{dn}(x)}{1 - p \text{sn}^2 x \text{sn}^2 y} \quad (4.6)$$

$$\text{cn}(x \pm y) = \frac{\text{cn}(x)\text{cn}(y) \mp \text{sn}(x)\text{dn}(x)\text{sn}(y)\text{dn}(y)}{1 - p \text{sn}^2 x \text{sn}^2 y} \quad (4.7)$$

$$\text{dn}(x \pm y) = \frac{\text{dn}(x)\text{dn}(y) \mp \text{sn}(x)\text{cn}(x)\text{sn}(y)\text{cn}(y)}{1 - p \text{sn}^2 x \text{sn}^2 y} \quad (4.8)$$

$$\frac{d}{dx}(\text{sn}(x)) = \text{cn}(x)\text{dn}(x) \quad (4.9)$$

$$\frac{d}{dx}(\text{cn}(x)) = -\text{sn}(x)\text{dn}(x) \quad (4.10)$$

$$\frac{d}{dx}(\text{dn}(x)) = -p \text{sn}(x)\text{cn}(x) \quad (4.11)$$

If $x \ll 1$, then the following relations hold.

$$\operatorname{sn}(x) \approx x - (1+p) \frac{x^3}{3!} + (1+14p+p^2) \frac{x^5}{5!} \quad (4.12)$$

$$\operatorname{cn}(x) \approx 1 - \frac{x^2}{2!} + (1+4p) \frac{x^4}{4!} \quad (4.13)$$

$$\operatorname{dn}(x) \approx 1 - p \frac{x^2}{2!} + p(4+p) \frac{x^4}{4!} \quad (4.14)$$

If $p \ll 1$, then the following relations hold.

$$\begin{aligned} \operatorname{sn}(x) &\approx \sin(y) + \frac{p}{4} \sin(y) \cos^2(y) \\ &\approx \sin(x) - \frac{p}{4} [x - \sin(x) \cos(x)] \cos(x) \end{aligned} \quad (4.15)$$

where $y = \frac{x}{1+p/4}$.

$$\begin{aligned} \operatorname{cn}(x) &\approx \cos(y) - \frac{p}{4} \cos(y) \sin^2(y) \\ &\approx \cos(x) + \frac{p}{4} [x - \sin(x) \cos(x)] \sin(x) \end{aligned} \quad (4.16)$$

$$\begin{aligned} \operatorname{dn}(x) &\approx 1 - \frac{p}{2} \sin^2(y) - \frac{p^2}{4} \left[\sin^2(y) - \frac{1}{2} \sin^4(y) \right] \\ &\approx 1 - \frac{p}{2} \sin^2(x) - \\ &\quad \frac{p^2}{4} \left[\sin^2(x) - \frac{1}{2} \sin^4(x) - x \sin(x) \cos(x) \right] \end{aligned} \quad (4.17)$$

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