MASTER

Physical Processes in Collapse Driven Supernova

**R. W. Mayle
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Physical Processes in Collapse Driven Supernova

By

Ronald William Mayle

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PHYSICAL PROCESSES IN COLLAPSE DRIVEN SUPERNOVA

Ronald W. Mayle

ABSTRACT

Supernova are believed to occur when a massive star, unable to continue burning nuclear fuel in its interior, collapses, and subsequently produces a spectacular explosion. The understanding of how this occurs has been the subject of much research over the years, yet the exact scenario has been elusive. James R. Wilson, in 1982, discovered what may be the solution to this problem. The explosive mechanism he found relies on the fact that the central part of these massive stars evolve into neutron stars, and in so doing, emit about 10^{53} ergs of energy in neutrinos. About one second after the core of the star has collapsed to nuclear densities, thermodynamic conditions in the region behind the accretion shock (produced as the core halted and bounced) and outside the neutrinosphere, are favorable for free baryons to absorb the hot core neutrinos, and gain thermal energy. This gain of energy produces material pressure that pushes on the accretion shock, and helps it to move into the stellar envelope. Work that continues the investigation of these late time supernova (so called because it was previously thought that, after bounce, an explosion would occur on timescales of tens of milliseconds) is reported in this thesis. The method of neutrino transport being used in a numerical simulation is very important in the study of late time supernova, as the explosive mechanism depends on neutrino heating of the material behind the shock. Neutrino transport is discussed, and results are presented that indicate flux limited neutrino transport does an adequate job of following the neutrino evolution. The core region of these exploding stars becomes unstable to convective motions during the supernova evolution. Convective mixing allows more neutrinos to escape from under the neutrinosphere, and thus increases the amount of heating by neutrinos. An approximate method of incorporating convection is described, and some results of including convection in a computer model is presented. Another phenomena is seen in computer simulations of supernova, oscillations in the neutrino luminosity and mass accretion rate onto the proto-neutron star. The last topic discussed in this thesis describes the attempt to understand this oscillation by perturbation of the steady state solution to equations approximating the complex physical processes occurring in the late time supernova.

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***This work is dedicated to my
good friend and teacher
James R. Wilson***

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Chapter I

A SHORT HISTORY OF SUPERNOVA

In 1934 Baade and Zwicky suggested that the bright optical displays (10^{49} ergs total output) seen in a small region of a galaxy, over a time scale of 100's of days, resulted from the explosive death of a star. (Later, it was discovered that about 10^{51} ergs of kinetic energy contained in matter motion is also released during the catastrophic event.) Today, it is said that Baade and Zwicky probably made the correct association.

Hoyle (1946) recognized that a highly evolved star ($M \geq 10M_{\odot}$) with an iron core, would be gravitationally unstable to collapse. In this 1946 paper he proposed that such a star would become rotationally unstable as the collapse continued, due to angular momentum conservation. Material would be thrown off as the centrifugal forces overcame gravity and thus produce a supernova explosion.

Burbidge, Burbidge, Fowler and Hoyle (1957) proposed a different mechanism for the explosion of a supernova. They assumed the gravitational instability proposed previously by Hoyle (1946) would trigger the collapse, and argued that the dynamical implosion accompanying collapse could result in the rapid compression and heating of the lighter elements surrounding the iron core, and thereby initiate a thermonuclear explosion in this region.

Present day workers in the field of supernova theory probably would agree that some supernova are triggered by a gravitationally unstable iron core, while others explode due to nuclear energy generated by elements lighter than iron. However, the supernova mechanisms mentioned above are not considered likely, even though the exact scenario is still not completely certain today.

Observers have found that the majority of supernova can be classified as belonging to one of two distinct types. Type I supernova have no hydrogen lines in a spectrum taken at maximum brightness. They are seen in all galaxy types and their light curves tend to be homogeneous. Type II supernova possess hydrogen lines in the spectrum taken at maximum brightness, are not seen in elliptical galaxies and, as a class, have light curves that are not as homogeneous as the Type I light curves.

From the observations above, a reasonable association of red giant stars (with iron cores, unable to stably burn nuclear fuel further) as progenitors of Type II supernova, and white dwarf stars (accreting material from a massive companion and approaching the limiting mass for degenerate electron pressure to withstand collapse) as progenitors of Type I supernova

can be made. The fact that red giants contain a hydrogen envelope and are young compared to stars in an elliptical galaxy, while white dwarfs have shed their hydrogen envelope, are end products of stellar evolution, and are found in any galaxy, would be consistent with the observations.

As of the present, a common consensus has been arrived at (see Woosley and Weaver (1986) and references contained therein) that the explosive energy produced by an unstable white dwarf, and which drives the explosion itself, is due to the nuclear reactions that convert the light elements found in the core of a white dwarf (i.e. C, O, Ne) to iron group elements. The exact method of this burning (i.e. detonation or deflagration) is still undecided. Also agreed is that the energy powering a collapse driven supernova is gravitational energy, as the remnant produced after the explosion, had, in some way, delivered part of its gravitational binding energy to the material blown away. In the collapse driven supernova, the mechanism for delivery of the binding energy to the stellar envelope is still not agreed upon.

Colgate and White (1966) published results of a numerical calculation that obtained explosions from a collapse driven supernova. Neutrinos exiting the core (produced by the capture of electrons on the core baryons) deposited enough energy (during rare interactions with matter) to give the envelope an escape velocity. This burst of neutrinos lasted about .01 second. They did not use radiation transport to follow the neutrinos, but used instead entropy source and sink terms in the matter internal energy equation to model the effect of neutrinos.

Wilson (1971) published results of numerical work which included neutrino transport in a fully consistent general relativistic calculation modeling a collapse driven supernova. He showed that the neutrino burst was not strong enough to eject material, but that the rebound from collapse could, in low mass, stars start a shock wave that would eject the envelope.

Around the late 60's and into the 70's, the understanding of the weak interaction was undergoing change. The Weinberg-Salam model of the electroweak interaction became the standard theory, and showed new possibilities for neutrino interaction with matter. Neutral weak current interactions were first observed in the laboratory in 1973 (see Commins and Bucksbaum (1983), for an introduction to weak interactions), and are predicted by the Weinberg-Salam model. For the supernova problem, coherent scattering of neutrinos through neutral currents (cross-section proportional to the number of scatterers squared)

would enhance the coupling between the neutrinos and the elements (i.e. Si, O, C) found in the mantle surrounding the iron core (see Freedman (1974)). It was hoped this increased cross-section would revive the Colgate and White scenario for supernova explosions. Many workers (Wilson(1974), Bruenn(1975), Schramm and Arnett(1975)) included the new weak interaction cross-sections, but found that explosions (for the finally accepted neutrino-nucleon coupling (see Weinberg (1972))) still did not result.

Although it was long recognized that a central part of the star ($\rho \gtrsim 10^{11} \frac{g}{cc}$) forms where neutrinos do not escape rapidly due to the small neutrino mean free path, it was now becoming apparent that one could gain understanding of the supernova evolution by separating the stellar structure into two parts, the material inside the neutrinosphere, and the material in the rest of the star (see Bethe et al. (1979)).

Attention was also being paid to the possibility that the hydrodynamic bounce of the core would drive a shock wave out into the mantle, and so blow off the envelope of the star, as was seen by Wilson (1971) for low mass stars. The bounce motivated much research into the equation of state of matter near and above nuclear densities (see Bethe et al. (1979)).

Numerical modelers (Bowers and Wilson(1982 a,b), Cooperstein (1982)), however, were finding the shock wave stalling on its exit from the core, becoming an accretion shock. This was due to the loss of energy to neutrinos produced by the high temperature post-shocked material, and by the energy needed to dissociate the heavy elements passing through the shock to free baryons or helium. (See Baron, Cooperstein, and Kahana (1985) for recent work on the prompt hydrodynamic bounce supernova and the equation of state.)

The subject of this thesis concerns a new mechanism for the explosion produced in a collapse driven supernova. Wilson (1985) discovered that, at late times (i.e. about one second after bounce) the accretion shock can be revived. Energy is deposited in the region behind the shock by neutrinos leaving the neutrinosphere, thereby strengthening the shock and allowing it to proceed out into the stellar envelope.

Chapter II gives a qualitative description of the evolution of a supernova, in particular a late time supernova.

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Chapter II

QUALITATIVE DESCRIPTION OF A COLLAPSE DRIVEN SUPERNOVA

Figure 1 shows a Hertzsprung-Russell diagram with the solid line the main sequence, along which stars are stably burning hydrogen in their central regions. A $15 M_{\odot}$ red giant is shown in the upper right hand corner of the diagram, and its prior and subsequent evolution is described below.

STRUCTURE OF A SUPERNOVA PROGENITOR

The $15 M_{\odot}$ star remained on the main sequence while burning hydrogen to helium in the inner 10 % of its mass, where thermodynamic conditions were favorable for the production of helium. It remained on the main sequence approximately 10^7 years. This lifetime can be estimated using the fact that 7 Mev per nucleon of nuclear binding energy is liberated when 4 hydrogen nuclei become one He nucleus (probably mediated by the CNO cycle in this massive star). Using $L \approx 10^4 L_{\odot}$, from figure 1:

$$t_{MS} = \left(\frac{.1M}{m_B} \right) \left(\frac{7 \text{ Mev}}{L} \right) = 1.7 \times 10^7 \text{ yr.}$$

where m_B is the mass of a proton. After exhausting hydrogen, the central regions contract, raising temperatures and densities to the point where helium can burn to carbon.

For stars greater than about $8 M_{\odot}$, thermonuclear burning begins when the electrons are non-degenerate. Since the core is in quasi-equilibrium, the pressure at any radius is determined by the amount of mass above that radius. If the electrons are nondegenerate then $P \propto \rho T$. As the burning begins, if T increases, ρ will decrease (the pressure must stay relatively constant). This lowers the reaction rate, and thereby allows the star to adjust to the new burning conditions without a major disruption. If the electrons were degenerate, $P \propto \rho^{\gamma}$ ($\gamma \geq 1$), and when thermonuclear burning begins, there is no stabilizing feedback to allow the reaction rate to adjust to the stars energy production needs. Thermonuclear burning under degenerate conditions can result in a major disruption of the star.

During the central core contraction of late stage burning, the stellar radius increases due to the increased luminosity heating and pushing on the outer layers of the star. However, the luminosity increase is only slight, and massive stars ($M \gtrsim 10 M_{\odot}$) tend to track in the Hertzsprung-Russell diagram in a roughly horizontal manner (see Weaver and Woosley (1980)).

After exhausting helium, the star goes through a core contraction phase until carbon burning can begin. In fact, the star begins to burn elements up to the iron group nuclei, each in succession, as it exhausts its current fuel, and core contracts until the next available element can be burned. Since heavier elements release less binding energy per nucleon, these further stages become much shorter than the main sequence lifetime.

When silicon burning is going on in the core, temperatures are around .5 Mev (see Clayton (1968)). The iron group nuclei produced come into nuclear statistical equilibrium as temperatures are now so high that nuclear reaction rates are rapid enough to produce equilibration. During this burning stage to iron, neutrino losses lead to values of temperature and density such that the iron core becomes mildly degenerate before collapse. (In the most recent $15 M_{\odot}$ star evolutionary calculation done by Weaver and Woosley, the inner $1.3 M_{\odot}$ is in NSE at the onset of collapse.)

Prior to collapse, a hot iron white dwarf size core exists at the center of the star, surrounded by concentric shells of lighter elements, remnants of the earlier burning stages. These shells still are producing energy, but not enough to compensate for the energy lost in radiation from the stellar surface. The main source of pressure support for the iron white dwarf is electron degeneracy pressure.

From energy considerations:

$$\langle \epsilon_e \rangle > \frac{\rho \bar{Z}}{m_B} + \frac{3}{2} \frac{\rho k_B T}{56 m_B} \approx \frac{\rho G M}{R} \quad (1)$$

where $\langle \epsilon_e \rangle$ is the average energy per electron, and $\bar{Z}$ is the number of matter electrons per baryon. (Any nuclear energy generated during a collapse to radius R is assumed radiated away.)

Now $\langle \epsilon_e \rangle \approx \frac{3}{4} \mu_e$, where μ_e is the electron chemical potential. One finds from (1):

$$\frac{3}{4} \mu_e \bar{Z} + \frac{3}{2} \frac{k_B T}{56} \approx \frac{G M m_B}{R} \quad (2)$$

Using $\mu_e = (3\pi^2)^{\frac{1}{3}} \frac{\hbar}{m_e c} \left(\frac{\rho \bar{Z}}{m_B} \right)^{\frac{1}{3}} m_e c^2 = 11.1 (\rho_{10} \bar{Z})^{\frac{1}{3}} \text{ Mev}$ ($\rho_N = 10^{-N} \rho$ with ρ in $\frac{\text{g}}{\text{cc}}$), where the expression for μ_e is accurate in the zero temperature limit, and neglecting the second term on the left hand side of (2) results in:

$$\mu_e \approx \frac{4}{3} \frac{G M m_B}{R \bar{Z}} \quad (3)$$

Taking $R \approx 10^8 \text{ cm}$ (a typical number from evolutionary calculations), $\bar{Z} \approx .46$ for iron,

and $M \approx M_{CH}$, the limiting mass for a white dwarf (i.e. $5.76 \bar{Z}^2 M_{\odot}$), the value found for μ_e is about 5.0 Mev.

Now the temperature is about .5 Mev (silicon burning temperature) and so $\frac{5kT}{\mu_e} \approx \frac{1}{10}$. Thus the approximation used in going from equations (2) to (3) is justified. The density can then be determined using $\rho_{10} = \frac{1}{2} \left(\frac{\mu_e}{11.1} \right)^3$ and $\mu_e = 5$ Mev. This gives a value of about $2 \times 10^9 \frac{g}{cc}$ for the central density.

Summarizing this section, a $15 M_{\odot}$ star evolves to a point where an iron white dwarf core has formed. Central temperatures are about .5 Mev and central densities are around $2 \times 10^9 \frac{g}{cc}$. Iron will not burn to produce energy, as it is the most tightly bound nucleus. At this time, the character of the evolution of the star changes, due to this inability of iron to burn exothermically.

The next section describes the first event that occurs when the $15 M_{\odot}$ red giant reaches the point where nuclear energy generation in the core is impossible.

COLLAPSE

A collapse is triggered by the loss of pressure support. Two mechanisms contribute to this loss, iron thermal disintegration and electron capture onto protons. Iron thermal disintegration is discussed first.

Iron Thermal Disintegration

where (4) is a simplification of a large network of equilibrium reactions. ${}^{56}Fe^*$ indicates an excited state of the iron nucleus. (Note, to change one Fe nucleus into 13 He nuclei and 4 neutrons requires about 125 Mev.)

Since reactions (4) occur in equilibrium, entropy is not changed by thermal disintegration of iron. However the adiabatic index, γ , which would be $\frac{4}{3}$ (pressure support by degenerate electrons), drops below this value because of the thermal disintegration. The energy needed to excite or dissociate the iron becomes unavailable for thermal pressure support.

The internal energy of the core is dominated at the onset of collapse by the semi-degenerate electron gas. From equation (15) in Chapter V, one finds the value of the internal energy to be about $2 \times 10^{18} \frac{erg}{gram}$, where $\rho = 7.8 \times 10^9 \frac{g}{cc}$, $T = .6$ Mev and $\bar{Z} = .42$

are used (these values of ρ, T , and $\bar{Z}$ are typical of evolutionary calculations). Again using (5.15), the internal energy per gram contained in the electron gas at $T=6$ Mev minus the value at $T=0$, implies there is $4 \times 10^{17} \frac{\text{erg}}{\text{gram}}$ in thermal energy available for the Fe thermal dissociation. Since around 1.65×10^{19} ergs of energy is needed to convert one gram of iron into one gram of helium, a rough estimate of the amount of iron breakup needed to completely exhaust the thermal energy in the core is 25 %.

The Saha equation for reaction (4) can be written as:

$$\frac{n_{\alpha}^{13} n_N^4}{n_{Fe}} = \frac{2^{43}}{(56)^{\frac{1}{2}} g_{Fe}} \left(\frac{m_B k_B T}{2\pi \hbar^2} \right)^{24} \exp \left(-\frac{125}{T_{\text{Mev}}} \right) \quad (5)$$

where g_{Fe} is the statistical weight of the iron. Using $n_N = \frac{4}{13} n_{\alpha}$, and introducing the mass fractions X_{α} , X_N , and X_{Fe} , (5) can be rewritten as:

$$\frac{X_{\alpha}^{17}}{1 - \frac{14}{13} X_{\alpha}} = \frac{(2.47 \times 10^{62})}{\rho_0^{16} g_{Fe}} T_{\text{Mev}}^{24} \exp \left(-\frac{125}{T_{\text{Mev}}} \right) \quad (6)$$

where $X_N + X_{\alpha} + X_{Fe} = 1$ has been used.

Figure 2 shows the (ρ, T) plane and curves of constant X_{α} . Also shown is the track of the temperature and density of the inner zone found in a numerical calculation of the collapse of a $15 M_{\odot}$ star, parameterized by time. At these temperatures, g_{Fe} can be much greater than unity, due to the many excited states of iron. The expression used for g_{Fe} in constructing figure 2 is taken from Bowers and Wilson (1982b).

The track shown in figure 2 can be understood as follows. Since the iron breakup is endothermic, the energy needed to fuel (4) is lost to pressure support (γ drops below $\frac{4}{3}$ and $P \propto \rho^{\gamma}$). In order to try and compensate for the pressure deficit, the star begins to contract in order to increase the density and temperature and thus reduce the deficit. This, however, leads to more iron thermal disintegration which forces the star to contract further, and thus the nature of the instability becomes apparent. Note, for the evolutionary track shown, the density increases by a factor of 100 while the temperature only increases by a factor of 4, reflecting the fact that the major pressure support is electron degeneracy pressure.

The appreciable breakup of helium by thermal disintegration into two protons and two neutrons requires higher temperatures than reactions (4) at similar densities. In the breakup up of helium, three new particles are produced and 28.3 Mev of energy is required. This gives 9.5 Mev per new particle produced. The breakup of iron to 13 helium nuclei and 4 neutrons produces 16 new particles and requires 125 Mev, giving an average of 7.8 Mev per

new particle produced. The new particles must have some positive energy, however small. Therefore, there is a range in temperature where iron can be dissociated while helium break up is not likely. However, a very small fraction of free protons will be present, due to some helium breakup, which will be important for electron capture.

Electron Capture on Protons

Electrons can be captured on protons in nuclei, or on free protons. This statement can be translated into the following reactions:

$$e^- + p \Rightarrow n + \nu_e \quad (7)$$

$$e^- + N(Z, A) \Rightarrow N^*(Z-1, A) + \nu_e \quad (8)$$

with (8) followed by:

$$N^*(Z-1, A) \Rightarrow N(Z-1, A) + \gamma \quad (9)$$

where $N(Z, A)$ represents a nucleus with A nucleons and Z protons, and $N^*(Z, A)$ indicates an excited state of $N(Z, A)$.

The reaction rate for (7) or (8) can be written using Fermi's Golden Rule (assuming a given initial state) as:

$$d\Gamma = \frac{2\pi}{\hbar} \left(\frac{1}{2} \sum_{\text{spins}} |H_{fi}|^2 \right) (1 - f_\nu) \frac{d^3 p_\nu}{(2\pi\hbar)^3} \delta(E_\nu + Q - E_e) \quad (10)$$

where $(1 - f_\nu)$ is a blocking factor that inhibits (7) or (8) if the final neutrino state is already occupied. H_{fi} is the interaction Hamiltonian, p_ν is the neutrino momentum, E_ν is the neutrino energy, E_e is the electron energy, and Q is the difference in energies of the final and initial nuclear states. At the temperatures seen in a supernova collapse, the neutrons and protons or heavy nuclei never become relativistic, so the use of Fermi's golden rule in the non-relativistic limit is justified.

Consequently, applying (10) to (7). At the energies seen in a supernova (i.e. less than 50 Mev), the weak interaction can be considered to be a contact interaction, and in the rest frame of the proton, the neutron produced remains almost at rest. Assuming the dipole approximation in calculating $|H_{fi}|$ (i.e. $\exp(\frac{i}{\hbar}(\vec{p}_e + \vec{p}_\nu) \cdot \vec{x}) \approx 1$) one sees that $|H_{fi}|^2$ must only depend on weak interaction constants (i.e. no energy dependence, so it is just a number). Equation (10) can be rewritten as:

$$d\Gamma = \frac{\sigma_0}{2} \left(\frac{1 + 3\alpha^2}{4} \right) c(1 - f_\nu) \left(\frac{E_\nu}{m_e c^2} \right)^2 \delta(E_\nu + Q - E_e) dE_\nu \quad (11)$$

where $\sigma_0 = \frac{4G_F^2 m_e^2 \hbar^2}{\pi c^2} = 1.7 \times 10^{-44} \text{ cm}^2$, $\alpha \approx 1.2$, and G_F is the weak interaction constant.

Early in the collapse, the neutrinos escape from the star and so rob the core of internal energy and pressure support. To find the neutrino luminosity from (7), integrate (11) over the initial electron states after first multiplying by E_ν and the number of protons. Since $\frac{\mu_e}{k_B T} \approx 10$ in the core at the onset of collapse, and $\mu_e \geq Q \approx 1.4 \text{ Mev} \geq m_e c^2 = .5 \text{ Mev}$, the integration simplifies, and one finds:

$$L_{\nu_e} = (4.9 \times 10^{13}) X_P \rho \left(\frac{\mu_e}{m_e c^2} \right)^6 \frac{\text{erg}}{\text{g cm}^3} \quad (12)$$

where X_P is the mass fraction of protons.

The average energy of the emitted neutrino can be found using (11). The expression becomes:

$$\langle E_\nu \rangle = \frac{\int_0^{\mu_e} dE_e E_e^2 (E_e - Q)^3}{\int_0^{\mu_e} dE_e E_e^2 (E_e - Q)^2} \approx \frac{5}{6} \mu_e \quad (13)$$

Now:

$$\mu_e - \langle E_\nu \rangle = \mu_N - \mu_P = \bar{\mu} \quad (14)$$

and:

$$\frac{n_P}{n_N} = e^{-\bar{\mu}/k_B T} \quad (15)$$

Using (12), (13), (14), (15), and $X_N = \frac{1}{13} X_\alpha$, one finds:

$$\frac{L_{\nu_e}}{\rho} = (3.8 \times 10^{12}) X_\alpha \exp \left(-\frac{\mu_e}{6k_B T} \right) \left(\frac{\mu_e}{m_e c^2} \right)^6 \frac{\text{erg}}{\text{g sec}} \quad (16)$$

Since $X_\alpha \leq .2$ during the collapse phase, and $\frac{\mu_e}{k_B T}$ does not vary much (see figure 2, and use $\mu_e \approx 11.1(\rho_{10} \bar{Z})^{\frac{1}{3}} \text{ Mev}$ during collapse (so take $\frac{\mu_e}{k_B T} \approx 10$) to find:

$$\frac{L_{\nu_e}}{\rho} \leq (2 \times 10^{11}) \left(\frac{\mu_e}{m_e c^2} \right)^6 \approx (2 \times 10^{17}) \frac{\text{erg}}{\text{g sec}} \quad (17)$$

As previously mentioned, there is about $2 \times 10^{18} \frac{\text{erg}}{\text{gram}}$ of internal energy contained in the electron gas at the start of the collapse. Using (17), reaction (7) would exhaust this energy store in about 10 seconds.

Electron capture on nuclei is more complicated due to the fact that the neutron produced must be able to find an available state. Since heavy nuclei such as Fe have more neutrons than protons, part of the energy of the captured electron must go into boosting the proton

to a higher energy neutron state. Bethe et al. (1979) argue that the daughter nucleus is excited about 3 Mev above a lower energy state. From a shell model of the nucleus, the proton cannot reach the lowest energy neutron state available since angular momentum would not be conserved. They base their analysis on the following reaction:

which they take as representative of electron capture during collapse.

It should be pointed out that, as electron capture occurs, the typical nucleus has $\frac{Z}{A}$ less than $\frac{26}{56} = .46$, the value for iron, and will also begin to acquire neutrons as the high electron chemical potential inhibits neutron decay.

Fuller et al. (1982a,b) and Fuller (1982) have done detailed calculations of electron capture rates in nuclei and find the Bethe et al. (1979) estimate of 3 Mev to be good. Also discovered is that when $A - Z \approx 40$, the proton must begin filling another shell that is much higher in energy than the shell being filled when $A - Z$ is less than 40. This means the capture rate will be reduced at this point, by about a factor of $\frac{1}{10}$ (private communication from Jim Wilson).

Note that Baym et al. (1971) argue that the atomic number A , of the average heavy nucleus is given by:

$$A = 194(1 - \bar{Z})^2(1 - .236\rho_{12}^{\frac{1}{3}})^{-1}$$

and can become large as the density increases towards nuclear density.

A Fermi gas model of a nucleus can be used, along with the recognition that the daughter nucleus is in an excited state of 3 Mev to find an approximate capture rate or neutrino luminosity for (18) (see Bethe et al. (1979) for this type of calculation). An easy way to estimate the luminosity is to use the luminosity for (7), and assume that perhaps 10 % of the protons in each nucleus are energetic enough to capture (i.e. those near the Fermi energy in a statistical model of the nucleus). Equation (12) becomes:

$$L_{\nu_e}^A = \left(\frac{.1Z}{A}\right) (4.9 \times 10^{13}) \rho \left(\frac{\mu_e}{m_e c^2}\right)^6 \frac{\text{erg}}{\text{g sec}} \quad (19)$$

Assuming $\frac{Z}{A} \approx .46$ (for iron) and $\mu_e \approx 5$ Mev, the neutrino luminosity per gram can be found from (19) to be:

$$\frac{L_{\nu_e}^A}{\rho} \approx (2 \times 10^{18}) \frac{\text{erg}}{\text{g sec}}$$

Since the internal energy is about $2 \times 10^{18} \frac{\text{erg}}{\text{gram}}$, the reaction (8) can deplete this internal energy on 1 second time scales.

Note that electron capture, unlike iron thermal disintegration, is not an equilibrium process, and thus can change the entropy per baryon. The neutrino can escape from the system, and lower the entropy, but since capture rates lag behind equilibrium, entropy can increase (i.e. bulk viscosity). In practice, the entropy remains almost unchanged.

This section can be concluded by recognizing that when pressure support is lost through iron thermal disintegration and electron capture on protons, the iron core must collapse on a freefall timescale. An estimate of this collapse time is thus:

$$t_{ff} \approx \frac{1}{\sqrt{G\rho}} \approx .1 \text{ sec.}$$

Actually, since pressure support is not completely absent, the time for collapse is several times longer.

Also a burst of neutrinos is produced when the collapse accelerates, as the electron capture on nuclei reduces the value of $\bar{Z}$, the number of matter electrons per baryon, to about .33. The neutrinos cease escaping freely as the density reaches values where the neutrino mean free path is on the order of the size of the radius. Using equation (11), an estimate of the mean free path of a neutrino is:

$$\frac{1}{\lambda_\nu} = \frac{\sigma \rho}{m_B} \approx \sigma_0 \left(\frac{E_\nu}{m_e c^2} \right)^2 \frac{\rho}{m_B}$$

Therefore $\lambda_\nu \approx 10^6$ cm when $E_\nu \approx 3k_B T \approx 10$ Mev, and $\rho \approx 2.5 \times 10^{12} \frac{\text{g}}{\text{cc}}$.

Figure 3 shows the mass point trajectories of the iron core during the collapse phase, and the bounce, which is the subject of the next section.

BOUNCE AND SHOCK FORMATION

Figure 4 shows a sequence of infall velocity curves, parameterized by the central density, and covering a .001 second span in time. The last curve, with $\rho = 1.3 \times 10^{14} \frac{\text{g}}{\text{cc}}$, is just before bounce and shock formation.

Evolutionary calculations give a value for s , the entropy per baryon in units of Boltzmann's constant, around 1 at the start of the collapse. Bethe et al. (1979) point out that s cannot change appreciably during collapse. This implies that most nucleons stay inside the heavy nuclei, and thus the nuclear contribution to the pressure is small during collapse. In fact, most nucleons stay inside the heavy nuclei until the heavy nuclei touch and a uniform

sea of nuclear matter forms. At densities near but still less than nuclear density, the nuclear pressure is negative due to the attractive nuclear force. At densities above nuclear density, the nuclear force becomes repulsive and the nuclear pressure will dominate all other sources of pressure.

The structure of the curves in figure 4 can be understood as follows. By the above discussion, the major pressure support during collapse is degenerate electrons, which gives $P = K\rho^\gamma \approx K\rho^{\frac{4}{3}}$ (as previously noted, γ is slightly less than $\frac{4}{3}$ due to iron thermal disintegration and electron capture). A $\gamma = \frac{4}{3}$ gas is neutrally stable, and there is no natural length scale that can be constructed from K, G , and M . What happens is that material inside the sonic point (i.e. the point in the star where the sound speed equals the infall velocity) can stay in communication and collapse homologously in what are successive $n=3$ polytrope-like configurations. The material outside the sonic radius collapses in quasi-free fall. Considerable analytic work has been done on this phase of collapse by Goldreich and Weber (1980) and Yahil and Lattimer (1982). The results of this work produced a prediction for the size of the homologous core, which turns out to be very close to the Chandrasekhar mass. Another prediction is that the material outside the homologous core has a density that varies as $\frac{1}{R^3}$.

Since the value of $\bar{Z}$ is now around .33 due to the electron capture, M_H , the mass of the homologous core is about:

$$M_H \approx M_{CH} = 5.76 \bar{Z}^2 M_\odot = .63 M_\odot$$

In realistic calculations, M_H is found to be about .55 $M_\odot$. It is smaller than the predicted value due to the negative nuclear pressure at densities near but still less than nuclear. The prediction that the density varies as $\frac{1}{R^3}$ is seen to be satisfied in numerical simulations.

When the central density climbs to $1.9 \times 10^{14} \frac{g}{cc}$, nuclear density has been reached (since $\bar{Z} \approx .33$, nuclear density is less than $2.7 \times 10^{14} \frac{g}{cc}$, the value for $\bar{Z} = .5$). At this point γ begins to rise above $\frac{4}{3}$ due to the near incompressible nature of nuclear matter above nuclear densities, and the collapse halts. The experimental value of $K_N = 9m_B \frac{dP}{d\rho}$, the nuclear compressibility modulus, is found to be 220 Mev (see Blaziot et al. (1976)). (Note, at densities many times nuclear density, γ must be less than 2, if the condition that the sound speed is less than the speed of light is fulfilled (see Lightman et al. (1975)).)

Brown et al. (1982) argue that pressure waves travel from the halted central regions

outward, carrying information of the impending bounce to the still falling material at larger radii. The pressure waves cannot pass far beyond the sonic point without a shock wave forming. The homologous core bounces as a unit and the rebound launches the shock wave into the infalling material. The shock weakens as heavy nuclei passing through the shock are dissociated into neutrons and protons. When the shock moves through and beyond the neutrinosphere, the post shocked heated material radiates neutrinos that can escape the star and further weaken the shock.

The question of the shock promptly moving outward is a thorny one. Baron, Cooperstein and Kahana (1985) exploit the uncertainty in the supra-nuclear equation of state and find that a soft enough supra-nuclear equation of state (i.e. taking K_N , the compressibility modulus to be around 120-140 Mev) and a slightly stiffer sub-nuclear equation of state (thus lessening the effect of the attractive part of the nuclear pressure, and increasing the size of the homologous core) produces a shock initially energetic enough to reach a radius where the energy losses to neutrinos and the dissociation of heavy nuclei have become small. These losses become small when the shock weakens to the point where it is unable to dissociate nuclei, and if the post shock temperature is small enough so that neutrino production falls. The softer equation of state allows a deeper bounce and a large homologous core reduces the amount of material the shock must dissociate in its exit from the core.

The equation of state we are using has a compressibility modulus more in line with the laboratory value of 220 Mev found on measurements made with such nuclei as ^{208}Pb which has $\bar{Z} = .394$ (see Blaziot et al. (1976)). Using $K_N = 220$ Mev produces a much stiffer equation of state than using a value around 140 Mev. Consequently the rebound is not as helpful in launching an energetic shock, as found by Baron, Cooperstein and Kahana (1985).

In the following, the bounce mechanism is assumed to fail in producing an explosion. Figure 5 extends figure 3 to later times. As can be seen, an explosion occurs, and a proto-neutron star of baryon mass $1.6 M_\odot$ is left as a remnant. The next section discusses the physics important to the late time explosion.

LATE TIME SUPERNOVA

An important feature of figure 5 is that the region outside the neutrinosphere is composed of neutrons and protons. The existence of these free nucleons is due to the dissociation of heavier elements by the shock.

The presence of neutrons and protons outside the neutrinosphere is important because they are able to absorb and emit neutrinos and antineutrinos by the following reactions:

$$n + \nu_e \rightleftharpoons p + e^- \quad (20)$$

$$p + \bar{\nu}_e \rightleftharpoons n + e^+ \quad (21)$$

The gravitational binding energy of a cold neutron star is about $.1Mc^2$. Using $M = 1.6M_\odot$, one finds about 3×10^{53} ergs of energy that must be radiated away (mainly in neutrinos of all types). The possibility exists that nucleons outside the neutrinosphere may absorb energy from the electron neutrinos and electron antineutrinos produced by the forming neutron star, and so heat the material under the shock. The heating increases the material pressure behind the shock, which helps the shock to move outward.

An estimate of the composition of the star behind the shock can be made independent of a computer simulation, using an energy conservation argument. Assuming iron is dissociated to neutrons and protons by the shock, the following equation holds:

$$\frac{GMm_{Fe}}{R} \approx (8 \text{ Mev} + \frac{3}{2}T_{Mev})(56) \quad (22)$$

or, taking $M=1.6M_\odot$:

$$\frac{22}{R_7} = 8 + \frac{3}{2}T_{Mev} \quad (23)$$

where $R_7 = 10^{-7} \times R$ with R in cm. Assuming iron dissociates to helium (needing approximately 2 Mev per nucleon) one finds:

$$\frac{22}{R_7} = 2 + \frac{3}{2}T_{Mev} \quad (24)$$

Finally, assuming the shock is not strong enough to dissociate the iron, results in:

$$\frac{22}{R_7} = \frac{3}{2}T_{Mev} \quad (25)$$

In figure 6 equations (23),(24),and (25) are graphed, and the regions where the shock can dissociate the material into baryons, helium, or no dissociation can be seen. The actual temperature would be an interpolation of the disjointed segments.

This simple construction reproduces the general composition features seen in figure 5, after the bounce and shock formation. The temperature profile is also roughly in agreement with the numerical simulations.

The net rate of heating of the stellar material outside the neutrinosphere can be estimated. In the region near $R = 10^7$ cm, at late times, the densities are more similar to the outer regions of the original iron white dwarf core, and are around $10^8 \frac{g}{cc}$. From figure 6, temperatures are 1-2 Mev. In this region, electrons are relativistic and quasi-degenerate (i.e. $\frac{\mu_e}{k_B T} \approx 1.7$). Equations (20) and (21) will be written as:

$$b + l \rightleftharpoons b + \nu_l \quad (26)$$

where $\frac{\mu_e}{k_B T} \approx 0$ is assumed (this implies that the number density of the electrons is identical to the number density of the positrons). Also, the mass difference between the neutron and proton, and the type of neutrino (particle or anti-particle) will be ignored.

The matter cooling rate per baryon due to $b + l \Rightarrow b + \nu_l$ can be written using (11) as:

$$\dot{E}_{cool} = \int E_\nu \frac{E_l^2}{\pi^2 (\hbar c)^3} f_l dE_l d\Gamma \quad (27)$$

Assuming:

$$f_l = \frac{1}{\exp \frac{E_l}{k_B T} + 1} \quad (28)$$

the integral (with the neglect of blocking factors) can be done to find:

$$\dot{E}_{cool} = 8.1 \sigma_o c \left(\frac{m_e c}{\hbar} \right)^3 m_e c^2 \left(\frac{k_B T}{m_e c^2} \right)^6 \quad (29)$$

The matter heating rate per baryon due to $b + \nu_l \Rightarrow b + l$ is calculated in a similar manner as follows:

$$\dot{E}_{heat} = \int E_\nu \frac{E_\nu^2}{8\pi^3 (\hbar c)^3} f_\nu dE_\nu d\Omega d\Gamma^* \quad (30)$$

where:

$$d\Gamma^* = \sigma_o c \left(\frac{1 + 3\alpha^2}{4} \right) (1 - f_l) \left(\frac{E_l}{m_e c^2} \right)^2 \delta(E_l - E_{\nu_e}) \quad (31)$$

Figure (7) illustrates some notation, and shows the physical picture that is being assumed in the derivation.

Since the region of interest is outside the neutrinosphere, the neutrino distribution function, f_ν , is assumed to remain constant along a beam, and have that value characteristic of neutrinos in equilibrium with the matter at temperature T_{sp} .

The angular integral in (30) can be approximately done giving:

$$\dot{E}_{heat} \approx \pi \left(\frac{R_{sp}}{R} \right)^2 \int \frac{E_\nu^3}{8\pi^3 (\hbar c)^3} f_\nu dE_\nu d\Gamma \quad (32)$$

Now completing the integrals in (32) using:

$$f_\nu = \frac{1}{\exp \frac{E_\nu}{k_B T} + 1}$$

one finds:

$$\dot{E}_{heat} = 8.1 \sigma_{oc} \left(\frac{m_e c}{\hbar} \right)^3 m_e c^2 \left(\frac{R_{sp}}{2R} \right)^2 \left(\frac{k_B T_{sp}}{m_e c^2} \right)^6 \quad (33)$$

Measuring temperatures in Mev, the net heating rate per baryon becomes:

$$\dot{E}_{net} = (3.3 \times 10^{-6}) \left(\left(\frac{R_{sp}}{2R} \right)^2 T_{sp}^6 - T^6 \right) \frac{\text{erg}}{\text{baryon sec}} \quad (34)$$

where temperatures are measured in Mev. For an alternate way of obtaining (34), see Bethe and Wilson (1985).

$\dot{E}_{net}$ is positive provided:

$$\frac{R_{sp}}{2R} \geq \left(\frac{T}{T_{sp}} \right)^3$$

Using $T_{sp} \approx 5$ Mev, $R_{sp} \approx 3 \times 10^6$ cm, one finds:

$$R \approx 2 \times 10^7 \text{ cm}$$

The gravitational potential energy per baryon at $R = 2 \times 10^7$ cm is:

$$E_{grav} = \frac{GMm_B}{R} \approx 1.8 \times 10^{-5} \frac{\text{erg}}{\text{baryon}} \quad (35)$$

where $M = 1.6M_\odot$ has been taken.

Equations (34) and (35) define a timescale for the net heating to overcome the gravitational binding as:

$$t_{Exp} = \frac{E_{grav}}{\dot{E}_{net}} = .03 \text{ sec.}$$

where $T_p = 5$ Mev, $T = 1$ Mev, $R_p = 3 \times 10^6$ cm, and $R = 3 \times 10^7$ cm have been taken in (34). These are typical values seen in a collapse calculation.

Assuming an explosion does occur, by neutrino heating and the subsequent revival of the shock, the density in the region between the shock and the proto-neutron star will drop as the envelope begins to separate from the remnant. The density is dominated by neutrons and protons, formally the major source of interaction for the neutrinos. As the density falls, electron-positron pairs become the major opacity source for neutrino interaction. As will be seen in Chapter IV, the opacity for neutrino interaction with electrons can be written as:

$$k_e^0 = (\text{const}) \frac{\sigma_0 E_\nu k_B T}{m_B (m_e c^2)^2}$$

in regions of high temperature and low density (i.e. $\frac{E_\nu T}{\mu_e} \geq 1$). It is possible for the material in the bubble behind the shock to interact with the core neutrinos and absorb energy through scattering with the electron-positron pairs.

Thus the bubble region can be heated by the neutrinos and so prevent the temperature from rapidly falling, as it would without some heating mechanism, when the shock moves outward during the explosion.

Photons and electron-positron pairs will provide pressure support (i.e. $P = \frac{11}{12} a T^4$, where $a = \frac{4\sigma_B}{c}$, with σ_B the Stephan-Boltzmann constant), even though the density (dominated by the baryons) is falling.

Figure 8 shows the results of the numerical computation for a $25 M_\odot$ progenitor star. Here, another phenomena is clearly seen. The neutrino luminosity can oscillate in time, as well as the mass accretion rate. Chapter VI will report preliminary results in the study of this neutrino oscillator.

The last subject to discuss in Chapter II is the possible effects of magnetic fields or rotation. The median rotational period, τ_{rot} , of the observed pulsars is about .67 sec (see Shapiro and Teukolsky (1983)). The shortest period is 1.558 ms (PSR 1937+214; Backer et al. (1982)). The vibrational period, τ_{vib} , of a neutron star is about .5ms (i.e. $\tau_{vib} \approx \sqrt{\frac{R^3}{GM}}$). $\frac{\tau_{vib}}{\tau_{rot}} \approx .3$, even in the most extreme example, hence the dynamic effects of rotation are negligible, and must even have been of less importance in the formation of a neutron star.

Since $B \approx 10^{12}$ gauss for the majority of observed pulsars, $P_B = \frac{1}{8\pi} B^2 \approx 4 \times 10^{22}$ cgs. This is completely negligible to matter pressures seen in a collapse. At the onset of collapse,

the pressure in degenerate electrons is approximately 4×10^{27} cgs, as can be seen using equation (5.15) and the fact that the electron pressure is about $\frac{1}{3}$ the electron energy density. During the subsequent evolution, the material pressure reaches much higher values.

Therefore, for the majority of stars, spherical collapse calculations must be able to explain the supernova explosions.

A numerical calculation of a supernova including magnetic fields and rotation was done by LeBlanc and Wilson (1970) and later repeated with a new equation of state and neutrino physics by Sympalasty (1984). Both of these calculations found, for extreme values of magnetic field and rotation, a jet of about $.07M_{\odot}$ in neutronized matter escapes along the rotation axis. This may be a possible site for the production of the r process nuclei. However, only a small fraction of exceptional supernovae are needed to account for the r process elements if neutron jets are produced. These papers concerned the early times of a supernova (i.e. late times supernova were not investigated), and the reader is referred to them for more quantitative details of the effects of magnetic fields and rotation on the evolution of a supernova.

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Chapter II Figure Captions

Fig. 1 Hertzsprung-Russell Diagram adapted from Shu (1982).

Fig. 2 Curves of constant Helium mass fraction on a (ρ, T) plane.

Fig. 3 Collapse and bounce of a 15 solar mass star. The symbol B indicates free baryons.

Fig. 4 Velocity versus radius for the 15 solar mass star during the final .001 second before bounce.

Fig. 5 Late time behavior of the 15 solar mass star. This figure continues figure 3.

Fig. 6 Approximate temperature and composition profile behind the shock.

Fig. 7 Illustration of the physical conditions important to understanding the late time neutrino heating mechanism.

Fig. 8 A late time supernova of a 25 solar mass progenitor. The phenomena of oscillations in neutrino luminosity and mass accretion rate is clearly seen.

FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4

FIGURE 5

FIGURE 6

FIGURE 7

FIGURE 8

Chapter III

EQUATIONS TO MODEL A SUPERNOVA

INTRODUCTION

The supernova code I inherited is described in Bowers and Wilson (1982a,b), hereafter denoted by BW. In that paper, the equations and numerical methods used were discussed. This code approximated some aspects of general relativity (i.e. the force equation is correct in the static limit).

It has long been known that taking into account the full hydrodynamic effects of general relativity does make a difference in some details of the model supernova evolution (see May and White(1967), Wilson(1971)). One difference is that a deeper bounce is found with general relativity than with Newtonian physics. However, the inclusion of general relativity is not the determining factor in producing supernova explosions. (However, see Baron, Cooperstein, and Kahana(1985) for some recent work on the combined effects of general relativistic hydrodynamics and various parameterizations of the high density equation of state).

As mentioned above, the supernova model described in BW has an enhanced gravitational force (due to inclusion of static general relativity effects), that gives a deep bounce, and therefore larger possible infall velocities. An observer far away from the star (and therefore not significantly influenced by the gravitational field produced by the star) may ask if special relativistic effects are important in a supernova collapse.

The question can be answered by looking at a computer simulation (or, see below for an estimate without appealing to computer output). The v/c ratio can become as large as .3; however, at any time, no more than $0.01M_{\odot}$ achieves such high velocities. During most of the evolution, $.2c$ is a more typical peak infall velocity. Special relativistic effects can therefore be estimated to be on the order of 4 % (i.e. $(v/c)^2 \leq .04$).

In the neutrino transport used by BW, some order v/c effects were taken into account, but the gravitational redshift was not. From computer calculations, it is found that at bounce, the redshift, z , is .25 at the center of the star, rising to .55 at the end of the simulation (about 1 second after bounce). However, at the neutrinosphere, $z=.05$ at bounce, rising to $z=.20$ at the end of the same calculation. Since, inside the neutrinosphere, neutrinos are trapped, z at the neutrinosphere radius is the important number to consider in

estimating the effect of redshift on a late time supernova.

As discussed in Chapter II, the heating rate of the matter behind the shock due to the neutrinos exiting the neutrinosphere is given by:

$$\dot{E}(R) = k_a(\epsilon_\nu(R), T) \frac{L(R)}{4\pi R^2} \quad (1)$$

where $L(R)$ is the neutrino luminosity at radius R .

In Chapter II it was noted that the absorptive opacity in the temperature and density range found behind the shock can be expressed as:

$$k_a = a_1 \epsilon_\nu^2 + a_2 \epsilon_\nu T$$

where a_1 and a_2 depend on the composition.

In terms of values at the neutrinosphere radius, R_{sp} , the luminosity at infinity can be expressed as:

$$L(\infty) = \frac{L(R_{sp})}{(1+z)^2}$$

and the energy of a neutrinos emitted at the neutrinosphere and observed at infinity is:

$$\epsilon_\nu(\infty) = \frac{\epsilon_\nu(R_{sp})}{(1+z)}$$

Since the neutrinosphere is defined to be the radius where the optical depth to infinity is $2/3$, there is still a probability for a neutrino to interact outside the neutrinosphere (this fact, of course, is of fundamental importance to late time supernova). For a neutrino observed at infinity, and emitted at the neutrinosphere:

$$z \approx \frac{GM}{R_{sp}c^2} \leq .2$$

if $M=1.4 M_\odot$ and $R_{sp}=10$ km are taken (see the value of z quoted on the first page of this chapter, for z at the end of a calculation).

Taking into account that fact that a neutrino absorbed before escaping feels only a fraction of the redshift at infinity, and that the region important for absorption is typically a few times the neutrinosphere radius, a rough estimate of the average redshift at absorption is:

$$\langle z \rangle = \frac{2}{3} z$$

(this would correspond to emission at R_{sp} and absorption at $3R_{sp}$, a radius chosen as representative for absorption.) Using $\langle z \rangle$, (1) can be rewritten as:

$$\dot{E}(R) = k_s \left(\frac{\epsilon_\nu}{1 + \langle z \rangle}, T \right) \frac{L(R_{sp})}{4\pi R_{sp}^2} \left(\frac{1}{1 + \langle z \rangle} \right)^2$$

Since k_s has both a quadratic and linear dependence on ϵ_ν , the effect of the redshift on the heating rate can be written as:

$$\dot{E}(R, \langle z \rangle) = \left(\frac{1}{1 + \langle z \rangle} \right)^n \dot{E}(R, 0)$$

with $n=3$ or 4 , depending on which term dominates the absorptive opacity. Assuming $n=4$, $\langle z \rangle = .13$, the effect of the redshift is to drop the heating rate by 40 % below the value found without redshifting the neutrinos.

From the discussion so far, it is clear that the most important physical process neglected (as far as late time supernova are concerned) is the gravitational redshift of the neutrinos exiting the neutrinosphere.

Another effect of importance (for late time supernova) which could increase the neutrino luminosity exiting the core is convection (material mixing would dredge up trapped neutrinos from below the neutrinosphere and allow them to escape). This topic will be the concern of Chapter V.

Originally, a completely new computer code was planned, and Appendix A contains two complete sets of fully general relativistic equations, each of which could be used for such a code (see below). However, new stellar input models (provided by Stanford Woosley) were becoming available at the time a new code was being planned. This led to a change in direction of research, as new codes take approximately three years of development and debugging, and we felt that it was important to run the new models as soon as possible. Therefore, of the defects mentioned above, the redshift of the neutrinos was felt to be the most important to remedy in some fashion, and this could be done without too drastic a change to the present code. In this chapter, the approximate equations now used in the supernova code will be presented. Some attempt will be made to justify any approximation that is made.

Two complete sets of equations are given in Appendix A, one is more appropriate for an Eulerian (fixed coordinate system) solution of the equations, the other is appropriate for a Lagrangian (coordinates tied to matter) solution of the equations. The supernova code I

inherited uses a Lagrangian system of coordinates, and the equations to be found in this chapter will start from the Lagrangian equations in Appendix A.

HYDRODYNAMIC ACCELERATION

The following five equations are repeated from Appendix A:

$$\frac{1}{a} \frac{\partial U}{\partial t} = - \left(1 + \frac{U_{tot} + P_{tot}}{c^2} \right) \left(\frac{\partial P_{tot}}{\partial m} + \frac{2H}{R} \frac{\partial R}{\partial m} + \frac{1}{c^2} \frac{\partial}{\partial t} \left(\frac{\tilde{G}b}{a} \right) \right) - \frac{GM}{R^2} \left(1 + \frac{4\pi P_{tot} R^3}{Mc^2} \right) \quad (\text{A.44})$$

$$M = 4\pi \int_0^R \left(\rho + \frac{e_{tot}}{c^2} + \frac{\tilde{G}U}{c^4 \Gamma} \right) R^2 dR \quad (\text{A.43})$$

$$\Gamma = \left(1 + \frac{U^2}{c^2} - \frac{2GM}{Rc^2} \right)^{\frac{1}{2}} = \frac{1}{b} \frac{\partial R}{\partial m} \quad (\text{A.40})$$

$$b = \frac{1}{4\pi R^2 \rho} \quad (\text{A.37})$$

$$U = \frac{1}{a} \frac{\partial R}{\partial t} \quad (\text{A.42})$$

where $e_{tot} = \tilde{e} + e$, $P_{tot} = \tilde{P} + P$, $H = \frac{1}{2}(\tilde{e} - 3\tilde{P})$ and:

e = matter energy density

P = matter pressure

$\tilde{e}$ = neutrino energy density

$\tilde{P}$ = neutrino pressure

$\tilde{G}$ = neutrino energy flux

As mentioned in Appendix A, Γ is a relativistic generalization of a special relativistic gamma, and $U \approx \Gamma v$ is the radial component of the matter 4 velocity as measured by an Eulerian observer. Using v , Γ can be rewritten as:

$$\Gamma = \frac{1}{\sqrt{1 - (v/c)^2}} \left(1 - \frac{2GM}{Rc^2} \right)^{\frac{1}{2}}$$

Now, changing derivatives with respect to m into derivatives with respect to R (using equation (A.40)), and replacing U with Γv , equation (A.44) becomes:

$$\begin{aligned} & \left(1 + \frac{e_{tot} + P_{tot}}{\rho c^2}\right) \rho \left(\frac{1}{a} \frac{\partial v}{\partial t} + \frac{v}{a} \frac{\partial \ln \Gamma}{\partial t}\right) \\ &= -\Gamma \left(\frac{\partial P_{tot}}{\partial R} + \rho \left(1 + \frac{e_{tot} + P_{tot}}{\rho c^2}\right) \frac{GM}{R^2} \left(1 + \frac{4\pi P_{tot} R^3}{Mc^2}\right) \frac{\left(1 - \left(\frac{v}{c}\right)^2\right)}{\left(1 - \frac{2GM}{Rc^2}\right)}\right) \\ & \quad - \frac{1}{c^2 b} \frac{\partial}{\partial t} \left(\frac{\tilde{G} b}{a}\right) - \frac{2\Gamma H}{R} \end{aligned} \quad (2)$$

An estimate of $\bar{P}/P$ can be made. When the relativistic corrections in (2) are important (in the central part of the star), the core will have bounced (after reaching nuclear densities) and the neutrinos will be trapped, and in equilibrium with the matter. Since both electrons and neutrinos are relativistic degenerate fermions in the central regions (of the possible neutrino types, only ν_e are present in appreciable numbers due to the degeneracy of the electrons):

$$\frac{\bar{P}}{P_e} = \frac{\frac{1}{4} n_\nu \mu_\nu}{\frac{1}{4} n_e \mu_e} = \frac{1}{2} \left(\frac{\mu_\nu}{\mu_e}\right)^3 \frac{\mu_\nu}{\mu_e} = \frac{1}{2} \left(\frac{\mu_\nu}{\mu_e}\right)^4$$

In the core, the reaction:

gives the relation:

$$\mu_N + \mu_\nu = \mu_e + \mu_P$$

Since ρ is about nuclear density (2.5×10^{14} g/cc), $\bar{Z} = .33$, $\mu_e = 11.1(\rho_{10} \bar{Z})^{\frac{1}{3}}$ Mev = 230 Mev, and $\mu_N, \mu_P = 30$ Mev (see Bethe et al. (1979)), one finds $\mu_\nu = 200$ Mev. Using these values:

$$\frac{\bar{P}}{P_e} = .30$$

Now the baryons can be taken as a nonrelativistic degenerate fermion gas for purposes of the estimate being made here. So:

$$P_B = \frac{1}{5} \frac{\hbar^2}{m_B} (3\pi^2)^{\frac{2}{3}} (n_N^{\frac{5}{3}} + n_P^{\frac{5}{3}})$$

where n_N is the number of neutrons/vol. and n_P is the number of protons/vol. Using $P_e = \frac{1}{4} n_e \mu_e$, one finds:

$$\frac{\bar{P}}{P} = \frac{\bar{P}}{P_e + P_B} = \frac{\frac{P}{\bar{P}_e}}{1 + \frac{P}{\bar{P}_e}} = .14$$

The ratio $\bar{e}/e$ should be about the same, and so $\bar{e}$ and $\bar{P}$ will be neglected as compared to e and P in the mass correction terms in (2). Therefore (2) can be rewritten as:

$$\begin{aligned} & \left(1 + \frac{e+P}{\rho c^2}\right) \rho \left(\frac{1}{a} \frac{\partial v}{\partial t} + \frac{v}{a} \frac{\partial \ln \Gamma}{\partial t}\right) \\ &= -\Gamma \left(\frac{\partial P}{\partial R} + \rho \left(1 + \frac{e+P}{\rho c^2}\right) \frac{GM}{R^2} \left(1 + \frac{4\pi PR^3}{Mc^2}\right) \frac{\left(1 - \left(\frac{v}{c}\right)^2\right)}{\left(1 - \frac{2GM}{Rc^2}\right)}\right) \\ & \quad - \Gamma \frac{\partial \bar{P}}{\partial R} - \frac{1}{c^2 b} \frac{\partial}{\partial t} \left(\frac{\bar{G} b}{a}\right) - \frac{2\Gamma H}{R} \end{aligned} \quad (3)$$

An estimate of the infall velocities can be made consistent with that quoted previously. When the supernova collapse begins, matter in the envelope surrounding the dense white dwarf central core of a red giant, previously in quasi-hydrostatic equilibrium, but suddenly without pressure support, begins to free fall. If it were to continue to free fall to the position of the original core surface, it would achieve velocities such that:

$$\frac{v}{c} \approx \sqrt{\frac{2GM}{R}} \approx .2$$

where $M = 1.4 M_\odot$ and $R = 10^7$ cm are assumed. This is descriptive of pre-bounce conditions. If material could continue to free fall to a typical neutron star radius, the above estimate would increase by a factor of 3. The collapse evolution is not simple free fall, however, as the rebounding shock wave produced when the core ceases to collapse, moves into the falling material, decelerating it. Also, in late times, the heating of matter behind the shock wave by neutrinos produces material pressure that further reduces the value of v/c from the extreme estimate given above. As stated previously, peak values from the computer output indicate v/c is usually less than .2, and special relativistic effects are on the order of 4 %.

With the neglect of $(v/c)^2$ compared to unity, the right hand side of (3) can be written as:

$$R.H.S. = -\Gamma \left(\frac{\partial P}{\partial R} + \rho \left(1 + \frac{e+P}{\rho c^2}\right) \frac{GM}{R^2} \left(1 + \frac{4\pi PR^3}{Mc^2}\right) \frac{1}{\left(1 - \frac{2GM}{Rc^2}\right)}\right) + \text{neu. accel.} \quad (4)$$

and:

$$\Gamma = \left(1 - \frac{2GM}{Rc^2}\right)^{\frac{1}{2}}$$

Γ can now be interpreted as giving the relation between increments in areal radius, and radius measured perpendicular to a sphere centered on the star.

In the static limit, when there is no neutrino acceleration and no matter motion, (4) determines the structure of a neutron star (see Weinberg (1972)), and will give the exact structure (consistent with a given equation of state) of a central remnant, as the numerical evolution proceeds.

Γ is unity at the center of the star, and reaches a minimum near the surface of the central remnant. Then it increases back to unity as R increases outward. Using typical values of a neutron star mass and radius:

$$\Gamma_{\min} \approx .8$$

Time derivatives of Γ during collapse are negligible as Γ stays bracketed by $.8 \leq \Gamma \leq 1.$, and so the terms in (3) involving $\frac{\partial \Gamma}{\partial t}$ will be dropped.

From (A.43):

$$M = 4\pi \int_0^R \left(\rho + \frac{e_{tot}}{c^2} + \frac{\tilde{G}v}{c^4} \right) R^2 dR \approx 4\pi \int_0^R \left(\rho + \frac{e}{c^2} + \frac{\tilde{G}v}{c^4} \right) R^2 dR$$

An estimate of the size of $\tilde{G}v/\rho c^4$ can be made. This ratio should be largest just outside the neutrinosphere where the neutrinos begin to free stream. Now the mean free path of neutrinos can be estimated (see Lamb and Pethick (1976)):

$$\frac{\lambda}{10^6 \text{ cm}} = \left(\frac{10^{12} \frac{\text{g}}{\text{cc}}}{\rho} \right) \left(\frac{10 \text{ Mev}}{\epsilon_\nu} \right)^2$$

So λ is about the neutron star radius (10^6 cm) when $\rho \approx 10^{12} \text{ g/cc}$ and $\epsilon_\nu \approx 3T_{\text{matter}} \approx 10 \text{ Mev}$.

Assuming the neutrino emission is blackbody, $\tilde{G} = \frac{21}{8} \sigma_B T^4$ (note the $\frac{21}{8}$ comes from the fact that 6 neutrino types are present, and a fermion with one spin is worth $\frac{7}{16}$ of a photon, for blackbody radiation). Taking v/c about .1, and with the above values of ρ and T , one finds:

$$\frac{\tilde{G}v}{\rho c^4} \approx 10^{-3}$$

To a very good approximation:

$$M = 4\pi \int_0^R \left(\rho + \frac{e}{c^2} \right) R^2 dR$$

The hydrodynamic equations now used in the supernova code are:

$$\begin{aligned} \left(1 + \frac{e+P}{\rho c^2} \right) \rho \left(\frac{1}{a} \frac{\partial v}{\partial t} \right) = & -\Gamma \left(\frac{\partial P}{\partial R} + \rho \left(1 + \frac{e+P}{\rho c^2} \right) \frac{GM}{R^2} \left(1 + \frac{4\pi P R^3}{M c^2} \right) \frac{1}{\left(1 - \frac{2GM}{R c^2} \right)} \right) \\ & -\Gamma \left(\frac{\partial \tilde{P}}{\partial R} + \frac{1}{c^2 b \Gamma} \frac{\partial}{\partial t} \left(\frac{\tilde{G}b}{a} \right) + \frac{2H}{R} \right) \end{aligned} \quad (5)$$

$$M = 4\pi \int_0^R \left(\rho + \frac{e}{c^2} \right) R^2 dR \quad (6)$$

$$\frac{1}{a} \frac{\partial R}{\partial t} = \Gamma v \quad (7)$$

Equations (5),(6),(7) differ from the equations in BW with the inclusion of the a and Γ factors, plus the first factor on the left hand side of (5).

From Appendix A:

$$a = \exp \int_R^\infty \frac{1}{c^2} \left(\left(\frac{\partial P_{tot}}{\partial m} + \frac{2H}{R} \frac{\partial R}{\partial m} + \frac{1}{c^2} \frac{\partial}{\partial t} \left(\frac{\tilde{G}b}{a} \right) \right) \frac{dm}{\left(\rho + \frac{e_{tot}+P_{tot}}{c^2} \right)} \right) \quad (A.38)$$

or changing from an integral over dm to one over dR , (A.38) becomes:

$$a = \exp \int_R^\infty \frac{1}{c^2} \left(\left(\frac{\partial P_{tot}}{\partial R} + \frac{2H}{R} + \frac{1}{c^2 \Gamma b} \frac{\partial}{\partial t} \left(\frac{\tilde{G}b}{a} \right) \right) \frac{dR}{\left(\rho + \frac{e_{tot}+P_{tot}}{c^2} \right)} \right) \quad (8)$$

Making similar approximations in (8) as in (5) results in:

$$a = \exp \int_R^\infty \frac{1}{c^2} \left(\left(\frac{\partial P_{tot}}{\partial R} + \frac{2H}{R} + \frac{1}{c^2 \Gamma b} \frac{\partial}{\partial t} \left(\frac{\tilde{G}b}{a} \right) \right) \frac{dR}{\left(\rho + \frac{e+P}{c^2} \right)} \right) \quad (9)$$

In the static weak field limit:

$$\frac{\partial P}{\partial R} \approx -\frac{\rho GM}{R^2}$$

Assuming $R \geq \bar{R}$ where the mass outside $\bar{R}$ is negligible compared to the total mass of the system, then:

$$a \approx \exp \int_R^\infty \frac{1}{c^2} \left(-\frac{GM}{R^2} \right) dR = \exp \left(\frac{-GM}{Rc^2} \right) = e^{\frac{\phi}{c^2}} \approx 1 + \frac{\phi}{c^2}$$

where $\frac{\phi}{c^2} \ll 1$, and ϕ is the Newtonian gravitational potential. (Notice that in this case $\Gamma \simeq a$, i.e. the Schwarzschild metric.)

Thus it is seen that a represents a generalization of the gravitational slowing down of clocks to the dynamic case. In the supernova code, a is normalized to unity at the outer boundary of the star, in this way, the time measured on the outer boundary is proper time. The equation used in the supernova code for a is:

$$a = \exp \int_R^{R_{\max}} \frac{1}{c^2} \frac{\partial P}{\partial R} \frac{dR}{\left(\rho + \frac{4\pi R^2 P}{c^2} \right)} \quad (10)$$

In (10), the neutrino acceleration terms have been neglected since they are small compared to the matter pressure acceleration (see the previous estimate of the ratio $\bar{P}/P$ in the central part of the star).

MASS CONSERVATION

Equation (A.45) is rewritten below:

$$\frac{1}{a} \frac{\partial}{\partial t} \ln(\rho R^2) + \frac{\partial U}{\partial m} / \frac{\partial R}{\partial m} = 0 \quad (\text{A.45})$$

using:

$$\frac{\partial U}{\partial m} / \frac{\partial R}{\partial m} = \frac{\partial U}{\partial R}$$

$$U \approx \Gamma v$$

Equation (A.45) becomes:

$$\frac{1}{a} \frac{\partial \rho}{\partial t} + \frac{\rho}{R^2} \frac{\partial}{\partial R} (R^2 \Gamma v) = 0 \quad (11)$$

This differs from the BW equation with the retention of the factors of a and Γ .

ENERGY CONSERVATION

From Appendix A:

$$\frac{1}{a} \frac{\partial(e/\rho)}{\partial t} + \frac{P}{a} \frac{\partial}{\partial t} \left(\frac{1}{\rho} \right) = - \int ck(B-F) d\mu \frac{dQ}{a} \quad (\text{A.47})$$

This can be rewritten as:

$$\frac{1}{a} \frac{\partial(e/\rho)}{\partial t} + \frac{P}{a} \frac{\partial}{\partial t} \left(\frac{1}{\rho} \right) = \frac{T}{a} \frac{\partial S}{\partial t} \quad (12)$$

with:

$$\frac{T}{a} \frac{\partial S}{\partial t} = - \int ck(B-F) d\mu \frac{dQ}{a}$$

The right hand side of (12) is identified with $\frac{T}{a} \frac{\partial S}{\partial t}$, giving the change in entropy for the matter in terms of the neutrinos. The left hand side of (12) is used as it stands in the supernova code (again the $\frac{1}{a}$ factors are additions to the code described in BW). The opacities are discussed later in Appendix B.

In a study of antineutrino production during gravitational collapse (see Woosley, Wilson, and Mayle (1985)), a 500000 $M_{\odot}$ star was followed both with the supernova code using (5),(6),(7),(10),(11),(12), and a fully general relativistic collapse code. (This code is a 1-dimensional version of a 2-dimensional collapse code used to study the production of gravitational waves. It was kindly provided by Charles Evans (1984).) This large star has a high entropy going into collapse (entropy/baryon $\approx 200k_B$), and the neutrino production had a negligible effect on the dynamics of collapse, so an adiabatic collapse (used in the fully general relativistic code) is a good approximation. Both codes showed the trapped surface forming at the same interior mass, indicating that at least in this case, the approximation in the supernova code is very good.

NEUTRINO TRANSPORT

Equation (A.46) is rewritten below:

$$\begin{aligned} \frac{1}{ca} \frac{\partial F}{\partial t} = & - \frac{\mu \Gamma}{a R^2} \frac{\partial}{\partial R} (a R^2 F) - \Gamma \left(\frac{1}{R} - \frac{\partial}{\partial R} \ln a \right) \left(\frac{\partial}{\partial \mu} (F(1 - \mu^2)) \right) \\ & + \frac{F}{ca \rho} \frac{\partial \rho}{\partial t} + \frac{R}{c} \frac{\partial}{\partial R} \left(\frac{U}{R} \right) \left(\frac{\partial}{\partial \mu} (\mu(1 - \mu^2) F) + \mu^2 Q \frac{\partial F}{\partial Q} \right) \\ & + \frac{Q}{ca} \frac{\partial F}{\partial Q} \left(\frac{\partial}{\partial t} \ln \frac{R}{a} \right) + k \rho (B - F) \end{aligned}$$

Now defining:

$$\begin{aligned}E_q &= \int_{-1}^1 F d\mu \\ \tilde{G}_q &= \int_{-1}^1 c F \mu d\mu \\ P_q &= \int_{-1}^1 F \mu^2 d\mu\end{aligned}$$

and integrating (A.46) over $d\mu$ results in:

$$\begin{aligned}\frac{1}{a} \frac{\partial E_q}{\partial t} &= -\frac{\Gamma}{aR^2} \frac{\partial}{\partial R} (aR^2 \tilde{G}_q) - Q \frac{\partial E_q}{\partial Q} \frac{\partial \ln a}{a \partial t} + \frac{E_q}{a\rho} \frac{\partial \rho}{\partial t} \\ &+ R \frac{\partial}{\partial R} \left(\frac{U}{R} \right) Q \frac{\partial P_q}{\partial Q} + Q \frac{\partial E_q}{\partial Q} \frac{U}{R} + \int_{-1}^1 c k \rho (B - F) d\mu\end{aligned}\quad (13)$$

Now replace in the above:

$$R \frac{\partial}{\partial R} \left(\frac{U}{R} \right) Q \frac{\partial P}{\partial Q} + Q \frac{\partial E_q}{\partial Q} \frac{U}{R}$$

with:

$$Q \frac{\partial}{\partial Q} \left(\frac{\tilde{E}_q}{R^2} \frac{\partial}{\partial R} \left(\frac{D}{\lambda c} r^2 U \right) \right)$$

where D is the diffusion coefficient (see following discussion).

This is part of an approximation called flux limited diffusion. It is a means (somewhat ad hoc, but see Levermore (1979)) to close the moment equations in such a way that in the diffusion limit, $\frac{D}{\lambda c} = \frac{1}{3}$ (equivalent to $E_q = 3P_q$, in which the replacement above is an exact equality, λ is the neutrino mean free path). In the free streaming limit:

$$\frac{D}{\lambda c} = \frac{1}{\lambda \left| \frac{\partial \ln E_q}{\partial R} \right|}$$

The free streaming limit is dictated by the following identification:

$$\tilde{G}_q = -D \frac{\partial E_q}{\partial R} \quad (14)$$

using:

$$D = \frac{c}{\left| \frac{\partial \ln E_q}{\partial R} \right|} = -\frac{c \tilde{E}_q}{\frac{\partial E_q}{\partial R}}$$

then:

$$\tilde{G}_q = \frac{cE_q}{\frac{\partial E_q}{\partial R}} \frac{\partial E_q}{\partial R} = cE_q$$

This is the exact result for free streaming. In the diffusion limit $D = \frac{1}{3}\lambda c$ so:

$$\tilde{G}_q = -\frac{1}{3}\lambda c \frac{\partial E_q}{\partial R}$$

The form chosen for D is arbitrary, but must have the appropriate asymptotic behavior.

One simple choice is:

$$D = \frac{\lambda c}{3 + \lambda \left| \frac{\partial \ln E_q}{\partial R} \right| \xi} \quad (15)$$

with:

$$\xi = 1 + \frac{3}{1 + \frac{\xi}{2} + \frac{\xi^2}{6}}$$

$$x = \lambda \left| \frac{\partial \ln E_q}{\partial R} \right|$$

More is said about the validity of this approximation in Chapter IV.

With the flux limited diffusion approximation, (13) becomes:

$$\begin{aligned} \frac{1}{a} \frac{\partial \tilde{E}_q}{\partial t} = & -\frac{\Gamma}{aR^2} \frac{\partial}{\partial R} (aR^2 \tilde{G}_q) - Q \frac{\partial \tilde{E}_q}{\partial Q} \frac{\partial \ln a}{a \partial t} + \frac{E_q}{a\rho} \frac{\partial \rho}{\partial t} \\ & + Q \frac{\partial}{\partial Q} \left(\frac{\tilde{E}_q}{R^2} \frac{\partial}{\partial R} \left(\frac{D}{\lambda c} R^2 \Gamma v \right) \right) + \int_{-1}^1 ck\rho(B-F)d\mu \end{aligned} \quad (16)$$

Since:

$$\tilde{e} = \int_0^\infty \tilde{E}_q \frac{dQ}{a}$$

$$\tilde{G} = \int_0^\infty \tilde{G}_q \frac{dQ}{a}$$

$$\tilde{P} = \int_0^\infty \tilde{P}_q \frac{dQ}{a}$$

integration of both sides of (16) over dQ and dividing both sides by a , one finds:

$$\begin{aligned} \frac{1}{a} \frac{\partial \tilde{e}}{\partial t} = & -\frac{\Gamma}{a^2} \frac{1}{R^2} \frac{\partial}{\partial R} (a^2 R^2 \tilde{G}) + \frac{\tilde{e}}{a\rho} \frac{\partial \rho}{\partial t} - \int \left(\frac{E_q}{R^2} \frac{\partial}{\partial R} \left(\frac{R^2 D \Gamma v}{\lambda c} \right) \right) dQ \\ & + \int_0^\infty \int_{-1}^1 ck\rho(B-F)d\mu \frac{dQ}{a} \end{aligned} \quad (17)$$

Note that in the static free streaming limit one finds from (17):

$$\tilde{G} = \frac{\text{const}}{a^2 R^2} \quad (18)$$

The a^2 factor in the denominator of (18) is the expected redshift for the energy luminosity of free streaming neutrinos.

Now, the flux limited diffusion approximation determines the neutrino acceleration of the matter if the demand of energy and momentum conservation is required. First, I will write the expression for matter acceleration, then show that it is the correct one to be consistent with flux limited diffusion. Equation (5) becomes:

$$\begin{aligned} \left(1 + \frac{e+P}{\rho c^2}\right) \frac{\rho}{a} \frac{\partial v}{\partial t} = -\Gamma \left(\frac{\partial P}{\partial R} + \rho \left(1 + \frac{e+P}{\rho c^2}\right) \frac{GM}{R^2} \left(1 + \frac{4\pi R^3 P}{Mc^2}\right) \frac{1}{\left(1 - \frac{2GM}{Rc^2}\right)} \right) \\ + \Gamma \left(\int_0^\infty \frac{\tilde{G}_q}{\lambda c} \frac{dQ}{a} + \frac{\partial \ln a}{\partial R} \int_0^\infty \frac{D}{\lambda c} \tilde{E}_q \frac{dQ}{a} \right) \end{aligned} \quad (19)$$

An equation for total energy density (i.e. $\frac{1}{2}\rho v^2 + e + \tilde{e}$, with the form of the kinetic energy in the Newtonian limit) should be constructed using (11), (12), (17), and (19). However, since the general relativistic mass correction term on the left hand side of (19) is only important in the very center of the star, where $v/c \leq .001$ and neutrinos are trapped, for purposes of this derivation only, (19) will be replaced by equation (20) below:

$$\frac{\rho}{a} \frac{\partial v}{\partial t} = -\Gamma \left(\frac{\partial P}{\partial R} + F_{grav} - \int_0^\infty \frac{\tilde{G}_q}{\lambda c} \frac{dQ}{a} - \frac{\partial \ln a}{\partial R} \int_0^\infty \frac{D}{\lambda c} \tilde{E}_q \frac{dQ}{a} \right) \quad (20)$$

with:

$$F_{grav} = \rho \left(1 + \frac{e+P}{\rho c^2}\right) \frac{GM}{R^2} \left(1 + \frac{4\pi R^3 P}{Mc^2}\right) \frac{1}{\left(1 - \frac{2GM}{Rc^2}\right)}$$

The right hand sides of (19) and (20) are identical.

Now:

$$\begin{aligned} \frac{1}{a} \frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 \right) &= \frac{v^2}{2a} \frac{\partial \rho}{\partial t} + \frac{\rho v}{a} \frac{\partial v}{\partial t} \\ &= -\frac{v^2 \rho}{2} \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 \Gamma v) - v \Gamma \frac{\partial P}{\partial R} + v \Gamma \int_0^\infty \frac{\tilde{G}_q}{\lambda c} \frac{dQ}{a} \\ &\quad + v \Gamma \frac{\partial \ln a}{\partial R} \int_0^\infty \frac{D}{\lambda c} \tilde{E}_q \frac{dQ}{a} - \Gamma v F_{grav} \end{aligned} \quad (21)$$

Using (12), (17), and (21), one finds (where the time derivative is now locally Eulerian, not Lagrangian):

$$\begin{aligned}
& \frac{\partial}{\partial r} \left(\frac{1}{2} \rho v^2 + e + \bar{e} \right) + \frac{1}{R^2} \frac{\partial}{\partial R} \left(\Gamma R^2 \left(v \left(\frac{1}{2} \rho v^2 + e + \bar{e} + P + \int \frac{D\bar{E}_r}{\lambda c} \frac{dQ}{a} \right) + \bar{G} \right) \right) \\
& = -\Gamma v F_{\nu\mu} - \bar{G} \Gamma \frac{\partial \ln \frac{1}{\kappa}}{\partial R} \quad (22) \\
& = \text{work done by gravity on matter and neutrinos}
\end{aligned}$$

From (22) it is thus seen that the approximation to general relativity used is not Post Newtonian, but one in which the static effects of general relativity are done correctly (i.e. gravitational force, redshift of neutrinos), but the matter kinetic terms are treated in the Newtonian approximation.

The above discussion assumed only one type of neutrino. However, six different types of neutrinos are present, the electron neutrino, mu neutrino, tau neutrino, and their antiparticles. The transport equation (17) will apply to each type, with different source and sink terms for the possible interactions with matter. There will be a contribution to (12), the matter internal energy equation, and (19), the matter acceleration for each type of neutrino.

A brief discussion of opacity, and some differences between the opacities used in BW, and what is presently used in the supernova code will be found in Appendix B.

LEPTON NUMBER CONSERVATION

Finally, equation (A.48) is repeated below:

$$\frac{\rho}{m_F} \frac{1}{a} \frac{\partial \bar{Z}}{\partial t} = - \int_0^\infty \int_{-1}^1 (B - F) \frac{dQ}{Q} d\mu$$

This equation is now used in the supernova code, with contributions from the electron neutrino and antineutrino on the right hand side.

SUMMARY

In this chapter the new equations (the old equations are found in Bowers and Wilson (1982 a,b)) used to model a supernova are presented. The new physics that is being added is the gravitational redshift of the neutrinos produced in the collapse. Estimates are made of the importance of its inclusion to the late time supernova explosion. (Other improvements in the physics now used in the supernova code are presented in Appendix B.)

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Chapter IV

NEUTRINO TRANSPORT

In Chapter III, the flux limited diffusion approximation for neutrino transport was introduced. This chapter is concerned with the validity of replacing the Boltzmann equation with the flux limited diffusion approximation.

Neutrino Transport (Flux Limited Diffusion Approximation)

The supernova computer code transports electron neutrinos using multigroup flux limited diffusion. The following equation determines the neutrino evolution:

$$\frac{dF}{dt} = \Gamma \nabla \cdot (a D \nabla F) + \frac{dF}{dt}_{cap} + \frac{dF}{dt}_{comp} + \frac{dF}{dt}_{scatt} + \frac{dF}{dt}_{em} + \frac{dF}{dt}_{rad} + \frac{dF}{dt}_{grav} \quad (1)$$

where the energy density of electron neutrinos at radius R is $\int F \frac{dQ}{Q}$ and D is the diffusion coefficient. D contains a flux limiter that forces $F(\frac{Q}{a}, R)$ to be proportional to $\frac{1}{R^2}$ in the regions where neutrinos are free streaming. Q is the energy of a neutrino as measured by an observer at infinity, while $\frac{Q}{a}$ is the locally measured energy (see Chapter III). The form used for D is as follows:

$$D = \frac{\lambda c}{3 + \lambda \left| \frac{d \ln F}{dR} \right| \xi}$$

where:

$$\xi = 1 + \frac{3}{1 + \frac{x}{2} + \frac{x^2}{8}}$$

and:

$$x = \lambda \left| \frac{d \ln F}{dR} \right|$$

This particular form for D was arrived at by Wilson (private communication) when photon radiation transport was being studied.

The terms on the right hand side of (1) include changes in the neutrino field due diffusion, electron capture, changes in volume, scattering, emission and absorption, and work done on the matter. The last term in (1), as well as the factors of a and Γ , are due to general relativity (see Chapter III).

Looking at equation (16) in Chapter III, one can make the following identifications with (1) of this chapter:

$$\begin{aligned}
\frac{dF}{dt}_{\text{cap}} + \frac{dF}{dt}_{\text{scat}} + \frac{dF}{dt}_{\text{ss}} &\equiv a \int_{-1}^1 ck_p(B-F)d\mu \\
\frac{dF}{dt}_{\text{rad}} &= aQ \frac{\partial}{\partial Q} \left(\frac{F}{R^2} \frac{\partial}{\partial R} \left(\frac{D}{\lambda_c} R^2 \Gamma_0 \right) \right) \\
\frac{dF}{dt}_{\text{comp}} &= \frac{F}{\rho} \frac{\partial \rho}{\partial t} \\
\frac{dF}{dt}_{\text{grav}} &= -Q \frac{\partial F}{\partial Q} \frac{\partial a}{\partial t}
\end{aligned}$$

and the same symbol, F , has been used for $F(\frac{Q}{a}, R, t, \mu)$ and $F(\frac{Q}{a}, R, t) = \int F(\frac{Q}{a}, R, t, \mu) d\mu$.

The electron antineutrinos and mu-tau neutrinos and antineutrinos are similarly treated except that the capture term in (1) is not included.

Boltzmann Transport Equation

Let f represent the distribution function of electron neutrinos such that:

$$\int f(\vec{x}, \vec{p}, t) \frac{d^3p}{h^3} = \text{number density of neutrinos}$$

In the limit of complete thermodynamic equilibrium:

$$f(\vec{x}, \vec{p}, t) = \frac{1}{\exp\left(\frac{E - \mu}{k_B T}\right) + 1}$$

The transport equation for $f(\vec{x}, \vec{p}, t)$ is taken to be:

$$\begin{aligned}
\frac{1}{c} \frac{\partial f}{\partial t} + \hat{p} \cdot \nabla f &= k_a \rho(b - f) - f \int (1 - f') \rho \kappa_s(E, \Omega \rightarrow E', \Omega') \frac{d^3p'}{h^3} \\
&+ (1 - f) \int f' \rho \kappa_s(E', \Omega' \rightarrow E, \Omega) \frac{d^3p'}{h^3}
\end{aligned} \quad (2)$$

On the right hand side of (2), the first term represents absorption and emission of neutrinos by free nucleons, and b is a Fermi-Dirac distribution function representing emission with $\mu_\nu = \epsilon_N - \epsilon_p + \mu_e$. The second and third terms represent scattering by free nucleons, electrons and positrons, and helium nuclei.

Since the accuracy of the neutrino transport is most important to the late time supernova explosions at late times, the comparison with the flux limited diffusion approximation is made after bounce, at which time only the above sources of opacity are considered important (i.e. the heavy nuclei are all dissociated). This occurs in the region of the star from the center to the shock position.

Equation (2) neglects Doppler shift, Doppler aberration effects, advection, and gravitational redshift. After bounce, the core is quasi-static and all the above effects are small except the gravitational redshift (see Chapter III). The gravitational redshift is taken into account after the final spectrum is solved for, by using the energy shift that neutrinos would undergo if emitted from the neutrinosphere. In (2), the term $\frac{\partial f}{\partial t}$ will be neglected, and the equation solved in the static approximation.

The construction of the opacities for use in (2) is discussed next. First, an estimate of the importance of electron-positron scattering opacity as compared to the total opacity is made. Helium will be ignored as an opacity source for purposes of making this estimate.

The ratio of electron-positron scattering to total opacity is written as:

$$\frac{k_e^s}{k_{tot}} = \frac{n_e - \sigma_{e-\nu} + n_{e+} \sigma_{e+\nu}}{n_N \sigma_{N\nu}^a + n_N \sigma_{N\nu}^s + n_P \sigma_{P\nu}^s + n_e - \sigma_{e-\nu} + n_{e+} \sigma_{e+\nu}} \quad (3)$$

where:

$$\sigma_{N\nu}^a \approx \left(\frac{1 + 3\alpha^2}{4} \right) \sigma_o \left(\frac{E}{m_e c^2} \right)^2$$

$$\sigma_{N\nu}^s \approx \sigma_o (1 - C_A)^2 \left(\frac{E}{m_e c^2} \right)^2$$

$$\sigma_{P\nu}^s \approx \frac{\sigma_o}{4} ((C_V - 1)^2 + 3\alpha^2 (C_A - 1)^2) \left(\frac{E}{m_e c^2} \right)^2$$

and $\sigma_o = 1.7 \times 10^{-44} \text{ cm}^2$, E is the neutrino energy. C_V and C_A are weak interaction constants and $\alpha \approx 1.2$ is a strong interaction constant. The cross-sections are taken from Tubbs and Schramm (1975), hereafter denoted by TS.

If $C_V \approx 1$, $C_A = .5$, then $\sigma_{P\nu}^s \approx \sigma_{N\nu}^s$ (since $\alpha \approx 1.2$). For the number of pairs per unit volume the following holds:

$$n_{pairs} = n_{e+} = \int \frac{1}{\exp\left(\frac{E + \mu_e}{k_B T}\right) + 1} \frac{d^3 p}{h^3} \approx \frac{1}{\pi^2} \left(\frac{m_e c}{h} \right)^3 \left(\frac{k_B T}{m_e c^2} \right)^3 F_2^-(\eta)$$

where the final expression is good in the limit $\frac{m_e c^2}{k_B T} \ll 1$. The symbol $F_2^-(\eta)$ is defined as:

$$F_2^-(\eta) = \int_0^\infty \frac{x^2 dx}{e^{x+\eta} + 1}$$

with $\eta = \frac{\mu_e}{k_B T}$. $F_2^-(\eta) \approx 2e^{-\eta}$ is an approximation good to within 10 %.

In the limit of degenerate electrons with $C_V = 1$ and $C_A = .5$:

$$\sigma_{e^{-}\nu} = \frac{7}{32} \sigma_0 \frac{E \mu_e}{(m_e c^2)^2}$$

This cross-section is taken from TS.

If the electrons are degenerate, the number of positrons is negligible. In this limit:

$$\frac{k_e^0}{k_{tot}} = \frac{\frac{7}{32} \bar{Z}}{(1.33(1 - \bar{Z}) + .25) \frac{E}{\mu_e} + \frac{7}{32} \bar{Z}}$$

In this case $\langle E \rangle \approx \mu_e$, and taking $\bar{Z}$, the number of matter electrons per baryon, to be about $\frac{1}{3}$, one finds:

$$\frac{k_e^0}{k_{tot}} = .06$$

Therefore electron scattering is completely negligible in the degenerate limit.

If electrons and positrons are relativistic and non-degenerate then (taking $C_V = 1$ and $C_A = .5$):

$$\begin{aligned} \sigma_{e^{-}\nu} &= \frac{7}{8} \sigma_0 \frac{E k_B T}{(m_e c^2)^2} \\ \sigma_{e^{+}\nu} &= \frac{3}{8} \sigma_0 \frac{E k_B T}{(m_e c^2)^2} \end{aligned}$$

where the cross-sections are taken from TS.

Therefore in the limit of relativistic non-degenerate electrons and positrons:

$$\frac{k_e^0}{k_{tot}} = \frac{\frac{5.5}{\rho_7} T^3 e^{-\eta} + \frac{7}{8} \bar{Z}}{(1.33(1 - \bar{Z}) + .25) \frac{E}{\rho_7} + \frac{5.5}{\rho_7} T^3 e^{-\eta} + \frac{7}{8} \bar{Z}}$$

where $\rho_7 = \rho \times 10^{-7}$ with ρ in $\frac{g}{cc}$, T and E are in Mev, and the approximation for $F_2^-(\eta)$ mentioned above has been used.

There is another effect that should be accounted for in the above. The scattering opacities should be transport opacities (i.e. opacities weighted by the factor $1 - \cos \theta$, where θ is the scattering angle). Using data taken from TS, a fit for $\cos \theta$ can be made, and (3) can be changed to include this factor. The fit for $\cos \theta$ used is:

$$\cos \theta = \frac{E}{4k_B T + \mu_e + E}$$

It can be seen that is $E \gg 4k_B T + \mu_e$, then the electron positron scattering will also not be important.

To estimate the value of (3) use the fact that $\bar{Z} \leq \frac{1}{3}$ in the region of interest, and that the electrons are slightly degenerate ($\eta \approx \frac{1}{3}$) to find:

$$\frac{k_e^*}{k_{tot}} = \frac{\frac{4.3}{\rho_1} T^3 + .29}{3.4 + \frac{4.3}{\rho_1} T^3 + .29}$$

where the most restrictive value of $\bar{Z}$ (i.e. $\frac{1}{3}$) has been taken. Demanding that this be less than .20 results in the condition $\rho_1 \geq 8T^3$, with T in Mev. Looking at a computer simulation, this relation is generally seen to hold during the late time evolution in the region from the center to near the shock position.

Since nucleons dominate the opacity, the electron-positron scattering contribution may be approximated without greatly effecting the calculated spectrum.

From the above estimate for electron-positron scattering opacity, it is seen that:

$$k_e^*(\mu_e, T, E) = EH(\mu_e, T, E)$$

where $H(\mu_e, T, E)$ is a more slowly varying function of E than $k_e^*(\mu_e, T, E)$. Figure 1a and 1b show graphs of $\frac{\kappa_e}{ET}$ versus $\log(\frac{k_B T}{\mu_e})$ constructed using the results of TS. In the high temperature limit $H(\mu_e, T, E)$ becomes independent of E , as can be seen from figures 1a and 1b.

Now write:

$$f \int (1 - f') \kappa_e^*(E, \Omega \rightarrow E', \Omega') \frac{d^3 p'}{h^3} = k_e^* < 1 - f' > f$$

where $< 1 - f' >$ is some average blocking factor.

The other term involving electron-positron scattering will be approximated in such a way that, in the limit $f = e^{\frac{\mu_e - E}{k_B T}}$, the electron-positron scattering terms will cancel (with the neglect of the blocking factors). This condition results in:

$$(1 - f) \int f' \kappa_e^*(E', \Omega' \rightarrow E, \Omega) \frac{d^3 p'}{h^3} = \frac{E}{4\pi k_B T} (1 - f) \int_E^\infty \int_{4\pi} k_e^*(\mu_e, T, E') \frac{f'}{E'} dE' d\Omega'$$

To show this is correct, the integrals on the right hand side of the above equation need to be done with $f = e^{\frac{\mu_e - E}{k_B T}}$. To do the energy integral, take $H(\mu_e, T, E)$ outside the integral so that:

$$\frac{E}{4\pi k_B T} \int_E^\infty \int_{4\pi} k_e^*(\mu_e, T, E') e^{\frac{\mu_e - E'}{k_B T}} \frac{dE'}{E'} d\Omega' = \frac{H(\mu_e, T, E) E}{k_B T} \int_E^\infty e^{\frac{\mu_e - E'}{k_B T}} dE' = k_e^* f$$

This is a very good approximation in the limit $\frac{k_s^2}{k_B^2 T} \ll 1$, since in this case, $k_s^2 \propto E$ and $H(\mu_s, T, E)$ is independent of E . This is also the region where $f = e^{\frac{E-\mu_s}{k_B T}}$, if the neutrinos are in equilibrium.

The nucleon scattering terms are simpler in form since the scattering is conservative. The proton differential cross-section has an angular dependence such that:

$$\frac{d\sigma}{d\Omega} \propto (1 - \frac{1}{3} \cos \theta)$$

and the neutron cross-section is isotropic (see TS). The nucleon scattering terms will be written as:

$$-(1-f)k_s^{N+P}f + (1-f)k_s^{N+P} \frac{1}{4\pi} \int_{4\pi} f' d\Omega'$$

where the angular dependence of the cross-section has been averaged over 4π , weighted by $(1-\cos \theta)$ in order to produce a transport cross section. Notice that this is another approximation as the angular dependence of the cross-section really should appear inside the angular integral over f .

The scattering cross-section for helium is taken from TS and is given by the following expression:

$$\sigma_{He} = \frac{8}{3} \sigma_o \left(\frac{E}{m_e c^2} \right)^2 \sin^4 \theta_W$$

where θ_W is the Weinberg angle. The above has been averaged over 4π steradians weighted by $(1-\cos \theta)$.

Since both helium and the neutron-proton scattering opacity have the same dependence on the neutrino energy, the scattering opacity for these particles may be added to find a total conservative scattering opacity for use in the transport equation.

The Boltzmann equation is rewritten below:

$$\begin{aligned} \hat{p} \cdot \nabla f = & \rho k_s^{N+P} (b - f) - \rho k_s^2 < 1 - f' > f + \frac{E(1-f)}{4\pi k_B T} \int_E^\infty \int_{4\pi} \rho k_s^2 \frac{f'}{E'} dE' d\Omega' \\ & -(1-f) \rho k_s^C f + (1-f) \rho k_s^C \frac{1}{4\pi} \int_{4\pi} f' d\Omega' \end{aligned}$$

where:

$$k_s^C = k_s^{N+P} + k_s^{He}$$

Next, introduce a new distribution function $F(\vec{x}, \vec{p}, E)$ such that:

$$F(\vec{x}, \vec{p}, E) = \left(\frac{E}{hc}\right)^3 f(\vec{p}, \vec{x})$$

This is done since the supernova code uses the angular average of $F(\vec{x}, \vec{p}, E)$ as the neutrino distribution function (i.e. $F_{code} = \int F(\vec{x}, \vec{p}, E) d\Omega$).

The transport equation for $F(\vec{x}, \vec{p}, E)$ becomes:

$$\begin{aligned} \vec{p} \cdot \nabla F = \rho k_a^{N+P} (B - F) - \rho k_e^s <1 - f' > F + \frac{E^4}{4\pi k_B T} (1 - f) \int_E^\infty \int_{4\pi} \rho k_e^s \frac{F'}{E'^4} dE' d\Omega' \\ - (1 - f) \rho k_e^C F + (1 - f) \rho k_e^C \frac{1}{4\pi} \int_{4\pi} F' d\Omega' \end{aligned} \quad (4)$$

where $B = \frac{E^3}{(hc)^3} b$.

In order to solve the Boltzmann transport equation, the density, temperature, and composition profile must be provided so that the numerical value of the opacity can be found. One approximation is to use the density, temperature, and composition profile from the output of a computer simulation generated using the flux limited diffusion approximation. This should be an acceptable procedure, to the extent that the two methods of neutrino transport give similar results.

The blocking factors $<1 - f' >$ and $(1 - f)$ are also evaluated using the value of f obtained from the supernova calculation. The supernova code used 16 energy bins and the following are taken for the blocking factors:

$$<1 - f' > = \frac{\sum_{bins} \left(1 - \frac{F(E', \vec{x})}{4\pi E'^3} (hc)^3\right) E'^2 \Delta E'_{bin}}{\sum_{bins} E'^2 \Delta E'_{bin}}$$

and:

$$1 - f = 1 - \frac{F(E, \vec{x})}{4\pi E^3} (hc)^3$$

One collapse problem, a $15 M_\odot$ star, was run with 32 energy bins and weak sensitivity to bin number was found in the calculated spectrum.

Equation (4) is still difficult to solve since it is a differential-integral equation for $F(\vec{x}, \vec{p}, E)$. In the spirit of the proceeding approximations, the spectrum produced by the supernova simulation is used to evaluate the integral involving the electron-positron scattering since, as was shown earlier, it is much less important than the nuclear contribution. Equation (4) can be rewritten as:

$$F(\vec{x}, \hat{p}, E) = \left(\frac{E}{hc}\right)^3 f(\vec{p}, \vec{x})$$

This is done since the supernova code uses the angular average of $F(\vec{x}, \hat{p}, E)$ as the neutrino distribution function (i.e. $F_{code} = \int F(\vec{x}, \hat{p}, E) d\Omega$).

The transport equation for $F(\vec{x}, \hat{p}, E)$ becomes:

$$\begin{aligned} \hat{p} \cdot \nabla F = & \rho k_a^{N+P} (B - F) - \rho k_e^2 <1 - f'> F + \frac{E^4}{4\pi k_B T} (1 - f) \int_E^\infty \int_{4\pi} \rho k_e^2 \frac{F'}{E'^4} dE' d\Omega' \\ & - (1 - f) \rho k_e^C F + (1 - f) \rho k_e^C \frac{1}{4\pi} \int_{4\pi} F' d\Omega' \end{aligned} \quad (4)$$

where $B = \frac{E^3}{(hc)^3} b$.

In order to solve the Boltzmann transport equation, the density, temperature, and composition profile must be provided so that the numerical value of the opacity can be found. One approximation is to use the density, temperature, and composition profile from the output of a computer simulation generated using the flux limited diffusion approximation. This should be an acceptable procedure, to the extent that the two methods of neutrino transport give similar results.

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$$<1 - f'> = \frac{\sum_{bins} \left(1 - \frac{F(E', \vec{x})}{4\pi E'^3} (hc)^3\right) E'^2 \Delta E'_{bin}}{\sum_{bins} E'^2 \Delta E'_{bin}}$$

and:

$$1 - f = 1 - \frac{F(E, \vec{x})}{4\pi E^3} (hc)^3$$

One collapse problem, a $15 M_\odot$ star, was run with 32 energy bins and weak sensitivity to bin number was found in the calculated spectrum.

Equation (4) is still difficult to solve since it is a differential-integral equation for $F(\vec{x}, \hat{p}, E)$. In the spirit of the proceeding approximations, the spectrum produced by the supernova simulation is used to evaluate the integral involving the electron-positron scattering since, as was shown earlier, it is much less important than the nuclear contribution. Equation (4) can be rewritten as:

The beams are followed until they come within a distance $R_{\min}$ from the center of the star. The beams are then picked up on the opposite side. $R_{\min}$ is chosen well inside the neutrinosphere so that in the omitted region, the neutrinos are in equilibrium and collisions should not change F . This also helps to remove sensitivity to the electron-positron scattering opacity, since in this region the neutrinos are degenerate, and the approximation used for this opacity is not good for degenerate neutrinos. The errors introduced by the numerical integration are negligible compared to the uncertainty in the density, temperature, and composition profile used in the opacity construction.

Figures 2a and 2b are polar plots of the angular distribution for F_ν and $F_{\bar{\nu}}$ (i.e. the electron neutrino and antineutrino distribution function) at a radius of 270 km at a time late in the collapse of a $25 M_\odot$ star. The dependence on angle is consistent with the interpretation that most of the emission is coming from a region near the surface of the neutrinosphere.

Electron Antineutrinos

Electron antineutrinos are treated similarly to the electron neutrinos, the only difference being in the opacities. Electron-positron scattering is more important for electron antineutrinos than neutrinos since, while both types scatter from nuclei, antineutrinos are absorbed only by protons, neutrinos only by neutrons, and the value of Y_P , the proton fraction, can drop as low as .1 in the region near the neutrinosphere. Therefore, the approximate form for the electron-positron scattering term in the Boltzmann equation will have a greater effect on the electron antineutrino spectrum than on the electron neutrino spectrum. However, the supernova code includes the effects of electron-positron scattering with a more consistent approximation than that used in the Boltzmann equation, and relatively good agreement is obtained with the flux limited diffusion calculation and the calculation done with the Boltzmann equation.

Mu Tau Neutrinos and Antineutrinos

Mu tau neutrinos and antineutrinos (which can be treated as four essentially identical neutrinos when $C_V = 1$ and $C_A = .5$ for ν_e scattering) are scattered but not absorbed by nucleons. The electron-positron opacity is now comparable to the baryon opacity. No attempt to solve for a spectrum of these neutrinos with the Boltzmann equation is made since the electron-positron scattering opacity used is only approximate. However, with good

agreement of the electron antineutrino spectrum calculated with the supernova code and the Boltzmann equation, and the fact that electron-positron scattering can be an important source of opacity for electron antineutrinos, the supernova code calculated mu tau spectrum is done adequately.

SUMMARY

Figure 3(a-e) show comparasions of the electron neutrino spectrum calculated with the flux limited diffusion approximation and with the Boltzmann equation. Figure 4(a-e) show the electron antineutrino spectrum as calculated by both methods. Five times are chosen during the evolution to show general trends. Figures 3a and 4a are .1 second after bounce, while figures 3b and 4b are just before the explosion. Figures 3c and 4c are at the explosion time. These three times show the most discrepancy between the two methods, and are all taken when there is unsteady motion in the atmosphere. Figures 3d,3e,4d,and 4e show times after explosion, when matter motion is fairly steady on one second timescales. Here agreement is very good. The $25M_{\odot}$ model C progenitor star, as listed in Wilson et al. (1985), is the stellar model that produced the spectra seen in figures 3 and 4.

Good agreement is found between the spectrum as calculated by the Boltzmann equation and by the flux limited diffusion approximation, as average energies calculated using both spectra were within 20 % of each other. The luminosities, however, were in better agreement, between 5 to 10 %. The peak discrepancies are at the times which had the most motion in the atmospheres. (If the flux from the Boltzmann equation were actually fed back into the evolution equations for the star, the system would adjust so as to lower this discrepancy. For example, too high a flux would lower the temperature, thereby bringing down the flux.)

One weakness in the above calculation is in the electron-positron scattering cross-section. The form used is certainly not good in the region where the neutrinos are degenerate, but at this time the blocking factors help to reduce the importance of the electron-scattering cross-section. In fact, inside the neutrinosphere near the star center, where electrons are very degenerate, the agreement between the two methods of calculation is only fair, but the spectra and flux exiting the neutrinosphere is within the estimates given above.

It must be remembered that the supernova code finds a spectrum by the time integration of the flux limited diffusion equation, while the Boltzmann equation solved in this chapter assumes the static limit. The flux limited distribution function contains a history of the supernova evolution and when the atmosphere is unsteady, disagreement between the two

methods of calculation should be expected. Even though $v/c \leq .2$ (see Chapter III), the time rate of change of v/c and the distribution function can be important in determining the spectrum. In fact, at $t=.4695$ sec., or the time figures 3a and 4a are taken, the structure of the star was frozen, and the supernova flux limited diffusion subroutines were used to evolve the neutrinos until a steady state situation was reached. This test resulted in improved agreement of the spectra and luminosities. However, repeating this procedure at $t=.7835$ sec. (figures 3b and 4b) failed to improve agreement.

Some further results concerning the neutrino spectra and luminosity for the $25M_{\odot}$ model C star (see Wilson et al. (1985)) is now presented. Figures 5(a-e) in this chapter show the sum of the spectra for the mu, tau neutrinos and their associated antineutrinos, as calculated by the flux limited diffusion approximation. Five times during the collapse are chosen, and correspond to the same times shown in figures 3 and 4. Figures 5(f-h) show the time integrated spectrum for the electron neutrinos, electron antineutrinos, and the mu-tau group of neutrinos. Figures 5i and 5j show the electron neutrino and electron antineutrino luminosity versus time. They are similar, except that the electron capture burst seen in the electron neutrino luminosity during collapse is not seen in the electron antineutrino luminosity (there is no corresponding process). Figure 5k shows the electron neutrino, electron antineutrino and one of the mu tau group of neutrinos luminosity superimposed onto the same graph. Figure 5l shows the structure of the electron capture burst on a magnified time scale. Figures 5m and 5n show average neutrino energies for the various neutrino types as a function of time. Since mu and tau neutrinos and antineutrinos only scatter from baryons, their mean free path tends to be longer than for electron neutrinos and antineutrinos, which are emitted and absorbed by baryons, as well as scattered. This means that average energies for the mu tau group of neutrinos are generally larger than the electron neutrinos and antineutrinos. This is the case since the mu tau group of neutrinos can escape from the star at a smaller radius, where the temperature is higher than at the radius where electron neutrinos and electron antineutrinos escape. After explosion, however, due to the drop in the proton fraction near the outer part of the proto-neutron star (see Chapter V), the average electron antineutrino energy increases to about the value of the mu tau group of electrons. There are few protons to participate in emission and absorption reactions with $\bar{\nu}_e$. Therefore, the electron antineutrino has a mean free path close to one of the mu tau group of neutrinos. Figures 6(a-o) show the time integrated neutrino spectra for some of the

other stars studied in Wilson et al. (1985). Figures 7(a-o) show the luminosities obtained from some of the stars studied in Wilson et al. (1985). The burst signal from electron capture is very similar for all iron core collapse driven supernova, since it is produced by the deleptonization of approximately the inner $.6 M_{\odot}$ of these collapsing stars. Figure 8 shows the curve of binding energy versus gravitational mass of a neutron star used to obtain the remnant masses for the supernovae found in Wilson et al.(1985). This binding energy is also used to calculate the total luminosity in neutrinos that must be produced by the remnant as it becomes a cold neutron star.

Chapter IV References

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Chapter IV Figure Captions

Fig. 1 Electron neutrino scattering cross-section constructed using the results of Tubbs and Schramm (1975). T, μ_e , and the neutrino energy are all measured in Mev. Cross-sections are measured in units of 10^{-44} cm^2 .

Fig. 2 Angular distribution of the electron neutrinos with energy 8 Mev and angular distribution of the electron antineutrinos with energy 16 Mev at a point 270 km from the star center at late times. The neutrinosphere is inside the point where the angular distribution is shown. The star is the 25 model C (see Wilson et al. (1985)).

Fig. 3 Comparasion between the flux limited diffusion approximation calculated spectrum (FLD) and the Boltzmann equation (BE) for the neutrinos at 5 different times. The star is the 25 model C. Integrating the area under a curve gives the luminosity in Mev/sec, or $1.6 \times 10^{-6} \int I d\epsilon = L \frac{c}{v} \frac{d\epsilon}{d\epsilon}$.

Fig. 4 The same comparasion is made for the electron antineutrinos as was made for the electron neutrinos (see fig. 3 caption).

Fig. 5(a-e) The combined spectrum from the 4 members of the mu tau group of neutrinos as calculated by the flux limited diffusion approximaton. The five times chosen agree with the times in figures 3 and 4.

Fig. 5(f-h) Time integrated spectrum, taken at three different times for the electron neutrinos and antineutrinos and for the 4 members of the mu tau group. The star is the 25 Model C.

Fig. 5(i-l) Luminosity versus time curves for for the electron neutrino, electron antineutrino, and one of the members of the mu tau group, call it the muon neutrino. Note the sharp peak of electron neutrinos at the early times. This is the burst from electron capture. For ease in seeing other structure, the electron capture burst is drawn one half it's actual height. This does not apply to figure 5l which is a magnified view of the structure of the electron capture burst.

Fig. 5(m-n) The time dependence of the average neutrino energies seen in the collapse of 25 Model C star. Fig 5n shows the comparasion between the average energy as given by the flux limited diffusion, denoted by the subscript C, and by the Boltzmann equation generated spectrum, denoted by the subscript S. Figure 5m shows the effect of redshift on the average energies.

Fig. 6 Time integrated spectra from many of the stars studied in Wilson et al. (1985).

Fig. 7 Luminosities from many of the stars studied in Wilson. et al. (1985)

Fig. 8 Neutron star binding energy as a function of gravitational mass. This curve is used to find the total amount of energy that must be radiated away in neutrinos as a hot neutron star cools to a final cold state.

FIGURE 1a

FIGURE 1b

FIGURE 2a

FIGURE 2b

FIGURE 3a

FIGURE 3b

FIGURE 3c

FIGURE 3d

FIGURE 3e

FIGURE 4a

FIGURE 4b

FIGURE 4c

FIGURE 4d

FIGURE 4e

FIGURE 5a

FIGURE 5b

FIGURE 5c

FIGURE 5d

FIGURE 5e

FIGURE 5f

FIGURE 5g

FIGURE 5h

FIGURE 5i

FIGURE 5j

FIGURE 5k

FIGURE 51

FIGURE 5m

FIGURE 5n

FIGURE 6a

FIGURE 6b

FIGURE 6c

FIGURE 6d

FIGURE 6e

FIGURE 6f

FIGURE 6c

FIGURE 6h

FIGURE 6i

FIGURE 6j

FIGURE 6k

FIGURE 61

FIGURE 6m

FIGURE 6n

FIGURE 6o

FIGURE 7a

FIGURE 7b

FIGURE 7c

FIGURE 7d

FIGURE 7e

FIGURE 7f

FIGURE 7E

FIGURE 7b

FIGURE 7i

FIGURE 7j

FIGURE 7k

FIGURE 71

FIGURE 7m

FIGURE 7n

FIGURE 7o

for number sequence only

FIGURE 3

Chapter V

CONVECTION IN SUPERNOVA

INTRODUCTION

After bounce, and the subsequent launching of the accretion shock into the outer core, the physical variables, Y_L , the lepton number per baryon, and S , the entropy per baryon, typically show a dependence on the mass coordinate similar to the one shown in figure 1.

The dip in Y_L near the neutrinosphere is due to the following reaction:

with the neutrino escaping, this part of the core can deleptonize early. The bump in Y_L at $.7 M_\odot$ is what remains of a discontinuity in Y_L produced when the shock began its outward motion.

The current position of the shock is seen as the spike in entropy. It originated around $.7 M_\odot$ and, in moving to its present position, has dissociated heavy nuclei into free baryons and helium. Interior to $.7 M_\odot$, the density is so high that a uniform sea of baryons exists. (A profile of the proton fraction is similar in shape to Y_L .) At earlier times, heavy nuclei exist inside $.7 M_\odot$, since the density at the earlier times is not yet high enough to squeeze the nuclei into a uniform sea of nuclear matter.

Y_L increases outside the neutrinosphere and beyond the shock position. It reaches a value of about .5 and stays relatively constant at .5 throughout the rest of the star. (The neutrino burst from the electron capture during collapse will appear as a bump in Y_L approximately 10^{10} cm from the star center.) The Y_L profile is more smooth than the entropy profile near the shock since heavy nuclei outside the shock can participate in electron capture, although only a few protons in any nuclei can capture due to nuclear shell effects. Also true, is that the timescale for electron capture is much longer than the dynamical timescale at the densities near the shock. The heavy nuclei pass through the shock and are dissociated before they can capture appreciably.

Entropy is low in the central regions of the star initially, and the inner $.7 M_\odot$ homologous core remains unshocked during the evolution. Therefore, entropy remains low inside $.7 M_\odot$. From $.7$ to $2 M_\odot$, the entropy originally produced by the shock has decreased due to the cooling that results from the loss of neutrinos. This cooling has produced a region from $1.5 M_\odot$ to $2 M_\odot$ where entropy decreases with increasing mass.

The core, in this configuration, is unstable in the sense that a redistribution of the material could result in a configuration with lower total energy.

Mixing can occur as the motion of various size parcels of material from one radial position to another. If a parcel, on moving downward, became more dense than the average surrounding material, it would continue to move downward, until it reaches a radius where its density was no greater than the surroundings (assuming no convective overshoot).

Macroconvection describes the limit where most parcels have a size comparable to the dimensions of the system being considered. In this case, numerical study must take into account the non-spherical nature of the problem due to the large parcel size. Two studies of core overturn have been done (Livio, Buchler, and Colgate (1980), and Smarr, Wilson, Barton, and Bowers (1981)) using two-dimensional computer codes (axial symmetry is assumed). These studies concentrated on the early times in a supernova explosion (i.e. a few milliseconds after bounce). Both studies found a portion of the core overturned on a timescale of 5-20 ms. In particular, Smarr et al. (1981) found the region from .7 to $1.25 M_{\odot}$ overturning, but not enough neutrinos were released to achieve an explosion.

Microconvection describes the limit where most parcels have a size much smaller than the entire region of instability. This type of convection can be studied within the context of a spherical computer code by means of a mixing length theory. This chapter is concerned with microconvection, and the manner in which it is incorporated in a computer model for a supernova.

CONVECTION

Stars, in quasi-static equilibrium, can be unstable, in that a parcel of matter, if moved from its position downward (upward) may become more (less) dense than the surrounding matter, and so be driven even further downward (upward). This type of motion is called convection.

A criterion for the onset of convection is thus:

$$\left. \frac{\partial \rho}{\partial R} \right|_{disp} - \left. \frac{\partial \rho}{\partial R} \right|_{star} \leq 0 \quad (1)$$

where $\left. \frac{\partial \rho}{\partial R} \right|_{disp}$ is the rate of change of density of the parcel with respect to radius, and $\left. \frac{\partial \rho}{\partial R} \right|_{star}$ is the star's density gradient.

Different possibilities for convection occur depending on how the parcel is moved. If the parcel is moved adiabatically, but always in pressure and composition equilibrium, the

Schwarzschild criterion for the onset of convection is found.

Using the following figure to illustrate the notation, the Schwarzschild criterion can be derived:

Equation (1) becomes:

$$\rho(P', s', Y_L') - \rho(P', s, Y_L') \geq 0$$

Using a Taylor series expansion results in:

$$\rho(P', s', Y_L') - \rho(P', s, Y_L') = \left(\frac{\partial \rho}{\partial s} \right)_{P, Y_L} l \frac{ds}{dR} \geq 0$$

Now:

$$\frac{\partial \rho}{\partial s} \Big|_{P, Y_L} = \frac{T}{C_P} \left(\frac{\partial \rho}{\partial T} \Big|_{P, Y_L} \right) \frac{ds}{dR} \geq 0 \quad (2)$$

For most equations of state $\frac{\partial \rho}{\partial T} \Big|_{P, Y_L} \leq 0$, and the Schwarzschild criterion becomes:

$$\frac{ds}{dR} \leq 0$$

Returning to figure 1, it is seen that the region from 1.2 to 1.8 $M_{\odot}$ is unstable according to this criterion.

The Ledoux criterion is found by moving the parcel adiabatically, with no composition change but in pressure equilibrium. Then equation (1) becomes:

$$\rho(P', s', Y_L') - \rho(P', s, Y_L) \geq 0$$

Again using a Taylor series expansion the above equation becomes:

$$\rho(P', s', Y_L') - \rho(P', s, Y_L) = \left(\frac{\partial \rho}{\partial s} \right)_{P, Y_L} \left(\frac{ds}{dR} \right) + \left(\frac{\partial \rho}{\partial Y_L} \right)_{P, s} \left(\frac{dY_L}{dR} \right) \geq 0 \quad (3)$$

using (2) in (3), the Ledoux criterion becomes:

$$\left(-\frac{T}{C_P} \frac{\partial \rho}{\partial T} \right)_{P, Y_L} \frac{ds}{dR} + \left(-\frac{\partial \rho}{\partial Y_L} \right)_{P, s} \frac{dY_L}{dR} \leq 0$$

where as before $\frac{\partial \rho}{\partial T} \big|_{P, Y_L} \leq 0$ for most materials. Also $\frac{\partial \rho}{\partial Y_L} \big|_{P, s} \leq 0$ since increasing Y_L will result in a density decrease at constant P due to the extra portion of the pressure now carried by the leptons.

In figure 1 it is seen that the decrease in Y_L is very large in the region from the center out to 2 $M_{\odot}$. Even though $\frac{ds}{dR} \geq 0$ from the center out to 1.2 $M_{\odot}$, the composition gradient is large enough that the region is unstable according to the Ledoux criterion.

Another possibility exists that models the salt finger instability. This instability is discussed in more detail below. For now, move the parcel in temperature and pressure equilibrium, but not composition equilibrium.

The criterion for convection to occur becomes:

$$\rho(P', T', Y_L') - \rho(P', T', Y_L) \geq 0$$

or:

$$\rho(P', T', Y_L') - \rho(P', T', Y_L) = \left(\frac{\partial \rho}{\partial Y_L} \right)_{P, T} \left(\frac{dY_L}{dR} \right) \geq 0$$

Again $\frac{\partial \rho}{\partial Y_L} \big|_{P, T}$ is generally negative, and the criterion for the salt finger instability to occur is:

$$\frac{dY_L}{dR} \leq 0$$

From figure 1 it is seen that the region of the star from the center to $2 M_{\odot}$ is unstable to salt fingers.

Convection in collapsed stellar cores that results from the gradient in Y_L was first suggested by Epstein (1979).

MODEL FOR CONVECTION

A standard mixing length theory is used (see Cox and Giuli (1968)) to model time dependent convection.

A convective velocity is assumed to develop such that:

$$\frac{1}{2} \rho v_C^2 = g \delta \rho l \quad (4)$$

where:

$$\delta \rho = - \left(\frac{\partial \rho}{\partial R} \right)_{\text{disp}} - \frac{\partial \rho}{\partial R} \Big|_{\text{star}} l$$

$$g = - \frac{1}{\rho} \frac{dP}{dR} \Big|_{\text{star}} = \frac{P}{\rho l_P}$$

$$l_P = - \frac{dR}{d \ln P} \Big|_{\text{star}}$$

l = distance parcel moves before mixing

Using:

$$v_S = \left(\frac{\gamma P}{\rho} \right)^{\frac{1}{2}}$$

the expression for v_C becomes:

$$v_C = \left(\frac{2l}{\gamma l_P} \right)^{\frac{1}{2}} \left(\frac{\delta \rho}{\rho} \right)^{\frac{1}{2}} v_S \quad (5)$$

Equation (4) is written down by assuming that the change in kinetic energy of a parcel is equal to the work done by the buoyancy force.

The following equations are used as a guide in the construction of numerical model equations:

$$\rho \frac{\partial \bar{Z}}{\partial \tau} = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \Gamma D \rho \frac{\partial \bar{Z}}{\partial R} \right) \quad (6)$$

$$\rho \frac{\partial \epsilon}{\partial \tau} = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \Gamma D \rho \frac{\partial \epsilon}{\partial R} \right) \quad (7)$$

$$\rho \frac{\partial Y_\nu}{\partial \tau} = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \Gamma D \rho \frac{\partial Y_\nu}{\partial R} \right) \quad (8)$$

where:

$$\bar{Z} = \frac{m_B}{\rho} (n_{e^-} - n_{e^+})$$

$$Y_\nu = \frac{m_B}{\rho} (n_\nu)$$

ϵ = matter internal energy/mass

ρ = matter rest mass density

$$D = \frac{1}{3} v_C l$$

Equation (8) is really 6 equations, one for each neutrino type. The time derivative is a local Eulerian time derivative, and Γ is a relativistic correction factor defined in Chapter III and Appendix A.

NUMERICAL METHOD

This section describes the method in which convection is introduced into the supernova code. Equations (6), (7), and (8) will be used as a guide in the following. Before proceeding further, the assumption that $l = l_P$ is made (or that the parcel moves no further than the pressure scale height). Since mixing length theory cannot be expected to give definitive answers but only good guesses as to the effect of the actual physical process, there is some uncertainty in deciding on the choice for l , the mixing length.

Once a choice for l is made, a rest mass is defined such that:

$$\delta m = \frac{\Delta \tau}{\Delta R} \rho D (4\pi R^2) = \frac{\Delta \tau}{\Delta R} \rho (4\pi R^2) \frac{1}{3} v_C l \quad (9)$$

This has the physical interpretation of being the amount of mass that is exchanged between different zones in time $\Delta \tau$.

Figure 2 illustrates the zoning convention used in the supernova code. Using subscripts k , to indicate the zonal position, (9) is rewritten as:

$$\delta m_k = \Delta \tau \frac{\rho_k + \rho_{k+1}}{R_{k+1} - R_{k-1}} 4\pi R_k^2 \frac{1}{3} v_{Ck} l_k$$

Equation (6) then becomes:

$$\bar{Z}'_k - \bar{Z}_k = \frac{1}{m_k} [\Gamma_k \delta m_k (\bar{Z}_{k-1} - \bar{Z}_k) - \Gamma_{k+1} \delta m_{k+1} (\bar{Z}_k - \bar{Z}_{k+1})] \quad (10')$$

where m_k is the baryon rest mass in zone k , and $\bar{Z}'_k$ is the value of $\bar{Z}_k$ at the next time step. Similar equations are written for (7) and (8).

The convection equations are solved explicitly (as seen from (10)), in the split operator approximation (i.e. only the convection equations are solved, the other evolution equations are not used. After the new state variables are found from the convection equations, the other evolution equations are solved in turn). If more than 25 % of the mass in any zone needs to be mixed in one timestep (typically 10^{-4} second), time evolution is halted while the material is moved in a sequence of steps, each of which moves no more than 25 % of the zone mass. After each step, the equation of state routine is called to determine the thermodynamic variables.

Since total neutrino number is conserved by (8), but not total neutrino energy, any difference in total neutrino energy found after the convection subroutine is called is placed in the matter internal energy. In the region where convection is important, the neutrinos are strongly coupled to the matter. The neutrino fields will thus adjust to the correct neutrino distribution, and any errors introduced in the distribution function by the convection subroutine should be corrected, as the other subroutines are called in a time step.

Since convection using the Schwarzschild or Ledoux criterion is more traditional an approach than using the salt finger instability, the salt finger instability is discussed in more detail in the following.

SALT FINGERS

The salt finger mechanism can be illustrated as follows. Imagine a region of hot salty water overlaying a region of cold fresh water. Should a parcel of salt water begin to sink, heat will diffuse from the parcel more quickly than salt (this is just due to the different molecular means of transport of energy by collisions, and transport of salt by particle diffusion). Thus, the salt water can reach temperature equilibrium with the environment sooner than composition equilibrium. However, at the same temperature as fresh water, salt water is more dense due to the salt content, and so will sink further. It will continue to sink in a finger like fashion, producing what is known as a salt finger.

Now, a similar thought experiment can be done for a supernova. Imagine a region of neutrons over a layer of protons (with electrons to conserve charge). Of course, this is in the presence of a gravitational field with the neutrons in a region of larger gravitational potential. Should a parcel of neutrons begin to sink, collisions with neutrinos of all types will allow the parcel to come into temperature equilibrium with the environment more quickly than composition equilibrium.

The composition changes more slowly since it is determined by the following equation:

$$\frac{\rho}{m_B} \frac{\partial Y_L}{\partial \tau} = \frac{\partial}{\partial \tau} \int (F_{\nu_e} - F_{\bar{\nu}_e}) \frac{dE}{E} \quad (11a)$$

while the energy exchange rate is found from:

$$\frac{\partial e_m}{\partial \tau} = \frac{\partial}{\partial \tau} \int (F_{\nu_e} + F_{\bar{\nu}_e} + F_{\nu_\mu} + F_{\bar{\nu}_\mu} + F_{\nu_\tau} + F_{\bar{\nu}_\tau}) dE \quad (11b)$$

where τ is the local time and E is the local neutrino energy.

Note that mu and tau neutrinos and their antiparticles cannot change composition (for example, $n + \nu_\mu \rightleftharpoons p + \mu^-$ or $n + \nu_\tau \rightleftharpoons p + \tau^-$) since muons are too heavy to be produced (i.e. $\frac{m_\mu c^2}{k_B T} \approx 20$) in the region in which neutron fingers is important. Tau neutrinos cannot change composition as the mass of a tau particle is 1784 Mev and will not be found in a supernova. The matter and neutrinos are in quasi-equilibrium and the difference in the electron neutrino and antineutrino fluxes determine the nuclear composition. This difference is about $\frac{1}{10}$ the total neutrino flux in all 6 neutrino types which exchange energy with the nuclear matter by scattering as well as by emission absorption reactions (i.e. $n + \nu_e \rightleftharpoons p + e^-$ or $p + \bar{\nu}_e \rightleftharpoons n + e^+$).

Once the neutron parcel is in temperature and pressure equilibrium with the environment, it will be more dense (electron pressure being the difference), as such, the neutron parcel can continue to sink further in a finger like fashion.

The analogy with the salt finger case will be carried further. Linear stability analyses of the salt finger instability (and other doubly diffusive phenomena) have been carried out by many authors (see Baines and Gill (1969) for a good summary). For the salt finger case, the wavelength of the most unstable mode is given by (see Turner (1974)):

$$l_F = \left(\frac{\kappa_T \eta d}{g \alpha \Delta T} \right)^{\frac{1}{4}} \quad (12)$$

where:

g = gravitational acceleration

$$\alpha = -\frac{1}{\rho} \frac{\partial \rho}{\partial T} |_{P, comp}$$

d = height of region

ΔT = temperature difference across d

κ_T = thermometric conductivity

η = kinematic viscosity

l_P can be taken as an estimate of the finger width.

Simple kinetic theory arguments can be used to find:

$$\kappa_T = \frac{1}{3} \lambda_\nu c \quad (13)$$

and:

$$\eta = \frac{1}{3} \lambda_\nu c \frac{\rho_\nu}{\rho} \quad (14)$$

In (13) and (14) the following notation is used:

λ_ν = neutrino mean free path

$c^2 \rho_\nu$ = neutrino energy density

and $\frac{\lambda_\nu}{R_{sp}} \leq 1$ (where R_{sp} is the neutrinosphere radius) is being assumed.

Figure 3 shows the inner $2 M_\odot$ of a supernova core at late time on a (ρ, T) plane. Most of this region has $\bar{Z}$, the number of matter electrons per baryon, approximately equal to 1.5, as convection has been included in the evolution. Without convection $\bar{Z} \approx 33$ would be a more likely value in the central core. Nuclear density has been reached at about $10^{14} \frac{g}{cc}$, at which time nuclear forces come into play, and the temperature stops rising as pressure support from the repulsive nuclear force begins to become important. At $\rho \approx 3.2 \times 10^{14} \frac{g}{cc}$, the major pressure support is essentially temperature independent, and the temperature falls as the density increases. Contours of constant $\mu_e/k_B T$ and $\mu_\nu/k_B T$ are shown. Over most of the star $\mu_\nu/k_B T \approx 3$ (or $\mu_e/k_B T \approx 4$) is an indicator of the star track. μ_e and μ_ν are related by:

$$\mu_e - \mu_{\bar{e}} = \mu_N - \mu_P$$

μ_{ν_e} and μ_e differ more and more as the density increases owing to the increasing difference in μ_N and μ_P . The baryons are well described as an ideal gas outside $.95 M_\odot$ and in this region an estimate of l_F can be made.

The energy of a relativistic lepton-antilepton gas can be written as (see Bludmann and Van Riper (1977)):

$$E_l = \frac{7}{8} a T^4 \left(1 + \frac{30}{7} \left(\frac{\mu_l}{\pi k_B T} \right)^2 + \frac{15}{7} \left(\frac{\mu_l}{\pi k_B T} \right)^4 \right) \quad (15)$$

where only one spin state has been assumed. For a relativistic electron-positron gas the factor $\frac{7}{8}$ should be replaced by $\frac{7}{4}$.

Using (15) and the ideal gas expression for energy density of a free baryon gas, the condition:

$$\rho = (1.7 \times 10^9) T_{MeV}^3$$

can be shown to imply that the neutrinos carry approximately 20 % of the total internal energy. This track in the (ρ, T) plane also falls close to the star track outside $.95 M_\odot$.

For estimating l_F , use $\alpha = -\frac{1}{\rho} \frac{\partial \rho}{\partial T} |_{P, comp} = \frac{1}{T}$, the ideal gas result. Replace $\frac{E}{\rho}$ with $\frac{GM}{R^2} = \frac{1}{\tau_{vib}}$, and equation (12) becomes:

$$l_F = \left(\frac{\lambda_{\nu} c \tau_{vib}}{3} \right)^{\frac{1}{2}} \left(\frac{\rho_{\nu}}{\rho} \frac{T}{\Delta T} \right)^{\frac{1}{4}} \quad (16)$$

Now:

$$\frac{\rho_{\nu}}{\rho} = \frac{E_{\nu}}{\rho c^2} = (3 \times 10^{-4}) T_{MeV} \approx .01$$

where $T = 30$ Mev has been taken. Also take:

$$\frac{\Delta T}{T} = 1$$

and:

$$\tau_{vib} \approx \frac{1}{2} \times 10^{-3} \text{ sec.}$$

to find:

$$l_F = (1 \times 10^3) \lambda_F^{\frac{1}{2}} \text{ cm} \quad (17)$$

In order to estimate the time for a finger of size l_F to come into temperature equilibrium with the surroundings, consider figure 4. Solve:

$$\frac{\partial E}{\partial t} + \nabla \cdot G_\nu = 0 \quad (18)$$

inside the cylinder, with the surface held at a constant temperature, where E is the total internal energy density, and G_ν is the neutrino flux. Assume the diffusion approximation, and use (15) along with the fact that $\mu_\nu/k_B T \approx$ constant to find:

$$G_\nu = -\frac{4}{3} \lambda_\nu c \frac{E_\nu}{T} \frac{\partial T}{\partial r} \quad (19)$$

In (18), since the baryons make up at least 80 % of the total energy density, use:

$$\frac{\partial E}{\partial t} \approx \frac{E}{T} \frac{\partial T}{\partial t}$$

and write:

$$\frac{E}{T} \frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{4}{3} \lambda_\nu c \frac{E_\nu}{T} \frac{\partial T}{\partial r} \right) \quad (20)$$

Since only an estimate is needed, replace (20) with:

$$\frac{\partial T}{\partial t} = \frac{4}{3} \lambda_\nu c \frac{E_\nu}{E} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \quad (21)$$

Equation (21) is a diffusion equation with the diffusion coefficient:

$$D = \frac{4}{3} \lambda_\nu c \frac{E_\nu}{E}$$

The time for the cylinder to come into equilibrium is about:

$$t_{diff} = \left(\frac{l_F}{2.4} \right)^2 \frac{1}{D} \quad (22)$$

where 2.4 is the first zero of the Bessel function J_0 . Using (17) in (22), a value of t_{diff} is found:

$$t_{diff} = 1.2 \times 10^{-5} \text{ sec.}$$

From the computer model of salt finger convection, convective velocities are found to be about $3 \times 10^3 \frac{\text{cm}}{\text{hr}}$ and the mixing length (pressure scale height) is about $3 \times 10^5 \text{ cm}$, implying a lifetime of a salt finger before mixing to be:

$$t_{\text{life}} = 10^{-3} \text{ sec.}$$

Therefore, the isothermal model for parcel interchange is a good one to use for salt fingers.

Figure 5 shows l_F and l_P for the same core shown in figure 3 and at the same time.

More on convection will be found in Chapter VII.

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Chapter V Figure Captions

Fig. 1 The lepton number per baryon and the entropy per baryon as a function the star mass.

Fig. 2 The zoning conventions used in the supernova code.

Fig. 3 The track of the central region of a 25 solar mass star at the late times on a (ρ, T) plane. Convection was modeled in the evolution of this star.

Fig. 4 A skematic showing the neutron finger instability.

Fig. 5 The pressure scale height and the size of a neutron finger as a function of radius.

FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4

FIGURE 5

Chapter VI

RADIATIVE ACCRETION SHOCKS

INTRODUCTION

Figure 1a shows the mass point trajectories for a $100 M_{\odot}$ star at late times in a gravitational collapse (see Wilson et al. (1985)). Figure 1b shows the neutrino luminosity and mass accretion rate for the same star. An oscillatory phenomena is superimposed over a steady accretion of matter onto the central remnant. To study this oscillation, a simpler problem can be constructed that models the more complex physics contained in the supernova code.

Going to a simpler problem allows the investigation of the stability of a steady state solution to small radial perturbations. If the system is found to be unstable, then one may be able to understand the late time supernova in terms of this instability. A simpler problem also allows a smaller number of variables to be used in the description of the physics, thereby isolating the effects thought to be most important.

The simpler problem must contain an oscillatory shock, and a non-degenerate gas of neutrons and protons occupying the region between the shock and the neutrinosphere. Possible interaction of neutrinos and the matter behind the shock must be allowed, along with the possibility of neutrinos being produced by the matter as it reaches the neutrinosphere, due to the impact of the matter onto the remnant. A simple heating-cooling function can model the interaction of neutrinos with the matter, and a boundary condition at the neutrinosphere can replace the surface of the remnant. Strong shock boundary conditions can be used at the shock position.

The physics chosen for the simpler problem is motivated by the present understanding of the oscillations. A lump of matter, as it is stopped by the central remnant, produces neutrinos. Kinetic energy of motion is transformed into thermal energy, which then finds its way into neutrino energy as the heated matter radiates neutrinos. The neutrinos already exiting the neutrinosphere, which originate from within the central remnant, are joined by this accretion burst of neutrinos and increase the heating rate of the neutrons and protons behind the shock. This increases the material pressure behind the shock and pushes the shock wave outwards. The outward movement of the shock reduces the mass accretion rate and reduces the amount of neutrinos produced by accretion. This allows the matter pressure to drop behind the shock, since the heating rate has fallen, and thus the shock begins to

move inward again, allowing the mass accretion rate to increase. The cycle then can repeat, starting with another accretion burst of neutrinos. This scenario is rather complex, and the simplified version is still a difficult problem to handle.

A steady state solution can be found to the simpler problem, then perturbation methods can be used to test stability. In this chapter, the results found in the construction of steady state solutions for two geometries, planar and spherical is presented (see figure 2). The perturbation equations will also be given for both the planar case and the spherical case. The numerical solution of these perturbed equations is not done for this thesis, but will be the topic of a future paper.

COMPLETE SET OF EQUATIONS FOR THE PLANAR CASE

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \quad (1)$$

$$\rho \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) = - \frac{\partial P}{\partial x} - \rho g \quad (2)$$

$$\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho e v) + P \frac{\partial v}{\partial x} = \Lambda_{HC} \quad (3)$$

$$P = \frac{\rho k_B T}{\mu} \quad (4)$$

$$\frac{P}{\rho e} = \gamma - 1$$

$$\Lambda_{HC} = \frac{\sigma}{m_B} \rho \left(\frac{k_B}{m_e c^2} \right)^2 \sigma_B (T_{sp}^4 - T^4) \quad (5)$$

In the above T_{sp} is the temperature of the neutrinosphere, σ_B Stephan-Boltzmann constant, σ is a constant with the dimensions of cm^2 , k_B is the Boltzmann constant and μ is the mean molecular weight. In (2), g is the gravitational acceleration and is assumed constant, as the mass contained in the region from $0 \leq x \leq x_S$ (see figure 2) is assumed small compared to the mass of the compact object. Λ_{HC} is constructed to model equation (34) of Chapter II.

BOUNDARY CONDITIONS

At the shock, strong shock boundary conditions are used, and are as follows:

$$P|_{z=z_s} = \frac{2}{\gamma+1} \rho_a u^2 \quad (6)$$

$$v|_{z=z_s} = -\frac{\gamma-1}{\gamma+1} |u| \quad (7)$$

$$\rho|_{z=z_s} = \frac{\gamma+1}{\gamma-1} \rho_a \quad (8)$$

where u is the infall velocity into the shock front, and ρ_a is the ambient density of the matter falling into the shock.

At the neutrinosphere, the following boundary condition is used:

$$\frac{7}{8} \sigma_B T_p^4 = \frac{1}{2} \rho v^2 c + F_C \quad (9)$$

where the factor of $\frac{7}{8}$ is due to considering equal contributions from electron neutrinos and antineutrinos ($\frac{7}{16}$ for each separately, due to Fermi-Dirac statistics rather than Bose-Einstein). Equation (9) implies that the kinetic energy of the matter is transformed into blackbody neutrino emission after reaching the neutrinosphere. F_C is a core flux, taken to be independent of the material outside the core. For the steady state case, this condition just determines the numerical value of F_C to be used in the perturbed problem. Other boundary conditions could be used, such as:

$$\frac{7}{8} \sigma_B T_p^4 = \frac{1}{2} \rho v^2 c + \frac{\gamma}{\gamma-1} \frac{cP}{\rho} + F_C$$

This takes into account the fact that the matter must be brought to rest at the neutrinosphere, and contains internal, as well as kinetic energy, that could be converted to neutrinos. As mentioned above, work on the perturbed problem is still in progress, and other boundary conditions must still be tried at $x=0$, in order to test the sensitivity of the results to this inner boundary condition.

STEADY STATE SOLUTION OF THE PLANAR PROBLEM

In the case where time derivatives are zero, equations (1), (2) and (3) become:

$$\frac{d}{dx}(\rho v) = 0 \quad (10)$$

$$\rho v \frac{dv}{dx} = -\frac{dP}{dx} - \rho g \quad (11)$$

$$v \frac{dP}{dx} + \gamma P \frac{dv}{dx} = (\gamma-1) \Lambda_{HC} \quad (12)$$

where the equation of state has been used in (3) to obtain (12). Equation (10) is easily solved to find:

$$\rho v = -\rho_a |u| = \text{constant} \quad (13)$$

Using (11) and (12) to solve for $\frac{dP}{dx}$ and $\frac{dv}{dx}$ results in:

$$\frac{dv}{dx} = \frac{1}{\frac{\gamma P}{\rho v^2} - 1} \left(\frac{g}{v} + \frac{(\gamma - 1)\Lambda_{HC}}{\rho v^2} \right) \quad (14)$$

$$\frac{dP}{dx} = -\frac{1}{\frac{\gamma P}{\rho v^2} - 1} \left(\frac{\gamma P g}{v^2} + \frac{(\gamma - 1)\Lambda_{HC}}{v} \right) \quad (15)$$

Since insight into the possible solutions to (13), (14), and (15) can be found by dividing (15) by (14) to find:

$$\frac{dP}{dv} = -\frac{\frac{\gamma P g}{v^2} + \frac{(\gamma - 1)\Lambda_{HC}}{v}}{\frac{g}{v} + \frac{(\gamma - 1)\Lambda_{HC}}{\rho v^2}} \quad (16)$$

Since neither side has an x dependence, a plot of various solutions to (16) (i.e. $P(v)$) can be made using various properties of $\frac{dP}{dv}$. For example, all points (P, v) such that $\frac{dP}{dv} = 0$, have a curve that passed horizontally through (P, v) . Before constructing the plot, it is useful to introduce the following dimensionless variables:

$$\begin{aligned} \bar{P} &= \frac{P}{\rho_a u^2} \\ \bar{w} &= -\frac{v}{|u|} \\ \bar{\rho} &= \frac{\rho}{\rho_a} = -\frac{|u|}{v} \\ \alpha &= \frac{\sigma}{m_B} \frac{k_B^2 \sigma_B}{g|u|} \frac{T_{sp}^6}{(m_e c^2)^2} \\ \beta &= \alpha \left(\frac{\mu u^2}{k_B T_{sp}} \right)^6 \end{aligned}$$

Note that $l = x \frac{g}{u^2}$ introduces l , a dimensionless distance.

Using dimensionless variables (16) becomes:

$$\frac{d\bar{P}}{d\bar{w}} = -\frac{\gamma \bar{P} - (\gamma - 1)(\alpha - \beta(\bar{P}\bar{w})^6)}{\bar{w} - (\gamma - 1)(\alpha - \beta(\bar{P}\bar{w})^6)} \quad (17)$$

Figure 3 shows a plot of various solutions to (17). The arrows indicate the direction of increasing x .

It is also useful to introduce new dependent variables, the sound speed and mach number:

$$c^2 = \frac{\gamma P}{\rho} \quad (18)$$

$$m^2 = \frac{v^2}{\frac{\gamma P}{\rho}} \quad (19)$$

In terms of dimensionless variables (18) and (19) become:

$$\bar{c}^2 = \left(\frac{c}{u}\right)^2 = \gamma \bar{P} \bar{w} \quad (20)$$

$$\bar{m}^2 = m^2 = \frac{\bar{w}}{\gamma \bar{P}} \quad (21)$$

Using $\bar{c}$ and $\bar{m}$, equation (17) becomes:

$$\frac{d\bar{c}}{d\bar{m}} = -\frac{\bar{c}}{\bar{m}} \frac{\bar{m}\bar{c}(\gamma-1) - (\gamma-1)(\alpha - \beta \frac{\bar{c}^{12}}{\gamma^2})(\gamma\bar{m}^2 - 1)}{\bar{m}\bar{c}(\gamma+1) - (\gamma-1)(\alpha - \beta \frac{\bar{c}^{12}}{\gamma^2})(\gamma\bar{m}^2 + 1)}$$

$\bar{c}$ and $\bar{m}$ are useful since there is a linear relationship between $\log \bar{c}$, $\log \bar{m}$ and $\log \bar{P}$ and $\log \bar{w}$:

$$2 \ln \bar{c} = \ln \bar{P} + \ln \bar{w} + \ln \gamma$$

$$2 \ln \bar{m} = -\ln \bar{P} + \ln \bar{w} - \ln \gamma$$

This is basically a rotation of 45 degrees in the $\log \bar{P}$, $\log \bar{w}$ plane.

Figure 3 can be divided into 4 areas, classified as subsonic or supersonic, and net heating greater than zero or net heating less than zero. Along the line $\bar{m} = 1$, $\frac{d\bar{P}}{d\bar{w}}$ and $\frac{d\bar{w}}{d\bar{P}}$ are both infinite, except for one point, a critical point where both derivatives are finite. The location of this point depends on α and β , the two dimensionless parameters previously defined.

Since the accretion shock is stationary, the infall material velocity must be supersonic and the shocked material velocity subsonic. This restricts attention to the region of figure 3 representing subsonic flow.

The important fact to be seen in figure 3 is that, given an initial point on a possible solution curve, and following the curve against the arrows, puts one on a track asymptotic to the net heating equals zero curve. If the post shock temperature is less than the neutrinosphere

temperature, the initial point is in the net heating ≥ 0 part of figure 3. In this case, the post shocked material can initially be heated by neutrinos, but as the matter moves closer to $x = 0$, and passes over the zero net heating line, it cools and asymptotically approaches the neutrinosphere temperature. The neutrinosphere temperature thus acts as a clamp on the matter temperature, not allowing the matter temperature to rise appreciably above the neutrinosphere temperature during infall. If the post shocked material temperature is greater than the neutrinosphere temperature, the initial point in figure 3 is in the net heating ≤ 0 region. In this case the matter temperature monotonically decreases towards the value of the neutrinosphere temperature as the matter moves toward the neutrinosphere, and no heating region forms.

Figures 4a-4g show some results for the numerical integration of (13), (14) and (15), assuming that the post-shock temperature is less than the neutrinosphere temperature. Note figure 4f, a P versus $-v$ diagram, and compare it to figure 3. The neutrinosphere temperature was chosen to be 4 Mev, and figure 4d shows the matter temperature versus distance, first rising behind the shock, then asymptotically approaching 4 Mev.

PERTURBATION OF SHOCK—PLANAR CASE—

The manner in which the perturbation of the time independent solution is carried out will now be described. A new coordinate y , is used such that the value of y at the shock position will always have the numerical value x_{s0} , or the unperturbed shock position. A grid velocity is thus introduced with a set of grid observers which move so that $y = x_{s0}$ at the shock. The relevant equations with a grid velocity, v_G , introduced become:

$$\frac{\partial \rho}{\partial t}|_y + \frac{\partial}{\partial x}(\rho(v - v_G))|_t + \rho \frac{\partial v_G}{\partial x}|_t = 0 \quad (22)$$

$$\rho \left(\frac{\partial v}{\partial t}|_y + (v - v_G) \frac{\partial v}{\partial x}|_t \right) = - \frac{\partial P}{\partial x}|_t - \rho g \quad (23)$$

$$\frac{\partial}{\partial t}(\rho e)|_y + \frac{\partial}{\partial x}(\rho e(v - v_G))|_t + \rho e \frac{\partial v_G}{\partial x}|_t + P \frac{\partial v}{\partial x}|_t = \Lambda_{HC} \quad (24)$$

Note that if $v_G = 0$, Eulerian hydrodynamics is recovered, and if $v_G = v$, Lagrangian hydrodynamics is obtained. The shock position is taken as:

$$x_s = x_{s0} + x_{s1} e^{i\omega t} \quad (25)$$

where x_{s1} and ω are in general complex numbers, and $\frac{|x_{s1}|}{x_{s0}}$ is assumed a small number. The choice for y is:

$$y = x - x \frac{x_{s1}}{x_{s0}} e^{\omega t} \quad (26)$$

It can be seen that when $x = x_s$, the shock position, $y = x_{s0}$, to second order in $\frac{x_{s1}}{x_{s0}} = \epsilon$, the smallness parameter used in deriving the perturbation equations.

Introduce the following perturbed variables P_1 , ρ_1 , and v_1 such that:

$$\begin{aligned} P &= P_0(y) + \epsilon P_1(y) e^{\omega t} \\ \rho &= \rho_0(y) + \epsilon \rho_1(y) e^{\omega t} \\ v &= v_0(y) + \epsilon v_1(y) e^{\omega t} \end{aligned} \quad (27)$$

where P_0 , ρ_0 , and v_0 are the unperturbed variables. From (26) the derivatives transform as:

$$\frac{\partial}{\partial x}|_t = (1 - \epsilon e^{\omega t}) \frac{\partial}{\partial y}|_t \quad (28a)$$

$$\frac{\partial}{\partial t}|_x = \frac{\partial}{\partial t}|_y - \epsilon \omega y e^{\omega t} \frac{\partial}{\partial y}|_t = \frac{\partial}{\partial t}|_y - v_G \frac{\partial}{\partial y}|_t \quad (28b)$$

Identifying the correct grid velocity using (28) results in:

$$v_G = \epsilon \omega y e^{\omega t}$$

Defining s as ωy , the following perturbation equations result when (22), (23) and (24) are expanded using (27) and only terms first order in ϵ are retained:

$$v_0 \frac{d\rho_1}{dy} + \rho_1 \left(\omega + \frac{dv_0}{dy} \right) = -\rho_0 \frac{dv_1}{dy} - v_1 \frac{d\rho_0}{dy} + s \frac{d\rho_0}{dy} \quad (29)$$

$$v_0 \frac{dv_1}{dy} + v_1 \left(\omega + \frac{dv_0}{dy} \right) = -\frac{1}{\rho_0} \left(\frac{dP_1}{dy} - \frac{\rho_1}{\rho_0} \frac{dP_0}{dy} \right) + s \frac{dv_0}{dy} - g \quad (30)$$

$$v_0 \frac{dP_1}{dy} + P_1 \left(\omega + \gamma \frac{dv_0}{dy} \right) = (\gamma - 1)(\Lambda_0 + \Lambda_1) - v_1 \frac{dP_0}{dy} - \gamma P_0 \frac{dv_1}{dy} + s \frac{dP_0}{dy} \quad (31)$$

The neutrinosphere temperature is allowed to oscillate, with the following equation defining T_{P_0} and T_{P_1} :

$$T_{sp} = T_{P_0} + \epsilon T_{P_1} e^{i\omega t}$$

The function Λ_{HC} is written as:

$$\Lambda_{HC} = \Lambda_0 + \epsilon \Lambda_1 e^{i\omega t}$$

where:

$$\Lambda_0 = \frac{\sigma}{m_B} \left(\frac{k_B}{m_e c^2} \right)^2 \sigma_B \rho_0 (T_{P_0}^6 - T_0^6)$$

$$\Lambda_1 = \frac{\sigma}{m_B} \left(\frac{k_B}{m_e c^2} \right)^2 \sigma_B (6\rho_0 (T_{P_1} T_{P_0}^5 - T_1 T_0^5) + \rho_1 (T_{P_0}^6 - T_0^6))$$

Using $\rho_0 T_1 = \frac{\mu}{k_B} P_1 - \rho_1 T_0$ found from $P = \frac{\epsilon k_B T}{\mu}$, one finds:

$$\Lambda_1 = \frac{\sigma}{m_B} \left(\frac{k_B}{m_e c^2} \right)^2 \sigma_B \left(6\rho_0 T_{P_1} T_{P_0}^5 - 6T_0^5 \frac{P_1 \mu}{k_B} + \rho_1 (T_{P_0}^6 + 5T_0^6) \right)$$

where T_0 is the unperturbed matter temperature, and T_1 is the perturbed matter temperature.

Equation (29), (30), and (31) can be written in matrix form as:

$$A \frac{dD}{dy} + \Omega D + QD = H \quad (32)$$

where:

$$A = \begin{pmatrix} v_0 & \rho_0 & 0 \\ 0 & v_0 & \frac{1}{\rho_0} \\ 0 & \gamma P_0 & v_0 \end{pmatrix}$$

$$\Omega = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

$$Q = \begin{pmatrix} \frac{dv_0}{dy} & \frac{d\rho_0}{dy} & 0 \\ -\frac{1}{\rho_0^2} \frac{dP_0}{dy} & \frac{dv_0}{dy} & 0 \\ -(\gamma-1) \frac{\sigma}{m_B} \left(\frac{k_B}{m_e c^2} \right)^2 \sigma_B (T_{P_0}^6 + 5T_0^6) & \frac{dP_0}{dy} & \gamma \frac{dv_0}{dy} + 6(\gamma-1) \frac{\sigma}{m_B} \left(\frac{k_B}{m_e c^2} \right)^2 \sigma_B \frac{\mu}{k_B} T_0^5 \end{pmatrix}$$

$$D = \begin{pmatrix} \rho_1 \\ v_1 \\ P_1 \end{pmatrix}$$

$$H = \begin{pmatrix} s \frac{d\rho_0}{dy} \\ s \frac{dv_0}{dy} - g \\ s \frac{dP_0}{dy} + 6(\gamma - 1) \frac{\sigma}{m_B} \left(\frac{k_B}{m_B c^2} \right)^2 \sigma_B \rho_0 T_{P_1} T_{P_0}^5 + (\gamma - 1) \Lambda_0 \end{pmatrix}$$

Perturbation of the strong shock boundary conditions (6), (7) and (8) result in:

$$\begin{aligned} v_{1,y=x_{s0}} &= \frac{2}{\gamma + 1} \omega x_{s0} \\ P_1|_{y=x_{s0}} &= \frac{4}{\gamma + 1} \rho_s |u| \omega x_{s0} \\ \rho_1|_{y=x_{s0}} &= 0 \end{aligned} \quad (33)$$

Perturbation of the inner boundary condition (9) at $x = y = 0$ gives the following equation relating ρ_1 , v_1 , and T_{P_1} :

$$\frac{7}{8} \frac{4\sigma_B}{c} T_{P_0}^3 T_{P_1} = \frac{1}{2} \rho_1 v_0^2 + v_1 \rho_0 v_0 \quad (34)$$

In solving (32), the value for A^{-1} will be needed and is given by:

$$A^{-1} = \frac{1}{v_0 \left(v_0^2 - \frac{\gamma P_0}{\rho_0} \right)} \begin{pmatrix} v_0^2 - \frac{\gamma P_0}{\rho_0} & -\rho_0 v_0 & 1 \\ 0 & v_0^2 & -\frac{v_0}{\rho_0} \\ 0 & -v_0 P_0 \gamma & v_0^2 \end{pmatrix}$$

This problem is not really an eigenvalue problem for finding ω , since any value of ω will yield a possible solution, given the boundary conditions (33) and (34). This occurs since T_{P_1} , the perturbed neutrinosphere temperature can be adjusted to give a solution, once a value of ω is chosen.

In preliminary work done on the numerical solution of the perturbed equations, what has become apparent is that T_{P_1} can be thought of as a function of ω . The system resonates when the value of T_{P_1} becomes a maximum, at those values of ω that are the natural frequencies of the system.

COMPLETE SET OF EQUATIONS FOR THE SPHERICAL CASE

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \rho v) = 0 \quad (35)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} - g \quad (36)$$

$$\frac{\partial}{\partial t}(\rho e) + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \rho e v) + \frac{P}{r^2} \frac{\partial}{\partial r}(r^2 v) = \Lambda_{HC} \quad (37)$$

$$\frac{\partial L}{\partial r} = -4\pi r^2 \Lambda_{HC} \quad (38)$$

$$g = \frac{GM}{r^2} \quad (39)$$

$$P = \frac{\rho k_B T}{\mu} \quad (40)$$

$$\Lambda_{HC} = \frac{\sigma}{m_B} \rho \left(\frac{k_B T_{sp}}{m_e c^2} \right)^2 \left(\frac{L}{4\pi r^2} f(r) - \frac{7}{8} a c T^4 \left(\frac{T}{T_{sp}} \right)^2 \right) \quad (41)$$

The form chosen for Λ_{HC} is suggested by equation (34) in Chapter II, but is more general since it allows L , the neutrinosphere luminosity, to vary with position (see Bethe and Wilson (1985)).

The factor $f(r)$ in (41) takes into account the fact that integration over the solid angle subtended by the neutrinosphere at position r , only approximately done in obtaining (2.34) must be done correctly as r approaches r_{sp} , the neutrinosphere radius. However, even if $f(r) = 2\left(\frac{r}{r_{sp}}\right)^2(1 - \sqrt{1 - \left(\frac{r_{sp}}{r}\right)^2})$, the value found when exact solid integration is used, Λ_{HC} fails to become zero when $r = r_{sp}$, $T = T_{sp}$, and $L = 4\pi r_{sp}^2 \left(\frac{7}{8} \sigma_B T_{sp}^4\right)$, as it should. The reason for the discrepancy is due to the approximation of the neutrinosphere as a sharp surface, with a discontinuous change from an optically thick region to an optically thin one. The real neutrinosphere occupies a finite region in the radial coordinate. A form for $f(r)$

is chosen to force Λ_{HC} to be zero at the neutrinosphere, but not effect the correct large r behavior. The form chosen for $f(r)$ is as follows:

$$f(r) = \left(1 + 3 \left(\frac{r_{sp}}{r}\right)^2\right) \quad (42)$$

The amount of matter in the region $r_{sp} \leq r \leq r_s$ is assumed small as compared to the central remnant mass, and so g , the gravitational acceleration is given by (39) with M the approximately constant central mass. The only previously undefined constant appearing in (35) to (41) is a , which is defined as $\frac{4\epsilon_B}{c}$.

BOUNDARY CONDITIONS

The infalling material is assumed to be composed of iron, with 8 Mev per nucleon in binding energy. The shock is assumed strong enough to dissociate the iron to free baryons, and the energy associated with this breakup must be included in the shock jump conditions. At the shock, the boundary conditions are:

$$k_B T|_{r=r_s} = \frac{3}{32} m_B u^2 \left(1 + \sqrt{1 + \frac{32}{9} \frac{\epsilon_B}{m_B u^2}}\right) - \frac{\epsilon_B}{2} \quad (43)$$

$$v|_{r=r_s} = -|u| \left(\frac{5}{8} - \frac{3}{8} \sqrt{1 + \frac{32}{9} \frac{\epsilon_B}{m_B u^2}}\right) \quad (44)$$

$$\rho|_{r=r_s} = \frac{\rho_a}{\frac{5}{8} - \frac{3}{8} \sqrt{1 + \frac{32}{9} \frac{\epsilon_B}{m_B u^2}}} \quad (45)$$

where $P|_{r=r_s}$ can be found using (40). These jump conditions are done for a $\gamma = \frac{5}{3}$ gas only. ϵ_B is 1.28×10^{-5} erg (8 Mev), u is the infall velocity into the shock, and ρ_a is the ambient density of the infalling material. The choice $|u| = \sqrt{\frac{2GM}{r_s}}$ suggests itself as a free fall velocity to be used at the outer boundary.

A discussion of general time dependent boundary conditions at $r = r_s$ will not be attempted in this thesis. The boundary conditions for the steady state problem at $r = r_{sp}$ are:

$$\begin{aligned} T &= T_{sp} \\ L &= \frac{7}{8} G_8 T_{sp}^4 (4\pi r_{sp}^2) \end{aligned} \quad (46)$$

STEADY STATE SOLUTION OF THE SPHERICAL PROBLEM

The time derivatives in equations (35), (36) and (37) are set to zero, and these three equations are manipulated to give:

$$r^2 \rho v = \text{constant} \quad (47)$$

$$\frac{dv}{dr} = \frac{1}{\frac{\gamma P}{\rho v^2} - 1} \left(\frac{GM}{r^2 v} + \frac{(\gamma - 1) \Lambda_{HC} - 2\gamma \frac{P_e}{r}}{\rho v^2} \right) \quad (48)$$

$$\frac{dP}{dr} = -\frac{1}{\frac{\gamma P}{\rho v^2} - 1} \left(\frac{\gamma P G M}{r^2 v^2} + \frac{(\gamma - 1) \Lambda_{HC} - 2\gamma \frac{P_e}{r}}{v} \right) \quad (49)$$

Using the time independent form of (35), (36) and (37), along with (38) allows the following equation to be constructed:

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \left(\rho v \left(e + \frac{P}{\rho} + \frac{1}{2} v^2 - \frac{GM}{r} \right) + \frac{L}{4\pi r^2} \right) \right) = 0 \quad (50)$$

Equation (50) is a statement of total energy flux conservation, and can be integrated to give:

$$L = l_0 + \dot{M} \left(\frac{\gamma}{\gamma - 1} \frac{P}{\rho} + \frac{1}{2} v^2 - \frac{GM}{r} \right) \quad (51)$$

where l_0 is a constant, and $\dot{M} = -4\pi r^2 \rho v$ is the constant mass accretion rate (see equation (47)).

Results of the numerical integration of the steady state spherical problem are seen in figures 5a through 5g, which show plots similar to figures 4a through 4g. Figure 5h shows the luminosity versus radius curve.

From the results of the numerical integration, near the neutrinosphere, the matter temperature, as a function of radius, can be approximated by demanding $\Lambda_{HC} \approx 0$, or:

$$T = \left(\frac{T_{sp}^2}{\frac{7}{8} a c} \frac{L}{4\pi r^2} f(r) \right)^{\frac{1}{4}}$$

Comparing figures 5a through 5h to the output of the supernova code taken at a late time, but before explosion, shows remarkable agreement with the steady state results found here. Even the decrease, and then increase of the density behind the shock (see figure 5c) was reproduced in the supernova code output.

Figure 5g, the curve of Λ_{HC} versus radius shows the matter, after being dissociated can absorb energy from the neutrinos. The matter is much colder than it would have been if dissociation had not occurred, since 8 Mev per nucleon of the infall kinetic energy went into the iron breakup and is thus unavailable for shock heating. This implies the pressure is low and the material quickly begins to accelerate under the influence of gravity. Since the matter is being compressed as it falls, due at first to geometric effects, and later to the increasing pressure, it heats up and begins to radiate neutrinos. It soon radiates more energy than it absorbs and a region of cooling exists as the material falls to the neutrinosphere, now decelerating due to the high pressure gradients.

The temperature of the neutrinosphere thus acts as a clamp on the matter temperature at late times when baryons occupy the region outside the neutrinosphere up to the shock position.

PERTURBATION OF SHOCK—SPHERICAL CASE—

A similar procedure was used to derive perturbation equations for the spherical case, as for the planar case. The resulting formula appear below:

$$A \frac{dD}{dR} + \Omega D + QD = H \quad (52)$$

where:

$$A = \begin{pmatrix} v_0 & \rho_0 & 0 & 0 \\ 0 & v_0 & \frac{1}{\rho_0} & 0 \\ 0 & \gamma P_0 & v_0 & \frac{\gamma-1}{4\pi R^2} \\ 0 & 0 & 0 & \frac{1}{4\pi R^2} \end{pmatrix} \quad (53)$$

$$\Omega = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (54)$$

$$Q = \begin{pmatrix} \frac{dv_0}{dR} + 2\frac{v_0}{R} & \frac{d\rho_0}{dR} + 2\frac{\rho_0}{R} & 0 & 0 \\ -\frac{1}{\rho_0^2} \frac{dP_0}{dR} & \frac{dv_0}{dR} & 0 & 0 \\ 0 & \frac{dP_0}{dR} + 2\frac{P_0}{R} & \frac{\gamma dv_0}{dR} + 2\frac{\gamma v_0}{R} & 0 \\ \left(\frac{\Gamma}{\rho_0} \left(\frac{Lof(R)}{4\pi R^2} + \frac{35}{8} ac T_0^4 \left(\frac{T_0}{T_{F0}} \right)^2 \right) \right) & 0 & -\frac{\Gamma \mu}{\rho_0 \mu_0} \left(\frac{21}{4} ac T_0^3 \left(\frac{T_0}{T_{F0}} \right)^2 \right) & \frac{Ff(R)}{4\pi R^2} \end{pmatrix} \quad (55)$$

$$D = \begin{pmatrix} \rho_1 \\ v_1 \\ P_1 \\ L_1 \end{pmatrix} \quad (56)$$

$$H = \begin{pmatrix} \tilde{s} \frac{d\rho_0}{dR} - \frac{2\rho_0 v_0}{R^2} \frac{r_{s0} r_{sp}}{r_{s0} - r_{sp}} \\ \tilde{s} \frac{dv_0}{dR} + g \frac{r_{s0}}{r_{s0} - r_{sp}} (1 - 2 \frac{r_{sp}}{R}) \\ \tilde{s} \frac{dP_0}{dR} - \frac{2\gamma v_0 P_0}{R^2} \frac{r_{sp} r_{s0}}{r_{s0} - r_{sp}} - (\gamma - 1) \frac{2\tilde{R}}{R} \Lambda_0 \\ - \left(\frac{3r_{sp} - \frac{2}{R} r_{sp} r_{s0}}{r_{s0} - r_{sp}} \right) \Lambda_0 - \frac{\Gamma f(R) L_0}{4\pi R^2} \left(\frac{2\tilde{R}}{R} \left(\frac{1}{f(R)} - 2 + \frac{RT_{P0}}{RT_{P0}} \right) \right) \end{pmatrix} \quad (57)$$

$$A^{-1} = \frac{1}{v_0^2 - \frac{\gamma P_0}{\rho_0}} \begin{pmatrix} \frac{v_0^2 - \frac{\gamma P_0}{\rho_0}}{v_0} & -\rho_0 & \frac{1}{v_0} & -\frac{\gamma-1}{v_0} \\ 0 & v_0 & -\frac{1}{\rho_0} & \frac{\gamma-1}{\rho_0} \\ 0 & -\gamma P_0 & v_0 & -(\gamma-1)v_0 \\ 0 & 0 & 0 & \left(v_0^2 - \frac{\gamma P_0}{\rho_0} \right) 4\pi R^2 \end{pmatrix}$$

and new variables appearing in equations (52) through (57) are defined as follows:

$$\Gamma = \frac{\sigma}{m_B} \left(\frac{k_B T_{P0}}{m_e c^2} \right)^2 \rho_0$$

$$\tilde{s} = \left(\frac{R - r_{sp}}{r_{s0} - r_{sp}} \right) \omega r_{s0}$$

$$\tilde{R} = \left(\frac{R - r_{sp}}{r_{s0} - r_{sp}} \right) r_{s0}$$

$$r = R + \left(\frac{R - r_{sp}}{r_{s0} - r_{sp}} \right) \epsilon r_{s0} e^{i\omega t}$$

$$L = L_0 + \epsilon L_1 e^{i\omega t}$$

Note that the variables in equations (52) through (57) are complex in general. Equation (52) is thus a 4×4 matrix equation and is equivalent to two 4×4 matrix equations containing real variables. Of course, this is also true in the planar case previously presented where the equation corresponding to (52) is a complex 3×3 matrix equation equivalent to two 3×3 real matrix equations.

BOUNDARY CONDITIONS

The perturbed boundary conditions are given for the outer boundary only, $R = r_{s0}$, and are as follows:

$$\frac{k_B T_1}{m_B} = \frac{3}{32} |u| \omega r_{s0} \left(2 + \zeta + \frac{1}{\zeta} \right)$$

$$v_1 = \frac{3}{8} \omega r_{s0} \left(1 + \frac{1}{\zeta} \right)$$

$$\rho_1 = 8\rho_a \frac{\omega r_{s0}}{|u|} \left(\frac{32}{9m_B u^2} \epsilon_B \right) \left(\frac{1}{(5 - 3\zeta)^2 \zeta} \right)$$

where $\zeta = \sqrt{\left(1 + \frac{32}{9} \frac{\epsilon_B}{m_B u^2} \right)}$.

SUMMARY AND SPECULATIONS

From a study of the steady state solution to the problem described in the introduction, it has been found that in the region between the neutrinosphere and the shock, the matter temperature cannot rise above the neutrinosphere temperature as the matter falls onto the remnant, if the post-shock temperature is less than the neutrinosphere temperature. If the post-shock temperature is higher than the neutrinosphere temperature, the temperature of the matter monotonically decreases to the neutrinosphere temperature as the material moves to the central remnant. This occurs because the matter can cool by neutrino emission if the temperatures try to rise above the neutrinosphere temperature. Therefore, for a heating region to form, the post-shock temperature must be less than the neutrinosphere temperature. In the spherical case, this is almost guaranteed to occur, since the dissociation of heavy elements robs infall kinetic energy that could otherwise go into shock heating. In the planar case considered, it was easy to find solutions where the post shock temperature was higher than the neutrinosphere temperature and no heating region formed. No dissociation was assumed to take place for the planar case. Of course, it is also possible to force no heating behind the shock in the spherical case, but this requires the infall velocity to be much greater than $\sqrt{\frac{2GM}{r_s}}$ and it is not easy to see a way to avoid $|u| \approx \sqrt{\frac{2GM}{r_s}}$ being the case.

I wish to speculate on the possible future results of a perturbation analysis using the steady state solution. Looking at figure 5g, it seems likely that a perturbation at a frequency that produces a Λ_1 (the perturbed heating-cooling function) with a spatial "wavelength"

that fits one wavelength between the neutrinosphere and the shock will produce the fundamental mode of oscillation seen in the computer simulation. If this is the case, it implies the oscillation in neutrino luminosity is not produced by material near the surface of the core, but by all the material in the region between the neutrinosphere and the shock. I hope this is the case, as it implies the solutions to the perturbation equations may not be overly sensitive to the boundary conditions at the neutrinosphere.

Chapter VI References

Wilson, J.R., Mayle, R., Woosley, S.E, and Weaver, T.A. 1985,
XII Texas Symp. Rel. Ap. , in press.

Chapter VI Figure Captions

Fig. 1 Mass point trajectories for a 100 solar mass star undergoing gravitational collapse. The luminosity and mass accretion rate are shown in the accompanying figure.

Fig. 2 Schematic drawing of the simplified problem of a radiative shock in the presence of neutrinos exiting the neutrinosphere.

Fig. 3 Pressure versus velocity solutions for the planar problem.

Fig. 4 Results of numerical integration of the steady state equations of motion for the planar problem (a γ of $\frac{4}{3}$ was used).

Fig. 5 Results of numerical integration of the steady state equations of motion for the spherical problem (a γ of $\frac{5}{3}$ was chosen).

FIGURE 1a

FIGURE 1b

FIGURE 2

FIGURE 3

FIGURE 4a

FIGURE 4b

FIGURE 4c

FIGURE 4d

FIGURE 4e

FIGURE 4f

FIGURE 4c

FIGURE 5a

FIGURE 5b

FIGURE 5c

FIGURE 5d

FIGURE 5e

FIGURE 5f

FIGURE 5g

FIGURE 5h

Chapter VII

SUPERNOVA CODE RESULTS

INTRODUCTION

In this chapter, work I did in collaboration with other authors will be discussed. The papers that resulted may be consulted for a more complete report of the research done. In the second section of this chapter, some brief remarks on preliminary results from the inclusion of convection in the supernova code are made. Also presented is the result from a computer run in which a neutrino redshift was included in the supernova evolution. Finally, a summary of my contributions to supernova research will be made, along with my opinions on the state of the understanding of collapse driven supernova.

Collaborative Efforts

Gravitational Collapse and the Cosmic Antineutrino Background

Woosley, Wilson and Mayle 1985, Ap. J., accepted for publication

The possibility exists that current large detectors being used to search for proton decay could also be used to detect supernova. Antineutrinos produced during gravitational collapse could be detected on the earth by the reaction $\bar{\nu}_e + p \Rightarrow e^+ + n$. With the detection of both the positron and the neutron, low fluxes of antineutrinos could be measured. This fact led to an investigation of the electron antineutrino spectrum produced by collapsing stars in general, whether or not an explosion occurred, since in the process of collapse, neutrinos of all types are produced due to the high temperatures and densities involved.

The higher the antineutrino energy, the larger the cross-section for interaction, so spectral information is important. The supernova code, with multi-group flux limited diffusion, could provide this spectral information.

A study was undertaken of the antineutrino spectrum produced by the collapse of stars ranging in mass from $10 M_{\odot}$ to $5 \times 10^5 M_{\odot}$. The hardest neutrinos were found to be produced by stars less than $100 M_{\odot}$ in mass, probably in the mass range $8 M_{\odot} \leq M \leq 100 M_{\odot}$. The average antineutrino energy produced is about 15 Mev. However, stars in this mass range produce rare supernova events, so an actual detection of a single supernova exploding is unlikely, while the time integrated production of electron antineutrinos is also limited for these stars by nucleosynthetic constraints.

The larger mass stars, in the mass range $M \geq 100M_{\odot}$, produce average antineutrino energies less than about 5 Mev. The main reason for the drop in average energy, as compared to the lower mass stars, is that the neutrinosphere, if it develops, is eaten by the black hole that forms in the center of the star. The lower mass stars, upon collapse, produce dense cores that may or may not collapse further to black holes. The cores last long enough to provide for much neutrino emission, even if a black hole eventually forms. The antineutrinos escaping from the neutrinosphere in the stars in the lower mass range have energies about 3 times the neutrinosphere temperature in energy units. The neutrinos produced in the stars with $M \geq 100 M_{\odot}$ produce a large fraction of the antineutrinos from volume emission in optically thin regions.

There is a terrestrial background of antineutrinos produced by the natural radioactivity of the earth below about 3 Mev. This makes detection of antineutrinos from extraterrestrial sources impossible for energies below about 5 Mev. The conclusion was reached that present day detectors would be hard pressed to identify antineutrinos from gravitational collapse. However, in the design of future detectors, this work is useful in that it produced some numbers for experimenters to consider in their planning.

In order to do this research, the supernova code needed modification. Previous to this time, the supernova code had studied collapsing stars in the mass range $10M_{\odot} \leq M \leq 25M_{\odot}$. These stars form dense cores that may or may not collapse to black holes. For the $150M_{\odot}$, $500M_{\odot}$ and $5 \times 10^5 M_{\odot}$ stars, black holes form at the center and continue to accrete the stellar envelope.

I was involved in modifying the supernova code to approximate the formation of a black hole, and in finding suitable boundary conditions to apply at the boundary of the black hole and the rest of the star. Many different ideas were tried, but I will only describe the method of approximation we finally decided gave the best representation of the actual problem.

A black hole is declared to exist at that time when any sphere of total mass M moves inside twice the Schwarzschild radius for that mass (i.e. $r \leq 2r_s = \frac{4GM}{c^2}$). From then on, the central zone becomes the "black hole" (i.e. zone 1). The adjacent zone (i.e. zone 2) is merged into the central zone on each time cycle in which any mass zone in the star is inside twice the Schwarzschild radius for the enclosed mass. The merging process was conducted such that the resulting pressure between zones 1 and the new zone 2 (i.e. old zone 3) was less than or equal to the old pressure between the old zone 2 (merged into zone 1) and old

zone 3 (new zone 2). This becomes the boundary condition, that the "black hole" produces no pressure on the accreting material. Each time a new zone is accreted, the problem time step is reduced by 40 %, in order to help insure the numerical errors introduced by the abrupt merging would not produce instabilities.

The manner of testing the approximation is discussed next. As mentioned in Chapter III, the approximations used on the supernova code, including gravitational redshift and an enhanced gravitational force (see Chapter III), as well as the black hole accretion were checked against a fully general relativistic code provided by Chuck Evans. Both codes ran the $5 \times 10^5 M_{\odot}$ star and the results were compared. Doing the comparison was part of my contribution to the research. It is not a trivial operation, as some modifications to the general relativistic collapse code needed to be made in order to make this type of calculation. The results of this comparison convinced us that the approximations in the supernova code were very good.

I was also involved with Jim Wilson in using a known solution to steady state matter accretion onto a black hole, in order to check the approximations used in the supernova code. The known solution is the general relativistic analogue to the Bondi accretion solution, called Michel accretion (see Michel (1972)).

All three authors worked in the semi-analytic formulae presented in the resulting paper, to which I refer the reader for complete details.

Stellar Core Collapse and Supernova

Wilson, Mayle, Woosley and Weaver 1985 XII Texas Symp. Rel. Ap., in press

For the first time, stars in the mass range $11_{\odot} \leq M \leq 100M_{\odot}$, with the possible exception of the $100_{\odot}$ star, were shown to explode at the end of their stable evolution. These supernovae were all late time supernovae, where neutrino heating of the matter behind the shock produces material pressure that helps the shock move outward into the stellar envelope. Explosive energies are found to be consistent with the experimental value of 10^{51}erg . Also, the formation of a neutron star remnant of about $1.4M_{\odot}$ is seen to be produced by stars in the mass range less than about $20M_{\odot}$. Stars in the mass range above this may produce blackholes as well as a supernova display. Three progenitor stars, all of initial mass $25M_{\odot}$, but with widely varying iron core masses going into collapse, became supernova. The variance in the size of the iron cores reflected differences in the physics used

to evolve the stars up to the point of collapse. The late time heating mechanism is thus shown to be fairly insensitive to the exact details of the stellar evolutionary calculations. The nucleosynthesis produced by one the supernova, the most recent calculation of the $25 M_{\odot}$ star done Weaver and Woosley, was found to be in good agreement with solar abundances.

The oscillations in the neutrino luminosity and the mass accretion rate first seen by Jim Wilson (see Wilson (1985)) were again seen in some of the supernovae studied for the Texas Symposium. A qualitative explanation of the origin of the instability is also given in the Texas paper.

The scenario for the late time supernova can be found discussed in Chapter II of this thesis, and the qualitative explanation of the neutrino luminosity can be found in the introduction to Chapter VI.

Neutrinos from Gravitational Collapse

Schramm, Mayle and Wilson 1985 Proc. First International Symp. on Underground Phys., in press

Neutrinos from Gravitational Collapse

Mayle, Wilson and Schramm 1985 Ap. J., submitted for publication

The first paper listed contains some preliminary results on the production of neutrinos of all types by collapsing stars in the mass range $12M_{\odot} \leq M \leq 100M_{\odot}$. The second paper presents the results of spectral calculations, as well as a discussion of the method of calculation (i.e. flux limited diffusion). The neutrino production is of interest to particle experimenters that are planning to attempt detection of neutrinos from extraterrestrial sources. If detection of neutrinos from supernova becomes possible, it is pointed out in the first paper, that the Wilson delayed supernova mechanism, could be verified, if oscillations in the neutrino luminosity were observed on time scales of .1 second.

Part of the material in Chapter IV is contained in these papers, and the reader is referred to this chapter for a more thorough discussion of the neutrino production found during collapse.

PRELIMINARY RESULTS

The calculations that were done for the XII Texas Symposium on Relativistic Astrophysics did not include a gravitational redshift or a model for convection. Figures 1a through 1c show mass point trajectories for a $25M_{\odot}$ star (identified as 25 C in Wilson et al. 1985). Figure 1a shows the evolution without convection or redshift. Figure 1b is the evolution of the same star with a convection model included in the physics but no redshift. The model for salt finger convection was used. Figure 1c is the $25 M_{\odot}$ with a redshift included but no convection. As expected, the model with convection has an enhanced neutrino emission due to the dredge up of neutrinos from underneath the neutrinosphere. Figure 2 shows a graph of Y_P , the proton fraction for the stars shown in figure 1a and 1b. As can be seen, the negative gradient in Y_P in the model without convection is smoothed out in the model with convection. The neutrino enhancement is only modest, however, with about a 30-50 % increase in luminosity. The explosive energy of 2.0×10^{51} ergs was correspondingly about 30 % higher than the model without convection, which gave 1.6×10^{51} ergs when the additional opacity due to electron-positron pair scattering was included. The run of the $25M_{\odot}$ with a redshift of the neutrinos, is, as expected, a weaker explosion. The explosive energy of $.7 \times 10^{51}$ ergs is roughly 40 % less than the model without redshift.

SUMMARY

Contributions in Supernova Research

The material in Chapters IV, V, and VI is essentially my own research, as well as the material in Chapter III covering the approximate manner in which the general relativistic red shift is taken into account. I also contributed to the collaborative work summarized in the first part of this chapter. Other contributions have been made in improving or adding additional physics to an already established computer code. These include: improving the electron-positron equation of state, adding pair and plasma source and sink terms to the electron neutrino and antineutrino transport, improving the electron-positron scattering formula by refitting the results of Tubbs and Schramm (1975) with the newest value of $\sin^2\theta_W=.21$, and recognizing that the computer code I inherited did not correctly count the total number of electrons and positrons as scattering centers for neutrino interaction.

Collapse Driven Supernova

Unless our understanding of the equation of state of matter above nuclear densities is mistaken, and the compressibility modulus is around 140 Mev rather than near the laboratory experimental value of 220 Mev, the discovery of the explosive mechanism for collapse

driven supernova has been found. Agreement with observed output energy, nucleosynthesis, and remnant size strengthens the case for the Wilson heating mechanism.

There are uncertainties in the stellar evolutionary models, such as nuclear reaction rates or electron capture rates that can change the evolutionary scenario (see Wilson et al. 1985) and the resulting progenitor configuration. This in turn can effect the final nucleosynthesis or the output energy; however, uncertainties in stellar evolution do not seem to drastically effect the chances of an explosion occurring by late time neutrino heating.

The late time supernova mechanism, in which heating of the envelope behind the shock is important to the success of the explosion, can be studied without dealing with uncertain physics (i.e. a poorly known equation of state). The physical processes are well known in the heating region. This then brings up the question of numerics.

Full multi-group multi-angle solution of the Boltzmann equation for neutrino transport takes about an order of magnitude more computer time than multi-group flux limited diffusion. For some of the computer simulations reported in this chapter, a run of 2 hours on a Cray was necessary to follow about 1 second of supernova evolution. The neutrino transport subroutines currently use about 60 % of the CPU time consumed in a run. At present, computer time constraints force approximate transport methods.

The work reported in Chapter VI is a step in the direction of trying to understand the late time heating mechanism without a complete supernova time evolution. I believe this is a good approach, and hope further work will result in a better understanding of late time supernova.

Chapter VII References

Michel, F.C. 1972, *Ap. and Spac. Sci.*, v15. 153.

Wilson, J.R. 1985 , in *Numerical Astrophysics*, ed. J. Centralla, J. LeBlanc, and R. Bowers, (Jones and Bartlett:Boston), p. 422.

Chapter VII Figure Captions

Fig. 1 Mass point trajectories for the 25 solar mass supernova. These figures show the effect of convection and a gravitational redshift.

Fig. 2 Proton fraction versus interior mass in the 25 solar mass star. The figure compares the evolution with and without convection.

FIGURE 1a

FIGURE 1b

FIGURE 1c

FIGURE 2

Appendix A

INTRODUCTION

The complete set of equations needed to describe the evolution of a supernova include:

1. Einstein's equations
2. Equations that guarantee the conservation of:
 - (a) matter + neutrino stress energy
 - (b) electric charge
 - (c) baryon number
 - (d) lepton number
3. Boltzmann equation for neutrinos
4. Matter equation of state

The purpose of this appendix is to translate the necessary equations described above into mathematical form. In the first section, the evolution equations are displayed and briefly discussed. The only assumption made is that matter is described by an ideal fluid. In the second section, the additional assumption of spherical symmetry is introduced.

SECTION I

In the following, letters at the beginning of the Latin alphabet, such as a, b, c, denote spacetime indices of tensors. Letters near the middle of the Latin alphabet, such as i, j, k represent space indices. At the end of this appendix, a list of the notation, in the approximate order in which it was introduced, can be found.

EINSTEIN EQUATIONS

In units where $G = \frac{1}{8\pi}$, $c=1$, the Einstein equations are:

$$G_{ab} = T_{ab} + t_{ab} \quad (1)$$

where T_{ab} is the matter stress energy tensor, and t_{ab} is the neutrino stress energy tensor. Four of the 10 equations (1) are not evolution equations, but constraint equations, as they contain only first derivatives of the metric with respect to the timelike coordinate (see

Weinberg (1972)). If the initial data satisfy the constraint equations at one time, the 6 evolution equations guarantee the constraint equations are satisfied thereafter.

The 3 + 1 formalism of Arnowitt, Deser, and Misner described in Misner, Thorne, and Wheeler (1973) (hereafter denoted by MTW), and further refined by York (1979), provide a natural way to separate the 4 constraint equations from the 6 dynamical equations. The idea is to break spacetime into space plus time, and then write g_{ab} and G_{ab} in terms of geometrical objects describing the particular way spacetime is sliced up into 3 dimensional hypersurfaces.

From differential geometry, two 3 dimensional tensors describe an embedded surface in a 4 dimensional space. A_{ij} , the metric tensor in the slice, determines the intrinsic curvature of the surface, independent of the surrounding manifold. K_{ij} , the extrinsic curvature tensor, describes the embedding of the surface in 4 dimensional spacetime.

The 3 dimensional surface determines a vector field, n^a , of unit timelike normals (i.e. $n_a n^a = -1$). n^a can represent the 4 velocity of a physical observer, called the Eulerian observer, who is at rest in the slice. In the following, the way in which g_{ab} and G_{ab} are expressed in terms of K_{ij} , A_{ij} , and n^a is sketched.

Λ_{ab} is defined as follows:

$$\Lambda_{ab} = g_{ab} + n_a n_b$$

Λ_{ab} is really a spatial metric since $\Lambda_{ab} n^b = 0$. In fact, since $\Lambda_{ab} = \Lambda_{ba}$ and $\Lambda_{ab} n^b = 0$, the 16 components of Λ_{ab} are all expressible in terms of only 6, which are taken as A_{ij} .

G_{ab} is constructed with ${}^4R_{bcd}$, the 4 dimensional Riemann tensor. The 256 components of ${}^4R_{bcd}$ can be expressed in terms of only 20 independent components (see MTW). Fourteen of the 20 can be expressed in terms of A_{ij} , K_{ij} , and their spatial derivatives (i.e. D_i , the covariant derivative compatible A_{ij}), using the equations of Gauss and Codazzi. (The Gauss-Codazzi equations are integrability conditions that A_{ij} and K_{ij} must satisfy if they are to describe a surface that can actually be constructed in spacetime.)

These 14 components of ${}^4R_{bcd}$ are all that is needed to form the following 4 components of G_{ab} :

$$n^a n^b G_{ab} = \frac{1}{2} \left({}^3R + (K^i_i)^2 + K_{ij} K^{ij} \right) \quad (2a)$$

$$-\Lambda^a_i n^b G_{ab} = D_j K^j_i - D_i K^j_j \quad (2b)$$

where 3R is the Ricci scalar compatible with Λ_{ij} . Now using $G_{ab} = T_{ab} + t_{ab}$ in the above, the 4 Einstein constraint equations are found (see York (1979) for a more complete discussion):

$${}^3R + (K_i^i)^2 - K_{ij}K^{ij} = 2\rho_H = \text{Hamiltonian constraint} \quad (3a)$$

$$D_j K_i^j - D_i K_j^j = J_i = \text{momentum constraint} \quad (3b)$$

where:

$$\rho_H = n^a n^b (T_{ab} + t_{ab})$$

$$J_i = -\Lambda_i^a n^b (T_{ab} + t_{ab})$$

ρ_H is the energy density and J_i is the momentum density as seen by the Eulerian observer.

Before writing the remaining 6 evolution equations, the manner in which the spatial coordinates are to be moved from slice to slice must be specified. This is called a gauge freedom.

Define a vector field t^a such that:

$$t^a = \alpha n^a + \beta^a$$

with $\beta^a n_a = 0$ (i.e. β^a is really a spatial vector). The trajectory of constant spacelike coordinates will travel along t^a , as is illustrated in figure 1. β^i represents the freedom to perform a special relativistic Lorentz transformation relative to the Eulerian observer, and α adjusts coordinate time so that αdt is the proper time as measured by the coordinate observers.

α is known as the lapse function, and β^i is the shift vector. Together, they form 4 kinematical degrees of freedom, and must be specified independently of Einstein's equations (they are just telling the coordinate observers how to move.)

The 4 dimensional line element can then be written as:

$$ds^2 = -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \Lambda_{ij} dx^i dx^j \quad (4)$$

The remaining 6 components of ${}^4R_{abcd}$ can now be expressed in terms of Λ_{ij} , K_{ij} , α , β^i and their derivatives. G_{ab} is then determined, and the 6 second order in time Einstein evolution equations can be written down.

However, it is more convenient to use the following 12 first order in time equations, equivalent to the 6 evolution equations:

$$\begin{aligned} \frac{\partial K_{ij}}{\partial t} = & -D_i D_j \alpha + \alpha \left({}^3 R_{ij} - 2K_{il} K_j^l + K_i^l K_{lj} - S_{ij} - \frac{1}{2} \Lambda_{ij} (\rho_H - S) \right) \\ & + \beta^l D_l K_{ij} + K_{il} D_j \beta^l + K_{lj} D_i \beta^l \end{aligned} \quad (5)$$

$$\frac{\partial \Lambda_{ij}}{\partial t} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \quad (6)$$

where:

$$S_{ij} = \Lambda_i^a \Lambda_j^b (T_{ab} + t_{ab})$$

S_{ij} is thus the spatial stresses as measured by the Eulerian observer. S is defined to be S_i^i .

For completeness, the constraint equations are repeated below. As stated previously, the above 12 first order evolution equations guarantee the constraints are satisfied, provided they are satisfied at one time.

$${}^3 R + (K_i^i)^2 - K_{ij} K^{ij} = 2\rho_H$$

$$D_j K_i^j - D_i K_j^j = J_i$$

BARYON NUMBER CONSERVATION

Defining n_B as the number of baryons per unit volume as measured in a frame where the matter is locally at rest, the equation that implies baryon number conservation is:

$$(n_B u^a)_{;a} = 0 \quad (7)$$

where u^a is the matter 4 velocity.

Neglecting the electron mass as compared to the baryon mass, the matter rest mass density, ρ , is proportional to n_B , the constant of proportionality being the baryon mass. Therefore, (7), can be replaced by:

$$(\rho u^a)_{;a} = 0 \quad (8)$$

Equation (8) will be rewritten in a more physically transparent form, using D , the density measured by the Eulerian observer, and $v^i = \frac{u^i}{\alpha} = \frac{dx^i}{dt}$, the 3 velocity of the fluid as measured by the coordinate observer (see Wilson(1979), Shapiro and Teukolsky(1980)).

The fluid 3 velocity, v_E^i , as measured by the Eulerian observer is found to be:

$$v_E^i = \frac{v^i + \beta^i}{\alpha}$$

The figure 2 illustrates the above. Now defining $u = -n^a u_a$, it can be shown (using $u_a = g_{ab} u^b$ and $u^a u_a = -1$) that:

$$u = \frac{1}{\sqrt{1 - \frac{(v_i + \beta_i)(v^i + \beta^i)}{\alpha^2}}} = \frac{1}{\sqrt{1 - v_E^i v_{Ei}}}$$

u is thus a generalization of the special relativistic gamma that relates physical quantities measured in two frames, the fluid frame and the Eulerian frame.

Therefore, $D = u\rho$, and equation (8) becomes:

$$\frac{\partial D}{\partial t} + D \frac{\partial \ln \Upsilon^{\frac{1}{2}}}{\partial t} + \frac{1}{\Upsilon^{\frac{1}{2}}} \frac{\partial}{\partial x^i} (\Upsilon^{\frac{1}{2}} v^i D) = 0 \quad (9)$$

where $\Upsilon^{\frac{1}{2}} \equiv \sqrt{\text{Det } h_{ij}}$ (note that $\sqrt{-\text{Det } g_{ab}} = \alpha \Upsilon^{\frac{1}{2}}$).

In the nonrelativistic limit (9) reduces to:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

STRESS ENERGY CONSERVATION

The matter stress energy tensor, T_{ab} , will be taken in the ideal fluid approximation:

$$T^{ab} = (\rho + \epsilon_m + P_m) u^a u^b + P_m g^{ab} \quad (10)$$

where ϵ_m is the matter internal energy density and P_m is the matter pressure, as seen by an observer at rest with respect to the matter fluid. This is an excellent approximation for the supernova collapse, as viscosity and heat conduction due to the finite mean free path of the matter particles is only important in shock fronts, where material properties can change over distances on the order of the mean free path.

The neutrino stress energy tensor, t_{ab} , is defined as:

$$t^{ab} = \int p^a p^b f dP \quad (11)$$

where f is the neutrino distribution function, dP is the invariant 3 momentum volume, and p^a is the neutrino 4 momentum (see Lindquist (1966)). Six neutrino fields are considered, $\nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau$, and t_a , will represent the total contribution from all 6 fields.

The equation for stress energy conservation is:

$$(T_a^b + t_a^b)_{;b} = 0 \quad (12)$$

This is a set of 4 equations which split into an energy conservation equation, and Euler's equation. The energy conservation is discussed first. It is taken to be:

$$u^a (T_a^b + t_a^b)_{;b} = 0 \quad (13)$$

Introducing the energy density, $E = u c_m$, as measured by the Eulerian observer, equation (13) becomes:

$$\begin{aligned} \frac{\partial E}{\partial t} + E \frac{\partial \ln \Gamma^{\frac{1}{2}}}{\partial t} + \frac{1}{\Gamma^{\frac{1}{2}}} \frac{\partial}{\partial x^i} (E v^i \Gamma^{\frac{1}{2}}) + \frac{P_m}{\Gamma^{\frac{1}{2}}} \left(\frac{\partial (u \Gamma^{\frac{1}{2}})}{\partial t} + \frac{\partial}{\partial x^i} (u v^i \Gamma^{\frac{1}{2}}) \right) \\ = \alpha u^a t_{a;b} \end{aligned} \quad (14)$$

In the nonrelativistic limit, the above becomes:

$$\frac{\partial E}{\partial t} + \nabla \cdot (\vec{v} E) + P_m \nabla \cdot \vec{v} = -\text{change in neutrino energy}$$

The Euler equation is:

$$(T_i^b + t_i^b)_{;b} = 0 \quad (15)$$

A new 4 vector, $S_\mu = (\rho + e_m + P_m) u_\mu = (D + E + P_m) u_\mu$ is useful in writing (15) in a physically transparent form. Using S_μ , equation (15) becomes:

$$\begin{aligned} \frac{\partial S_i}{\partial t} + S_i \frac{\partial \ln \Gamma^{\frac{1}{2}}}{\partial t} + \frac{1}{\Gamma^{\frac{1}{2}}} \frac{\partial}{\partial x^j} (S_i v^j \Gamma^{\frac{1}{2}}) + \alpha \frac{\partial P_m}{\partial x^i} \\ + \frac{1}{2} \frac{\partial g^{ab}}{\partial x^i} \frac{S_a S_b}{S^i} = -\alpha t_{i;b} \end{aligned} \quad (16)$$

In the nonrelativistic limit $S_i = \rho v^i$, and using the nonrelativistic mass conservation equation, (16) becomes, in the absence of a gravitational field:

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} + \nabla P = \text{neutrino acceleration}$$

LEPTON NUMBER CONSERVATION

Lepton number is defined as the number of leptons minus the number of antileptons. The leptons present in a supernova collapse are the electron, electron neutrino, muon neutrino, and tau neutrino. All the antiparticles of the 4 leptons above are also present. The muon, whose rest mass is 105 Mev would only be produced in the very central regions of a supernova where the equation of state is poorly known, so its presence is neglected in the supernova model studied here. The tau lepton has a rest mass of 1784 Mev and will not be present in a supernova collapse (see the Particle Properties Data (1984) for the masses quoted).

The equation for lepton number conservation is:

$$(n_0 u^a)_{;a} + N^a_{;a} = 0 \quad (17)$$

where $n_0 = n_{e-} - n_{e+}$ is the difference in the number densities of electrons and positrons as measured in the fluid rest frame. $N^a = N^a_{\nu_e} - N^a_{\bar{\nu}_e}$, where $N^a_{\nu_e} = \int p^a f_{\nu_e} dP$ is the number flux in neutrinos of species a . N^a is thus the difference in the number flux of electron neutrinos and electron antineutrinos. Mu and Tau neutrinos and their antiparticles are not able to change lepton number since the production of tau particles is not likely in a supernova, and any muons produced are in a region where matter is in thermal equilibrium with the neutrinos. In the above f_{ν} is the generalized Fermi-Dirac distribution function for the " s^{th} " species of neutrino.

Now introducing $\bar{Z}$ such that:

$$n_0 = \frac{\rho \bar{Z}}{m_B}$$

and using $(\rho u^a)_{;a} = 0$, (17) can be written as:

$$\frac{D}{m_B} \left(\frac{\partial \bar{Z}}{\partial t} + v^i \frac{\partial \bar{Z}}{\partial x^i} \right) = -\alpha N^a_{;a} \quad (18)$$

In the nonrelativistic limit, (18) becomes:

$$\frac{\rho}{m_B} \left(\frac{\partial \bar{Z}}{\partial t} + \vec{v} \cdot \nabla \bar{Z} \right) = \text{change in lepton number}$$

CHARGE CONSERVATION

The equation expressing conservation of electric charge is:

$$(n_0 u^a)_{;a} = (n_P u^a)_{;a} \quad (19)$$

where n_p is the number of protons per unit volume as measured in the fluid frame.

Introducing Z such that:

$$n_p = \frac{\rho Z}{m_B}$$

equation (19) becomes:

$$\frac{\partial \bar{Z}}{\partial t} + v^i \frac{\partial \bar{Z}}{\partial x^i} = \frac{\partial Z}{\partial t} + v^i \frac{\partial Z}{\partial x^i} \quad (20)$$

BOLTZMANN EQUATION FOR NEUTRINOS

A generally covariant Boltzmann equation has been derived by Lindquist (see Lindquist(1966)).

An abbreviated derivation follows.

Call $f_{\nu_s}(x^a, p^a)$ the generalized Fermi Dirac distribution function that describes neutrinos of species s . Then:

$$df_{\nu_s} = \frac{\partial f_{\nu_s}}{\partial x^a} dx^a + \frac{\partial f_{\nu_s}}{\partial p^a} dp^a = \delta f_{\nu_s, \text{collisions}}$$

Now assuming the neutrinos move along geodesics between collisions:

$$\frac{dx^a}{d\tau} = p^a$$

$$\frac{dp^a}{d\tau} = -\Gamma_{bc}^a p^b p^c$$

where the variable τ parameterizing the geodesics is normalized so that $\frac{dx^a}{d\tau} = p^a$ holds.

The Boltzmann equation becomes:

$$p^a \frac{\partial f_{\nu_s}}{\partial x^a} - \Gamma_{bc}^a p^b p^c \frac{\partial f_{\nu_s}}{\partial p^a} = \delta f_{\nu_s, \text{collisions}} \quad (21)$$

EQUATION OF STATE

The equation of state is discussed in Appendix B. For now, the equation of state is symbolically written as:

$$P_m = P_m(\rho, \epsilon_m)$$

The supernova equations have been written in a general form in the above. In Section II, the assumption of spherical symmetry is introduced, and the supernova equations are rewritten using this additional assumption.

SECTION II

A 4 dimensional line element that respects spherical symmetry can be written as (see Wilson(1979)):

$$ds^2 = -(\alpha^2 - \beta_r \beta_r) dt^2 + 2\beta_r dr dt + A^2 dr^2 + B^2 r^2 d\Omega^2 \quad (22)$$

Two different gauge choices will be made, and the complete set of supernova equations will be written for each choice.

Before proceeding further, some counting of gravitational dynamical degrees of freedom can be done. Of the 10 Einstein equations, 4 are constraint equations and involve no dynamics. There is also a gauge freedom, in that the 4 coordinates, x^a , can be chosen arbitrarily. So Einstein's equations can describe at most 2 degrees of freedom (i.e. the 2 polarization states of a gravitational wave). By Birkhoff's theorem (see Weinberg (1972)), a spherical object cannot radiate gravitational waves (there is no preferred direction with which to assign directions of polarization). This means that it should be possible to find a coordinate system in which the metric is determined only by the 4 constraint equations (metric instantaneously responding to matter and neutrinos), since there are no gravitational degrees of freedom present.

Isotropic Gauge

Einstein's Equations

For a more complete discussion of Einstein's equations in the form written below, see Wilson(1979)). A sketch of the derivation follows.

In equation (22), use β_r to force $A = B$ on each slice, and determine α by "maximal slicing", or $K_i^i = 0$. (In equilibrium, and neglecting gravity, a soap film stretched across a closed wire loop, will assume a shape such that $K_i^i = 0$. In this case K_{ij} is the extrinsic curvature of the soap film (see Landau and Lifshitz (1959)).

It will be shown below that $K_i^i = 0$ implies $\alpha = 1 + \phi$, in the limit of weak gravity, where ϕ = Newtonian gravitational potential.

In order to find the equation determining α , use (5) in this appendix. Raise one index with Λ^{ij} , and take the trace of both sides of the equation. The following equation results:

$$\frac{\partial K_i^i}{\partial t} = -D_i D^i \alpha + \alpha \left(K_{ij} K^{ij} + \frac{1}{2}(\rho_H + S) \right) + \beta^i \frac{\partial K_j^j}{\partial x^i} \quad (23)$$

Demanding $\frac{\partial K_i^i}{\partial t} = 0$ and, $K_i^i = 0$ in (23), the slicing condition for α is found. In the gauge chosen, (23) becomes:

$$\frac{1}{A^3 r^2} \frac{\partial}{\partial r} \left(r^2 A \frac{\partial \alpha}{\partial r} \right) = \alpha \left((\rho + \epsilon_m) \left(u^2 - \frac{1}{2} \right) + P_m \left(u^2 + \frac{1}{2} \right) + \frac{3}{2} (K_r^r)^2 + \rho_\nu \right) \quad (24)$$

In the above $\rho_\nu = t_{ab} n^a n^b$, and using the definition of $K_{ij} = -\frac{1}{2} (A_{ij,t} n^t + A_{tj} n^t_{,i} + A_{it} n^t_{,j})$ (see MTW), it can be shown $K_\theta^\theta = K_\phi^\phi$. With this result, and $K_i^i = 0$, the term in (23) $K_{ij} K^{ij}$, becomes $\frac{3}{2} (K_r^r)^2$ in (24).

The condition $K_i^i = 0$ can also be written down using the definition of K_{ij} mentioned above:

$$K_i^i = K_r^r + K_\theta^\theta + K_\phi^\phi = K_r^r + 2K_\theta^\theta = 0$$

implies:

$$3 \left(\frac{\partial A}{\partial t} - \beta^r \frac{\partial A}{\partial r} \right) - \frac{A}{r^2} \frac{\partial}{\partial r} \left(r^2 \beta^r \right) = 0 \quad (25)$$

Equation (25) will be further discussed after the constraint equations are written down. The Hamiltonian constraint equation (3a) becomes:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial A^{\frac{1}{2}}}{\partial r} \right) = -\frac{A^{\frac{1}{2}}}{4} \left((\rho + \epsilon_m + P_m) u^2 - P_m + \rho_\nu + \frac{3}{4} (K_r^r)^2 \right) \quad (26)$$

The momentum constraint equation (3b) has only one nonzero component, and is written as:

$$\frac{1}{r^3 A^3} \frac{\partial}{\partial r} \left(r^3 A^3 K_r^r \right) = S_r - j_r + \frac{\beta_r}{\alpha} \rho_\nu \quad (27)$$

where:

$$j_r = t_r^c n_c$$

As mentioned previously, β^r is used to keep $A = B$ throughout the evolution. This means:

$$\frac{\partial}{\partial t} (r^2 \Lambda_{rr}) = \frac{\partial}{\partial t} \Lambda_{\theta\theta}$$

and:

$$\frac{\partial}{\partial t} (\sin^2 \theta \Lambda_{\theta\theta}) = \frac{\partial}{\partial t} \Lambda_{\phi\phi}$$

must be true. Using (6) in this appendix, a coordinate condition on β^a can be found that guarantees $A = B$. It can be shown to be:

$$\frac{\partial \beta^r}{\partial r} - \frac{\beta^r}{r} - \frac{3}{2} \alpha K_r^r = 0 \quad (28)$$

Now using (28) and the equation from the set (6) for $\frac{\partial A}{\partial t}$, equation (25) can be reproduced. This means the choice for β^r is consistent with the choice $K_i^i = 0$ for maximal slicing.

Four equations have been found to determine $K_r^r, \alpha, A, \beta^r$ without using all of Einstein's equations. The complete set must be satisfied, and it will not be shown, but the equations determining $K_r^r, \alpha, A, \beta^r$ in terms of spatial derivatives only, are consistent with the evolution equations determining K_r^r and A in terms of time derivatives. (For example, solve (5) for $\frac{\partial K_r^r}{\partial t}$, and use (27) for $\frac{\partial K_r^r}{\partial r}$. If (27) is consistent with (5), then $\frac{\partial}{\partial t} \frac{\partial K_r^r}{\partial r} = \frac{\partial}{\partial r} \frac{\partial K_r^r}{\partial t}$ using (27) in the left hand side, and (5) in the right hand side. To show the equality, $(T_a^b + t_a^b)_{;b} = 0$ is also needed.)

The equations determining the metric variables are rewritten below:

momentum constraint

$$K_r^r = \frac{1}{r^3 A^3} \int_0^r r^3 A^3 \left(S_r - j_r + \frac{\beta_r}{\alpha} \rho_\nu \right) \quad (27)$$

coordinate condition

$$\beta^r = -\frac{3}{2} r \int_r^\infty \frac{\alpha K_r^r}{r} dr \quad (28)$$

Hamiltonian constraint

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial A^{\frac{1}{2}}}{\partial r} \right) = -\frac{A^{\frac{1}{2}}}{4} \left((\rho + e_m + P_m) u^2 - P_m + \rho_\nu + \frac{3}{4} (K_r^r)^2 \right) \quad (26)$$

maximal slicing

$$\frac{1}{A^3 r^2} \frac{\partial}{\partial r} \left(r^2 A^{\frac{1}{2}} \frac{\partial \alpha}{\partial r} \right) = \alpha \left((\rho + e_m) \left(u^2 - \frac{1}{2} \right) + P_m \left(u^2 + \frac{1}{2} \right) + \frac{3}{2} (K_r^r)^2 + \rho_\nu \right) \quad (24)$$

Thus the isotropic gauge, along with maximal slicing, allows the metric variables to be found by solving ordinary differential equations, explicitly demonstrating the lack of gravity waves in a spherical geometry.

Finally in the limit of weak gravity, where the source terms in (24) and (26) are dominated by the rest mass matter density, it can be seen by direct substitution that $\alpha = 1 + \phi$ and $A = 1 - \phi$, where ϕ is the Newtonian gravitational potential determined from Poisson's equation:

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \phi) = 4\pi G \rho = \frac{\rho}{2}$$

Neutrino Boltzmann Equation

Next, the neutrino Boltzmann equation, and the construction of the neutrino stress energy tensor will be considered. Only one species will be done as an example; however, 6 Boltzmann equations must actually be found, and t_{ab} really contains contributions from 6 species.

Now the neutrino momentum can be written in terms of the tetrad unit vectors (those unit vectors that are locally Minkowskian in the spacelike slice):

$$\vec{p} = p^0 \vec{e}_0 + p^1 \vec{e}_1 + p^2 \vec{e}_2 + p^3 \vec{e}_3$$

where:

$$(\vec{e}_0)^a = n^a = \text{normal to spacelike slice}$$

$$(\vec{e}_1)^r = \frac{1}{A} ; (\vec{e}_1)^a = 0, a \neq r$$

$$(\vec{e}_2)^\theta = \frac{1}{Ar} ; (\vec{e}_2)^a = 0, a \neq \theta$$

$$(\vec{e}_3)^\phi = \frac{1}{Ar \sin \theta} ; (\vec{e}_3)^a = 0, a \neq \phi$$

and:

$$p^0 = E$$

$$p^1 = E \cos \eta = E \mu$$

$$p^2 = E \sin \eta \cos \xi = E(1 - \mu^2)^{\frac{1}{2}} \cos \xi$$

$$p^3 = E \sin \eta \sin \xi = E(1 - \mu^2)^{\frac{1}{2}} \sin \xi$$

Figure 3 defines the angles used in the above. Following Wilson (unpublished notes), the Boltzmann equation takes on a relatively simple form if, instead of E (the energy measured by the Eulerian observer), $W = EA$ is used. If this is so, then:

$$\begin{aligned} p^t &= \frac{W}{\alpha A} \\ p^r &= \frac{W}{A} \left(\frac{\mu}{A} - \frac{\beta^r}{\alpha} \right) \\ p^\theta &= \frac{W}{A^2} \frac{(1 - \mu^2)^{\frac{1}{2}}}{r} \cos \xi \\ p^\phi &= \frac{W}{a^2} \frac{(1 - \mu^2)^{\frac{1}{2}}}{r \sin \theta} \sin \xi \end{aligned}$$

are the contravariant components of p^μ , while the covariant components are:

$$\begin{aligned} p_t &= \beta^r \mu W - \frac{W}{A} \alpha \\ p_r &= \mu W \\ p_\theta &= r(1 - \mu^2)^{\frac{1}{2}} W \cos \xi \\ p_\phi &= r(1 - \mu^2)^{\frac{1}{2}} W \sin \xi \sin \theta \end{aligned}$$

With this momentum parameterization, $p_r = \mu W$, the radial component of the covariant momentum, takes on a simple form.

Also introduce $\dot{H} = r^2 W^2 f$, which is a kind of neutrino number luminosity, and the Boltzmann equation in conservation form becomes:

$$\begin{aligned} \frac{\partial \dot{H}}{\partial t} + \frac{\partial}{\partial r} \left(H \left(\frac{\alpha \mu}{A} - \beta^r \right) \right) + \frac{\partial}{\partial W} \left(\dot{H} W \left(\frac{(1 - \mu^2) \beta^r}{r} - \mu \left(\frac{\partial}{\partial r} \left(\frac{\alpha}{A} \right) - \mu \frac{\partial \beta^r}{\partial r} \right) \right) \right) \\ + \frac{\partial}{\partial \mu} \left(H \left(\left(\frac{\alpha}{A} - \mu \beta^r \right) \frac{1 - \mu^2}{r} - (1 - \mu^2) \left(\frac{\partial}{\partial r} \left(\frac{\alpha}{A} \right) - \mu \frac{\partial \beta^r}{\partial r} \right) \right) \right) \\ = \alpha r^2 W^2 \frac{A}{W} (\text{collisions terms}) \end{aligned} \quad (29)$$

where:

$$\text{collision terms} = \kappa \rho (I - f)$$

Lindquist defines κ so that it is an invariant or scalar quantity. This means knowledge of κ in the fluid frame allows the calculation in any frame. Lindquist finds:

$$\kappa = \frac{\sigma_a(\epsilon_f, T, \mu_e, \dots)}{m_B} \left(\frac{p_f}{\epsilon_f} \right) \epsilon_f = \frac{\sigma_a(-\vec{p} \cdot \vec{u}, T, \mu_e, \dots)}{m_B} \left(1 - \frac{(\vec{p} \cdot \vec{p})^2}{(\vec{u} \cdot \vec{p})^2} \right)^{\frac{1}{2}} (-\vec{p} \cdot \vec{u}) \quad (30)$$

where the rightmost equality shows κ is a scalar quantity. p_f is the magnitude of the neutrino momentum as measured by the fluid observer ($p_f = \epsilon_f$ for a massless particle) and all dot products are 4 vector dot products. The absorptive opacity is denoted by $\sigma_a(\epsilon_f, T, \mu_e, \dots)/m_B$. It depends in general on the neutrino energy, matter temperature, electron chemical potential, matter composition, etc. κ is thus seen to be (opacity)(momentum). The momentum factor comes from the fact that (21), the Boltzmann equation can be written more elegantly with a factor of momentum multiplying both sides of the equation.

As an example to illustrate the form of the collision term, consider pure absorption and emission processes. The energy of a neutrino in the fluid frame (as measured by the fluid observer), is found from:

$$\epsilon_f = -\vec{p} \cdot \vec{u}$$

Using (30), the right hand side of (29) becomes:

$$\alpha A r^2 W \kappa \rho (I - f) = \frac{\rho \sigma_a}{m_B} \alpha u \left(1 - \mu \frac{\beta^r + v^r}{\alpha} A \right) (\bar{B} - \bar{H}) \quad (31)$$

where $\bar{B}$ is an emission function. Assuming the emitting particles are in local thermodynamic equilibrium:

$$I = \frac{1}{\exp\left(\frac{\epsilon_f - \mu}{k_B T}\right) + 1}$$

and:

$$\bar{B} = \frac{r^2 W^2}{\exp\left(\frac{\epsilon_f - \mu}{k_B T}\right) + 1}$$

Also used in (31) is the fact:

$$\epsilon_f = -\vec{p} \cdot \vec{u} = \frac{uW}{A} \left(1 - \mu \frac{\beta^r + v^r}{\alpha} A \right)$$

The conservative form of Boltzmann's equation is useful for numerical computation as it facilitates construction of a finite difference form of the equation (see Roache (1972)).

To discuss limiting cases of the Boltzmann equation, it is more useful to return to f as an independent variable. In terms of f , (29) becomes:

$$\begin{aligned}
& \frac{1}{\alpha} \frac{\partial f}{\partial t} + \frac{1}{A} \left(\mu - \frac{\beta^r A}{\alpha} \right) \frac{\partial f}{\partial r} + W \frac{\partial f}{\partial W} \left(\frac{(1-\mu^2)\beta^r}{\alpha r} - \mu \left(\frac{1}{\alpha} \frac{\partial}{\partial r} \left(\frac{\alpha}{A} \right) - \frac{\mu}{\alpha} \frac{\partial \beta^r}{\partial r} \right) \right) \\
& + (1-\mu^2) \frac{\partial f}{\partial \mu} \left(\left(1 - \frac{\mu \beta^r A}{\alpha} \right) \frac{1}{Ar} - \left(\frac{1}{\alpha} \frac{\partial}{\partial r} \left(\frac{\alpha}{A} \right) - \frac{\mu^2}{\alpha} \frac{\partial \beta^r}{\partial r} \right) \right) \\
& = \frac{A}{W} \kappa \rho (I - f)
\end{aligned} \tag{32}$$

In the limit $\alpha = A = 1$, and $\beta^r = 0$, (32) becomes:

$$\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial r} + (1-\mu^2) \frac{\partial f}{\partial \mu} = \frac{\partial f}{\partial t} + \hat{p} \cdot \nabla f = \text{collisions}$$

This is the familiar radiation transport equation from photon radiation transport.

The terms involving α , A , and β^r contain relativistic effects. As mentioned before $\alpha = 1 + \phi$ and $A = 1 - \phi$ in the limit of weak gravity. Setting $\alpha = 1 + \phi$, $A = 1 - \phi$, $\beta^r = 0$, and assuming the static limit, (32) becomes:

$$\begin{aligned}
& \frac{\mu}{A} \frac{\partial f}{\partial r} + \frac{(1-\mu^2)}{Ar} \frac{\partial f}{\partial \mu} - \frac{\mu}{\alpha} W \frac{\partial f}{\partial W} \left(2 \frac{\partial \phi}{\partial r} \right) - \frac{(1-\mu^2)}{\alpha} \frac{\partial f}{\partial \mu} \left(2 \frac{\partial \phi}{\partial r} \right) \\
& = \text{collisions}
\end{aligned} \tag{33}$$

The third term in the above appears due to the redshift of the neutrino energy, and the fourth term takes into account the fact that the neutrino trajectory deviates from a straight line path in the presense of a strong gravitational field. The above can be written in a more suggestive form (neglecting ϕ compared to unity) as:

$$c\hat{p} \cdot \nabla f - \frac{2p}{c^2} \nabla \phi \cdot \frac{\partial f}{\partial \vec{p}} = \text{collision terms}$$

where dot products are ordinary three vector dot products, $p = W$ =neutrino energy, and c (i.e. speed of light), has been reintroduced. This looks like the nonrelativistic Boltzmann equation:

$$c\hat{p} \cdot \nabla f + \vec{F} \cdot \frac{\partial f}{\partial \vec{p}} = \text{collisions}$$

with $\vec{F} = \frac{2p}{c^2} (-\nabla \phi)$ as the force on a neutrino due to gravity.

Finally the terms in the left hand side of (32) involving β^r come from the fact that $\vec{E}$ and $\underline{\mu}$ are defined in the static observer frame and $-\frac{\beta^r}{\alpha}$ is a boost velocity from the coordinate

observer frame to the Eulerian frame. In the limit $\alpha = A = 1$, and $-\beta^r$ is the velocity of the Lorentz boost to the frame where the energy and angle are measured, the result of Mihalas (1978) is found. The neutrino stress energy tensor t_{ab} will now be constructed:

$$t^{ab} = \int p^a p^b f dP$$

Lindquist (1966) finds the following expression for the invariant 3 momentum volume:

$$dP = \sqrt{-\text{Det } g_{ab}} \frac{dp^r dp^\theta dp^\phi}{-p_t}$$

By doing a change of variables, dP can be expressed as:

$$dP = E dE d\mu d\xi = \frac{W dW}{A^2} d\mu d\xi$$

Now defining new variables:

$$\begin{aligned}\bar{\epsilon} &= \int f E^3 dE d\mu d\xi = \text{neutrino energy density} \\ \bar{G} &= \int f E^3 \mu dE d\mu d\xi = \text{neutrino energy flux} \\ \bar{P} &= \int f E^3 \mu^2 dE d\mu d\xi = \text{neutrino radial pressure}\end{aligned}\quad (34)$$

the following is found for t_a^b :

$$t_a^b = \begin{pmatrix} -\bar{\epsilon} + \frac{\beta^r A}{\alpha} \bar{G} & \beta^r (\bar{\epsilon} + \bar{P}) - \frac{A}{\alpha} \bar{G} \left((\beta^r)^2 + \frac{g^2}{\lambda^2} \right) & 0 & 0 \\ \bar{G} \frac{A}{\alpha} & \bar{P} - \frac{\beta^r A}{\alpha} \bar{G} & 0 & 0 \\ 0 & 0 & \frac{\bar{\epsilon} - \bar{P}}{2} & 0 \\ 0 & 0 & 0 & \frac{\bar{\epsilon} - \bar{P}}{2} \end{pmatrix}$$

where a denoted the row index and b the column index. Note that $t_a^a = 0$, since it describes a massless particle.

From t^{ab} , the neutrino contributions to the source terms in the gravitational equations (24), (26), (27), (28) can be found from:

$$\rho_\nu = t^{ab} n_a n_b$$

$$j_r = t_r^c n_c$$

For the energy equation (14) and, Euler's equation (16), $t_{a,b}^b$ will be needed. This could be formed using:

$$t_{a;b}^b = \frac{1}{\sqrt{-\text{Det } g_{ab}}} \left(\sqrt{-\text{Det } g_{ab}} t_{ab}^b \right)_{,b} - \Gamma_{ab}^c t_c^b$$

However, an alternate way is to use the transport equation:

$$t_{;b}^a = q^a$$

where:

$$q^a = \int dP p^a \kappa p (I - f)$$

This result follows by integrating (21) over dP after multiplying both sides of the equation by p^a (see Lindquist(1966)).

For the spherical problem, the components q_r and q_t are:

$$q_t = - \int E^3 dE d\mu d\xi \left(1 - \frac{\beta^r A}{\alpha} \mu \right) k p (I - f) \quad (35a)$$

$$q_r = \frac{A}{\alpha} \int E^3 dE d\mu d\xi k p (I - f) \quad (35b)$$

where $k = \frac{\sigma}{m_p}$ is the opacity measured in the fluid frame.

For the lepton conservation equation, $N_{;a}^a$ is needed. This can also be found from the transport equation (21). Multiplying both sides by dP and integrating results in:

$$N_{;a}^a = \int dP \kappa p (I - f) = \int E^2 dE d\mu d\xi k p (I - f)$$

Summary

Below is a summary of the complete set of evolution equations in the isotropic gauge. For simplicity, only one type of neutrino is assumed.

metric

$$ds^2 = - \left(\alpha^2 - \beta_r \beta^r \right) dt^2 + 2\beta_r dr dt + A^2 dr^2 + A^2 r^2 d\Omega^2$$

gravity

$$\begin{aligned}
 K_r^r &= \frac{1}{r^3 A^3} \int_0^r r^3 A^3 \left(S_r - \dot{r} + \frac{\beta_r}{\alpha} \rho_\nu \right) \\
 \beta^r &= -\frac{3}{2} r \int_r^\infty \frac{\alpha K_r^r}{r} dr \\
 \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial A^{\frac{1}{2}}}{\partial r} \right) &= -\frac{A^{\frac{1}{2}}}{4} \left((\rho + e_m + P_m) u^2 - P_m + \rho_\nu + \frac{3}{4} (K_r^r)^2 \right) \\
 \frac{1}{A^3 r^2} \frac{\partial}{\partial r} \left(r^2 A \frac{\partial \alpha}{\partial r} \right) &= \alpha \left((\rho + e_m) (u^2 - \frac{1}{2}) + P_m (u^2 + \frac{1}{2}) + \frac{3}{2} (K_r^r)^2 + \rho_\nu \right)
 \end{aligned}$$

neutrino Boltzmann equation

$$\begin{aligned}
 \frac{\partial \bar{H}}{\partial t} + \frac{\partial}{\partial r} \left(\bar{H} \left(\frac{\alpha \mu}{A} - \beta^r \right) \right) + \frac{\partial}{\partial W} \left(\bar{H} W \left(\frac{(1 - \mu^2) \beta^r}{r} - \mu \left(\frac{\partial}{\partial r} \left(\frac{\alpha}{A} \right) - \mu \frac{\partial \beta^r}{\partial r} \right) \right) \right) \\
 + \frac{\partial}{\partial \mu} \left(\bar{H} \left(\left(\frac{\alpha}{A} - \mu \beta^r \right) \frac{1 - \mu^2}{r} - (1 - \mu^2) \left(\frac{\partial}{\partial r} \left(\frac{\alpha}{A} \right) - \mu \frac{\partial \beta^r}{\partial r} \right) \right) \right) \\
 = \alpha r^2 W^2 \frac{A}{W} (\text{collisions terms})
 \end{aligned}$$

baryon conservation

$$\frac{\partial D}{\partial t} + D \frac{\partial \ln \Upsilon^{\frac{1}{2}}}{\partial t} + \frac{1}{\Upsilon^{\frac{1}{2}}} \frac{\partial}{\partial r} (\Upsilon^{\frac{1}{2}} v^r D) = 0$$

energy conservation

$$\begin{aligned}
 \frac{\partial E}{\partial t} + E \frac{\partial \ln \Upsilon^{\frac{1}{2}}}{\partial t} + \frac{1}{\Upsilon^{\frac{1}{2}}} \frac{\partial}{\partial r} (E v^r \Upsilon^{\frac{1}{2}}) + \frac{P_m}{\Upsilon^{\frac{1}{2}}} \left(\frac{\partial (u \Upsilon^{\frac{1}{2}})}{\partial t} + \frac{\partial}{\partial r} (u v^r \Upsilon^{\frac{1}{2}}) \right) \\
 = u(q_t + v^r q_r)
 \end{aligned}$$

Euler's equation

$$\begin{aligned}
 \frac{\partial S_r}{\partial t} + S_r \frac{\partial \ln \Upsilon^{\frac{1}{2}}}{\partial t} + \frac{1}{\Upsilon^{\frac{1}{2}}} \frac{\partial}{\partial r} (S_r v^r \Upsilon^{\frac{1}{2}}) + \alpha \frac{\partial P_m}{\partial r} \\
 + \alpha S_t \frac{\partial \alpha}{\partial r} - S_r \frac{\partial \beta^r}{\partial r} + \frac{1}{2} \frac{S_r^2}{S_t} \frac{\partial}{\partial r} \left(\frac{1}{A^2} \right) = -\alpha q_r
 \end{aligned}$$

lepton number conservation

$$\frac{D}{m_B} \left(\frac{\partial \bar{Z}}{\partial t} + v^r \frac{\partial \bar{Z}}{\partial r} \right) = - \int E^2 dE d\mu d\xi k_\rho (I - f)$$

charge conservation

$$\frac{\partial \bar{Z}}{\partial t} + v^r \frac{\partial \bar{Z}}{\partial r} = \frac{\partial Z}{\partial t} + v^r \frac{\partial Z}{\partial r}$$

from $u_a u^a = -1$, $S_a S^a = -(\rho + e_m + P_m)^2 u^2$, $n_a S^a = -(\rho + e_m + P_m) u^2$ and $u = -n^a u_a$, expressions for S^t , u , and v^r can be found:

$$S^t = \frac{1}{\alpha} \left((D + E + P_m U)^2 + \frac{S_r^2}{A^2} \right)^{\frac{1}{2}}$$

$$u = \frac{\alpha S^t}{D + E + P_m u}$$

$$v^r = \frac{u^r}{u^t} = \frac{S_r}{A^2 S^t} - \beta^r$$

A second choice of gauge is now presented.

Schwarzschild Gauge

In this section, a much briefer discussion of the appropriate equations will be given. The complete set has already been written by Wilson(1971).

To find the metric in this gauge, set $B = 1$ and $\beta^r = 0$ in (22). Changing to the metric signature used by Wilson (i.e. (+, -, -, -)) and to notation more in line with his notation, the metric becomes:

$$ds^2 = a^2 dt^2 - b^2 dm^2 - R^2 d\Omega^2 \quad (36)$$

The condition determining a is not maximal slicing, but that $u^t = \frac{1}{a}$, $u^r = 0$. The coordinate system is thus Lagrangian.

The total stress energy tensor (note $T_{ab} = (\rho + e_m + P_m)u_a u_b - P_m g_{ab}$) is:

$$T_c^d + t_c^d = \begin{pmatrix} -P_{tot} & 0 & 0 & -\frac{1}{2}\tilde{G} \\ 0 & -P_{tot} - H & 0 & 0 \\ 0 & 0 & -P_{tot} - H & 0 \\ \frac{1}{2}\tilde{G} & 0 & 0 & \rho + e_{tot} \end{pmatrix}$$

where c denotes the row index, d the column index in the above matrix representation, and:

$$H = \frac{1}{2}(\tilde{e} - 3\tilde{P})$$

where $\tilde{e}, \tilde{G}, \tilde{P}$ are neutrino energy, flux, and pressure, and $P_{tot} = \tilde{P} + P_m, e_{tot} = \tilde{e} + e_m$.

The equation for baryon number conservation, $(\rho u^a)_{;a} = 0$ results in:

$$\frac{\partial}{\partial t}(bR^2\rho) = 0$$

This is integrated to give:

$$b = \frac{1}{4\pi R^2\rho} \quad (37)$$

where the integration constant is chosen to be $\frac{1}{4\pi}$. With this choice, m is the total rest mass enclosed by a sphere with radius m , since:

$$m = \int \rho dV = \int_0^m \rho(4\pi R^2 b) dm = \int_0^m dm$$

The equation for a is determined from the conservation of stress energy:

$$(T_m^c + t_m^c)_{;c}$$

This equation becomes (after some simplification):

$$a = \exp \int_R^\infty \left(\left(\frac{\partial P_{tot}}{\partial m} + \frac{2H}{R} \frac{\partial R}{\partial m} + \frac{\partial}{\partial t} \left(\frac{\tilde{G}b}{a} \right) \right) \frac{dm}{(\rho + e_{tot} + P_{tot})} \right) \quad (38)$$

The energy conservation equation, $u^c(T_c^d + t_c^d)_{;d} = 0$ becomes:

$$\frac{1}{a} \frac{\partial(e_m/\rho)}{\partial t} + \frac{P_m}{a} \frac{\partial}{\partial t} \left(\frac{1}{\rho} \right) = -\frac{u^c}{\rho} t_{c;d}^d \quad (39)$$

The Hamiltonian constraint (a second order equation in m and t) becomes two first order equations:

$$\frac{1}{b} \frac{\partial R}{\partial m} = \left(1 + U^2 - \frac{2M}{R}\right)^{\frac{1}{2}} \equiv \Gamma \quad (40)$$

$$\frac{\partial M}{\partial R} = 4\pi \left(\rho + e_{tot} + \frac{\tilde{G}U}{\Gamma}\right) R^2 \quad (41)$$

where:

$$U = K_{\theta\theta} = \frac{1}{a} \frac{\partial R}{\partial t} \quad (42)$$

and the equation for $\frac{\partial M}{\partial R}$ can be integrated to give:

$$M = 4\pi \int_0^R \left(\rho + e_{tot} + \frac{\tilde{G}U}{\Gamma}\right) R^2 dR \quad (43)$$

M is seen to be a mass energy (including gravitational effects), and U is a component of the extrinsic curvature tensor. It can be shown (see May and White(1967)) that U is also the radial component of the fluid 4 velocity as measured by an Eulerian observer. In the limit gravity is neglected, $\Gamma = \frac{1}{\sqrt{1-U^2}}$. Thus Γ is a generalization of a special relativistic gamma (i.e. $U = \Gamma v$).

In this gauge, some of Einstein's evolution equations are used, instead of only using the constraint equations, as in the isotropic gauge. The resulting choice is motivated by ease of interpretation of the equations. (Again, in the case where symmetries are present, Einstein's equations are overdetermined, in that a smaller subset implies the whole set.)

The evolution equation for $K_{\theta\theta}$ results in:

$$\begin{aligned} \frac{1}{a} \frac{\partial K_{\theta\theta}}{\partial t} = \frac{1}{a} \frac{\partial U}{\partial t} = & - \frac{4\pi R^2 \Gamma}{\left(1 + \frac{e_{tot} + P_{tot}}{\rho}\right)} \left(\frac{\partial P_{tot}}{\partial m} + \frac{2H}{R} \frac{\partial R}{\partial m} + \frac{\partial}{\partial t} \left(\frac{\tilde{G}b}{a} \right) \right) \\ & - \frac{GM}{R^2} \left(1 + \frac{4\pi P_{tot} R^3}{M} \right) \end{aligned} \quad (44)$$

In the Newtonian limit, this equation looks like Force = (mass)(acceleration), even though it is one of Einstein's equations.

A final equation is needed, and $G^{mm} = T^{mm} + t^{mm}$ is chosen. This equation becomes:

$$\frac{1}{a} \frac{\partial}{\partial t} \ln(\rho R^2) + \frac{\partial U}{\partial m} \frac{\partial R}{\partial m} = 0 \quad (45)$$

In the Newtonian limit, this is the mass conservation equation, again giving a hydrodynamic equation from one of Einstein's equations.

For the neutrino Boltzmann equation, introduce Q such that $E = \frac{Q}{a}$ where $E = \epsilon_f = +\vec{p} \cdot \vec{u}$. Q is the energy an observer at infinity would measure for a neutrino emitted with energy E , which had escaped with no further interactions. (Note that with this metric signature $u^\alpha u_\alpha = +1$, and $\epsilon_f = +\vec{p} \cdot \vec{u}$.)

Also introduce a new dependent variable, F , such that:

$$\tilde{z} = \int f E^3 dE d\mu d\xi = \int F dE d\mu = \int F \frac{dQ}{a} d\mu$$

so:

$$F = 2\pi f E^3$$

The equation for neutrino transport becomes:

$$\begin{aligned} \frac{1}{a} \frac{\partial F}{\partial t} = & -\frac{\mu \Gamma}{a R^2} \frac{\partial}{\partial R} (a R^2 F) - \Gamma \left(\frac{1}{R} - \frac{\partial}{\partial R} \ln a \right) \left(\frac{\partial}{\partial \mu} (F(1 - \mu^2)) \right) \\ & + \frac{F}{a \rho} \frac{\partial \rho}{\partial t} + R \frac{\partial}{\partial R} \left(\frac{U}{R} \right) \left(\frac{\partial}{\partial \mu} (\mu(1 - \mu^2) F) + \mu^2 Q \frac{\partial F}{\partial Q} \right) \\ & + \frac{Q}{a} \frac{\partial F}{\partial Q} \left(\frac{\partial}{\partial t} \ln \frac{R}{a} \right) + k \rho (B - F) \end{aligned} \quad (46)$$

If simple absorption and emission processes are involved, then:

$$B = \frac{2\pi \epsilon_f^3}{\exp\left(\frac{\epsilon_f - \mu}{k_B T}\right) + 1}$$

where the emitting particles are assumed in thermodynamic equilibrium.

The equation for energy conservation becomes:

$$\frac{1}{a} \frac{\partial (e_m / \rho)}{\partial t} + \frac{P_m}{a} \frac{\partial}{\partial t} \left(\frac{1}{\rho} \right) = - \int k (B - F) d\mu \frac{dQ}{a} \quad (47)$$

The equation for lepton number conservation is found using (17):

$$\frac{1}{a} \frac{\partial \bar{Z}}{\partial t} = - \int (B - F) \frac{dQ}{Q} d\mu \quad (48)$$

And the equation for charge conservation (19) becomes:

$$\frac{1}{a} \frac{\partial \bar{Z}}{\partial t} = \frac{1}{a} \frac{\partial Z}{\partial t} \quad (49)$$

Summary

A summary of the supernova equations in the Schwarzschild gauge follows:

metric

$$ds^2 = a^2 dt^2 - b^2 dm^2 - R^2 d\Omega^2$$

gravity

$$M = 4\pi \int_0^R \left(\rho + e_{tot} + \frac{\tilde{G}U}{\Gamma} \right) R^2 dR$$

$$\frac{1}{b} \frac{\partial R}{\partial m} = \left(1 + U^2 - \frac{2M}{R} \right)^{\frac{1}{2}} \equiv \Gamma$$

$$\frac{1}{a} \frac{\partial K_{\theta\theta}}{\partial t} = \frac{1}{a} \frac{\partial U}{\partial t} = - \frac{4\pi R^2 \Gamma}{\left(1 + \frac{u_{tot} + P_{tot}}{\rho} \right)} \left(\frac{\partial P_{tot}}{\partial m} + \frac{2H}{R} \frac{\partial R}{\partial m} + \frac{1}{c^2} \frac{\partial}{\partial t} \left(\frac{\tilde{G}b}{a} \right) \right)$$

$$- \frac{GM}{R^2} \left(1 + \frac{4\pi P_{tot} R^3}{M} \right)$$

$$U = \frac{1}{a} \frac{\partial R}{\partial t}$$

$$\frac{1}{a} \frac{\partial}{\partial t} \ln(\rho R^2) + \frac{\partial U}{\partial m} \frac{\partial R}{\partial m} = 0$$

neutrino Boltzmann equation

$$\begin{aligned} \frac{1}{a} \frac{\partial F}{\partial t} = & - \frac{\mu \Gamma}{a R^2} \frac{\partial}{\partial R} (a R^2 F) - \Gamma \left(\frac{1}{R} - \frac{\partial}{\partial R} \ln a \right) \left(\frac{\partial}{\partial \mu} (F(1 - \mu^2)) \right) \\ & + \frac{F}{a \rho} \frac{\partial \rho}{\partial t} + R \frac{\partial}{\partial R} \left(\frac{U}{R} \right) \left(\frac{\partial}{\partial \mu} (\mu(1 - \mu^2) F) + \mu^2 Q \frac{\partial F}{\partial Q} \right) \\ & + \frac{Q}{a} \frac{\partial F}{\partial Q} \left(\frac{\partial}{\partial t} \ln \frac{R}{a} \right) + k \rho (B - F) \end{aligned}$$

baryon conservation

$$b = \frac{1}{4\pi R^2 \rho}$$

stress energy conservation

$$a = \exp \int_R^\infty \left(\left(\frac{\partial P_{tot}}{\partial m} + \frac{2H}{R} \frac{\partial R}{\partial m} + \frac{\partial}{\partial t} \left(\frac{\tilde{G}b}{a} \right) \right) \frac{dm}{(\rho + \rho_{tot} + P_{tot})} \right)$$

$$\frac{1}{a} \frac{\partial(e_m/\rho)}{\partial t} + \frac{P_m}{a} \frac{\partial}{\partial t} \left(\frac{1}{\rho} \right) = - \int k(B-F) d\omega \frac{dQ}{a}$$

lepton number conservation

$$\frac{1}{a} \frac{\rho}{m_B} \frac{\partial \overline{Z}}{\partial t} = - \int (B-F) \frac{dQ}{Q} d\mu$$

charge conservation

$$\frac{1}{a} \frac{\partial \overline{Z}}{\partial t} = \frac{1}{a} \frac{\partial Z}{\partial t}$$

Notation used in Appendix A (partial listing)

G_{ab} =Einstein tensor

T_{ab} =matter stress energy tensor

t_{ab} =neutrino stress energy tensor

g_{ab} =4 metric

n^a =unit vector to spacelike slice

K_{ij} =extrinsic curvature

Λ_{ij} =3 metric

${}^4R^a_{bcd}$ =4 dimensional Riemann tensor

D_i =covariant derivative compatible with Λ_{ij}

3R =Ricci scalar comparable with Λ_{ij}

$\rho_H = n^a n^b (T_{ab} + t_{ab})$

$J_i = -\Lambda_i^a n^b (T_{ab} + t_{ab})$

β^a =shift vector

α =lapse function

$t^a = \alpha n^a + \beta^a$

${}^3R_{ij}$ =Ricci tensor compatible with Λ_{ij}

$S_{ij} = \Lambda_i^a \Lambda_j^b (T_{ab} + t_{ab})$

$S = S_i^i$

n_B = baryon density measured in rest frame

u^a =matter 4 velocity

ρ =matter rest mass density

D =density measured by Eulerian observer

v^i =3 velocity of fluid

u =generalization of special relativistic gamma

$T^{\frac{1}{2}} = \sqrt{\text{Det}(\Lambda_{ij})}$

f =neutrino distribution function

dP =3 momentum volume element

p^a =neutrino 4 momentum

N^a = neutrino flux 4 vector

$\rho_\nu = t_{ab} n^a n^b$

$j_r = t_r^c n_c$

ϵ_f =neutrino energy measured by fluid observer

$$g^a = t^a_b$$

k =non-relativistic opacity

α =lapse function

Γ =generalization of special relativistic gamma

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Appendix A Figure Captions

Fig. 1 Illustration of the definition of the vector field t^a .

Fig. 2 Figure used in finding the fluid 3 velocity v_E^i , or the three velocity as measured by the Eulerian observer.

Fig. 3 Definition of the various angles used in neutrino transport.

FIGURE 1

FIGURE 2

FIGURE 3

Appendix B

Some miscellaneous topics will be covered in this appendix. They include the following:

1. Opacities for Neutrino Interactions
2. Thermonuclear Reaction Rates
3. Equation of State

OPACITIES FOR NEUTRINO INTERACTIONS

The following interactions are considered in the supernova code:

Emission and Absorption Processes

1. neutrino emission-absorption by interaction with:

(a) free baryons:

$$\text{i. } e^- + p \rightleftharpoons n + \nu_e$$

$$\text{ii. } e^+ + n \rightleftharpoons p + \bar{\nu}_e$$

(b) heavy nuclei: $e^- + N(Z, A) \rightleftharpoons \nu_e + N^*(Z-1, A)$

2. pair production: $e^- + e^+ \rightleftharpoons \nu + \bar{\nu}$

3. plasma process: $\gamma_{pl} \rightleftharpoons \nu + \bar{\nu}$

Scattering Processes

1. scattering on free baryons or heavy nuclei:

$$\text{(a) } \nu + b \rightleftharpoons \nu + b$$

$$\text{(b) } \nu + N(Z, A) \rightleftharpoons \nu + N(Z, A)$$

2. scattering on electrons and positrons:

$$\text{(a) } \nu + e^- \rightleftharpoons \nu + e^-$$

$$\text{(b) } \nu + e^+ \rightleftharpoons \nu + e^+$$

where the symbol ν can mean any of the three neutrinos (i.e. ν_e, ν_μ, ν_τ) or their antiparticles. The symbol b indicates either a neutron or a proton.

Note that the possible capture interaction, $e^+ + N(Z, A) \rightleftharpoons \bar{\nu}_e + N^*(Z+1, A)$, is not included in the supernova code since copious e^+ production needs temperatures greater than .511 Mev, and for conditions where this is true, either electrons are very degenerate (i.e. in the central regions), inhibiting positron production, or heavy nuclei have been dissociated by the outward moving shock wave, so few heavy nuclei are present for the reaction. A brief discussion of the cross-sections used for the neutrino interactions above will follow.

EMISSION AND ABSORPTION PROCESSES

Emission Absorption on Free Baryons

Using equation (2.31) from Chapter II, the cross-section for $\nu_e + n \rightleftharpoons p + e^-$ can be found. This cross-section is also calculated in Tubbs and Schramm (1975). The assumption is made that the nucleon is effectively at rest before and after the collision. The result for $\sigma(\nu_e n)$ becomes

$$\sigma(\nu_e n) = \sigma_0 \left(\frac{E_\nu + Q}{m_e c^2} \right)^2 I(E_\nu + Q) \left(\frac{1 + 3\alpha^2}{4} \right) \left(1 - \left(\frac{m_e c^2}{E_\nu + Q} \right)^2 \right)^{\frac{1}{2}}$$

where $I(z) = 1 - 1/(\exp(\frac{z - \mu_e}{k_B T}) + 1)$ is an inhibition factor that blocks the reaction if the final electron state is filled. Q is the difference between the initial and final nucleon energies. E_ν is the locally measured neutrino energy, T is the temperature, k_B is Boltzmann's constant, μ_e is the electron chemical potential, and $\alpha \approx 1.2$ (a strong interaction effect). $\sigma_0 = \frac{G_F^2 m_e^2 \hbar^2}{4\pi c^2} = 1.7 \times 10^{-44} \text{ cm}^2$, and is a typical neutrino interaction cross-section. G_F is the weak interaction constant.

Table I contains the baryon absorption-emission cross-sections as used in the supernova code. In Table I, $f_\nu = \frac{1}{4\pi} (\hbar c)^3 \int \frac{F}{E^2} d\mu$ is a generalized Fermi-Dirac distribution function for the neutrinos. The factor $(1 - f_\nu)$ is a blocking factor that inhibits the reaction if a final neutrino state is already occupied. F is similar to the intensity of brightness function for photon transport. The reactions listed in Table I are charged current weak interactions, and at the energies involved in a supernova, are basically unchanged from the original Fermi theory of weak interactions.

Emission and Absorption on Heavy Nuclei

In Chapter II, some of the considerations involved in the calculation of capture rates on heavy nuclei are discussed. During a supernova collapse, nuclei far from the line of beta stability, found at low temperatures and densities, are formed, due to the large value of the electron chemical potential. Measured values of the needed nuclear matrix elements do not exist for such elements, nor have theoretical calculations using the nuclear shell model been done, since work has heretofore concentrated on nuclei produced during stellar evolution (see Fuller (1982), Fuller et al. (1982ab)).

The approximation used in the supernova code assumes some fraction of the neutrons and protons in the typical heavy element interact with the neutrinos using the same cross-section as given in Table I. The equations used in the supernova code to model electron capture on heavy elements will be found in Bowers and Wilson (1982a), equations (4.2) to (4.12)

Pair and Plasma Processes

For the pair processes, the following Feynmann diagrams would be used to calculate the interaction cross-sections:

The Weinberg-Salam theory of electroweak interaction predicts the neutral current interaction, attributed to the exchange of a Z^0 boson. This allows the production of mu and tau neutrinos and antineutrinos by e^-e^+ annihilation.

Dicus (1972) calculated the energy production rates for the pair processes, assuming an equilibrium distribution of electrons and positrons. A fit to the Dicus (1972) results is used, as well as a fit to the results of Chiu (1968) for the energy production in neutrinos due to the plasma process (i.e. neutrino production by a "massive" photon that acquires an effective mass by interaction with the surrounding sea of matter particles), updated with the Weinberg-Salam coupling constants, to find the neutrino spectrum produced by these processes.

Since matter is in equilibrium, the spectrum of the neutrinos produced by both processes is assumed thermal, with zero neutrino chemical potential. Only in this case will the pair production be appreciable, since electron degeneracy inhibits the pair process. This assumption is not necessarily true for the neutrino spectrum produced by the plasma process. However, the same thermal spectrum (i.e. $f_\nu = 1/(\exp(\frac{E_\nu}{k_B T}) + 1)$) is assumed for both processes. For the manner in which the pair and plasma processes, as well as the inverse reactions, is incorporated in the neutrino transport, see Bowers and Wilson (1982a), equations (8.56) to (8.65).

The present supernova code now includes the pair and plasma processes in the ν_e and $\bar{\nu}_e$ neutrino transport. The code described by Bowers and Wilson (1982a) neglected these processes for the electron neutrino and electron antineutrino, including them only for $\nu_\mu, \bar{\nu}_\mu, \nu_\tau$ and $\bar{\nu}_\tau$. The manner of incorporation of the new source and sink terms for ν_e and $\bar{\nu}_e$ transport is in the same spirit as that used for the mu and tau neutrinos and their antiparticles described in Bowers and Wilson (1982a).

SCATTERING PROCESSES

Scattering in Free Baryons or Heavy Nuclei

The cross-sections used for scattering on free baryons and heavy nuclei are taken from Tubbs and Schramm (1975). They are listed in Table II.

Scattering on Electrons and Positrons

These scattering cross-sections are taken from a fit to numerical calculations done by Tubbs and Schramm (1975). This fit has been redone since Bowers and Wilson (1982a). Figures 1a and 1b of Chapter IV show new graphs of the cross-sections from the results

of Tubbs and Schramm (1975), using $\sin^2\theta_W=.21$. The previous fit assumed $\sin^2\theta_W=.35$. The new fit is chosen to reproduce figures 1a and 1b for the neutrino energies of 8 Mev and 16 Mev studied by Tubbs and Schramm (1975), and is thus extrapolated to other energies.

These curves can be used to find all possible electron-positron neutrino-antineutrino scattering cross-sections by multiplying the fit to the curves shown in figures 1a and 1b in Chapter IV by an appropriate coupling constant. See Bowers and Wilson (1982a) equation (7.3) for these coupling constants. The electron-positron scattering cross-sections are used in calculating the mean free path of the various neutrinos types. The mean free path is used in the construction of the flux limited diffusion constants (see Chapters III and IV).

Compton scattering of neutrinos of all types by electrons or positrons is also included in the supernova code using a Fokker-Planck approximation in the density region from $10^{10} \frac{g}{cc}$ to $10^{12} \frac{g}{cc}$ where the neutrinos interact significantly with the matter, but not strong enough to thermalize completely at the matter temperature. The results from a Monte Carlo calculation simulating the thermalization of neutrinos by Compton scattering with electrons, done by Tubbs (1978), is used to calibrate the Fokker-Planck equation. The Fokker-Planck equation can be derived from the Boltzmann equation only in the limit of small energy change between the initial and final neutrino energies. This is not necessarily true for neutrino Compton scattering, and so the calibration is necessary to justify the approximation. For densities less than $10^{10} \frac{g}{cc}$, a single scattering approximation is used to simulate the Compton scattering. For densities above $10^{12} \frac{g}{cc}$, the neutrinos are confined long enough for complete thermalization to occur, mainly by interaction with free nucleons, therefore the Compton scattering with electrons and positrons is neglected. See Bowers and Wilson (1982a) for a discussion of the flux limited diffusion to neutrino transport and the method the above cross-sections are used in the supernova code.

The present supernova code includes all electrons and positrons as scattering centers. The previous version of the supernova code only used the number of matter electrons (i.e. the number of electrons minus positrons) as possible scattering centers, and this could badly underestimate the electron-positron scattering effects.

THERMONUCLEAR BURN

Just prior to collapse, in massive stars (i.e. mass greater than $11M_{\odot}$), the core is primarily iron group nuclei with an onion skin-like structure of progressively lighter elements that dominate the composition as one moves out from the center.

For these stars, a simple nuclear reaction network is used to follow abundances of ^{12}C , ^{16}O , ^{28}Si and Fe. Once a zone has completely burned to Fe, that zone is considered to be in nuclear statistical equilibrium, with a Saha equation determining composition. It is found from numerical simulations that the energy released by thermonuclear reactions is not the determining factor in producing a supernova explosion, at least not in the launching of the shock and the matter heating behind the shock due to neutrino interactions. However, see Wilson et al. (1985) for a discussion of the further evolution of the shock wave as it proceeds out from $r \approx 10^9$ cm to $r \approx 10^{13}$ cm (i.e. the surface of the red giant), and the importance of energy released by thermonuclear reactions.

The energy rates per gram for $^{12}\text{C} \Rightarrow ^{16}\text{O}$, $^{16}\text{O} \Rightarrow ^{28}\text{Si}$, and $^{28}\text{Si} \Rightarrow \text{Fe}$ are taken from Clayton (1968). These equations and the numerical method used to implement them in the supernova code can be found in Bowers and Wilson (1982a).

For stars near the mass range $8-11M_{\odot}$, the core contains elements lighter than iron which undergo a collapse triggered by electron capture removing degenerate electron pressure support (see Wilson et al. (1975)). A more complete nuclear interaction network was provided by Stanford Woosley in order to study such progenitor stars, as nuclear energy generation may be important to the explosion.

EQUATION OF STATE

The matter particles that contribute to the pressure, energy density and rest mass density in the supernova code include: electrons, positrons, free neutrons and protons, helium nuclei, and one typical heavy nucleus. However, if a zone has elements present lighter than iron, nuclear statistical equilibrium does not hold, and the elements in this zone contribute in the ideal gas approximation. The massless particles that contribute to the pressure and energy density include the photons and the three neutrinos $\nu_e, \nu_{\mu}, \nu_{\tau}$ and their antiparticles. Muons, of rest mass 106 Mev, will only be present in the very central regions of the star, where the electron chemical potential can reach values of 250 Mev. A possible production reaction is $e^{-} \Rightarrow \mu^{-} + \nu_e + \bar{\nu}_{\mu}$. However, this is in a region where the pressure and energy density are dominated by nucleons near nuclear density, and the exact equation of state is uncertain. Muons are therefore neglected in the supernova code. The tau particle, of mass 1784 Mev, will not be produced in a supernova (see Particle Properties Data (1984) for the masses quoted).

All the matter particles and the photons are assumed in local thermodynamic equilib-

rium at one temperature. The neutrinos are treated using a flux limited diffusion approximation to neutrino transport, as previously mentioned.

The equation of state used can be found in Bowers and Wilson (1982ab), modified to agree with the Bethe, Brown, Cooperstein and Wilson (1993) equation of state near nuclear density.

The electron-positron equation of state listed in Bowers and Wilson (1982a) is not used in the present supernova code. The present code uses a tabulated electron-positron equation of state. Tables of pressure, energy density, and entropy were constructed by numerical integration, as functions of temperature and net electron number density (i.e. $n_e - n_{e^+}$). The accuracy of the numerical integration was checked by comparison with exact asymptotic results, plus comparison with other tables (i.e. Cox and Giuli (1968)). Also, a few points were carefully checked using error estimation formulae appropriate to the type of numerical integration used (i.e. simple midpoint integration), recognising the fact that error formulae apply only to well behaved functions, with no infinite derivatives. One must remove any integration domain that contains such points, and estimate the error for these domains separately. It was found that the tabulated values are within .1 % of the exact values. Logarithmic interpolation is used to find the physical quantities between the grid points. In order to check on the accuracy of the interpolation, points between the tabulated values were calculated in two ways, by numerical integration and by logarithmic interpolation. It was found that the interpolation gives the physical quantities within 1 per cent of their exact value. Most of the time the interpolation is good to within .5 %. Figure 1 shows the ratio of electron pressure to electron energy density as a function of temperature, parameterized by the density. It can be seen that at the lowest density, the ratio starts at $\frac{2}{3}$ as one expects. It then drops as the temperature rises. In the temperature range where pairs are forming, the ratio drops (as low as .13 in figure 1), since the pairs take energy to produce mass that does not show up in pressure. When the temperature reaches values where the mass in energy units is negligible compared to the temperature in energy units, the ratio becomes $\frac{1}{3}$. As the density increases, the value of $\frac{P_e}{E_e}$ drops below $\frac{2}{3}$ near $T=0$ Mev due to degeneracy effects. At the highest density shown the value of $\frac{P_e}{E_e}$ is $\frac{1}{3}$ for all the range of temperature shown. At this density, the electrons are effectively massless due to their extreme degeneracy.

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Table I. EMISSION ABSORPTION

REACTION	CROSS-SECTION
$n + \nu_e \Rightarrow p + e^-$	$\left(\frac{1+3g^2}{4}\right) \sigma_0 \left(\frac{E_\nu + Q}{m_e c^2}\right)^2 \left(1 - \frac{1}{\exp\left(\frac{E_\nu + Q - m_e c^2}{k_B T}\right) + 1}\right) \left(1 - \left(\frac{m_e c^2}{E_\nu + Q}\right)^2\right)^{\frac{1}{2}}$
$p + e^- \Rightarrow n + \nu_e$	$\left(\frac{1+3g^2}{8}\right) \sigma_0 \left(\frac{E_\nu}{m_e c^2}\right)^2 (1 - f_\nu)$
$p + \bar{\nu}_e \Rightarrow n + e^+$	$\left(\frac{1+3g^2}{4}\right) \sigma_0 \left(\frac{E_\nu - Q}{m_e c^2}\right)^2 \left(1 - \frac{1}{\exp\left(\frac{E_\nu - Q - m_e c^2}{k_B T}\right) + 1}\right) \left(1 - \left(\frac{m_e c^2}{E_\nu - Q}\right)^2\right)^{\frac{1}{2}}$
$n + e^+ \Rightarrow p + \bar{\nu}_e$	$\left(\frac{1+3g^2}{8}\right) \sigma_0 \left(\frac{E_\nu}{m_e c^2}\right)^2 (1 - f_{\bar{\nu}})$

* this cross-section is zero if $E_\nu \leq Q + m_e c^2$

Table II. SCATTERING

REACTION	CROSS-SECTION
$n + \nu \Rightarrow n + \nu$	$\sigma_0(1 - C_A)^2 \left(\frac{E_\nu}{m_e c^2}\right)^2$
$p + \nu \Rightarrow p + \nu$	$\frac{1}{4}\sigma_0((C_V - 1)^2 + 3\alpha^2(C_A - 1)^2) \left(\frac{E_\nu}{m_e c^2}\right)^2$
$N(Z, A) + \nu \Rightarrow N(Z, A) + \nu$	$\frac{1}{4}\sigma_0\left(a_0 A + \frac{1}{2}a_1(2Z - A)\right)^2 \left(\frac{E_\nu}{m_e c^2}\right)^2$

In the above table, the cross-sections have been integrated over scattering angle, $a_0 = \frac{1}{2}(C_A - C_V)$, $a_1 = 2 - C_V - C_A$, $C_V = \frac{1}{2} + 2\sin^2\theta_W$, $C_A = \frac{1}{2}$, and θ_W is the Weinberg angle. Reactions in table II are mediated by the Z^0 boson and are neutral current interactions.

Appendix B Figure Caption

Fig. 1 Ratio of electron-positron pressure to the energy density in electrons and positrons versus temperature for various densities.

FIGURE 1