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A POTENTIAL SOLVER FOR
SLOSHING-ION THERMAL BARRIERS

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Abstract

The quasineutrality equations at points (a) and (b) in a sloshing-ion thermal barrier are derived and an algorithm for their solution is given. The solution technique is sufficiently reliable and efficient to be used in a fluid code where it must be invoked at each time step. Circumstances under which the equations admit multiple solutions are noted and discussed.

1. INTRODUCTION

This report describes a numerical algorithm used to solve for the ambipolar potential in the barrier region of a tandem mirror machine with sloshing-ion thermal barriers. This algorithm was needed for use in a one-dimensional fluid model of such a machine,⁽¹⁾ for which this type of calculation must be done at each time step. Since any given run of such a fluid code might involve thousands of time steps, it was necessary that the algorithm be both reliable and efficient.

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The basic problem involves solving two nonlinear equations in two unknowns. The equations are the quasineutrality conditions at points (a) (the sloshing-ion injection point) and (b) (the magnetic field minimum) in the barrier region. The unknowns are the barrier potential depression, ϕ_b , and the potential difference between points (a) and (b), $\phi_a - \phi_b + \phi_c$ (see Fig. 1).

Attempts to use Newton-Raphson type root-finding algorithms on the problem were plagued with difficulties. Depending on the machine parameters, such methods might work 90 percent of the time, or even more, but in a fluid code calling the routine thousands of times in any one run, 90 percent of the time is not good enough. Furthermore, the nature of the failures and strange behavior that did occur suggested to us that there was potentially significant physics to be understood in tracking down the problems; it did not seem to be a solely numerical dilemma.

Specifically, one type of strange behavior that was observed was discontinuous jumps in the solution for the potentials as a result of continuous changes in the input parameters (known densities and temperatures at certain points in the machine). These jumps suggested to us that the equations might have multiple solutions, and that the searching technique might be jumping from one solution to another. This hunch was later verified, and, indeed, we found that under certain conditions the equations may admit as many as five distinct solutions.

For these reasons, we deemed it necessary to thoroughly analyze the two equations and the behavior of their solutions, in order to take the guesswork out of the solution technique. That analysis is presented in Chapter III of this report. Chapter II presents the equations used, and shows how they were derived.

II. THE QUASINEUTRALITY EQUATIONS AT POINTS (A) AND (B)

1. Introduction

Referring to Fig. 1 once again, we see that there are several different species of trapped particles in a sloshing-ion barrier region. Centered at point (b) there is a population of mirror-trapped "hot" electrons that are ECRH-heated, and a population of potential-trapped ions. The hot electron density at point (a) is less than that at point (b), and, indeed, it falls to zero at the outboard barrier mirror. The trapped ion density, on the other hand, falls to zero at point (a), since that is where the potential maximum occurs.

Centered at point (a), there is a population of potential-trapped "warm" electrons that are also ECRH-heated; this warm electron density falls to zero at point (b) since that is where a potential minimum occurs. Furthermore, a doubly-peaked axial density profile of "sloshing" ions has its outboard peak at point (a) as well. The minimum of the axial density profile of the sloshing-ions occurs at point (b), and the other maximum occurs at an "antinode" on the inboard side of point (b).

Finally, "passing" ions and electrons from the central cell pass through the barrier in the course of their transit through the machine. The passing ion density falls to zero at point (a), since that is where the absolute potential maximum occurs. The passing electron density, on the other hand, remains finite at both points (a) and (b), and does not fall to zero until the outboard barrier mirror, since that is where the absolute potential minimum occurs.

It is not our purpose to explain why one would want to set up this configuration, nor is it necessary here to delve into the dynamics of each of

these above-mentioned populations of particles. The reader who is not familiar with these points should consult one of the references.⁽²⁾ For the remainder of this report, we shall accept the above configuration as given, and examine relationships between potentials, densities, and temperatures.

From the above considerations, it is clear that the quasineutrality conditions at points (b) and (a) are of the form:

$$n_{ic}(b) + n_{it}(b) + n_{ih}(b) = n_{ec}(b) + n_{eh}(b) \quad (1)$$

and

$$n_{ih}(a) = n_{ec}(a) + n_{ew}(a) + n_{eh}(a) \quad (2)$$

respectively. Here $n_{ic}(b)$ is the passing ion density at point (b), $n_{it}(b)$ is the trapped ion density at point (b), $n_{ih}(b)$ is the sloshing-ion density at point (b), $n_{ec}(b)$ is the passing electron density at point (b), $n_{eh}(b)$ is the hot electron density at point (b), $n_{ih}(a)$ is the sloshing-ion density at point (a), $n_{ec}(a)$ is the passing electron density at point (a), $n_{ew}(a)$ is the warm electron density at point (a), and $n_{eh}(a)$ is the hot electron density at point (a).

We now need expressions for each of these densities in terms of known quantities, and the two unknown potentials. Our fluid code advances nine equations (among others) in time; these nine equations are listed in Fig. 2. Referring to this figure, we see that all the quantities in the right-hand column are furnished for us at each time step by the balance equations. Here, $n_{ic}(c)$ is the ion (and electron) density in the central cell, T_{ic} is the central cell ion temperature, T_{ec} is the central cell electron temperature, $\bar{E}_{eh}$ is the average hot electron energy at point (b), and T_{ew} is the warm electron temperature at point (a). We thus see that four of the nine densities that appear in Eqs. 1 and 2 are known to us from the start.

For the remainder, we must derive mapping equations by making an assumption about the distribution function for that population elsewhere in the machine, and then by considering the fraction of particles that are free to traverse the axial span between the point at which the distribution function is known and the point at which the density is desired. This has been done by several authors in slightly different ways. We shall compare our model with that of Pearlstein and Nevins.⁽³⁾

2. The Passing Ion Density at Point (b)

First we consider the calculation of $n_{ic}(b)$. The phase space for thermal ions is illustrated in Fig. 3. Those ions with values of energy, ϵ , and magnetic moment, μ , that fall in the shaded region are passing ions. Those below this region are trapped in the barrier, while those above this region are lost out the end of the machine. If we assume a Maxwellian distribution function in the central cell, then the passing ion density at point (b) is given by the following expression:

$$n_{ic}(b) = \iint_{R_3} \left[n_{ic}(c) \left(\frac{m_i}{2\pi T_{ic}} \right)^{3/2} \exp \left(- \frac{\epsilon - \phi_e}{T_{ic}} \right) \right] \frac{2\sqrt{2} \pi B_{ob} d\mu d\epsilon}{m_i^{3/2} \sqrt{\epsilon - \mu B_{ob} - (\phi_e - \phi_b)}} \quad (3)$$

where R_3 is the shaded region in Fig. 3, and B_{ob} is the magnetic field at point (b) (see Fig. 1). The first factor of the integrand (in square brackets) is the distribution function; the rest is the Jacobian for ϵ , μ coordinates.

The above integral can be evaluated in terms of complimentary error

functions, and this is done in a reference.⁽³⁾ For our purposes, however, several approximations are appropriate. First, we neglect the loss boundary cutoff of the distribution function; that is, we extend the upper boundary of the shaded region in Fig. 3 to infinity. This is justified as long as $\phi_c/T_{ic} \gg 1$. The integral is considerably simplified, and the result is:

$$n_{ic}(b) = n_{ic}(c) \left\{ \exp\left(\frac{\phi_b}{T_{ic}}\right) \operatorname{erfc}\left(\sqrt{\frac{\phi_b}{T_{ic}}}\right) - \sqrt{\frac{B_{mb} - B_{ob}}{B_{mb}}} \exp\left[\left(\frac{B_{mb}}{B_{mb} - B_{ob}}\right) \frac{\phi_b}{T_{ic}}\right] \operatorname{erfc}\left[\sqrt{\left(\frac{B_{mb}}{B_{mb} - B_{ob}}\right) \frac{\phi_b}{T_{ic}}}\right] \right\} \quad (4)$$

It can be verified that Pearlstein and Nevins' expression⁽³⁾ reduces to this one if we let their ϕ_c go to infinity.

We can further simplify Eq. 4 by applying the following approximation for the complimentary error function:

$$\operatorname{erfc}(z) \cong \frac{e^{-z^2}}{\sqrt{\pi z^2 + 1}} \quad (5)$$

This approximation goes to the correct limit as $z \rightarrow 0$, and is asymptotically correct as $z \rightarrow \infty$. In between, it is accurate to roughly 20%. Applying it to Eq. 4, we get:

$$n_{ic}(b) \cong n_{ic}(c) \left\{ \sqrt{\frac{T_{ic}}{T_{ic} + \pi\phi_b}} - \left(\frac{B_{mb} - B_{ob}}{B_{mb}}\right) \sqrt{\frac{T_{ic}}{\left(\frac{B_{mb} - B_{ob}}{B_{mb}}\right) T_{ic} + \pi\phi_b}} \right\} \quad (6)$$

Finally, if $\phi_b/T_{ic} \gg 1$ and $B_{mb}/B_{ob} \gg 1$, we can expand the above in powers of:

$$\sqrt{\frac{T_{ic}}{T_{ic} + \pi\phi_b}} \quad \text{and} \quad \frac{B_{ob}}{B_{mb}}$$

and their products, and retain only the lowest term. The result is:

$$n_{ic}(b) = n_{ic}(c) \frac{B_{ob}}{B_{mb}} \sqrt{\frac{T_{ic}}{T_{ic} + \pi\phi_b}} \quad (7)$$

This is the expression we use for the passing ion density at point (b).

3. The Sloshing-Ion Density at Point (b)

The phase space for sloshing-ions is shown in Fig. 4. The sloshing-ions occupy the shaded area. To make the problem analytically tractable, we assume a Maxwellian distribution for these ions at point (a). In actuality, the sloshing-ion distribution function at point (a) is heavily weighted in the perpendicular direction, and this is an aspect of this model that we shall return to shortly. For now, however, we get $n_{ih}(b)$ from the integral:

$$n_{ih}(b) = \iiint_{R_4} \left[n_{ih}(a) \left(\frac{m_i}{2\pi\bar{E}_{ih\perp}} \right)^{3/2} \exp \left(- \frac{\epsilon - \phi_e - \phi_c}{\bar{E}_{ih\perp}} \right) \right] \frac{2\sqrt{2} \pi B_{ob} d\mu d\epsilon}{m_i^{3/2} \sqrt{\epsilon - \mu B_{ob} - (\phi_e - \phi_b)}} \quad (8)$$

where R_4 is the shaded region in Fig. 4, and $\bar{E}_{ih\perp}$ is the average perpendicular energy of the sloshing-ions at point (a).

The integral is straightforward, and the result is:

$$\begin{aligned}
 n_{ih}(b) = n_{ih}(a) & \left\{ \sqrt{\frac{B_{ma} - B_{ob}}{B_{ma}}} \exp \left[\left(\frac{\phi_e + \phi_c}{\bar{E}_{ih\perp}} \right) - \left(\frac{B_{ma}}{B_{ma} - B_{ob}} \right) \left(\frac{\phi_e - \phi_b}{\bar{E}_{ih\perp}} \right) \right] \right. \\
 & \cdot \operatorname{erfc} \left[\sqrt{\frac{B_{ma}}{B_{ma} - B_{ob}}} \cdot \sqrt{\left(\frac{B_{ma} - B_{ob}}{B_{ma} - B_{oa}} \right) \left(\frac{\phi_e + \phi_c}{\bar{E}_{ih\perp}} \right) - \left(\frac{\phi_e - \phi_b}{\bar{E}_{ih\perp}} \right)} \right] \\
 & - \sqrt{\frac{B_{oa} - B_{ob}}{B_{oa}}} \exp \left[\left(\frac{B_{oa}}{B_{oa} - B_{ob}} \right) \left(\frac{\phi_e + \phi_c}{\bar{E}_{ih\perp}} \right) - \left(\frac{B_{oa}}{B_{oa} - B_{ob}} \right) \left(\frac{\phi_e - \phi_b}{\bar{E}_{ih\perp}} \right) \right] \\
 & \left. \cdot \operatorname{erfc} \left[\sqrt{\frac{B_{oa}}{B_{oa} - B_{ob}}} \cdot \sqrt{\left(\frac{B_{ma} - B_{ob}}{B_{ma} - B_{oa}} \right) \left(\frac{\phi_e + \phi_c}{\bar{E}_{ih\perp}} \right) - \left(\frac{\phi_e - \phi_b}{\bar{E}_{ih\perp}} \right)} \right] \right\} \quad (9)
 \end{aligned}$$

We can now use Eq. 5 to approximate the complimentary error function, and we can let B_{ma} go to infinity. Under these approximations, the result simplifies to:

$$n_{ih}(b) \cong n_{ih}(a) \left(\frac{B_{ob}}{B_{oa}} \right) \sqrt{\frac{\bar{E}_{ih\perp}}{\bar{E}_{ih\perp} + \pi \delta \phi_a}} \quad (10)$$

Reference 3 does not attempt to compute the axial variation of sloshing-ion density in the above way; rather, it simply assumes a certain axial density profile. So there is nothing there to which the above result may be compared. Reference 2, however, gives the equation:

$$n_{ih}(b) \cong n_{ih}(a) \left(\frac{B_{ob}}{B_{oa}} \right)^\zeta \sqrt{\frac{E_{inj,a}}{E_{inj,a} + \pi \delta \phi_a}} \quad (11)$$

where $E_{inj,a}$ is the sloshing-ion injection energy at point (a), which, it is reasonable to assume, is equal to $\bar{E}_{ih\perp}$. The power ζ , to which the field

ratio (B_{ob}/B_{oa}) is raised in Eq. 11, is a parameter introduced to model the effect of charge exchange pumping on the axial density profile;² increasing charge exchange pumping relative to scatter loss increases ζ .

Also noteworthy is the lack of the factor π multiplying $\delta\phi_a$ in Eq. 11. This is an attempt by the author of Ref. 2 to model the non-Maxwellian character of the sloshing-ion distribution function. Fokker-Planck results indicate that leaving the factor of π in Eq. 10 predicts much too large a density depression for the sloshing-ions.⁽⁴⁾ Probably the most reasonable approach would be to assume neither π nor 1 as the multiplier, but rather to leave it as a thus far undetermined parameter which we shall call s_i . Like ζ , it may depend upon the charge exchange rate. Both ζ and s_i can be determined from future Fokker-Planck studies.

So the equation we finally use for $n_{ih}(b)$ is:

$$n_{ih}(b) \cong n_{ih}(a) \left(\frac{B_{ob}}{B_{oa}} \right)^\zeta \sqrt{\frac{\bar{E}_{ih\perp}}{\bar{E}_{ih\perp} + s_i \delta\phi_a}} \quad (12)$$

4. The Cold Electron Density at Point (b)

The phase space for thermal electrons is shown in Fig. 5. There are three shaded regions labelled I, II, and III. The passing electrons occupy region I. We assume that regions II and III will also fill up with cold, thermal (though not passing) electrons, and we decide to include these in $n_{ec}(b)$. Further, we assume a Maxwellian distribution for thermal electrons in the central cell. We thus get $n_{ec}(b)$ from the integral:

$$n_{ec}(b) = \iint_{R_5} \left[n_{ec}(c) \left(\frac{m_e}{2\pi T_{ec}} \right)^{3/2} \exp \left(- \frac{\epsilon + \phi_e}{T_{ec}} \right) \right] \frac{2\sqrt{2} \pi B_{ob} d\mu d\epsilon}{m_e^{3/2} \sqrt{\epsilon - \mu B_{ob} + (\phi_e - \phi_b)}} \quad (13)$$

where R_5 includes all three shaded regions. If we now neglect the loss boundary cutoff of the distribution function by extending the upper boundary of the shaded region in Fig. 5 to infinity, then the integral becomes easy and the result is:

$$n_{ec}(b) = n_{ec}(c) \exp \left(- \frac{\phi_b}{T_{ec}} \right) \quad (14)$$

Pearlstein and Nevins⁽³⁾ give the exact expression with the cutoff. Their expression reduces to ours as their ϕ_e goes to infinity.

5. The Cold Electron Density at Point (a)

To calculate $n_{ec}(a)$, we refer to Fig. 5 once again. This time we must integrate only over regions I and II, since electrons in region III cannot exist at point (a). The distribution function is the same that we used to calculate $n_{ec}(b)$, but now the Jacobian is different. Thus:

$$n_{ec}(a) = \iint_{R_{5I} + R_{5II}} \left[n_{ec}(c) \left(\frac{m_e}{2\pi T_{ec}} \right)^{3/2} \exp \left(- \frac{\epsilon + \phi_e}{T_{ec}} \right) \right] \frac{2\sqrt{2} \pi B_{oa} d\mu d\epsilon}{m_e^{3/2} \sqrt{\epsilon - \mu B_{oa} + (\phi_e + \phi_c)}} \quad (15)$$

Once again, we neglect the loss region. The result is:

$$n_{ec}(a) = n_{ec}(c) \left\{ \exp\left(\frac{\phi_c}{T_{ec}}\right) \operatorname{erfc}\left(\sqrt{\frac{\phi_b + \phi_c}{T_{ec}}}\right) + \frac{2}{\sqrt{\pi}} \sqrt{\frac{B_{oa} - B_{ob}}{B_{ob}}} \exp\left(-\frac{\phi_b}{T_{ec}}\right) D\left(\sqrt{\frac{B_{ob}}{B_{oa} - B_{ob}}} \left(\frac{\phi_b + \phi_c}{T_{ec}}\right)\right) \right\} \quad (16)$$

where $D(x)$ is Dawson's integral, defined by:

$$D(x) \equiv e^{-x^2} \int_0^x e^{s^2} ds \quad (17)$$

We now use the approximation, Eq. 5, for the complimentary error function, and the following approximation for $D(x)$:

$$D(x) \cong \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{\pi x^2 + 1}} \quad (18)$$

This approximation is asymptotically correct as x goes to infinity, but goes to the wrong limit as x goes to zero (the real $D(x)$ vanishes). It is thus only applicable for large x . The result is:

$$n_{ec}(a) \cong n_{ec}(c) \exp\left(-\frac{\phi_b}{T_{ec}}\right) \left\{ \sqrt{\frac{T_{ec}}{T_{ec} + \pi(\phi_b + \phi_c)}} + \left(\frac{B_{oa}}{B_{ob}} - 1\right) \sqrt{\frac{T_{ec}}{\left(\frac{B_{oa}}{B_{ob}} - 1\right) T_{ec} + \pi(\phi_b + \phi_c)}} \right\} \quad (19)$$

Then, for $(B_{oa}/B_{ob} - 2)^2 (T_{ec}/\phi_a) \ll 1$, this is well approximated by:

$$n_{ec}(a) \cong n_{ec}(c) \exp\left(-\frac{\phi_b}{T_{ec}}\right) \frac{B_{oa}}{B_{ob}} \sqrt{\frac{T_{ec}}{T_{ec} + \pi(\phi_b + \phi_c)}} \quad (20)$$

which is the expression we shall use.

6. The Hot Electron Density at Point (a)

Finally, consider the phase space region for hot electrons in Fig. 5. In calculating $n_{eh}(a)$, we integrate only over region II, since only thermal passing electrons occupy region I, and electrons in region III cannot exist at point (a). We assume for now a Maxwellian distribution function for the hot electrons at point (b), although we shall return to this point later. The actual distribution function is heavily weighted in the perpendicular direction. Under these assumptions, we may write:

$$n_{eh}(a) = \iint_{R_{ehII}} \left[n_{eh}(b) \left(\frac{m_e}{2\pi \bar{E}_{ehI}} \right)^{3/2} \exp \left(- \frac{\epsilon + (\phi_e - \phi_b)}{\bar{E}_{ehI}} \right) \right] \frac{2\pi \bar{E}_{oh} d\mu d\epsilon}{m_e^{3/2} \sqrt{\epsilon - \mu B_{oh} + (\phi_e + \phi_c)}} \quad (21)$$

The result of this calculation is:

$$n_{eh}(a) = n_{eh}(b) \left\{ \sqrt{\frac{B_{mb} - B_{oa}}{B_{mb}}} \exp \left[\left(\frac{B_{mb}}{B_{mb} - B_{oa}} \right) \frac{\phi_c}{\bar{E}_{ehI}} + \frac{\phi_b}{\bar{E}_{ehI}} \right] \right. \\ \cdot \operatorname{erfc} \left[\sqrt{\left(\frac{B_{mb}}{B_{mb} - B_{oa}} \right) \frac{\phi_c}{\bar{E}_{ehI}} + \frac{\phi_b}{\bar{E}_{ehI}}} \right] + \frac{2}{\sqrt{\pi}} \sqrt{\left(\frac{\phi_b + \phi_c}{\bar{E}_{ehI}} \right) - \frac{B_{oa}}{B_{mb}} \left(\frac{\phi_b}{\bar{E}_{ehI}} \right)} \\ \left. - \frac{2}{\sqrt{\pi}} \sqrt{\left(\frac{\phi_b + \phi_c}{\bar{E}_{ehI}} \right)} + \frac{2}{\sqrt{\pi}} \sqrt{\frac{B_{oa} - B_{ob}}{B_{ob}}} D \left(\sqrt{\left(\frac{B_{ob}}{B_{oa} - B_{ob}} \right) \left(\frac{\phi_b + \phi_c}{\bar{E}_{ehI}} \right)} \right) \right\} \quad (22)$$

If we now take $B_{mb} \gg B_{oa}$, and use our usual approximations for the complimentary error function and Dawson's integral, we finally get:

$$n_{eh}(a) \cong n_{eh}(b) \frac{B_{oa}}{B_{ob}} \sqrt{\frac{\bar{E}_{eh\perp}}{\bar{E}_{eh\perp} + \pi(\phi_b + \phi_c)}} \quad (23)$$

Though there is no analogous result with which to compare this in Ref. 3, Ref. 2 quotes the result:

$$n_{eh}(a) \cong n_{eh}(b) \left(\frac{E_{mb} - B_{oa}}{E_{mb} - B_{ob}} \right) \sqrt{\frac{\bar{E}_{eh\parallel}}{\bar{E}_{eh\parallel} + (\phi_b + \phi_c)}} \quad (24)$$

Now Ref. 2 takes $E_{eh\perp} \cong 0.2 E_{eh}$ and $E_{eh\parallel} \cong 0.8 E_{eh}$. So the square root factor in Eq. 24 is essentially the same as that in Eq. 23 with the π replaced by 4. As we did with the sloshing-ions, we shall henceforth use neither π nor 4, but rather regard this number as presently undetermined and simply call it s_e . This parameter essentially models the non-Maxwellian character of the hot electron distribution function.

The magnetic field dependence in Eq. 24 is also clearly different from that in Eq. 23. There is a good reason for this. For $\bar{E}_{eh} \gg (\phi_b + \phi_c)$, the hot electron distribution function should approach that of a purely mirror-confined species. Several analytic models exist for such a distribution function;^(5,6) a reasonable model in this case is:

$$f(v, \theta) = \frac{n_0}{\pi} g(v) \left(\frac{B_{mb}}{B_{mb} - B_{ob}} - \frac{1}{\cos \theta} \right) \left\{ \frac{\frac{B_{mb}}{B_{ob}} \sin^2 \theta - 1}{\left(\frac{B_{mb}}{B_{ob}} \sin^2 \theta \right)} + \frac{1}{2} \arccos \left[\frac{\left(2 - \frac{B_{mb}}{B_{ob}} \sin^2 \theta \right)}{\frac{B_{mb}}{B_{ob}} \sin^2 \theta} \right] \right\} \quad (25)$$

where θ is the pitch angle:

$$\frac{\pi}{2} \geq \theta \geq \arcsin\left(\sqrt{\frac{B_{ob}}{B_{mb}}}\right)$$

v is the speed, and $g(v)$ is an arbitrary (appropriately normalized) function of v . The angular part of this distribution ($f/n_0 g$) is plotted in Fig. 6 for three different mirror ratios.

If we map the hot electron density from point (b) to point (a) using Eq. 25 for the distribution function, we get:

$$n_{eh}(a) = n_{eh}(b) \left(\frac{B_{mb} - B_{oa}}{B_{mb} - B_{ob}} \right) \quad (26)$$

Thus, this is the model distribution on which the field dependence in Eq. 24 is based. It is therefore reasonable to replace the magnetic field dependence in Eq. 23 by that in Eq. 26. In this way, the equation goes to the correct limit as $(\phi_b + \phi_c) \rightarrow 0$. We hope that eventually a more careful analysis can be done here using a model distribution function, based on Fokker-Planck results, that is good for finite $(\phi_b + \phi_c)$. For now we take:

$$n_{eh}(a) \cong n_{eh}(b) \left(\frac{B_{mb} - B_{oa}}{B_{mb} - B_{ob}} \right) \sqrt{\frac{E_{eh\perp}}{E_{eh\perp} + s_e(\phi_b + \phi_c)}} \quad (27)$$

as the hot electron mapping equation.

III. SOLVING THE QUASINEUTRALITY EQUATIONS

1. The Quasineutrality Equation at Point (b)

If we now take Eq. 1 and substitute Eqs. 7, 12 and 14 into it, we get the first of our two relationships between the two unknown potentials:

$$n_{ic}(c) \frac{B_{ob}}{B_{mb}} \sqrt{\frac{T_{ic}}{T_{ic} + \pi \phi_b}} + n_{it}(b) + n_{ih}(a) \left(\frac{B_{ob}}{B_{oa}} \right)^\zeta \sqrt{\frac{\bar{E}_{ih\perp}}{\bar{E}_{ih\perp} + s_i \delta \phi_a}} = n_{ec}(c) \exp\left(-\frac{\phi_b}{T_{ec}}\right) + n_{eh}(b) \quad (28)$$

Let us now define the dimensionless unknowns:

$$x \equiv \phi_b / T_{ec} \quad \text{and} \quad y \equiv \delta \phi_a / \bar{E}_{ih\perp}$$

and the parameter:

$$\Omega \equiv \sqrt{\frac{\bar{E}_{ih\perp}}{\bar{E}_{ih\perp} + s_i \delta \phi_a}}$$

So:

$$y(\Omega) = \frac{1}{s_i} \left(\frac{1}{\Omega^2} - 1 \right) \quad (29)$$

and:

$$\Omega(x) = D_1 + D_2 \left[e^{-x} - A(1 + Bx)^{-1/2} \right] \quad (30)$$

where:

$$D_1 \equiv \left[\frac{n_{eh}(b) - n_{it}(b)}{n_{ih}(a)} \right] \left(\frac{B_{oa}}{B_{ob}} \right)^\zeta \quad (31)$$

$$D_2 \equiv \left[\frac{n_c(c)}{n_{ih}(a)} \right] \left(\frac{B_{oa}}{B_{ob}} \right)^{\zeta} \quad (32)$$

$$A \equiv \left(\frac{B_{ob}}{B_{mb}} \right) \quad (33)$$

$$B \equiv \pi \left(\frac{T_{ec}}{T_{ic}} \right) \quad (34)$$

and $n_c(c) \equiv n_{ic}(c) = n_{ec}(c)$. Equation 30 gives Ω as a function of x , and Eq. 29 then gives y as a function of Ω .

First we examine the properties of $\Omega(x)$. Note:

$$\lim_{x \rightarrow 0} \Omega(x) = D_1 + D_2(1 - A) \equiv \Omega_0 \quad (35)$$

$$\lim_{x \rightarrow \infty} \Omega(x) = D_1 \equiv \Omega_{\infty} < \Omega_0 \quad (36)$$

Now, let x_{∞} be a value of x for which $\Omega(x) = \Omega_{\infty}$. Obviously, one solution is $x_{\infty} = \infty$. We now show that there is always another solution to:

$$D_1 + D_2 \left[e^{-x_{\infty}} - A (1 + Bx_{\infty})^{-1/2} \right] = \Omega_{\infty} = D_1$$

$$\text{or:} \quad 1 + \frac{B}{2} (2x_{\infty}) = A^2 e^{2x_{\infty}} \quad (37)$$

This equation is of the form:

$$1 + \beta z = \alpha e^z \quad (38)$$

where $z = 2x_{\infty}$, $\beta = B/2$, and $\alpha = A^2$. It is shown in Appendix I that Eq. 38 has one and only one finite solution if $0 < \alpha < 1$ and $\beta > 0$, which is the case here. Furthermore, Appendix I shows how to find that solution numerically.

We can learn more about $\Omega(x)$ by finding its extrema. Setting $d\Omega(x)/dx = 0$, we get:

$$-e^{-\tilde{x}} + \frac{A}{2} (1 + B\tilde{x})^{-3/2} B = 0$$

$$\text{or: } 1 + \frac{3B}{2} \left(\frac{2}{3} \tilde{x} \right) = \left(\frac{AB}{2} \right)^{2/3} e^{2\tilde{x}/3} \quad (39)$$

where $\tilde{x}$ is a value of x for which $\Omega(x)$ is an extremum. Equation 39 is also in the form of Eq. 38; this time with $z = 2\tilde{x}/3$, $\beta = 3B/2$, and $\alpha = (AB/2)^{2/3}$.

Now we have shown that there is always one and only one finite solution for x_∞ such that $\Omega(x_\infty) = \Omega(\infty) = 0_1$. Since $\Omega(x)$ is continuous and differentiable on $[x_\infty, \infty)$, we know (by Rolle's theorem) that $\tilde{x}$ will have at least one solution on $[x_\infty, \infty)$. Our analysis in Appendix I then tells us that if

$$0 < \alpha < 1 \quad (40a)$$

there will be only one solution, whereas if:

$$1 \leq \alpha < f(\beta) \equiv \beta \exp \left(\frac{1 - \beta}{\beta} \right) \quad (40b)$$

there will be two solutions, and finally if:

$$\alpha > f(\beta) \quad (40c)$$

there will be no solutions at all. For Eq. 39, this means that if:

$$0 < A < \frac{2}{B} \quad (41a)$$

there will be only one solution, whereas if:

$$\frac{2}{B} \leq A < F(B) \equiv \frac{3}{e^{3/2}} \sqrt{\frac{3}{2}} B^{1/2} \exp \left(\frac{1}{B} \right) \quad (41b)$$

there will be two solutions, and finally if:

$$A > F(B) \quad (41c)$$

there will be no solutions. Now:

$$\lim_{B \rightarrow 0} F(B) = \infty$$

$$\text{and } \lim_{B \rightarrow \infty} F(B) = \infty$$

and $F(B)$ has an absolute minimum on $(0, \infty)$ when $B = 2$. At that minimum, $F(2) = 3\sqrt{3}/e = 1.9115576\dots$. So, since $A < 1$, Eq. 41c will never be satisfied and there will always be at least one solution for $\tilde{x}$ (we expected this anyway by our Rolle's theorem argument).

Equations 41 then tell us that if:

$$0 < A < \frac{2}{B} \quad (42a)$$

there will be one solution for $\tilde{x}$ on $[x_\infty, \infty)$, and if

$$\frac{2}{B} < A < 1 \quad (42b)$$

there will be two solutions for $\tilde{x}$, at least one of which will be on $[x_\infty, \infty)$. We can now prove that in the latter case, in which there are two solutions for $\tilde{x}$, only one of the solutions lies on $[x_\infty, \infty)$, and the other must lie on $[0, x_\infty]$. We do this by considering $d\Omega(x)/dx$ at $x = 0$, thus:

$$\Omega'(0) = D_2 \left(\frac{AB}{2} - 1 \right) \quad (43)$$

So, if there are two solutions for $\tilde{x}$, then $\Omega'(0) > 0$. Recall, however, that $\Omega_\infty < \Omega_0$. If Ω is increasing at $x = 0$, this means that Ω must have a local maximum on $[0, x_\infty]$.

Everything we have learned about $\Omega(x)$ is illustrated in Figs. 7a and 7b. If Eq. 42a is satisfied--that is, if:

$$0 < \left(\frac{B_{ob}}{B_{mb}} \right) < \frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right) \quad (44a)$$

then there will be one solution for $\tilde{x}$ on $[x_\infty, \infty)$, and it will be a minimum; this is pictured in Fig. 7a. If, on the other hand, Eq. 42b is satisfied--that is, if:

$$\frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right) < \left(\frac{B_{ob}}{B_{mb}} \right) < 1 \quad (44b)$$

then $\tilde{x}$ will have one solution, $\tilde{x}_1$, on $[x_\infty, \infty)$ which will be a minimum, and another solution, $\tilde{x}_2$, on $[0, x_\infty]$ which will be a maximum; this is pictured in Fig. 7b. Note that Ω_{max} and Ω_{min} are defined as the absolute maximum and minimum of $\Omega(x)$ for $x > 0$, respectively.

Now that we know what $\Omega(x)$ looks like, we turn our attention to the behavior of y as a function of x . We can determine this by considering the application of Eq. 29 to each of the two curves pictured in Fig. 7. We first note, however, that $y(\Omega)$ has a singularity when $\Omega = 0$. Now we know that $\Omega_0 > \Omega_\infty > 0$, but Ω_{min} may very possibly dip below zero. So $y(x)$ may have singularities. Furthermore, Eq. 29 predicts $y < 0$ for $|\Omega| > 1$. We shall, however, refuse to consider negative values of y , and simply set $y = 0$ whenever this happens.

Keeping all this in mind, Figs. 8a-w picture all possible qualitative behavior of y as a function of x . Figures 8a-j correspond to $\Omega(x)$ having the form pictured in Fig. 7a, and Figs. 8k-w correspond to $\Omega(x)$ having the form pictured in Fig. 7b. These curves are taken to be the solution of the quasineutrality equation at point (b) for the two unknowns, x and y .

2. The Quasineutrality Equation at Point (a)

We now consider the quasineutrality equation at point (a). Fortunately, the behavior of the solution to this equation is far easier to analyze than that of the one at point (b). Take Eq. 2, and substitute Eqs. 20 and 27 into it:

$$\begin{aligned} n_{ih}(a) = n_{ec}(c) \exp \left(- \frac{\phi_b}{T_{ec}} \right) \frac{B_{oa}}{B_{ob}} \sqrt{\frac{T_{ec}}{T_{ec} + \pi(\phi_b + \phi_c)}} \\ + n_{ew}(a) + n_{eh}(b) \left(\frac{B_{mb} - B_{oa}}{B_{mb} - B_{ob}} \right) \sqrt{\frac{\bar{E}_{ehI}}{\bar{E}_{ehI} + s_e(\phi_b + \phi_c)}} \end{aligned} \quad (45)$$

This may be written:

$$F(x,y) \equiv e^{-x}(1 + Cy)^{-1/2} + D_3(1 + Dy)^{-1/2} = D_4 \quad (46)$$

where

$$D_3 \equiv \frac{n_{eh}(b)}{n_{ec}(c)} \left(\frac{B_{ob}}{B_{oa}} \right) \left(\frac{B_{mb} - B_{oa}}{B_{mb} - B_{ob}} \right) > 0, \quad (47)$$

$$D_4 \equiv \frac{B_{ob}}{B_{oa}} \left[\frac{n_{ih}(a) - n_{ew}(a)}{n_{ec}(c)} \right], \quad (48)$$

$$C \equiv \pi \left(\frac{\bar{E}_{ih}}{T_{ec}} \right), \quad (49)$$

$$D \equiv s_e \left(\frac{\bar{E}_{ih}}{\bar{E}_{ehI}} \right), \quad (50)$$

x and y are defined in part A of this section, and $F(x,y)$ is defined by the left-hand equality in Eq. 46.

We first note that $F(x,y)$ is a monotonically decreasing function of both x and y . This means that $F(x,y) = D_4$ gives y as a monotonically decreasing

function of x (or x as a monotonically decreasing function of y).

Furthermore, we may solve Eq. 46 for x :

$$x = \ln \left[\frac{(1 + Cy)^{-1/2}}{D_4 - D_3(1 + Dy)^{-1/2}} \right] \quad (51)$$

This shows that $x \rightarrow \infty$ as $y \rightarrow y_{\min}$, where $y_{\min}$ is defined:

$$y_{\min} \equiv \frac{1}{D} \left[\left(\frac{D_3}{D_4} \right)^2 - 1 \right] \quad (52)$$

and x is undefined for $y < y_{\min}$. Thus, for large x , y must be a concave-upward function of x (or x is a concave-upward function of y).

Now, from Eq. 51, it is straightforward to calculate:

$$\frac{dx}{dy} = -\frac{C}{2(1 + Cy)} - \frac{DD_3}{2} \cdot \frac{(1 + Dy)^{-3/2}}{[D_4 - D_3(1 + Dy)^{-1/2}]^2} \quad (53)$$

and:

$$\frac{d^2x}{dy^2} = \frac{C^2}{2(1 + Cy)^2} + \frac{DD_3}{2} \cdot \frac{\left\{ \frac{3}{2} D(1 + Dy)^{1/2} [D_4 - D_3(1 + Dy)^{-1/2}] + \frac{DD_3}{2} \right\}}{(1 + Dy)^3 [D_4 - D_3(1 + Dy)^{-1/2}]^2} \quad (54)$$

So, for $y > y_{\min}$, Eq. 53 tells us that $(dx/dy) < 0$ everywhere; we suspected this anyway. Also, for $y > y_{\min}$, Eq. 54 tells us that $(d^2x/dy^2) > 0$ everywhere. So the plot of y versus x is monotonically decreasing and concave-upward for all $y > y_{\min}$.

Next we consider the behavior of y as $x \rightarrow -\infty$. From Eq. 46, for $x \ll -1$, we get:

$$y \sim \frac{1}{C} \left(\frac{e^{-2x}}{D_4^2} - 1 \right) \quad (55)$$

So y blows up exponentially as $x \rightarrow -\infty$. Note that y does not asymptote for any finite value of x , so there is no minimum x (other than the physical requirement that x be greater than or equal to zero).

Figure 9 shows the qualitative behavior of y as a function of x for several different values of D_4 . Note that $y_{\min}$ may be either greater than or less than zero. These curves are taken to be the solution of the quasineutrality equation at point (a) for the two unknowns, x and y .

3. Determining the Intersection of the Two Solutions

The simultaneous solution to both quasineutrality equations has now been reduced to finding the points of intersection of curves like those in Fig. 9 with curves like those in Fig. 8. If the reader pictures monotone decreasing, concave-upward curves passing through each of the cases shown in Fig. 8, with different values of $y_{\min}$, then all the different possibilities become apparent. The cases shown in Figs. 8d and o will admit either one or zero solutions (for $x > 0$). Those in Figs. 8a-c, k-n will admit three, two, one or zero solutions. Those in Figs. 8e-j, p-w will admit five, four, three, two, one or zero solutions.

If there are multiple solutions, a way must be found to choose one from among them in a physically reasonable and consistent manner. To accomplish this and, at the same time, simplify the solution procedure, the following algorithm is used: for each of the curves in Fig. 8, the first simply-connected set of values of x for which $dy/dx > 0$ is found. Call this

set (x_1, x_2) . Then, the curves in Fig. 8 are modified so $y = y(x_1)$ for $x < x_1$, and $y = y(x_2)$ for $x > x_2$. The curve is not touched for $x \in (x_1, x_2)$. This has the effect of making all the curves in Fig. 8 nondecreasing functions of x . Finding the intersection between a nondecreasing function of x and a monotone decreasing function of x is then a relatively simple matter. It is accomplished in our subroutine using a regula falsi type algorithm.

We regard this solution technique as physically reasonable because if $\delta\phi_a$ is increased, the sloshing-ion density at point (b) decreases. If this happens, quasineutrality at (b) can be maintained by a concomitant decrease in passing electron density brought about by an increase in ϕ_b . We therefore expect the quasineutrality equation at point (b) to give y as an increasing function of x (or, at least, nondecreasing). We have thus insured that our solution technique will find the first solution obeying this condition.

We now consider what happens when the solution technique predicts no positive solutions for x and y . There are really only two ways that this can happen. The first is if $y_{\min} < 0$, and the curve of y versus x for the quasineutrality condition at point (a) has $y(x=0) < 0$. In this case, the code assumes that the solution technique is trying to make ϕ_b negative, so it sets ϕ_b to the smallest value physically possible--namely, zero. Physically, what is happening here is that there are too many sloshing-ions at point (a) for quasineutrality at (a) to be satisfied (because D_4 is too large). The subroutine's response to this problem--setting ϕ_b equal to zero-- has the effect of allowing both hot and passing electrons to pour into point (a) and, hence, to resolve the problem.

The other way that our solution technique will find no positive solutions for x and y is if $\Omega_{\min} > 0$ (so the solution to the quasineutrality condition at (b) has no asymptotes), and $y_{\min}$ is so large that the curve from Fig. 9 passes over that of Fig. 8 without intersecting it. In this case, the code assumes that the solution technique is trying to make ϕ_b go to infinity, so it sets ϕ_b to the largest value physically possible, namely $\delta\phi_a$. Physically, this happens when there are too few sloshing-ions at point (a) for quasineutrality at (a) to be satisfied (because D_4 is too small). The subroutine's response to this problem--setting ϕ_b equal to $\delta\phi_a$ and, hence, setting ϕ_c equal to zero--has the effect of destroying central cell confinement and thus dropping central cell density. This, in turn, increases D_4 and straightens out the problem by reducing the flow of passing electrons to point (a).

Finally, we point out that the solutions with $\phi_b > \delta\phi_a$ (and, hence, $\phi_c < 0$) are never allowed. If this happens, we set $\delta\phi_a = \phi_b$. This problem also tends to right itself by the mechanism described in the last paragraph.

IV. CONCLUSION

We have described a procedure for solving for the potentials ϕ_b and $\delta\phi_a$ in a sloshing-ion-type thermal barrier. The procedure involves finding the solutions to the quasineutrality equations at points A and B simultaneously, and choosing the physically meaningful solution from among them. The technique described is reliable and efficient and has been used successfully as a subroutine in a one-dimensional fluid code, called at each time step.

There are three main areas for future improvement to this work:

- 1) The mapping equations derived in part II could be improved by more correctly modeling the distribution functions of the trapped barrier species.
- 2) The mathematical approximations made in the mapping equations (such as those for the complimentary error function and Dawson's integral) could be removed, and the exact equations could be subjected to a similar analysis. This would be a difficult but worthwhile exercise because the exact equations might not yield the same multiplicity of solutions as the approximate ones.
- 3) A better criterion for which of the solutions to choose as the physical solution needs to be developed. The answer here might be that it depends on the past history of plasma densities and temperatures, but this is unclear.

It should be noted that all of the above three questions have been considered by various authors, but models with sufficiently reliable and efficient algorithms for their solution have yet to be developed.

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Appendix I

Solution to the equation $1 + \beta z = \alpha e^z$

The equation:

$$1 + \beta z = \alpha e^z \quad (A1)$$

is considered with $\alpha, \beta > 0$, and we are interested only in solutions with $z > 0$.

Clearly, if $\alpha < 1$, there is one and only one solution for z (called z_0). This can be seen from Fig. 10a. Let z_0 be the point at which $(1 + \beta z)$ and $\alpha \exp(z)$ have equal slopes. Then:

$$z_0 = \ln(\beta/\alpha) \quad (A2)$$

Note that for $\alpha < 1$, it must be that $z_0 < z_s$. The two curves are tangent when $z_0 = z_{0t}$:

$$1 + \beta z_{0t} = \alpha e^{z_{0t}} = \beta$$

or

$$\alpha = \beta \exp\left(\frac{1 - \beta}{\beta}\right) = f(\beta) \quad (A3)$$

Now:

$$\lim_{\beta \rightarrow 0} f(\beta) = \lim_{\beta \rightarrow \infty} f(\beta) = \infty \quad (A4)$$

and $f(\beta)$ has an absolute minimum when $\beta = 1$. Now $f(1) = 1$, so we can write:

$$f(\beta) \geq 1 \quad (A5)$$

So, we see that for $1 < \alpha < f(\beta)$, the equation has two solutions, and for $\alpha > f(\beta)$, the equation has no solutions. This is illustrated graphically in Figs. 10b and c.

An upper bound on both solutions can be found by approximating the exponential with a parabola (see dotted line in Fig. 10a). This gives:

$$z_{\max} = \left(\frac{\beta}{\alpha} - 1 \right) + \left[\left(\frac{\beta}{\alpha} - 1 \right)^2 - 2 \left(1 - \frac{1}{\alpha} \right) \right]^{1/2} \quad (\text{A6})$$

So, if there is one solution, it lies on $[0, z_{\max}]$. If there are two, the smaller lies on $[0, z_{\text{ot}}]$, and the larger lies on $[z_{\text{ot}}, z_{\max}]$. Knowing this, we can get the exact solution using the regula falsi algorithm.

Field and potential:

Ion density:

Electron density:

Figure 1

Equation		Relevant Quantity
1)	Central cell ion particle balance	$n_{ic}(c)$
2)	Central cell ion energy balance	T_{ic}
3)	Central cell electron energy balance	T_{ec}
4)	Trapped ion particle balance	$n_{it}(b)$
5)	Hot electron particle balance	$n_{eh}(b)$
6)	Hot electron energy balance	E_{eh}
7)	Sloshing ion particle balance	$n_{ih}(a)$
8)	Warm electron particle balance	$n_{ew}(a)$
9)	Warm electron energy balance	T_{ew}

Figure 2

72-103-0

Thermal ion phase space
Figure 3

72-104-0

Sloshing ions
Figure 4

72-105-0

Phase space for electrons
Figure 5

Angular part of model hot electron distribution
function for $E_{eh} \gg (\phi_b + \phi_c)$

$$(R \equiv B_{mb}/B_{ob})$$

Figure 6

Figure 7a

Figure 7b

$$0 < \left(\frac{B_{ob}}{B_{mb}} \right) < \frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right)$$

$$\Omega_{min} > 0$$

$$\Omega_{max} < 1$$

$$0 < \left(\frac{B_{ob}}{B_{mb}} \right) < \frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right)$$

$$\Omega_{min} > 0$$

$$\Omega_{\infty} < 1 < \Omega_{max}$$

$$0 < \left(\frac{B_{ob}}{B_{mb}} \right) < \frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right)$$

$$\Omega_{min} > 0$$

$$\Omega_{min} < 1 < \Omega_{\infty}$$

72-107-0

Figure 8a-c

$$0 < \left(\frac{B_{ob}}{B_{mb}} \right) < \frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right)$$

$$\Omega_{min} > 1$$

$$0 < \left(\frac{B_{ob}}{B_{mb}} \right) < \frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right)$$

$$-1 < \Omega_{min} < 0$$

$$\Omega_{max} < 1$$

$$0 < \left(\frac{B_{ob}}{B_{mb}} \right) < \frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right)$$

$$-1 < \Omega_{min} < 0$$

$$\Omega_{\infty} < 1 < \Omega_{max}$$

72-108-0

Figure 8d-f

$$0 < \left(\frac{B_{ob}}{B_{mb}} \right) < \frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right)$$

$$-1 < \lambda_{min} < 0$$

$$\Omega_{\infty} > 1$$

$$0 < \left(\frac{B_{ob}}{B_{mb}} \right) < \frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right)$$

$$\lambda_{min} < -1$$

$$\Omega_{max} < 1$$

$$0 < \left(\frac{B_{ob}}{B_{mb}} \right) < \frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right)$$

$$\lambda_{min} < -1$$

$$\Omega_{\infty} < 1 < \lambda_{max}$$

Figure 8g-i

$$0 < \left(\frac{B_{ob}}{B_{mb}} \right) < \frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right)$$

$$\omega_{min} < -1$$

$$\Omega_{\infty} > 1$$

$$\frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right) < \left(\frac{B_{ob}}{B_{mb}} \right) < 1$$

$$\omega_{min} > 0$$

$$\Omega_{max} < 1$$

$$\frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right) < \left(\frac{B_{ob}}{B_{mb}} \right) < 1$$

$$\Omega_{min} > 0$$

$$\Omega_0 < 1 < \Omega_{max}$$

72-110-0

Figure 8j-1

$$\frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right) < \left(\frac{B_{ob}}{B_{mb}} \right) < 1$$

$$i_{min} > 0$$

$$i_{co} < 1 < i_{do}$$

$$\frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right) < \left(\frac{B_{ob}}{B_{mb}} \right) < 1$$

$$i_{min} > 0$$

$$i_{min} < 1 < i_{co}$$

$$\frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right) < \left(\frac{B_{ob}}{B_{mb}} \right) < 1$$

$$i_{min} > 0$$

$$i_{min} > 1$$

72-111-0

Figure 8m-0

$$\frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right) < \left(\frac{B_{ob}}{B_{mb}} \right) < 1$$

$$-1 < \mu_{min} < 0$$

$$\mu_{max} < 1$$

$$\frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right) < \left(\frac{B_{ob}}{B_{mb}} \right) < 1$$

$$-1 < \mu_{min} < 0$$

$$\mu_0 < 1 < \mu_{max}$$

$$\frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right) < \left(\frac{B_{ob}}{B_{mb}} \right) < 1$$

$$-1 < \mu_{min} < 0$$

$$\mu_{\infty} < 1 < \mu_0$$

72-112-0

Figure 8p-r

$$\frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right) < \left(\frac{B_{ob}}{B_{mb}} \right) < 1$$

$$-1 < \omega_{\min} < 0$$

$$\omega_{\max} > 1$$

$$\frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right) < \left(\frac{B_{ob}}{B_{mb}} \right) < 1$$

$$\omega_{\min} < -1$$

$$\omega_{\max} < 1$$

$$\frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right) < \left(\frac{B_{ob}}{B_{mb}} \right) < 1$$

$$\omega_{\min} < -1$$

$$\omega_0 < 1 < \omega_{\max}$$

Figure 8s-u

$$\frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right) < \left(\frac{B_{ob}}{B_{mb}} \right) < 1$$

$$\lambda_{min} < -1$$

$$\lambda_{\infty} < 1 < \lambda_0$$

$$\frac{2}{\pi} \left(\frac{T_{ic}}{T_{ec}} \right) < \left(\frac{B_{ob}}{B_{mb}} \right) < 1$$

$$\lambda_{min} < -1$$

$$\lambda_{\infty} > 1$$

72-114-0

Figure 8v,w

72-115-0

The solution to the Quasineutrality equation at point (a)
Figure 9

72-116-0

Graphical solutions to the equation $1 + \beta Z = \alpha e^Z$

Figure 10a-c