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KINETIC THEORY AND SIMULATION OF MULTI-SPECIES PLASMAS IN TOKAMAKS EXCITED WITH ICRF MICROWAVES *

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This paper presents a description of a bounce-averaged Fokker-Planck quasilinear model for the kinetic description of tokamak plasmas. The non-linear collision and quasilinear resonant diffusion operators are represented in a form conducive to numerical solution with specific attention to the treatment of the boundary layer separating trapped and passing orbit regions of velocity space. The numerical techniques employed are detailed in so far as they constitute significant departure from those used in the conventional uniform magnetic field case. Examples are given to illustrate the combined effects of collisional and resonant diffusion.

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1. INTRODUCTION

It is generally the case in magnetic fusion devices that the magnetic field is nonuniform, but varies weakly on the scale of the gyromotion, the fastest nearly recurrent motion of the charged particles comprising the plasma. However, motion along the direction of the field inexorably carries the particle through finite variations of the field and the environment in which the charged particle is immersed as it executes its motion may vary significantly with these field variations. It is often necessary to consider this nonuniformity in order to involve salient features distinctive of a particular device in phenomena in which they play a significant role.

An example of such a feature is the presence of trapped particles in Tokamaks. Guiding center orbits which are nearly trapped and oppositely directed may be topologically adjacent in a Tokamak and thus may populate one another through diffusion (collisional or otherwise). In a uniform magnetic field calculation this could not occur. The only recurrent motion in that case is the gyromotion and oppositely directed orbits in the preceding sense are inaccessible to one another through small angle collisions or other similarly diffusive processes.

The electrical conductivity in the direction of the magnetic field in a Tokamak is affected by the presence of a population of trapped electrons (Hinton and Hazeltine¹, and Connor, et al.²). Since most trapped particles execute nearly recurrent *bounce* orbits along the direction of the ohmic electric field, the effect of the field on them is nullified, averaged away by their motion. Trapped particles thus carry no electrical current and can not contribute to the conductivity.

The use of cyclotron resonance heating in fusion devices and the production of plasma currents to allow steady-state operation in Tokamaks by the application of radio frequency microwave fields is currently of great interest. The nonuniformity of the magnetic field bears upon each of these phenomena.

One of the earliest evaluations of ion cyclotron resonance heating (ICRH) as a reliable mechanism for auxiliary heating from the standpoint of Fokker-Planck theory was done by Stix³. Although the groundwork for this approach had been laid earlier by Rosenbluth, et al.⁴ and Kennel and Englemann⁵ who developed the uniform field, infinite homogeneous media theory, a unified theory of particle-wave interactions had yet to be applied. The Stix approach was generalized for reactor studies by several authors, notably Kesner⁶, Scharer, et al.⁷, Blackfield and Scharer⁸, and more recently Hwang, et al.⁹. Each of these approaches is a variant of the uniform field infinite media calculation insofar as the kinetic theory is concerned, with various finite geometry (toroidal or mirror) corrections. More recently Mauel¹⁰ has modernized the quasilinear approach following the example of Berk¹¹ and Bernstein and Baxter¹².

Among the first to explore the possibilities of steady-state operation of Tokamaks with RF current drive were Fisch and Bers¹³; variations on this theme have been provided more recently by Fisch¹⁴, Fisch and Karney¹⁵, Harvey et al.¹⁶, and others.

In each case discussed, the interaction between particles and waves is localized by a resonance condition involving the particle motion, the fluctuating microwave field and the steady magnetic field. The strength of the interaction is determined by the time during which the particle motion and the fluctuating field are correlated. In a uniform field, decorrelation is effected by a finite width spectrum of waves, interference among members of the wave

ensemble decorrelating the interaction of the particle with any one single member. In a nonuniform field, the resonance is localized in physical space also through the dependence on the steady magnetic field. The particle motion itself thus provides a decorrelation mechanism which in many cases is overriding. Moreover, this provides a natural mechanism whereby a much broader population of orbits interact resonantly with the wave field.

A number of bounce-averaged Fokker-Planck codes have been written in recent years for the study of fusion plasma. The procedures, techniques, and arrays employed in the bounce-averaged code family **CQL** are generally similar to those used in their precursor family **FPPAC** designed to solve Fokker-Planck equations in a uniform magnetic field. In writing **CQL**, the authors drew heavily on the pool of ideas and methods previously developed by McCoy, et al.¹⁷, Cutler, et al.¹⁸, and Matsuda and Stewart¹⁹ have also developed bounce-averaged Fokker-Planck codes, though each with an approach different from that discussed here. The Cutler code is a non-conservation form code primarily designed to study mirror confined plasma. The Matsuda/Stewart code is a finite element code with a general elliptic solver for evaluating the Rosenbluth potentials. The **CQL** codes are designed primarily for Tokamaks in which there reside magnetically confined trapped and passing populations. The principal differences between the two families are associated with the bounce-averaging procedure and the maintenance of certain symmetries required by the orbit topology. Naturally, the distribution functions must be symmetric in the trapped region of velocity space: There is no distinction to be drawn between the populations on the left going and right going legs of a trapped orbit. The coefficients of the bounce-averaged Fokker-Planck operator must have the appropriate symmetries to maintain this property at every stage of the evolution of the bounce-averaged distributions. Goldston²⁰ has developed a code designed to treat tokamak plasmas and that code has been modified to conservative form and generalized by Hammett, et al.²¹.

This article is divided into three topical areas: Mathematical development, numerical analysis, and applications. In Sections 1.1 and 1.2 the partial differential operators modeling the effects of collisional and resonant diffusion are derived and reduced to a form amenable to numerical analysis. Section 1.3 is devoted to an examination of the boundary conditions in velocity space. Particular attention is given the boundary layer separating trapped and circulating orbits to allow meaningful numerical implementation down the road. Velocity space loss region models are discussed in Section 1.4.

The second part of the paper in combination with the appendices is devoted to numerical procedures employed to compute the relevant operator coefficients and to time advance the equation. The problem of density conservation and the treatment of the boundary layer between trapped and circulating orbit regions in velocity space are closely related. This is the subject of Section 2 and Appendix A. Appendix B contains a depiction of the time advancement algorithm. This procedure is a variant of the standard operator splitting scheme specifically tailored for particle density conservation in consideration of the orbit topology imposed by the presence of trapped and circulating particles. The method by which the resonant diffusion operator induced by RF fluctuating fields is computed is elucidated in Appendix C.

The last part of the article outlines some illustrative examples of the capabilities of **CQL**. Included are a unified calculation of neoclassical corrections to classical resistivity for finite aspect ratio. Section 3.1, a scenario for ICRF heating fusion reactivity enhancement. Section

3.2. and the simulation of a scanning charge-exchange spectrum analyzer diagnostic in an ICRF excited discharge, Section 3.3.

1.1. BOUNCE-AVERAGED COLLISION OPERATOR

Consider a multispecies magnetized plasma whose evolution is governed by the Fokker-Planck equation, in which the dominant diffusive mechanisms are small angle Coulomb collisions, wave-particle interactions within the purview of weak turbulence theory, and a small amplitude steady electric field:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \nabla_{\mathbf{v}} \cdot \left(\frac{q}{m} (\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}) f + \Gamma_c \right) = 0. \quad (1.1)$$

Here Γ_c represents the current density in velocity space due to small angle Coulomb collisions (Rosenbluth, et al.⁴). The additional complication of the field nonuniformity appears as a nontrivial coupling between the configuration and velocity space arguments of the field and test distribution functions f_i, f_j . The electric and magnetic fields in (1.1), $\mathbf{E}$ and $\mathbf{B}$, are composed of a fluctuating component arising principally due to the presence of antennae at the plasma periphery, and a steady component due to external coils. The method of choice is to solve for the linear response of the plasma (first order in the wave field amplitude) then evolve the time averaged colliding distributions, viewed as slowly varying functions of time, according to the resulting time averaged cooperative effects: The so called *quasilinear* effects are second order in the wave amplitude.

Denote the operation of time averaging on the wave period by $\langle Q \rangle_t = \hat{Q}$, where $Q = \hat{Q} + \tilde{Q}$; applying this operation to (1.1) there results the coupled pair of equations

$$\frac{\partial \hat{f}}{\partial t} + \mathbf{v} \cdot \nabla \hat{f} + \nabla_{\mathbf{v}} \cdot \left(\frac{q}{m} (\hat{\mathbf{E}} + \frac{\mathbf{v}}{c} \times \hat{\mathbf{B}}) \hat{f} + \langle \Gamma_c \rangle_t + \langle \Gamma_{qt} \rangle_t \right) = 0 \quad (1.2)$$

and

$$\frac{\partial \tilde{f}}{\partial t} + \mathbf{v} \cdot \nabla \tilde{f} + \nabla_{\mathbf{v}} \cdot \left(\frac{q}{m} (\tilde{\mathbf{E}} + \frac{\mathbf{v}}{c} \times \tilde{\mathbf{B}}) \tilde{f} \right) = -\nabla_{\mathbf{v}} \cdot \left(\frac{q}{m} (\tilde{\mathbf{E}} + \frac{\mathbf{v}}{c} \times \tilde{\mathbf{B}}) \hat{f} \right). \quad (1.3)$$

The quasilinear flux,

$$\Gamma_{qt} = \frac{q}{m} (\tilde{\mathbf{E}} + \frac{\mathbf{v}}{c} \times \tilde{\mathbf{B}}) \tilde{f}, \quad (1.4)$$

which appears in (1.2) is to be determined through the solution of (1.3), $\tilde{f}$, and a knowledge of the fluctuating fields.

Suppose for the moment that the fluxes, time averaged over a wave period are known. We are interested in the long term time averaged behavior of the distribution functions. Rapid fluctuations on intrinsic timescales such as wave, gyro, and bounce periods are viewed as subordinate in this respect. We therefore undertake to perform the appropriate time averages.

Since, unlike the wave frequency, the gyrofrequency varies from point to point, and bounce frequency varies from orbit to orbit, some care must be taken in the averaging procedure to preserve relevant detail. That is to say, we require the motion of the entity represented by the averaged equations to coincide with the average of the solutions to the exact equations: the latter being taken over an ensemble of initial conditions differing only with respect to the

initial phase of the nearly recurrent motion with respect to which the averaging is to apply. The general procedure is to average first over the highest frequency nearly recurrent motion, then proceed to lower frequency orbit structures until the relevant (collisional) time scale is reached.

We presume that, in order of magnitude, $\nabla_v \cdot \Gamma_c \sim \nu_c f$, $\nabla_v \cdot \Gamma_{ql} \sim \nu_{ql} f$ and $\Omega \gg \omega_B \gg \nu_c \sim \nu_{ql}$ where Ω is the gyrofrequency, ω_B is the bounce frequency, and ν_c , ν_{ql} are the corresponding interaction frequencies for particle-particle and wave-particle interaction.

In large gyration frequency theory we seek a solution $\hat{f}$ of (1.2) in the form of a series ordered in inverse powers of the large frequency Ω . Denoting the lowest order term in this expansion now by f and applying a standard gyrophase-averaging procedure to the time averaged Fokker-Planck equation (1.2) which we will indicate by $\langle \rangle_\phi$ (for details see Killeen, et al.²²) there results the gyrophase-averaged or gyrokinetic Fokker-Planck equation

$$\frac{\partial f}{\partial t} + v \cos \theta \hat{\mathbf{b}} \cdot \nabla f - \frac{q}{m} \hat{\mathbf{E}} \cdot \hat{\mathbf{b}} \frac{\partial f}{\partial v_{\parallel}} - \frac{1}{2} v \sin \theta \nabla \cdot \hat{\mathbf{b}} \frac{\partial f}{\partial \theta} + \langle \nabla_v \cdot \hat{\Gamma} \rangle_\phi = 0. \quad (1.5)$$

where $\hat{\mathbf{b}}$ is a unit vector in the direction of the magnetic field and v, θ are the velocity and pitch angle coordinates associated with a given gyration center.

Under most circumstances, field variation is sufficiently weak to assure that the changes in the parameters characterizing the unperturbed motion are adiabatic to requisite order. As a consequence there may exist nearly recurrent motion on an extended timescale (many gyroperiods) and certain invariants associated with motion along the field as well as the usual invariants associated with the gyromotion. We presume the presence of a magnetic well satisfying these conditions and nearly recurrent motion associated with this well of frequency ω_B greater than any other characteristic frequency of interest except the gyrofrequency. The bounce frequency, ω_B , the frequency with which a gyrocenter executes its nearly recurrent orbital motion in the varying field structure is given by the path integral

$$\frac{2\pi}{\omega_B} = \tau_B = \oint \frac{ds}{v \cos \theta} = \oint \frac{ds}{v_{\parallel}}, \quad (1.6)$$

where ds is the element of arclength along the magnetic field line associated with the gyrocenter motion. The relevant bouncephase infinitesimal is defined by $d\phi_B = ds \omega_B / v \cos \theta$. The magnetic moment $\mu = mv^2 \sin^2 \theta / 2B$ is an invariant of the unperturbed motion: Defining $\psi = B(s)/B(0)$, its invariance can be restated as $\sin^2 \theta = \psi \sin^2 \theta_0$ where θ_0 is the pitch angle coordinate at a fixed point $s = 0$ in the field through which all orbits pass. We adopt the notation that subscript ₀ hereafter will refer to quantities evaluated at this point which we call the midplane. In addition, we specialize to field structures which are periodic and piecewise monotonic about the minimum point at $s = 0$, continuous and continuously differentiable everywhere. This restriction is sufficiently lax to include the major features of a wide variety of magnetic fusion devices in which there exists a magnetic well. Collisions, interaction with microwave fields and ohmic electric fields are to be viewed as *perturbative** in

* *perturbative*: the energy gained or lost by a particle as a result of motion along the parallel electric field is viewed as small, i.e. of the order of the energy gained or lost in collisions with particles and interaction with waves.

this context so that to requisite order, $v = v_0$, the speed is also a constant of the unperturbed motion.

The large gyration frequency technique applied previously with respect to the gyromotion is next applied to the nearly recurrent bounce motion. As before, an expansion in inverse powers of the large frequency is presumed. This time the large frequency is ω_B and the lowest order (bouncephase independent) term is represented by $\mathcal{F}$. Phase averaging over the bouncephase variable ϕ_B in the system deriving from (1.5), (again, see Killeen, et al.²² for relevant details) we obtain an evolution equation for the bounce-averaged distribution function, $\mathcal{F}$:

$$\frac{\partial \mathcal{F}}{\partial t} = -\langle \langle \nabla_{\mathbf{v}} \cdot \hat{\Gamma} \rangle_{\phi_B} \rangle - \left\langle \frac{q}{m} \hat{\mathbf{E}} \cdot \hat{\mathbf{b}} \frac{\partial \mathcal{F}}{\partial v_{\parallel}} \right\rangle_{\phi_B}. \quad (1.7)$$

where t is now measured on a time scale slow with respect to a bounce period, viz many bounce periods in duration. $\mathcal{F}$ now contains no explicit dependence on ϕ_B ; any implicit dependence arises strictly through the orbit equations. That is to say, the equations of motion of the particle guiding centers provide the transformation relating the distribution function to be evolved at a given s or ϕ_B with its counterpart at any other s as follows:

$$\mathcal{F}_0(v_0, \theta_0) = \mathcal{F}(v(v_0, \theta_0; s), \theta(v_0, \theta_0; s)) \quad (1.8)$$

where we have previously defined v_0, θ_0 to be the speed and pitch angle in the equatorial midplane of a particle whose speed and pitch angle are v, θ at arclength s along the field.

Henceforth, for ease of notation, we will denote the double phase-average indicated in (1.7) simply by enclosing the expression to be averaged in $\langle\langle$ (double angle brackets).

Let us now consider the right hand side of this bounce-averaged Fokker-Planck equation. The gyrophase-averaged Fokker-Planck operator in a uniform magnetic field has been treated extensively elsewhere (cf. Rosenbluth, et al.⁴, Killeen, et al.²²). In the inhomogeneous magnetic field, the analogous local (fixed s) Fokker-Planck operator can be written in the same form:

$$\begin{aligned} \left(\frac{\delta \mathcal{F}}{\delta t} \right)_{cql} &= -\langle \nabla_{\mathbf{v}} \cdot \hat{\Gamma} \rangle_{\phi} \\ &= \gamma \left(\frac{1}{v^2} \frac{\partial}{\partial v} (A + B \frac{\partial}{\partial v} + C \frac{\partial}{\partial \theta}) + \frac{1}{v^2 \sin \theta} \frac{\partial}{\partial \theta} (D + E \frac{\partial}{\partial v} + F \frac{\partial}{\partial \theta}) \right) \mathcal{F} \\ &= \gamma \left(\frac{1}{v^2} \frac{\partial \mathcal{G}}{\partial v} + \frac{1}{v^2 \sin \theta} \frac{\partial \mathcal{H}}{\partial \theta} \right) \end{aligned} \quad (1.9)$$

where the flux $\hat{\Gamma}$ is composed of the three parts

$$\hat{\Gamma} = \hat{\Gamma}_c + \hat{\Gamma}_{ql} + \hat{\Gamma}_e \quad (1.10)$$

due respectively to the effects of Coulomb collisions, wave-particle interaction and ohmic electric field. The operator (1.9) is to be viewed as representing the time rate of change of the local distribution of particles in velocity space due to local interactions at orbit point s (or equivalently, at bouncephase ϕ_B). Equation (1.9) is not, however, an evolution equation for the bounce averaged distribution function; it is merely a representation of the operator which must first be bouncephase averaged then used in equation (1.7).

Regarding (1.7) and the comments thereafter, we define the notation

$$\begin{aligned} \left\langle \left(\frac{\delta \mathcal{F}_0}{\delta t} \right)_{cql} \right\rangle &= \frac{1}{\tau_B} \int_0^{s_B} \frac{ds}{v_{\parallel}} \left(\frac{\delta \mathcal{F}_0}{\delta t} \right)_{cql} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\delta \mathcal{F}_0}{\delta t} \right)_{cql} d\phi_B \\ &= \langle \nabla_{\mathbf{v}} \cdot \hat{\Gamma} \rangle = \frac{1}{v_0 \cos \theta_0 \tau_B} \nabla_{\mathbf{v}_0} \cdot \hat{\Gamma}_0. \end{aligned} \quad (1.11)$$

Here we take the bounce-average over one quarter of a cycle of the bounce motion and the analogous motion for transiting particles so that s_B in a tokamak is given by

$$\begin{aligned} \cos(\theta(v_0, \theta_0; s_B)) &= \cos(\theta_T) = 0; & \text{for trapped particles} \\ s_B = s_{mx} &= \pi q R_0; & q = r B_T / R_0 B_\theta; & \text{for circulating particles} \end{aligned} \quad (1.12)$$

and where q is the safety factor.

The operator (1.11) can be decomposed as was the flux $\hat{\Gamma}$ in (1.9); each part conserves (line density) N separately. That is to say, it can be shown²² that

$$\begin{aligned} 0 = \frac{\partial N}{\partial t} &= \int d^3 v_0 v_0 \cos \theta_0 \tau_B \left(\frac{\delta \mathcal{F}}{\delta t} \right)_{cql} = \int d^3 v_0 v_0 \cos \theta_0 \tau_B \langle \nabla_{\mathbf{v}} \cdot \hat{\Gamma} \rangle \\ &= \int d^3 v_0 \nabla_{\mathbf{v}_0} \cdot \hat{\Gamma}_0 \end{aligned} \quad (1.13)$$

where $\nabla_{\mathbf{v}_0}$ is the gradient operator in the midplane velocity coordinate system. $\hat{\Gamma}_0$ is found by rewriting (1.11) as

$$\begin{aligned} \left(\frac{\delta \lambda \mathcal{F}_0}{\delta t} \right)_{cql} &= -\nabla_{\mathbf{v}_0} \cdot \hat{\Gamma}_0 \\ &= \gamma \left(\frac{1}{v_0^2} \frac{\partial}{\partial v_0} (A_0 + B_0 \frac{\partial}{\partial v_0} + C_0 \frac{\partial}{\partial \theta_0}) + \frac{1}{v_0^2 \sin \theta_0} \frac{\partial}{\partial \theta_0} (D_0 + E_0 \frac{\partial}{\partial v_0} + F_0 \frac{\partial}{\partial \theta_0}) \right) \mathcal{F}_0 \\ &= \gamma \left(\frac{1}{v_0^2} \frac{\partial \mathcal{G}_0}{\partial v_0} + \frac{1}{v_0^2 \sin \theta_0} \frac{\partial \mathcal{H}_0}{\partial \theta_0} \right) \end{aligned} \quad (1.14)$$

where the coefficients $A_0 \rightarrow F_0$ can be found with prior knowledge of the coefficients $A \rightarrow F$ (1.9), and $\lambda = v_0 \cos \theta_0 \tau_B$.

Now since the particle transit or bounce time is small compared with a collisional relaxation time we evaluate changes in the bounce-averaged distribution $\mathcal{F}$ through an integration of its time rate of change along orbits in phase space. The procedure followed to obtain the coefficients $A_0 \rightarrow F_0$ in (1.14) thus involves two steps:

(A) Determine the Fokker-Planck coefficients at various points in s along a guiding-center orbit in a manner analogous to that given in the conventional uniform field treatment;

(B) Bounce-average these coefficients by accumulating a weighted sum (ie. perform the integral (1.11)) suitable for time advancing the distribution $\mathcal{F}_0$.

From (1.9) we write the local collision operator for species type α at orbit point s as

$$\left(\frac{\delta \mathcal{F}_\alpha}{\delta t} \right)_c = \gamma_\alpha \left(\frac{1}{v^2} \frac{\partial \mathcal{G}_\alpha}{\partial v} + \frac{1}{v^2 \sin \theta} \frac{\partial \mathcal{H}_\alpha}{\partial \theta} \right) \quad (1.15)$$

where we have selected the velocity space collisional flux components $\mathcal{G}_\alpha$ and $\mathcal{H}_\alpha$ in the form (1.9). Note that in the above a sum over species is implied, and $\gamma_\alpha = 4\pi Z_\alpha^4 e^4 / m_\alpha^2$.

These equations are identical in form to their analogues in uniform field theory, it being understood that $\mathcal{F}$ here is the bouncephase independent component of f , the gyrophase-averaged distribution, at the orbit point parameterized by arclength s , and velocity space coordinates $(v(v_0, \theta_0, s), \theta(v_0, \theta_0))$ (the subscript is the species identifier). Note that although $\mathcal{F}$ and therefore also $\mathcal{G}$ and $\mathcal{H}$ are implicitly dependent upon parameter s , they are nevertheless bouncephase independent, viz (1.8) applies.

The coefficients $A_\alpha \rightarrow F_\alpha$ are given by equations again identical in form to their analogues in uniform field theory. A complete representation of the coefficients for the uniform field case in terms of the Rosenbluth potentials and their derivatives can be found in Killeen, et al.²² or McCoy, et al.¹⁷. The mathematical procedure employed in the generation of the local (in s) Fokker-Planck coefficients is similar to that described there with suitable consideration given the fact there is an additional parameter (s).

The functions g_α and h_α , the local multispecies Rosenbluth potentials are now to be viewed as functions of parameter s through the implicit dependence of $\mathcal{F}$ on the arclength parameter (recall s bears a one-to-one correspondence with the poloidal angle). The integrals are over the entire domain in velocity space at point s .

The local Rosenbluth potentials and the distribution functions themselves may be represented by expansions in Legendre polynomials. For this purpose we let

$$\mathcal{F}_b(v, \theta, s; t) = \sum_{m=0}^{\infty} V_m^b(v, s; t) P_m(\cos \theta) \quad (1.16)$$

where the Legendre coefficients of the local distribution function is given by the projection integral

$$V_m^b(v, s; t) = \frac{2m+1}{2} \int_{-1}^{+1} d(\cos \theta) \mathcal{F}_b(v, \cos \theta, s; t) P_m(\cos \theta). \quad (1.17)$$

In practice, this integral is evaluated by transforming the variable of integration to the midplane system and using the appropriately transformed limits. The restricted domain of integration in the midplane system is that portion of (v_0, θ_0) space which maps into $(v(v_0, \theta_0, s), \theta(v_0, \theta_0))$ space at the relevant point s . Particles which bounce before reaching point s make no contribution to the collisions occurring there. Once $V_m^b(v, s; t)$ is determined, the procedure employed to generate the local coefficients is identical to that detailed in McCoy, et al.¹⁷. The efficiency of the calculation is optimized by computing the coefficients $A_\alpha(v, \theta, s; t) \rightarrow \bar{F}_\alpha(v, \theta, s; t)$ at values of θ for which there exists a $\theta_0(i)$ (on the midplane mesh) such that $\theta = \theta(\theta_0(i), s)$. Coefficients are determined along orbits piercing the midplane at mesh points; consequently, when the bounce-average is performed no interpolation is required. As mentioned previously,

the quadratures (eg. equation (1.17)) are transformed to the midplane, with the help of equation (1.8). The limits of integration are determined parametrically by s .

Using the notation introduced in equation (1.11) to denote the bounce averaging operation and the definition of λ following equation (1.14), as is shown in Killeen, et al.²², we can represent the coefficients $A_0 \rightarrow F_0$ in the bounce-averaged collision operator (1.14) in conservation form as

$$\begin{aligned} A_0 &= \lambda \langle \langle A \rangle \rangle; \quad B_0 = \lambda \langle \langle B \rangle \rangle; \quad C_0 = \lambda \langle \langle \frac{\cos \theta}{\sqrt{\psi} \cos \theta_0} C \rangle \rangle \\ D_0 &= \lambda \langle \langle \frac{\cos \theta}{\psi \cos \theta_0} D \rangle \rangle; \quad E_0 = \lambda \langle \langle \frac{\cos \theta}{\psi \cos \theta_0} E \rangle \rangle; \quad F_0 = \lambda \langle \langle \frac{\cos^2 \theta}{(\sqrt{\psi})^3 \cos^2 \theta_0} F \rangle \rangle \end{aligned} \quad (1.18)$$

In the event the effects of a DC parallel electric field are to be included the coefficients A_0 and D_0 must be augmented for passing orbits by

$$A_{E0} = -s^* E_{\parallel 0} \frac{q}{m} v_0^2 \cos \theta_0 \quad \text{and} \quad D_{E0} = s^* E_{\parallel 0} \frac{q}{m} v_0 \sin^2 \theta_0 \quad (1.19)$$

where $s^* = \int ds \psi$. The details of the derivation leading to this conclusion are given in Killeen, et al.²².

1.2. BOUNCE-AVERAGED RESONANT DIFFUSION

There is a growing body of literature dealing with the subject of quasilinear diffusion in magnetized plasma. The treatment of Bernstein and Baxter¹² is particularly instructive and the following analysis parallels its development to some extent.

In order to derive a representation for the quasilinear velocity space current density, $\hat{\Gamma}_{ql}$ it is necessary to solve (1.3), as is apparent from (1.4). Ideally, the wave field is known self consistently through a geometrical or physical optics calculation the sources for which are given by the appropriate moments of $\tilde{f}$. For the time being, however, suppose the wave field to be known.

Then, to solve (1.3), presume a solution to the unperturbed orbit equations

$$\frac{d\mathbf{r}'}{d\tau} = \mathbf{v}' \quad (1.20)$$

and

$$\frac{d\mathbf{v}'}{d\tau} = \frac{q}{m} (\hat{\mathbf{E}}(\mathbf{r}', \tau) + \frac{\mathbf{v}'}{c} \times \hat{\mathbf{B}}(\mathbf{r}', \tau)) \quad (1.21)$$

subject to the initial conditions $\mathbf{r}'(\mathbf{r}, \mathbf{v}, t) = \mathbf{r}$ and $\mathbf{v}'(\mathbf{r}, \mathbf{v}, t) = \mathbf{v}$. Equation (1.3) can therefore be expressed as the total time derivative along the orbit characteristics which solve (1.20) and (1.21).

$$\frac{d\tilde{f}}{d\tau} = \frac{\partial \tilde{f}}{\partial t} + \mathbf{v} \cdot \nabla \tilde{f} + \nabla_{\mathbf{v}} \cdot \left(\frac{q}{m} (\hat{\mathbf{E}} + \frac{\mathbf{v}}{c} \times \hat{\mathbf{B}}) \tilde{f} \right) = -\nabla_{\mathbf{v}} \cdot \left(\frac{q}{m} (\tilde{\mathbf{E}} + \frac{\mathbf{v}}{c} \times \tilde{\mathbf{B}}) \tilde{f} \right). \quad (1.22)$$

Choose the ansatz $\tilde{\mathbf{E}}' = \tilde{\mathbf{E}}(\mathbf{r}', \tau) = \mathbf{h}(\mathbf{r}') \exp(-i\omega\tau)$ for the fluctuating RF electric field along the unperturbed orbit so that the fluctuating magnetic field is given by $\tilde{\mathbf{B}}' =$

$-(ic/\omega)\nabla \times \mathbf{h} \exp(-i\omega\tau)$. The solution to (1.22), the first order fluctuating distribution function, can be represented as the time integral

$$\tilde{f}(\mathbf{r}, \mathbf{v}; t) = -\text{Re} \int_{-\infty}^t d\tau \frac{q}{m} (\mathbf{h} - \frac{i}{\omega} \mathbf{v} \times \nabla \times \mathbf{h}) \cdot \nabla_{\mathbf{v}} \tilde{f} e^{-i\omega\tau} = \text{Re}(e^{-i\omega t} g). \quad (1.23)$$

which is to be evaluated along the unperturbed orbits. The unperturbed orbits along which integral (1.23) is to be performed are helical paths generally following magnetic field lines, with a slow cross field drift superimposed. The method by which we proceed will be to ignore the slow drift and represent the motion as a high frequency (gyrofrequency) oscillatory motion about an oscillation center (gyrocenter) which moves along the magnetic field at a time varying pace.

To this end, we let $\mathbf{r} = \mathbf{R} + \rho$ where $\mathbf{R}$ is the position of the guiding center and recast the equations of motion (1.20)-(1.21) as equations governing the motion of the guiding center:

$$\frac{d\mathbf{R}'}{d\tau} = v'_{\parallel} \hat{\mathbf{b}}' + \frac{\hat{\mathbf{b}}'}{\Omega'} \times \left(\frac{\mu}{m} \nabla' B'_0 + v'_{\parallel} {}^2 \hat{\mathbf{b}}' \cdot \nabla' \hat{\mathbf{b}}' - \frac{q}{m} \mathbf{E}'_0 \right) = v'_{\parallel} \hat{\mathbf{b}}' + \mathbf{v}'_{\text{drift}} \quad (1.24)$$

where we will henceforth drop the slow drift term, and

$$\frac{dv'_{\parallel}}{d\tau} = -\hat{\mathbf{b}}' \cdot \nabla' (\mu B'_0) \quad (1.25)$$

where $\hat{\mathbf{b}}'$ is a unit vector in the direction of the magnetic field (at $\mathbf{R}'$). $\mathbf{v}' = v'_{\parallel} \hat{\mathbf{b}}' + \hat{\mathbf{b}}' \times (\mathbf{v}' \times \hat{\mathbf{b}}') = v'_{\parallel} \hat{\mathbf{b}}' + \mathbf{v}'_{\perp}$ is the particle velocity, and $\Omega' = qB'/mc$ is the local gyrofrequency. The gyroposition vector of the particle is given by

$$\rho = \sqrt{\frac{2\mu|\mathbf{B}|}{\Omega'^2 m}} \{ \hat{\mathbf{e}}_1 \sin(\phi + \Phi(\tau) - \Phi(t)) + \hat{\mathbf{e}}_2 \cos(\phi + \Phi(\tau) - \Phi(t)) \} \quad (1.26)$$

where $\hat{\mathbf{e}}'_1$ and $\hat{\mathbf{e}}'_2$ are mutually orthogonal unit vectors orthogonal to $\hat{\mathbf{b}}'$ at $\mathbf{R}'$ and

$$\Phi(t) = \int_0^t d\tau \Omega'(R'(\tau)). \quad (1.27)$$

Presently we shall perform the appropriate average eliminating the gyrophase dependence.

Since we have in mind eventually to parallel this calculation with a geometrical optics calculation of the wave fields, we choose $\mathbf{h}(\mathbf{r})$ to have the form

$$\mathbf{h}(\mathbf{r}) = \mathbf{a}(\mathbf{r}) e^{i\Lambda(\mathbf{r})} \quad (1.28)$$

where $\mathbf{a}(\mathbf{r})$ is a slowly varying (in space) amplitude and the local wavevector is $\mathbf{k} = \nabla\Lambda$.

Along the particle orbit the eikonal Λ varies according to

$$\frac{d\Lambda}{d\tau} = \frac{d\Lambda}{d\mathbf{r}'} \cdot \frac{d\mathbf{r}'}{d\tau} = \frac{d\Lambda}{d\mathbf{r}'} \cdot \frac{d}{d\tau} (\mathbf{R}' + \rho'). \quad (1.29)$$

Using (1.24) and (1.26) in (1.29) and neglecting the contributions of terms which have only slow variation along the orbit, (1.29) can be integrated and inserted in (1.23), the expression for the first order distribution g .

Representing the wavevector $\mathbf{k}$ in the local orthonormal basis following the guiding center $(\hat{\mathbf{b}}, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$

$$\mathbf{k} = k_{\parallel} \hat{\mathbf{b}} + k_{\perp} (\hat{\mathbf{e}}_1 \cos \delta + \hat{\mathbf{e}}_2 \sin \delta) \quad (1.30)$$

expression (1.23) for g is carried into

$$g = -\frac{q}{m} \sum_{\text{rays}} \sum_n e^{i(\Lambda - \mathbf{k} \cdot \rho - \chi + n\phi)} \int_{-\infty}^t d\tau e^{i(\chi' - n\delta')} \times \quad (1.31)$$

$$\mathbf{a}' \cdot \left\{ \hat{\mathbf{b}}' \frac{\cos \theta'}{\sin \theta'} J'_n \left[\sin \theta' \frac{\partial}{\partial v'} + \left(\frac{\cos \theta'}{v'} - \frac{1 - n\Omega'/\omega}{v' \cos \theta'} \right) \frac{\partial}{\partial \theta'} \right] \right.$$

$$\left. + \left[\hat{\mathbf{e}}'_+ e^{+i\delta'} J'_{n-1} + \hat{\mathbf{e}}'_- e^{-i\delta'} J'_{n+1} \right] \left[\sin \theta' \frac{\partial}{\partial v'} + \left(\frac{\cos \theta'}{v'} - \frac{k'_{\parallel}}{\omega} \right) \frac{\partial}{\partial \theta'} \right] \right\} \hat{\mathbf{f}}$$

where we have employed the identity

$$e^{iz \sin \alpha} = \sum_{n=-\infty}^{\infty} J_n(z) e^{in\alpha} \quad (1.32)$$

and the argument of J'_n is $k'_{\perp} \rho'$. The summation in (1.32) is over all n , $-\infty < n < +\infty$, and we have introduced the symbol χ to signify

$$\chi(n, t) = \int_0^t d\tau (n\Omega' + k'_{\parallel} v'_{\parallel} - \omega) = \int_0^t d\tau \nu(n, \tau). \quad (1.33)$$

Also introduced in (1.31) are the (complex) polarization vectors $\hat{\mathbf{e}}_{\pm} = (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)/2$. The time-averaged fluctuating distribution function g in the form (1.31) constitutes the solution of (1.3) sought for the purpose of representing the quasilinear velocity space flux (1.4) which appears in the quasilinear Fokker-Planck equation (1.2), whence, by a short calculation we find

$$\langle \hat{\Gamma}_{qt} \rangle_t = \frac{q}{4m} \sum_{\text{rays}} (\mathbf{a} + \frac{\mathbf{v}}{\omega} \times \mathbf{k} \times \mathbf{a})^* e^{-i\Lambda} g + \text{cc}. \quad (1.34)$$

Replacing the expression (1.31) for g in (1.34) and expanding $\exp(i\mathbf{k} \cdot \rho)$ as in (1.32) we come to an expression of the time averaged quasilinear velocity space flux which is directly amenable to phase-averaging. Lastly, we take velocity and pitchangle components of the flux (1.34), perform the appropriate averages over the oscillatory gyro and bounce motion, and then compare the result with the midplane flux components expressed through (1.14), and find it is possible to identify in the result the bounce-averaged quasilinear coefficients in the form (1.18) as $A_{0qt} = D_{0qt}$.

$$B_{0qt} = -\lambda \frac{q^2}{4m^2} \sum_{\text{rays}} \sum_n \left\{ \langle \langle \mathbf{a} \cdot \Theta_n v^2 \sin \theta e^{i\Psi} \rangle \rangle^* \Upsilon_{TB} \langle \langle \mathbf{a} \cdot \Theta_n \sin \theta e^{i\Psi} \rangle \rangle + \text{cc.} \right\} \quad (1.35)$$

$$C_{0,q_i} = -\lambda \frac{q^2}{4m^2} \sum_{\text{rays}} \sum_n \left\{ \langle \langle \mathbf{a} \cdot \Theta_n v^2 \sin \theta e^{i\Psi} \rangle \rangle^* \Upsilon_{\tau_B} \right. \\ \left. \times \langle \langle \mathbf{a} \cdot \left[\Theta_n \left(\cos \theta - \frac{k_{\parallel} v}{\omega} \right) - \eta \hat{\mathbf{b}} \frac{\cos \theta}{\sin \theta} J_n \right] e^{i\Psi} \frac{1}{v} \frac{\partial \theta_0}{\partial \theta} \rangle \rangle + cc. \right\} \quad (1.36)$$

$$E_{0,q_i} = -\lambda \frac{q^2}{4m^2} \sum_{\text{rays}} \sum_n \left\{ \langle \langle \frac{v \sin \theta}{\sqrt{\psi}} \frac{\partial \theta_0}{\partial \theta} \mathbf{a} \cdot \left[\Theta_n \left(\cos \theta - \frac{k_{\parallel} v}{\omega} \right) - \eta \hat{\mathbf{b}} \frac{\cos \theta}{\sin \theta} J_n \right] e^{i\Psi} \rangle \rangle^* \Upsilon_{\tau_B} \right. \\ \left. \times \langle \langle \mathbf{a} \cdot \Theta_n \sin \theta e^{i\Psi} \rangle \rangle + cc. \right\} \quad (1.37)$$

$$F_{0,q_i} = -\lambda \frac{q^2}{4m^2} \sum_{\text{rays}} \sum_n \left\{ \langle \langle \frac{v \sin \theta}{\sqrt{\psi}} \frac{\partial \theta_0}{\partial \theta} \hat{\mathbf{a}} \cdot \left[\Theta_n \left(\cos \theta - \frac{k_{\parallel} v}{\omega} \right) - \eta \hat{\mathbf{b}} \frac{\cos \theta}{\sin \theta} J_n \right] e^{i\Psi} \rangle \rangle^* \Upsilon_{\tau_B} \right. \\ \left. \times \langle \langle \mathbf{a} \cdot \left[\Theta_n \left(\cos \theta - \frac{k_{\parallel} v}{\omega} \right) - \eta \hat{\mathbf{b}} \frac{\cos \theta}{\sin \theta} J_n \right] e^{i\Psi} \frac{1}{v} \frac{\partial \theta_0}{\partial \theta} \rangle \rangle + cc. \right\} \quad (1.38)$$

In the preceding, we have abbreviated $\Psi = \chi - n\delta$ and introduced η , Θ_n where

$$\eta = \frac{\omega - k_{\parallel} v \cos \theta - n\Omega}{\omega \cos \theta} = \frac{\nu}{\omega \cos \theta} \quad (1.39)$$

vanishes at resonance, and

$$\Theta_n = \frac{\cos \theta}{\sin \theta} \hat{\mathbf{b}} J_n + \hat{\mathbf{e}}_- e^{-i\delta} J_{n+1} + \hat{\mathbf{e}}_+ e^{+i\delta} J_{n-1}. \quad (1.40)$$

For details of this calculation, the reader is referred to Killeen, et al.²²

The quantity referred to as Υ accounts for multiple bounce coherence effects. In actuality, processes such as collisional (extrinsic) or wave-particle interaction induced (intrinsic) gyrophase diffusion will assure Υ remains close to unity. The quantity χ given by (1.33) is a representation of the expected advance of the interaction phase, ν , along a given trajectory in a single bounce. This single bounce interaction phase is subject to stochastic variation due to collisions as well as due to the wave-particle interaction itself. Cohen, et al.²³ have shown the gyrophase diffusion varies as t^3 ; this provides an effective means of washing out multiple bounce resonances completely, and decorrelating single pass resonances to an extent roughly proportional to the cube of the number of gyroperiods separating them. In tokamaks the magnetic field connection length is quite long and in most cases of interest the plasma is sufficiently collisional to assure single bounce/transit effects are the only operant coherence effects of significance. These issues are considered in more detail in Appendix C.

The bounce-averages in (1.35)-(1.38) may be evaluated by the method of stationary phase due to the rapid variation with τ of the exponent Ψ . The eikonal Ψ is expanded about the

point where $d\Psi/dt$ vanishes (elsewhere the rapid oscillations provide cancelling contributions to the integrand). By (1.33), this implies

$$\frac{d\Psi}{dt} = \frac{d}{dt} \int_{\tau}^t dt' \nu(n, \tau) = \nu(n, t) = 0. \quad (1.41)$$

the condition of wave-particle resonance.

In the small banana width approximation, (vanishing $\mathbf{k} \cdot \mathbf{v}_{\text{drift}}$), η vanishes at the point of stationary phase, i.e. the wave-particle resonance. At points for which $dB/d\tau = 0$ coincidentally with resonances, not only is $\nu = 0$, but also $\dot{\nu} = 0$. The expansion of Ψ about these points must be extended to one higher order (including a term in $\ddot{\nu}$). For small $k_{\parallel}$, the same phenomena occur near orbit turning points. A detailed analysis of this procedure is presented in Appendix C. We can represent the phase integral in these cases mnemonically as

$$\int_0^{\tau_B} d\tau \Pi(n, \tau) e^{i\Psi} = \int_0^{\tau_B} d\tau \Pi \exp \left(-i \int_0^{\tau} d\tau' (\omega - k_{\parallel} v_{\parallel} - n\Omega)(\tau') \right) = \int_0^{\tau_B} ds \Pi \tau_c \delta(s - s_R) \quad (1.42)$$

where τ_c , which might be called an effective correlation time, is identified through evaluation of the trajectory integral; s_R is the position at which resonance occurs. Associated with the timelike quantity τ_c through the evaluation of Ψ at the point of stationary phase is a certain phase factor. It is the advance of this phase between subsequent resonances which determines the single bounce coherence of sequential wave-particle interactions. Orbit resonances which are too close to sense significant gyrophase diffusion can interfere constructively or destructively through the action of the relative interaction phase separating them. Details of the calculation pertaining to this phenomenon are discussed in Appendix C.

In reality the wave-particle resonant interaction is not confined to a single point (s_R), but rather occurs over an interval corresponding roughly to the distance the particle (gyrocenter) moves in a time of the order of $|\tau_c|$. We wish eventually to determine the spacial (poloidal) variation of RF power absorption so that the wave damping calculated through the quasilinear model can be used concurrently with a wave propagation calculation. To this end we introduce a resonance broadening model which, apart from a normalizing factor $|\tau_c|$, replaces the exponential in the integrand of (1.42) by a resonance weighting function $w(\tau)$ such that $\int d\tau w(\tau) = 1$, the weighting function is normalized; $\int d\tau \tau w(\tau) = \tau_R$, the weighting function is centered on the wave-particle resonance; and finally, $\int d\tau (\tau - \tau_R)^2 w(\tau) = |\tau_c|^2$, the width of the weighting function corresponds to the wave-particle correlation time. Clearly, the stationary phase method alone corresponds to a weighting function with point support, $w(\tau) = \delta(\tau - \tau_R)$.

This resonance broadening technique should be viewed more as a computational convenience than a theoretical generalization. We sacrifice no physical insight by neglecting the broadening in the description of the underlying structure of the theory. Thus, to simplify the discussion, let us restrict ourselves for the moment to the limiting case of point resonance. For finite $k_{\parallel}$, then, (1.42) is an adequate mnemonic representation everywhere except in the neighborhood of the tangent resonance points. At tangent resonance, while $\nu = 0$ and $\dot{\nu} = 0$,

$v_{\parallel} \neq 0$, and there results from the trajectory integral a variant of the form (1.42) involving $d\Pi/ds$.

In the vicinity of tangent resonance a gyrocenter trajectory passes through two resonances which are close to one another (in physical space). Since these resonances are close, there is a phase correlation between the two interactions which is, in general, not randomized. The role played by the Bessel functions in depicting wave-particle interaction interference (cf. 1.31, 1.32) at the gyro-orbit level is, roughly speaking, played by the Airy function at the bounce-orbit level. For a more mathematical statement of this effect, the reader is referred to Appendix C.

The method used to compute the quasilinear coefficients is outlined schematically in Figures 1, 2, and 3, and detailed in Appendix C. Figure 1 shows the phaseflow corresponding to the set of velocity space meshpoints chosen to represent the distribution $\mathcal{F}$. The abscissa is the arclength along the magnetic field measured from the midplane normalized to the arclength from the midplane (minimum B) to the throat (maximum B). The ordinate is the cosine of the pitchangle along a gyrocenter trajectory. The orbits above the separatrix correspond to co-passing orbits, those below the separatrix correspond to counter-passing orbits, and the elliptical trajectories correspond to trapped orbits. Since kinetic energy is a constant of the motion, these orbits are independent of v_0 . The vertical curves in Figure 1 represent the relation $\nu = 0$ in this space for a set of values v_0 . The solid curves are those for which $k_{\parallel} > 0$ and the dotted curves are those for which $k_{\parallel} < 0$. The intersections of the gyrocenter trajectories and the curves $\nu = 0$ are the resonances corresponding to positions of stationary wave-particle interaction phase. For $k_{\parallel} = 0$, the vertical curves all collapse into a single vertical line passing through the common intersection of the curves. That intersection becomes the turning point or bounce resonance point. For $k_{\parallel} \neq 0$ the loci of points in the phase space at which wave-particle resonance occurs coincidentally with $\nu = 0$, tangent resonance, is shown as the remaining curve in Figure 1. This curve, of course, also goes through the turning point resonance, since there is no parallel Doppler shift at that point.

Figure 2 shows the loci in the midplane velocity space of points corresponding to orbits all of which pass through resonance at the same arclength position (or equivalently, the same poloidal angle). Some velocity space points are covered by two resonance loci. These orbits pass through resonance twice, at two different arclength positions. Along the inner boundary of this region, the two resonance loci covering a given point are tangent to one another, and to a curve we will call the tangent resonance locus, shown more clearly in Figure 3. As $k_{\parallel} \rightarrow 0$ the tangent resonance curves become straight lines. These radials coincide with the asymptotes of the family of quasilinear diffusion characteristics depicted in Figure 4. Resonant diffusion for a definite wave frequency is one dimensional. Phase space density diffuses along these curves under the action of waves of a definite frequency, driven by density gradients locally tangent to the curves. This is in contrast to collisional diffusion which is two dimensional. For small $k_{\parallel}$ the asymptotes in Figure 4 correspond to orbits which bounce at resonance.

With the foregoing provisos, the quasilinear coefficients (1.35)-(1.38) can be cast in the mnemonic form

$$B_{0q} = \lambda S v^2 \sin^2 \theta \left. \frac{\tau_c^2}{\tau_B} \right|_R \quad (1.43)$$

$$C_{0,q_i} = \lambda S \sin \theta \left[\cos \theta - \frac{k_{\parallel} v}{\omega} \right] \frac{\partial \theta_0}{\partial \theta} \frac{\tau_c^2}{\tau_B} \Big|_R \quad (1.44)$$

$$E_{0,q_i} = \lambda S \frac{v}{\sqrt{\psi}} \sin^2 \theta \left[\cos \theta - \frac{k_{\parallel} v}{\omega} \right] \frac{\partial \theta_0}{\partial \theta} \frac{\tau_c^2}{\tau_B} \Big|_R \quad (1.45)$$

$$F_{0,q_i} = \lambda S \frac{\sin \theta}{\sqrt{\psi}} \left[\cos \theta - \frac{k_{\parallel} v}{\omega} \right]^2 \left[\frac{\partial \theta_0}{\partial \theta} \right]^2 \frac{\tau_c^2}{\tau_B} \Big|_R \quad (1.46)$$

where λ was defined in (1.14), the quantity S is given by

$$S = -\Upsilon \frac{q^2}{4m^2} \sum_{\text{rays}} \sum_n (\mathbf{a} \cdot \boldsymbol{\Theta}_n)^* (\mathbf{a} \cdot \boldsymbol{\Theta}_n) \quad (1.47)$$

and all quantities are to be evaluated at the appropriate resonance(s) as determined through (1.42).

The quasilinear coefficients of (1.43-1.46) are shown in Figure 5, where they are displayed as functions of $\cos \theta_0$ for various values of v_0 . The prominence in the energy diffusion coefficients occurs in the vicinity of tangent resonance. This particular case represents single frequency fundamental minority species excitation with rather small $k_{\parallel}$. The vertical lines coincide with the trapped/passing boundary. The energies represented are too small to see significant finite gyrophase interference effects, and the mesh spacing is too coarse to depict any but the first bouncephase interference fringe.

Figure 6 shows the ion distribution function of a minority species excited at its fundamental cyclotron harmonic. Figure 7 shows the velocity space fluxes driven by (a) the resonant diffusion, (b) the collisional diffusion, and (c) the combination of these. Notice the energy diffusion is maximal in the neighborhood of the tangent resonances, as described previously.

RF Power Absorption

The power absorbed on a flux surface per unit toroidal extent by resonant particles can be represented in terms of the quasilinear component of the Fokker-Planck operator:

$$P = \int d^3 v_0 \frac{1}{2} m v_0^2 \frac{\partial \lambda \mathcal{F}_0}{\partial t} = - \int d^3 v_0 \frac{1}{2} m v_0^2 \nabla_{\mathbf{v}_0} \cdot \hat{\Gamma}_{0,q_i}.$$

Using the expression for the divergence of the flux in the form (1.14), performing an integration by parts, and recognizing that

$$C_{0,q_i} = \frac{m \Omega_0 / \omega - \sin^2 \theta_0}{v_0 \sin \theta_0 \cos \theta_0} B_{0,q_i},$$

the RF power absorbed can be cast in the form

$$P_{RF} = -2\pi \int dv_0 \int d\theta_0 m v_0 \sin \theta_0 B_{0,q_i} \hat{\Delta}_0 \mathcal{F}_0 \quad (1.48)$$

where the differential operator $\hat{\Delta}_0$ is given by

$$\hat{\Delta}_0 = \frac{\partial}{\partial v_0} + \frac{m\Omega_0/\omega - \sin^2 \theta_0}{v_0 \sin \theta_0 \cos \theta_0} \frac{\partial}{\partial \theta_0}.$$

Following the discussion of resonance broadening we set

$$B_{0_i} = \bar{b}_0 \int d\tau w(\tau) \mathbf{E}^2(\tau)$$

or, equivalently,

$$B_{0_i} = \bar{b}_0 \int d\theta_p W(\theta_p) \mathbf{E}^2(\theta_p)$$

where the bouncephase interference factors discussed in Appendix C appear through the identification

$$\bar{b}_0 = b_0 \mathcal{T}(1 - e^{-D_0^2 \sin 2\Psi_R})$$

and the gyrophase interference is contained in b_0 through identification with (1.35), (1.40), (1.43), and (1.47). The quantities $\bar{b}_0$ and W are functions of orbit invariants v_0, θ_0 as well as wave field parameters $\omega, k_{\parallel}, k_{\perp}$ and wave polarization at resonance. In the present context, however, the field amplitude $\mathbf{E}^2$ for a given wave mode, is to be viewed as depending on spacial variable θ_p as well. This is done for purposes of distributing the power exchange spacially between the particles and the wave field.

The differential power absorption can now be represented as the integral

$$\frac{dP_{RF}}{d\theta_p} = -2\pi \int dv_0 \int d\theta_0 m v_0 \sin \theta_r \bar{b}_0 W \mathbf{E}^2 \hat{\Delta}_0 \mathcal{F}_0 \quad (1.49)$$

For finite $k_{\parallel}$ there arise situations in which ions can transit through two distinct resonances on a single (quarter) bounce or transit in the magnetic well. For well separated resonances far from tangent resonance, the power exchanged between the particles and the field must be divided between the resonances. The absolute interaction phase (the gyrophase at θ_p , say) is randomly distributed over the particle (gyro) ensemble; the appropriate average must be performed to eliminate any such dependence. The apportionment can depend only on the uncorrelated local interaction strengths: Whereas a single particle on any given bounce may absorb wave energy at one resonance and emit wave energy at another, the ensemble of particles on the same orbit will either absorb or emit as a whole, *but not both*. The net resulting power exchange will be distributed in θ_p according as the interaction strength associated with the local wave-particle resonance, ie. $\bar{b}_0 W \mathbf{E}^2$.

A word about applications: The model as set out in the foregoing is presently incorporated in the CQL family of codes. The philosophy with which the present version was developed centers on the mechanism of magnetic decorrelation. Though a charged particle must be in resonance to interact strongly with the exciting RF fields, it is not the case that the magnetic correlation time is the most rational measure of the strength of the interaction in all cases. The magnetic correlation time does, however, represent a persistent uniform underlying structure. In some cases of interest, finite bandwidth wave spectra give rise to wave-particle correlation times shorter than τ_c . Models which combine these effects in a rational manner are the subject of further study.

For ICRH studies, we adopt a simplified approach in specifying the field amplitudes a . For each wave the parallel index of refraction at resonance is specified. The cold plasma dispersion relation can be used to compute the perpendicular index of refraction and the relative amplitudes of the right and left circularly polarized components. Alternatively, rays can be traced to their source where their amplitude is specified. For details of the various wave models incorporated in the source the reader is referred to Killeen, et al.²².

1.3. VELOCITY SPACE BOUNDARY CONDITIONS

The peripheral boundary conditions enforced in the calculation are essentially identical to those in the 2-dimensional uniform field calculation (McCoy, et al.¹⁷), namely

$$\begin{aligned} \mathcal{F}_0(v_0 = 0, \theta_0) &\text{ is independent of } \theta, \\ \frac{\partial \mathcal{F}_0}{\partial v_0}(v_0 = 0, \theta_0 = \frac{\pi}{2}) &= 0, \\ \mathcal{F}_0(v_0 = \infty, \theta_0) &= 0, \\ \frac{\partial \mathcal{F}_0}{\partial \theta_0}(v_0, \theta_0 = 0) &= \frac{\partial \mathcal{F}_0}{\partial \theta_0}(v_0, \theta_0 = \pi) = 0, \text{ and} \\ \frac{\partial \mathcal{F}_0}{\partial \theta_0}(v_0, \theta_0 = \frac{\pi}{2}) &= 0. \end{aligned} \tag{1.50}$$

This last equation reflects the requirement that the bounce-averaged distribution be symmetric about $\pi/2$ in the trapped orbit region. This simply reiterates the fact that $\mathcal{F}$ is not an explicit function of s , and the two points in question signify the same orbit at ϕ_B and $\phi_B + \pi$.

There arises a complication at the trapped/passing boundary which does not appear in the uniform field theory: The bounce time diverges logarithmically there and the formal frequency ordering fails. Thus a valid set of jump conditions is required in the vicinity of the boundary layer separating trapped and circulating particle orbits in order to properly couple the solutions in the various neighboring regions. From (1.14) it is apparent that to satisfy conservation of particles (1.13), in addition to continuity of $\mathcal{F}_0$ at the trapped/passing boundary we must also require the normal flux of $\lambda \mathcal{F}_0$ to satisfy

$$0 = [[\hat{\Gamma}_0 \cdot \hat{e}_{\theta_0}]] = [(D_0 + E_0 \frac{\partial}{\partial v_0} + F_0 \frac{\partial}{\partial \theta_0}) \mathcal{F}_0] = [[\lambda_0]]. \tag{1.51}$$

Here the $[[]]$ means the jump (in the flux) between leaving the trapped region and entering the passing region. We note in passing that the orbit topology determines the form of the jump condition in its application (see Section 2). Here three distinct regions of velocity space are in contact: co-passing, counter-passing, and trapped orbits. The jump condition is merely a statement of the fact that all trapped particles which become passing particles, by collisions for example, must cross over the trapped/passing boundary and appear on the other side.

While the flux of $\lambda \mathcal{F}_0$ is not defined on the boundary, it is well defined to each side of the boundary. We must require that $\mathcal{F}_0$ be continuous but there may be a discontinuity in its normal derivative. Furthermore, discontinuity in the coefficients may arise as a consequence of the RF excitation.

The procedure by which (1.51) is implemented is described in detail in Appendices A and B.

1.4. VELOCITY SPACE LOSS REGION MODELS

It is often the case in magnetic fusion devices that various classes of charged particle orbits are not confined by the magnetic field structure; these orbits are called loss orbits. In the case of magnetic mirror traps, loss orbits are typically associated with small pitch angle and gyrocenters can freely stream out of the trap along field lines in a time of the order of a transit time. In toroidal devices field lines are generally confined to a limited volume and gyrocenters cannot leave the confinement region by following field lines for a single transit time. However, since the field lines are necessarily curved, there arise cross-field drifts which carry gyrocenters across flux surfaces and, in some cases, out of the region of useful confinement into a wall, limiter, gas layer, or divertor.

A toroidal loss model is used for tokamak calculations. Toroidal axisymmetry induces the conservation of toroidal canonical angular momentum. This fact is exploited to calculate the radial excursion from the bounce-averaged (or zero banana width) flux surface to first order in the parameter ρ_p/a , assumed small throughout. Here ρ_p is the gyroradius in the poloidal magnetic field and a is the minor radius of the plasma.

The interaction of the particle along its perturbed orbit with neutral gas near the wall or limiter structures is computed and used to establish the probability the particle is lost from its (unperturbed) orbit on any given bounce. In effect, a charge exchange Krook type extinction operator is bounce averaged along the first order gyrocenter drift orbits. A particle whose drift excursion brings it into contact with the wall is lost in a bounce time.

The toroidal canonical angular momentum is given to requisite order by

$$P_\zeta = mv_{\parallel} R \hat{\mathbf{b}} \cdot \hat{\mathbf{e}}_\zeta + \frac{q}{c} \Psi \quad (1.52)$$

where

$$\Psi = \Psi_0 - R_0 \int_0^r \frac{dr'}{r'} \int_0^{r'} dr'' r'' j_\zeta(r'') = \int_r^a dr R B_\theta \quad (1.53)$$

is the poloidal flux function and j_ζ is the toroidal electrical current density.

Since P_ζ is a constant of the motion, upon bounce-averaging (1.52) we have to first order in ρ_p/a ,

$$\langle\langle P_\zeta \rangle\rangle = P_\zeta = mv_{\parallel} R \hat{\mathbf{b}} \cdot \hat{\mathbf{e}}_\zeta + \frac{q}{c} \Psi = m \langle\langle v_{\parallel} R \rangle\rangle \hat{\mathbf{b}} \cdot \hat{\mathbf{e}}_\zeta + \frac{q}{c} \langle\langle \Psi \rangle\rangle, \quad (1.54)$$

which defines the bounce-averaged flux as

$$\langle\langle \Psi \rangle\rangle = \Psi + \frac{mc}{q} \hat{\mathbf{b}} \cdot \hat{\mathbf{e}}_\zeta (v_{\parallel} R - \langle\langle v_{\parallel} R \rangle\rangle). \quad (1.55)$$

We wish to identify $\langle\langle \Psi \rangle\rangle$ with the flux surface on which the zero banana width calculation takes place (zeroth order in ρ_p/a). This is in fact the operational definition of orbit ensemble used in throughout. This implies $\Psi = \langle\langle \Psi \rangle\rangle$ and as a direct result, regarding (1.54), that

$\langle\langle v_{\parallel} R \rangle\rangle = v_{\parallel} R$. Now for a trapped particle, to first order $\langle\langle v_{\parallel} R \rangle\rangle = 0$ so that the bounce-average of Ψ for a trapped particle is just the value of Ψ at the banana tips or turning points.

We proceed to use (1.54) in conjunction with (1.53) to calculate the radial excursion $\Delta(r/a)$ as a function of the bouncephase for a (trapped) orbit characterized by the initial conditions $\Psi(r/a) = \langle\langle \Psi \rangle\rangle$ at $v_{\parallel} = 0$. Figure 8 shows the result of this calculation for a set of orbits of increasing energy. The banana tips are all located on the same flux surface, $\langle\langle \Psi \rangle\rangle$. Having thus calculated the perturbed orbit (to lowest significant order) we can compute the probability that the particle survives a bounce by performing the attenuation integral

$$P = \exp \left(- \int dt / \tau(t) \right) \quad (1.56)$$

where $\tau = (n \langle \sigma v \rangle)^{-1}(t)$ is the mean interaction time along the orbit parameterized by t .

A similar procedure may be used to account for circulating particles with the exception that (1.55) must be solved to evaluate $\Psi(\phi_B)$, or alternatively $r(\phi_B)$ by inverting $\Psi(r) = \Psi(\phi_B)$. Figure 8 also shows a series of (projected) co-passing and counter passing orbits for a set of energies.

A Krook term involving the results of a neutral transport calculation (see Section 3.3) is also implemented. Figure 9 represents a contour map of the bounce-averaged charge exchange interaction rate as a function on the velocity (orbit invariant) space. The first order (in ρ_p/a) gyrocenter drift orbits have been used to perform this average. The solid bold contour is the image in this space of a limiter at $r/a = .9$; all orbits characterized by orbit invariants along this contour just touch the limiter (see also Figure 16 for the images of neighboring tori).

2. FLUX CONSERVATION AT THE TRAPPED/PASSING BOUNDARY

As mentioned earlier, the procedures, techniques, and arrays employed in the bounce-averaged code family **CQL** are generally similar to those used in their precursor family **FPPAC** designed to solve Fokker-Planck equations in a uniform magnetic field. In writing **CQL**, the authors drew whenever possible on the methods previously developed by McCoy, et al.¹⁷ and Cutler, et al.¹⁸. In this section we derive a boundary condition which reduces to condition (1.51) in the appropriate limit and which can be easily enforced numerically. The remainder of the numerical algorithm, insofar as it differs from the conventional technique for solving uniform field Fokker-Planck equations, is detailed in Appendix B.

We will let mesh indices itl and itu prescribe the pitchangles corresponding respectively to the co-moving and counter-moving legs of orbits on the trapped/passing boundary, *pitch* orbits; $\theta_{itu} = \pi - \theta_{itl}$. Now as the boundary layer about $\cos \theta_{itl} = -\cos \theta_{itu} = \mu_{0T}$ becomes vanishingly thin, condition (1.51) must hold; this implies that the total line density in the layer, ie. the field line density in the sense of (1.13), must vanish identically in time. However, since the finite difference mesh contains a boundary layer of finite thickness the line density in this boundary layer will neither vanish nor remain stationary. In any event, a proper treatment of the boundary layer must account for the logarithmic singularity in τ_B there. This subject is treated in detail in Appendix A. Here we simply relate the results of that analysis to its implementation in the numerical scheme.

Generally speaking, the procedure used to evolve the distribution function in the boundary layer represents the processes within the finite width layer as suitable averages over the regions adjacent to the layer. To clearly specify how this is to be done we first need to show what is meant by *volume element* in the boundary layer. Then *suitable average* can be described as a specific algorithm motivated by the considerations of Section 1.3 and Appendix A.

Boundary Layer: Volume Element

As was shown through (1.13) and discussed in more detail in Killeen, et al.²², the conserved quantity in the bounce-averaged system is the field line density. We write

$$N = \int_0^{s_{ms}} \frac{ds}{\psi} n(s) = \int d^3v v_{\parallel} \tau_B(v, \theta) \mathcal{F}(v, \theta) \cong \int_{\mathcal{R}} d^3v v_{\parallel} \tau_B(v, \theta) \mathcal{F}(v, \theta) + N_{bl}. \quad (2.1)$$

Here $\mathcal{R}$ denotes an integral whose domain is restricted away from the boundary layer defining the transition from passing to trapped particle orbits, and N_{bl} , the boundary layer line density, is given by

$$N_{bl} = 2\pi \int_{bl} dv_0 d\theta_0 v_0^2 \sin \theta_0 \lambda \mathcal{F}_0 = 4\pi \int_0^{\infty} dv_0 v_0^3 I_{bl} \quad (2.2)$$

where I_{bl} is derived in Appendix A and $\lambda = v_{\parallel 0} \tau_B$. The apparent extra factor 2 in (2.2) recognizes that the boundary layer has upper and lower halves, only one of which is represented

explicitly by I_{bl} . Taking the time derivative of (2.2) and restructuring using the notation of Appendix A. (A2)-(A4), in particular

$$\mathcal{F}_{0T} = \mathcal{F}_0(v_0, \theta_{il}) = \mathcal{F}_0(v_0, \theta_{iu}),$$

there results

$$\frac{\partial N_{bl}}{\partial t} = 4\pi \int_0^\infty dv_0 v_0^3 \frac{I_{bl}}{\mathcal{F}_{0T}} \frac{\partial \mathcal{F}_{0T}}{\partial t}. \quad (2.3)$$

Defining the quantity

$$\mathcal{L}_{bl} = \frac{2I_{bl}v_0}{\mathcal{F}_{0T}}, \quad (2.4)$$

(2.3) may be reexpressed as

$$\frac{\partial N_{bl}}{\partial t} = 2\pi \int_0^\infty dv_0 v_0^2 \mathcal{L}_{bl} \frac{\partial \mathcal{F}_{0T}}{\partial t} \quad (2.5)$$

and $\mathcal{L}_{bl}$ can be seen to play the role of $v_{||0T} \sin \theta_0 d\theta_0 \equiv \lambda \sin \theta_0 d\theta_0$.

Boundary Layer: Averaging

In Appendix A, the edges of the boundary layer are defined by $\mu_{0T}^\pm$. For purposes of the following we establish the following equivalences:

$$\theta_{il}^\mp = \arccos \mu_{0T}^\pm \quad (2.6)$$

are the *lower* pitchangle borders of the boundary layer, θ_{il}^- lying in the co-passing orbit region and θ_{il}^+ lying on the co-moving side of the trapped orbit region (see Figure 10). Also, $\theta_{iu}^\mp$ are the *upper* pitchangle borders of the boundary layer defined analogously. The upper and lower borders (in velocity) of a volume element centered at v_0 are located at $v_0^\pm = v_0 \pm \Delta v_0/2$. Let the element in the co-passing region adjacent to the boundary layer be centered at θ_0 . The pitchangle borders of this element (see Figure 10) are $\theta_0^\pm = \theta_0 \pm \Delta\theta_0/2$ where $\theta_0^+ \equiv \theta_{il}^-$.

The bounce-averaged Fokker-Planck equation (1.14) is valid at the point (v_0, θ_0) and we can integrate (1.14) across the element centered there to obtain

$$\frac{\partial N}{\partial t}(v_0, \theta_0) = 2\pi\gamma \left[\sin \theta_0 \Delta\theta_0 (\mathcal{G}_0(v_0^+, \theta_0) - \mathcal{G}_0(v_0^-, \theta_0)) + \Delta v_0 (\lambda_0(v_0, \theta_{il}^-) - \lambda_0(v_0, \theta_0^-)) \right].$$

This is simply a statement of the fact that the time rate of change of the number of particles in a volume element is the sum of all the relevant fluxes across the boundaries of the element.

The equation governing the evolution of $\mathcal{F}_{0T}$ must reflect the constraint that the total line density in the boundary layer change by exactly the sum of the θ_0 -fluxes of all the volume elements adjacent to the layer. In deference to this, v_0 -fluxes accorded by the *suitable average*

of v_0 -fluxes in elements adjacent to the layer can redistribute particles with respect to v_0 within the layer.*

The time rate of change of the total number of particles in a differential boundary volume element due to the θ_0 -flux from the co-passing region is $-2\pi\gamma\mathcal{H}_0(v_0, \theta_{iil}^-)\Delta v_0$. There are similar expressions for contributions from counter-passing and trapped particles. The proper sum of these contributions has the form

$$2\pi\gamma\Delta v_0(-\mathcal{H}_0(v_0, \theta_{iil}^-) + 2\mathcal{H}_0(v_0, \theta_{iil}^+) + \mathcal{H}_0(v_0, \theta_{iit}^-)).$$

With regard to the v_0 -fluxes, the relevant flux difference can be expressed as

$$2\pi\gamma(\langle\mathcal{G}_0\rangle_{bl}(v_0^+) - \langle\mathcal{G}_0\rangle_{bl}(v_0^-))$$

where it remains to define the averaged v_0 -flux, $\langle\mathcal{G}_0\rangle_{bl}$. While $\mathcal{F}_0$ and $\partial\mathcal{F}_0/\partial v_0$ are continuous across the boundary, $\partial\mathcal{F}_0/\partial\theta_0$ need not be. Furthermore, there may be discontinuities in the coefficients B_0 and C_0 . We may extend the definition of λ to include the boundary layer region through

$$\lambda(\theta_{iil}) = \mathcal{L}_{bl}/2 \sin \theta_{iil} \Delta\theta_{iil}$$

where $\Delta\theta_{iil}$ is the θ width of the boundary region. Using this notation we can define

$$\begin{aligned} A_0(\theta_{iil}) &= A_0(\theta_{iit}) \equiv \langle A_0 \rangle_{bl} \\ &\equiv \frac{1}{4}\lambda(\theta_{iil})\left(\frac{A_0}{\lambda}(\theta_{iil}^-) + \frac{A_0}{\lambda}(\theta_{iil}^+) + \frac{A_0}{\lambda}(\theta_{iit}^-) + \frac{A_0}{\lambda}(\theta_{iit}^+)\right) \end{aligned} \quad (2.7)$$

at velocity v_0 . The expression for $\langle B_0 \rangle_{bl}$ is similar. Terms of $\langle\mathcal{G}_0\rangle_{bl}$ in $\mathcal{F}_0$ and its v_0 derivative are thus unambiguously defined. Next, allowing for a discontinuity in $\partial\mathcal{F}_0/\partial\theta_0$ at θ_{iil} (and θ_{iit}), the associated average is

$$\begin{aligned} C_0(\theta_{iil})\frac{\partial\mathcal{F}_0}{\partial\theta_0}(\theta_{iil}) &= C_0(\theta_{iit})\frac{\partial\mathcal{F}_0}{\partial\theta_0}(\theta_{iit}) \equiv \left\langle C_0\frac{\partial\mathcal{F}_0}{\partial\theta_0} \right\rangle_{bl} \\ &\equiv \frac{1}{4}\lambda(\theta_{iil})\left(\frac{C_0}{\lambda}(\theta_{iil}^-)\frac{\partial\mathcal{F}_0}{\partial\theta_0}(\theta_{iil}^-) + \frac{C_0}{\lambda}(\theta_{iil}^+)\frac{\partial\mathcal{F}_0}{\partial\theta_0}(\theta_{iil}^+)\right. \\ &\quad \left.+ \frac{C_0}{\lambda}(\theta_{iit}^-)\frac{\partial\mathcal{F}_0}{\partial\theta_0}(\theta_{iit}^-) + \frac{C_0}{\lambda}(\theta_{iit}^+)\frac{\partial\mathcal{F}_0}{\partial\theta_0}(\theta_{iit}^+)\right). \end{aligned} \quad (2.8)$$

Using (2.7) and (2.8) we can now represent $\langle\mathcal{G}_0\rangle_{bl}$ as

$$\mathcal{G}_0(v_0, \theta_{iil}) = \mathcal{G}_0(v_0, \theta_{iit}) \equiv \langle\mathcal{G}_0\rangle_{bl} = \langle A_0 \rangle_{bl}\mathcal{F}_{0T} + \langle B_0 \rangle_{bl}\frac{\partial\mathcal{F}_{0T}}{\partial v_0} + \left\langle C_0\frac{\partial\mathcal{F}_{0T}}{\partial\theta_0} \right\rangle_{bl}. \quad (2.9)$$

* Since $\mathcal{G}_0$ in general may be discontinuous (ideal slip) across the layer, this averaging introduces a certain traction or mixing within the layer. However, since we are not concerned with the internal structure of the layer per se, this is inconsequential to the calculation as a whole.

Multiplying these fluxes by the appropriate finite differentials and combining produces a conservation law for the boundary layer which we write as

$$\begin{aligned} \frac{\partial N}{\partial t}(v_0) = 2\pi\gamma \Big[\Delta v_0 (-\mathcal{H}_0(v_0, \theta_{il}^-) + 2\mathcal{H}_0(v_0, \theta_{il}^+) + \mathcal{H}_0(v_0, \theta_{iu}^-)) \\ + (\sin \theta_{il} \Delta \theta_{il} + \sin \theta_{iu} \Delta \theta_{iu}) (\langle \mathcal{G}_0 \rangle_{bl}(v_0^+) - \langle \mathcal{G}_0 \rangle_{bl}(v_0^-)) \Big]. \end{aligned} \quad (2.10)$$

Finally, dividing both sides of (2.10) by the boundary layer volume element from (2.5), and passing to the limit as $\Delta v_0 \rightarrow 0$ there results the evolution equation for $\mathcal{F}_{0T}$:

$$\begin{aligned} \frac{\partial \mathcal{F}_{0T}}{\partial t}(v_0, \theta_{il}) = \frac{\gamma}{\mathcal{L}_{bl} v_0^2} \Big[(-\mathcal{H}_0(v_0, \theta_{il}^-) + 2\mathcal{H}_0(v_0, \theta_{il}^+) + \mathcal{H}_0(v_0, \theta_{iu}^-)) \\ + (\sin \theta_{il} \Delta \theta_{il} + \sin \theta_{iu} \Delta \theta_{iu}) \frac{\partial}{\partial v_0} \langle \mathcal{G}_0 \rangle_{bl}(v_0) \Big]. \end{aligned} \quad (2.11)$$

An examination of the limit of (2.11) as $\Delta \theta_{il} = \Delta \theta_{iu} = \theta_{il}^+ - \theta_{il}^- \rightarrow 0$ shows it reduces to the jump condition (1.51) as is required for consistency.

3. APPLICATIONS

In this section the application of CQL to two specific tokamak plasma problems is discussed. In the first of the three problems, neoclassical corrections to classical resistivity are calculated numerically for arbitrary inverse aspect ratio. The results are demonstrated to coincide with analytic expressions for the limiting cases in which such expressions exist in closed form. Then, turning attention to ICRF heating, we calculate the fusion reactivity enhancement due to second harmonic tail heating of a 50 - 50D - T mixture. In the third example, we describe a scanning charge exchange analyzer diagnostic simulation used to establish a direct correspondence between the model calculation and experimental data.

3.1. NEOCLASSICAL CORRECTIONS TO CLASSICAL RESISTIVITY

The presence of trapped particles in a tokamak plasma has a marked effect on the electrical conductivity. Analysis by Rosenbluth, et al.²⁵ yields the expansion in inverse aspect ratio ϵ

$$\sigma = \sigma_{sp} (1 - 1.95\sqrt{\epsilon} + O(\epsilon)) \quad (3.1)$$

in the limit $\epsilon \ll 1$. Here the classical Spitzer value for the conductivity is

$$\sigma_{sp} = 2 \frac{n_e e^2}{m_e} \tau_e \quad (3.2)$$

and the electron-ion momentum transfer time τ_e is

$$\tau_e = 3.44 \times 10^5 \frac{T_e^{3/2}}{n_e \lambda_{ei}} \quad (3.3)$$

Further analysis (Coppi and Sigmar²⁶) provides the $O(\epsilon)$ correction in (3.1) as

$$\sigma = \sigma_{sp} (1 - 1.95\sqrt{\epsilon} + .95\epsilon). \quad (3.4)$$

Connor, et al.² extended these estimates for finite ϵ . They express the distribution function in the form $f = f_s + \hat{f}$, where f_s is the distribution which would obtain in a uniform magnetic field calculation as described by Spitzer²⁷. The Spitzer-Härm collision operator (Spitzer and Härm²⁸) can be used to describe the bulk of the passing particles correctly while a model collision operator is applied to $\hat{f}$ under the assumption that $\hat{f}$ is highly localized in pitchangle. The analysis results in a value for the average conductivity, defined as $\bar{\sigma} = J_\phi / E_0$, where J_ϕ is the toroidal current density averaged over a magnetic surface and E_0 is the toroidal electric field at the magnetic axis. The value is given as

$$\bar{\sigma} = \sigma_{sp} I \left\{ \frac{1 - .039(1 - I)}{1 + .471(1 - I)} \right\} \quad (3.5)$$

where

$$I = \frac{3}{4} \langle B \rangle_{\theta_p} \int_0^{1/B_{max}} \frac{\lambda d\lambda}{\langle \sqrt{1 - \lambda B/B} \rangle_{\theta_p}} \quad (3.6)$$

Connor, et al.²⁸ make the point that as $\epsilon \rightarrow 1$ the distribution of passing particles becomes localized so that an accurate result can be gleaned in that case by just applying the model operator directly to f rather than to $\hat{f}$. The associated limit is $\bar{\sigma} = .93I\sigma_{sp}$ rather than the result of (3.5) which turns out to be $\bar{\sigma} = .71I\sigma_{sp}$. Those authors are confident in the accuracy of these results to within 20%.

The bounce-averaged Fokker-Planck code was used to compute resistivity enhancement $\bar{\eta}/\eta_{sp}$ due to toroidal effects over the full range of ϵ . The procedure employed was to choose an initial electron Maxwellian distribution and a fixed Maxwellian ion field distribution. The ions serve as a heat bath to allow the electrons to achieve a dynamic steady state. A small parallel electric field is applied, $E_0 \sim .03E_D$, where E_D is the Dreicer electric field, $E_D = m_e \nu_e v_e / 2e$, v_e is the electron thermal velocity, and ν_e is the electron self collision frequency, $\nu_e = 4\pi e^4 n_e \ln \Lambda_{ee} / m_e^2 v_e^3$. This choice of E_0 guarantees an exponentially small population of runaway electrons (electric force exceeds drag). Figure 11 shows the resistivity enhancement as a function of ϵ as determined by CQL (solid), as determined by (3.4) (upper dashed), and as determined by the Connor analysis (fine dashed). Note the universal agreement at the $\epsilon \rightarrow 0$ limit. The dashed line which starts at $\epsilon \sim .65$ represents the Connor result for the asymptotic case $\epsilon \rightarrow 1$. Agreement with the asymptotic limits is better than 1% in all cases in which agreement is expected.

For large the electric field, the parallel distribution, $\mathcal{F}$ integrated over $v_\perp$, exhibits a range of positive slope. This feature is characteristic of the *slideaway* regime described by Hui, et al.²⁹, and may be recognized in Figure 12 for which the values $E_\parallel = .07$ v/cm, $\epsilon = .09$ have been chosen. Figure 13 shows (a) the distribution $\mathcal{F}$ and (b) $\partial \mathcal{F} / \partial t$ after a time equivalent to about $50 \tau_{ei}$. Lastly, Figure 14 shows the relevant fluxes; the electric field flux (a) dominates the collisional flux (b) for most values of $v_\parallel > 0$ as evidenced by its similarity to the combined flux (c). Note the watershed at large positive $v_\parallel$ in the collisional flux. Collisional diffusion at this point is entirely pitchangle diffusion. As the electric field is increased, a larger proportion of electrons are found on runaway orbits. For $E_0 > .3E_D$ even a dynamic steady state becomes unachievable due to this effect in the absence of additional mitigating considerations. In fact, for these large electric fields, the concept of E_D becomes somewhat nebulous since it is defined in terms of the steady state.

3.2. ICRF INDUCED FUSION REACTIVITY ENHANCEMENT

We have examined the fusion reactivity enhancement by ICRF heating in large tokamaks by means of second harmonic excitation. The model consists of an excited species and a fixed Maxwellian field distribution. The steady state is achieved through a balance between resonant and collisional diffusion; energy is absorbed from the wave field by a deuterium distribution, then collisionally transferred to the fixed background tritium and electron distributions.

The reactivity enhancement is measured by dividing the resulting bounce-averaged fusion reaction rate for D-T fusion by that obtained using *equivalent* Maxwellian distributions in the same calculation. The equivalence here is defined by requiring that the collisional power transfer in the equivalent case from the Maxwellian replica of the excited ion distribution equal the RF power absorbed by the actual excited distribution. We present here an example of a calculation done as a part of a parameter study presented in Perkins et al.³⁰ and in greater

detail in Harvey, et al.³¹. The toroidal geometry in which the calculation takes place has the following dimensions: major radius $R_0 = 300$ cm, minor radius $a = 130$ cm, and flux surface minor radius $r = 65$ cm, yielding inverse aspect ratio $\epsilon = .217$. The tritium field distribution is a 4.5 keV Maxwellian of density $\sim 10^{14}$ cc⁻¹ at the midplane; the electrons preserve charge quasi-neutrality; the deuterium is originally thermal at 4.5 keV and 10^{14} cc⁻¹ at the midplane; the RF wave field chosen employs $k_{\parallel} = .0255$ cm⁻¹, and $k_{\perp} = .67$ cm⁻¹. The electric field has been adjusted dynamically so that the steady state absorbed power remains 1 watt/cc. The steady state excited distribution function which obtains is presented in Figure 15; the average energy is 6.9 keV.

The reactivity enhancement for this case was 5.9. The optimism which might otherwise accompany this result is mitigated by the rather low ratio of fusion power to RF power absorbed, $Q = .11$. For cases in which Q is closer to 1 the background tritium temperature is rather high, typically $T_i \sim 10$ keV, and the reactivity enhancement is generally less than 2. In this latter case a larger percentage of the plasma bulk populates regions of velocity space where fusion reactions are common, whereas in the former case virtually all fusion reactions which occur involve tail ions.

3.3. SCANNING CHARGE-EXCHANGE ANALYZER DIAGNOSTIC FOR TOKAMAKS

Tokamak fusion plasmas are generally considered to be fully ionized. However, a dilute population of neutral atoms usually coexists with the plasma in the confinement region. These neutrals may come from neutral beam sources used to heat the plasma and from Franck-Condon neutrals in the edge plasma due to refluxing at the limiter or wall. Driven by these sources, the velocity distribution of these primary neutrals and their progeny evolves through ionization and further charge-exchange, and through free streaming along straight line orbits.

The average velocity of a neutral (with the exception of the primary beam neutrals) is typically comparable to that of the ambient ions within a charge-exchange mean free path of the neutral's position. The time scale on which the neutral distribution relaxes to a quasi-steady state is thus a straight line thermal transit time across the plasma, $\tau_{n0} \sim a/v_{th}$. In a magnetic confinement device, this must be much faster than the fastest operant process working to change the ion distributions; i.e. slowing down, pitchangle diffusion, banana center diffusion. The neutral distribution can thus be viewed as determined by the ion distribution in the sense that following an initial transient the neutrals evolve quasistatically tracking the ions. These considerations suggest that measurement of the neutral spectrum at the plasma edge be used to monitor the ion distribution within the plasma. The presence of charge-exchange analyzer diagnostics on current experiments further motivated the construction of a numerical diagnostic for the Bounce-Averaged Fokker-Planck code. In this manner, it is possible not only to simulate the evolution of the plasma, but also the measurements taken, directly; the intent being that the diagnostic serve as a check to verify the adequacy of the model.

It is essential to the success of the diagnostic as a calibration tool that some realism be exercised in generating the background neutral distributions. However, the spectra of high energy neutrals appearing at the charge-exchange analyzer port are quite insensitive to the details of the neutral distribution within the plasma: Radially resolved profiles of background

neutral density and average energy are generally sufficient for accurate representation. To generate these profiles, we have used a neutral transport code developed by Burrell³² and modernized by Stockdale³³. The neutral transport calculation is performed in cylindrical geometry and includes the effects of wall (diffuse) reflection and volume neutral sources (neutral beam trapping, recombination). The details of the calculation have been treated elsewhere and will not be discussed further here.

The charge-exchange event leaves the velocities of the products essentially unchanged from their values prior to the interaction. This is the reason detection of neutrals can provide information about the ion velocity distribution. Unfolding the charge-exchange data gathered by the experimental diagnostic to recover the ion distribution is effectively the inverse of the calculation done in the numerical diagnostic where the ion distribution is known prior to the neutral spectra. Both calculations involve integrating the interaction (charge-exchange and ionization) operator over the neutral orbits from their birthplace to the detector. In one aspect the two exercises differ. The experimental apparatus generates a radially integrated composite signal, a line-averaged quantity. CQL provides radial resolution. Kaita, et al.³⁴ and Hammett, et al.²¹ have demonstrated RF signatures in charge-exchange analyzer data taken on PLT. It is gratifying to see that an effect which is understood on theoretical grounds should be so clearly discernable in experiment. This lends even more credence to the suggestion that a simulation of an experimental diagnostic such as this might well be used as a calibration tool to check the kinetic model employed in the calculation as a whole.

In the code calculation, we limit our model to the detection of neutrals born on and traveling in the equatorial plane of the torus. This simplifies the geometry and, moreover, in most cases also coincides with the experimental setup. The procedure uses the calculated small banana-width ion distribution as the ensemble of particles (in the finite banana-width case) all of which orbit the same bounce-averaged flux surface, $\Psi(r) = \langle \Psi \rangle = \bar{\Psi}$. Each $(v_0, \theta_0, \bar{\Psi})$ -orbit crosses the equatorial plane twice in a full bounce (or transit) period.

In order that a charge-exchange event be detectable, it must coincide with an equatorial plane crossing. The position of the crossing is obtained through the solution of (1.54), the condition for conservation of canonical toroidal angular momentum. An iterative scheme is employed to accomplish this. The equation is first expressed as a transcendental relation in $r = \Psi^{-1}(\Psi)$ at equatorial plane crossing. Solution then yields the minor radius position of the guiding center at those bouncephase points. Co-passing particles cross the equatorial plane outside the (toroidal) flux surface $\Psi = \bar{\Psi}$ at poloidal angle $\theta_p = 0$ and inside at $\theta_p = \pi$; for counter-passing particles the case is reversed; trapped particles cross outside on the co-moving leg and inside on the counter-moving leg, both crossings occurring for $\theta_p = 0$.

Figure 16 depicts the outcome of this calculation as contour maps in the midplane velocity system. All gyrocenter orbits which pierce the midplane velocity space (viewed as a Poincare map of the orbits) along a given contour will cross the equatorial midplane of the torus (a physical plane) at the same minor radius. In this sense, each contour can be viewed as the image in midplane velocity space of the locus of points in physical space all of a given minor radius at which detectable charge-exchange events can take place. Coincidentally, the contour corresponding to minor radius $r = a$ is also sometimes referred to as the *loss cone boundary*. The two figures differ in the passing orbit region of velocity space: (a) shows the

outboard equatorial crossings and (b) shows the inboard equatorial crossings. This distinction is unnecessary for trapped orbits.

The detection of neutrals born at these equatorial plane crossings occurs along the line of sight corresponding to the straight line orbit of the neutral subsequent to its birth. For purposes of the orbit integration, the charge-exchange event is assumed to occur at the position of the ion gyrocenter.

The line of sight is parameterized by R_T , the major radius at which it is tangent to a (toroidal) flux surface. This tangency radius, R_T , is related to the pitchangle of the gyromotion at equatorial plane crossing, $\theta_{ep}(v_0, \theta_0)$ and to the pitchangle of the magnetic field lines, β . A short exercise in geometry demonstrates that

$$R_T = (R_0 + r + \Delta r_B) \cos \theta_{ep} / \cos \beta$$

where R_0 is the major radius, r is the flux surface minor radius, $\Delta r_B(v_0, \theta_0)$ is the finite banana-width radial excursion at equatorial plane crossing, and β is given by

$$\cos \beta = \frac{B_T}{|B|} = \hat{e}_s \cdot \hat{b}$$

with $\theta_{ep} \geq \beta$. Orbits with insufficient $v_\perp$ according to this last relation, are not detectable by the analyzer, the resultant neutral velocity is not directed in the equatorial plane. Figure 17 shows contours of equatorial R_T on the midplane velocity space. Those orbits which underlie a given contour correspond to ions which upon charge exchange at equatorial plane crossing generate neutrals moving along lines of sight with equal R_T . Note there are no contours near $\theta_0 = 0, \pi$, for the reasons mentioned previously. Figure 18 shows the geometrical arrangement of the scanning charge exchange analyzer (in physical space).

The neutral flux is attenuated along the line of sight to the detector by interaction with the ambient plasma according to

$$\Phi = \Phi_0 \exp \left(- \int \frac{dl}{\lambda(l)} \right).$$

The interaction mean free path $\lambda(l)$ viewed as a function of orbit position is given by

$$v\lambda^{-1} = n_i \langle \sigma v \rangle_{cx} + \langle \sigma v \rangle_{ii} + n_e \langle \sigma v \rangle_{ei}.$$

Here the charge-exchange, ion and electron impact ionization reaction rates are averaged over a Maxwellian ambient background characterized by (radially resolved) profiles preset as mentioned earlier. λ is a function of neutral energy and orbit position. The attenuation integral calculation is done as a setup for each mesh point, stored in a *sensitivity* array, and used repeatedly to interpret current values of the ion distributions at meshpoints as neutral spectra in R_T . The calculation is arranged so different flux surfaces can be considered through parallel calculations in consistently reproduced ambient conditions. This allows direct correlation with the experimental diagnostic through summation. In view of Figure 17 and Figure 16, it is clear neutrals born of a given (average) flux surface ensemble whose subsequent trajectories are associated with a given R_T each suffer different attenuation on their journey to the analyzer, though all enter the analyzer from the same direction.

Figure 19 shows the spectral neutral flux at the charge exchange analyzer port as a function of R_T for various energies. These spectra are computed by the neutral transport package (Stockdale³³). That calculation presumes a cylindrically symmetric plasma; there are neither trapped particle effects nor finite banana width effects operant in the model. The spectra are the result of charge exchange events which occur in the equatorial plane between the ambient neutrals and the local Maxwellian ion background, attenuated along the line of sight to the charge exchange analyzer port, and (the result) integrated over the minor radius. At large R_T the neutrals entering the analyzer are neutrals which have been born at the plasma periphery, and therefore their spectra are characteristic of a lower average energy.

Figure 20 shows the results of a related calculation. Here the thermal neutral spectra arise through charge exchange of ions all in the same (average) flux surface ensemble with ambient neutrals: Finite banana width effects are included. For example, notice that charge exchange events which occur on the (outer) co-moving leg of trapped orbits produce larger neutral fluxes at the analyzer than those which occur on the (inner) counter-moving leg. This is a consequence of both larger neutral density at larger minor radii and the relatively smaller attenuation integral associated with outboard events. Figure 21 represents the same case as Figure 19 after the application of fundamental minority species heating RF excitation. The RF excited distribution mapped through this diagnostic procedure is that shown in Figure 6. The finite banana width equivalent to Figure 19 would be a sum of results like Figure 21, one from each range of flux surfaces.

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APPENDIX A: BOUNDARY LAYER DIAGNOSTIC

In order to implement the numerical solution of Fokker-Planck equations in non-uniform magnetic fields in conservation form (cf. (1.13-1.14)) it is necessary to compute integrals of the form

$$N = \int d^3v_0 \mu_0 \tau_B \mathcal{F}_0 = 2\pi \int dv_0 v_0^3 \int d\mu_0 \mu_0 \tau_B \mathcal{F}_0 \equiv 2\pi \int dv_0 v_0^3 I. \quad (A1)$$

This is problematic since τ_B diverges logarithmically in the neighborhood of the boundary separating trapped and circulating particle orbits. In magnetic fields of the form we consider here, this boundary is represented by the lines $\mu_0 = \pm\mu_{0T}$: In the more general field case, the more general boundary may have a more complicated form, and the velocity space orbit topology (cf. Section 1.3) may be considerably more complicated than that considered herein, yet there still arise integrals of the form (A1) with the same inherent difficulty.

The integral I in (A1) is well defined and finite and an algorithm for evaluating it accurately and efficiently is essential for problems in which one is concerned with treating fluxes between circulating and trapped orbits physically.

Let us limit the discussion which follows to a thin layer on either side of the boundary between trapped and circulating orbits. Further, let us define $\mu_0^- < \mu_0 < \mu_0^+$, where $\mu_0^- < \mu_{0T} < \mu_0^+$. By (1.51) and the discussion thereafter, we may approximate $\mathcal{F}_0$ as piecewise linear in μ_0 on this domain and take

$$\mathcal{F}_0 = \begin{cases} \alpha^- + \mu_0 \beta^- & \mu_0^- < \mu_0 < \mu_{0T} \\ \alpha^+ + \mu_0 \beta^+ & \mu_{0T} < \mu_0 < \mu_0^+. \end{cases} \quad (A2)$$

Also define

$$I_{bl} = \int_{\mu_0^-}^{\mu_0^+} d\mu_0 \mu_0 \tau_B \mathcal{F}_0 \quad (A3)$$

and for convenience let $\mathcal{F}_0(\mu_{0T}) \equiv \mathcal{F}_{0T}$, $\mathcal{F}_0(\mu_0^\pm) \equiv \mathcal{F}_0^\pm$, and $\mu_0 = \mu_{0T}(1 - \delta)$ where δ ranges over the values $-\Delta < \delta < \Delta$.

To get an analytic representation for I_{bl} it is necessary to choose a $\psi(s)$, magnetic field, so as to obtain an expression for τ_B in the neighborhood of $\mu_0 = \mu_{0T}$. To illustrate the calculation, let $\psi(s)$ be given by

$$\psi(s) = \frac{1 + \epsilon}{1 + \epsilon \cos \pi s / s_{mz}} \quad (A4)$$

where ϵ can be viewed as the inverse aspect ratio of an axisymmetric torus $\epsilon = r/R_0$ and $\pi s / s_{mz}$ can be viewed as the poloidal angle. The bounce time is then given by

$$\tau_B = \int_0^{s_B} \frac{ds}{v_{||}} = \int_0^{s_B} \frac{ds}{v_0 \sqrt{1 - \psi(s)(1 - \mu_0^2)}}. \quad (A5)$$

Using the identity $\cos \phi = 1 - 2 \sin^2(\phi/2)$ and the transformation $\zeta = \pi s/2s_{mx}$. (A5) can be recast as

$$\tau_B = \frac{2s_{mx}}{\pi v_0 \mu_0} \int_0^{\zeta_B} d\zeta \frac{\sqrt{1 - \mu_{0T}^2 \sin^2 \zeta}}{\sqrt{1 - (\mu_{0T}^2/\mu_0^2) \sin^2 \zeta}} \quad (A6)$$

where

$$\zeta_B = \begin{cases} \frac{1}{2} \arccos(1 - 2\mu_0^2/\mu_{0T}^2) & \mu_0^2 \leq \mu_{0T}^2 \\ \frac{1}{2} \pi & \mu_0^2 > \mu_{0T}^2 \end{cases} \quad (A7)$$

By expanding the numerator in (A6), a series expansion for τ_B can be generated each of whose terms is an integral which can be represented in terms of elliptic integrals:

$$\tau_B = \frac{2s_{mx}}{\pi v_0 \mu_0} \sum_{m=0}^{\infty} \alpha_m \mu_{0T}^{2m} \int_0^{\zeta_B} d\zeta \frac{\sin^{2m} \zeta}{\sqrt{1 - (\mu_{0T}^2/\mu_0^2) \sin^2 \zeta}} \quad (A8)$$

where the coefficients α_m are

$$\alpha_0 = 1, \alpha_1 = -\frac{1}{2}, \dots, \alpha_m = \frac{2m-3}{2m} \alpha_{m-1}. \quad (A9)$$

Using the fact that $\sin^2 \zeta_B = \mu_0^2/\mu_{0T}^2$, the series (A8) can be evaluated recursively as

$$\tau_B = \frac{2s_{mx}}{\pi v_0 \mu_0} \sum_{m=0}^{\infty} \alpha_m \mu_{0T}^{2m} J_{2m} \quad (A10)$$

where

$$J_{2m} = \frac{1}{2m-1} \left((2m-2) \left(1 + \frac{\mu_0^2}{\mu_{0T}^2} \right) J_{2m-2} - (2m-3) \frac{\mu_0^2}{\mu_{0T}^2} J_{2m-4} \right). \quad (A11)$$

To begin the recursion, values for J_0 and J_2 are required; a short calculation shows these functions are just

$$J_0 = \begin{cases} K\left(\frac{\mu_{0T}^2}{\mu_0^2}\right) & \mu_0^2 \geq \mu_{0T}^2 \\ \frac{\mu_0}{\mu_{0T}} K\left(\frac{\mu_0^2}{\mu_{0T}^2}\right) & \mu_0^2 \leq \mu_{0T}^2 \end{cases} \quad (A12)$$

and

$$J_2 = \begin{cases} \frac{\mu_0^2}{\mu_{0T}^2} \left[K\left(\frac{\mu_{0T}^2}{\mu_0^2}\right) - E\left(\frac{\mu_{0T}^2}{\mu_0^2}\right) \right] & \mu_0^2 \geq \mu_{0T}^2 \\ \frac{\mu_0}{\mu_{0T}} \left[K\left(\frac{\mu_0^2}{\mu_{0T}^2}\right) - E\left(\frac{\mu_0^2}{\mu_{0T}^2}\right) \right] & \mu_0^2 \leq \mu_{0T}^2 \end{cases} \quad (A13)$$

where K and E are complete elliptic integrals.

Using the Hastings polynomial approximation for K and E near modulus 1 and expanding the modulus in δ , the contribution to I from the boundary layer $-\Delta \leq \delta \leq \Delta$ can be written as

$$I_{bl} = \frac{2s_{mx}}{\pi v_0} \sum_{m=0}^{\infty} \alpha_m \mu_{0T}^{2m+1} \left\{ \int_{-\Delta}^0 d\delta (\alpha^- + \beta^- \mu_{0T}(1+\delta)) J_{2m}^-(\delta) + \int_0^{\Delta} d\delta (\alpha^+ + \beta^+ \mu_{0T}(1+\delta)) J_{2m}^+(\delta) \right\} \quad (A14)$$

where $J_{2m}^{\pm}$ are the Hastings approximations to the elliptic integrals indicated in (A12) and (A13). Performing the integrals and working to first order in Δ , we find

$$I_{bl} = \frac{4s_{mx} \mathcal{F}_{0T} \Delta}{\pi v_0} [\mu_{0T} \sqrt{1 - \mu_{0T}^2} (k' - k'' (\ln 2\Delta - 1)) - \sum_{m=0}^{\infty} \alpha_m \mu_{0T}^{2m+1} \beta_m + O(\Delta^2 \ln \Delta)] \quad (A15)$$

where k' and k'' are coefficients in the Hastings expansion for K , $K(x) = k' - k'' \ln x + O(x)$, and β_m is given by the recursion relation

$$\beta_m = [2(2m-2)\beta_{m-1} - (2m-3)\beta_{m-2}]/(2m-1) \quad (A16)$$

with $\beta_0 = 0$ and $\beta_1 = 1$.

A more general field model is also implemented in the code **CQL**. The technique just outlined is applied in a slightly modified form, again involving the integration of an expansion of the bounce time in the neighborhood of its singularity. The result is evaluated to first order in the boundary layer thickness: Details can be found in Killeen, et al.²²

APPENDIX B: NUMERICAL SOLUTION OF BOUNCE-AVERAGED FOKKER-PLANCK EQUATIONS

The procedure for bounce averaging the Fokker-Planck coefficients is streamlined by choosing the set of points $\{v, \theta\}$, given s_1 , at which to calculate the local Rosenbluth potentials. For a given orbit, it is a particular combination of derivatives of the local potentials (Killeen, et al.²²) which determines the local Fokker-Planck coefficients. Furthermore, it is a particular weighted sum of these local coefficients *along the given gyrocenter trajectory* which corresponds to the bounce-averages indicated in Eqs. (1.18). Therefore, the natural choice for the points in question is that given by (1.8): Namely, choose $\{v, \theta\} = \{v_j, \theta(\theta_i, s_1)\}$. This corresponds to choosing local θ mesh points which when traced back along an orbit to the midplane will correspond there to the predefined midplane mesh point θ_i . This removes the need to interpolate during the bounce-average procedure and expedites the calculation.

Implicit Time Advancement: Operator Splitting

The Fokker-Planck equation (1.7) with source and loss terms included may be written in the form (cf. (1.14))

$$\frac{\partial \lambda \mathcal{F}_0}{\partial t} = \gamma \left(\frac{1}{v_0^2} \frac{\partial \mathcal{G}_0}{\partial v_0} + \frac{1}{v_0^2 \sin \theta_0} \frac{\partial \mathcal{H}_0}{\partial \theta_0} \right) + \lambda \hat{K} \mathcal{F}_0 + \lambda S \quad (\text{B1})$$

where $\mathcal{G}_0$ and $\mathcal{H}_0$ incorporate collisional, quasilinear and ohmic terms. Here S is a generalized source of the type described in Killeen, et al.²² and $\hat{K}$ is a Krook operator representing various loss terms as described in Section 1.4.

For notational simplicity we suppress the subscript on variables $\mathcal{F}_0$, v_0 , and θ_0 for purposes of writing finite difference equations. It should be realized, however, that all the differencing is done in the midplane system.

We construct pitchangle and velocity meshes corresponding to $\{\theta_i, i = 1, I\}$; $\{v_j, j = 1, J\}$. While the mesh is required to be symmetric about $\frac{1}{2}\pi$, there is no mesh point at $\frac{1}{2}\pi$. We will use the convention in the following that θ_M denotes the largest θ mesh point less than $\frac{1}{2}\pi$.

Velocity differences come in the two varieties $\Delta v_j = \frac{1}{2}(v_{j+1} - v_{j-1})$ and $\Delta v_{j+\frac{1}{2}} = (v_{j+1} - v_j)$, quantities determined at half mesh points obey $B_{j+\frac{1}{2}} = \frac{1}{2}(B_j + B_{j+1})$, and we will represent the distribution function on the mesh by the abbreviation $\mathcal{F}_{i,j} = \mathcal{F}(v_j, \theta_i)$. Similarly, θ differences are written $\Delta \theta_i = \frac{1}{2}(\theta_{i+1} - \theta_{i-1})$ and $\Delta \theta_{i+\frac{1}{2}} = (\theta_{i+1} - \theta_i)$. The boundary regions about θ_{iM} and θ_{i1} are chosen as thin slivers typically 5% of the θ mesh spacing elsewhere. This aids the accuracy of the technique used to treat the boundary layer as described in Appendix A and Section 2.

An operator splitting scheme is used to time advance the kinetic equation (B1). While the technique is well known (Richtmyer²⁴), its application in this case is somewhat non-standard owing to the non-local nature of the differential operator due to the orbit topology (cf. Section 1.3). In this connection, certain features of the bounce-averaged system are unique and warrant description in their own right. Therefore, the procedure will be described in some detail here with particular emphasis on the manner in which density conservation is assured.

The conserved quantity, the numerical approximation to the field line density N (2.1), is given by

$$N = \sum_{i=1}^I \sum_{j=1}^J a_i b_j \lambda_i \mathcal{F}_{i,j} \quad (\text{B2})$$

where

$$a_i = \begin{cases} 2\pi \sin \theta_i \Delta \theta_i & \text{for } i \notin \{1, I, itl, itu\} \\ \pi \theta_{\frac{3}{2}}^2 & \text{for } i = 1 \\ \pi (\pi - \theta_{I-\frac{1}{2}})^2 & \text{for } i = I \\ \pi \mathcal{L}_{bl} & \text{for } i \in \{itl, itu\} \end{cases}$$

$$b_j = \begin{cases} v_j^2 \Delta v_j & \text{for } j \notin \{1, J\} \\ \frac{1}{3} v_{\frac{3}{2}}^2 & \text{for } j = 1 \end{cases}$$

and where we require that $\mathcal{F}$ vanish at the upper velocity extent of the mesh. $\mathcal{F}_{i,J} = 0$.

Further, we have defined

$$\lambda_i = \begin{cases} v_j \cos \theta_i \tau_B(v_j, \theta_i) & \text{for } i \notin \{itl, itu\} \\ \mathcal{L}_{bl}/2 \sin \theta_{itl} \Delta \theta_{itl} & \text{for } i \in \{itu, itl\}. \end{cases}$$

The numerical equations employed for each half of the operator splitting scheme are structured to conserve particle density exactly in the absence of particle sources and sinks. The equations describing the first half of the procedure, the split in the velocity v_0 direction are

$$\lambda_i \mathcal{F}_{i,j}^{n+\frac{1}{2}} = \begin{cases} \lambda_i \mathcal{F}_{i,j}^n \\ + \frac{\gamma \Delta t}{v_j^2 \Delta v_j} (g_{i,j+\frac{1}{2}}^{n+\frac{1}{2}} - g_{i,j-\frac{1}{2}}^{n+\frac{1}{2}}) \\ + \frac{\Delta t}{2} \lambda_i \hat{K}_{i,j}^n \mathcal{F}_{i,j}^{n+\frac{1}{2}} + \frac{\Delta t}{2} \lambda_i S_{i,j}^n \end{cases} \quad \text{for } \begin{cases} \forall i \\ 1 < j < J \end{cases}$$

for interior meshpoints, and

$$\lambda_i \mathcal{F}_{i,j}^{n+\frac{1}{2}} = \begin{cases} 0 & \text{for } \begin{cases} \forall i \\ j = J \end{cases} \\ \lambda_i \mathcal{F}_{i,1}^n \\ + \frac{3\gamma \Delta t}{v_{\frac{3}{2}}^3} g_{i,\frac{3}{2}}^{n+\frac{1}{2}} \\ + \frac{\Delta t}{2} \lambda_i \hat{K}_{i,1}^n \mathcal{F}_{i,1}^{n+\frac{1}{2}} + \frac{\Delta t}{2} \lambda_i S_{i,1}^n \end{cases} \quad \text{for } \begin{cases} \forall i \\ j = 1 \end{cases} \quad (B3)$$

for boundary points. Furthermore, the symmetry constraint in the boundary region forces

$$\mathcal{F}_{itu,j}^{n+\frac{1}{2}} = \mathcal{F}_{itl,j}^{n+\frac{1}{2}}; \quad \forall j. \quad (B4)$$

In the preceding, the v_0 -flux $\mathcal{G}$ is given in terms of the coefficients and divided differences as

$$\mathcal{G}_{i,j+\frac{1}{2}}^{n+\frac{1}{2}} = \begin{cases} \left(A^n \mathcal{F}^{n+\frac{1}{2}} \right)_{i,j+\frac{1}{2}} + B_{i,j+\frac{1}{2}}^n \frac{\mathcal{F}_{i,j+1}^{n+\frac{1}{2}} - \mathcal{F}_{i,j}^{n+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}} \\ + C_{i,j+\frac{1}{2}}^n \frac{\mathcal{F}_{i+1,j+\frac{1}{2}}^n - \mathcal{F}_{i-1,j+\frac{1}{2}}^n}{2\Delta\theta_i} & \text{for } i \notin \{itl, itu\} \\ \left(A^n \mathcal{F}^{n+\frac{1}{2}} \right)_{itl,j+\frac{1}{2}} + B_{itl,j+\frac{1}{2}}^n \frac{\mathcal{F}_{itl,j+1}^{n+\frac{1}{2}} - \mathcal{F}_{itl,j}^{n+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}} \\ + \frac{1}{4} \lambda_{itl} \left\{ \left(\frac{C}{\lambda} \right)_{itl-1,j}^n \frac{\mathcal{F}_{itl,j}^n - \mathcal{F}_{itl-1,j}^n}{\Delta\theta_{itl-\frac{1}{2}}} \right. \\ + \left(\frac{C}{\lambda} \right)_{itl+1,j}^n \frac{\mathcal{F}_{itl+1,j}^n - \mathcal{F}_{itl,j}^n}{\Delta\theta_{itl+\frac{1}{2}}} \\ + \left(\frac{C}{\lambda} \right)_{itu-1,j}^n \frac{\mathcal{F}_{itu,j}^n - \mathcal{F}_{itu-1,j}^n}{\Delta\theta_{itu-\frac{1}{2}}} \\ \left. + \left(\frac{C}{\lambda} \right)_{itu+1,j}^n \frac{\mathcal{F}_{itu+1,j}^n - \mathcal{F}_{itu,j}^n}{\Delta\theta_{itu+\frac{1}{2}}} \right\} & \text{for } i \in \{itl, itu\} \end{cases} \quad (B5)$$

where the averages have been performed as indicated by (2.7), (2.8), and (2.9) over the meshpoints nearest to θ_{itl} and θ_{itu} .

It can be shown that $C_{i,j+\frac{1}{2}} = 0$ for $i \in \{1, I\}$. Also, the superscripts $n + \frac{1}{2}$ and $n + 1$ are used to indicate time dependent quantities after the first (v) and second (θ) splits respectively. The cross derivatives are explicit and $\mathcal{F}$ is defined uniquely at the origin only after the second split.

That (B3) conserves density can be verified by forming the numerical derivative of (B2):

$$N^{n+\frac{1}{2}} - N^n = \sum_{i=1}^I \sum_{j=1}^J a_i b_j \lambda_i \left(\mathcal{F}_{i,j}^{n+\frac{1}{2}} - \mathcal{F}_{i,j}^n \right). \quad (B6)$$

Eliminating the source and sink terms in (B3) and substituting into (B6) gives

$$N^{n+\frac{1}{2}} = N^n.$$

The second half of the operator splitting scheme, now in the θ direction, is represented by the following: The interior points are advanced by

$$\lambda_i \mathcal{F}_{i,j}^{n+1} = \begin{cases} \lambda_i \mathcal{F}_{i,j}^{n+\frac{1}{2}} \\ + \frac{\gamma \Delta t}{v_j^2 \sin \theta_i \Delta \theta_i} \left(\mathcal{H}_{i+\frac{1}{2},j}^{n+1} - \mathcal{H}_{i-\frac{1}{2},j}^{n+1} \right) \\ + \frac{\Delta t}{2} \lambda_i \left(\hat{K}_{i,j}^n \mathcal{F}_{i,j}^{n+1} + S_{i,j}^n \right) \end{cases} \quad \text{for } \begin{cases} i \notin \{itl, itu\} \\ i \notin \{1, m, I\} \\ \forall j \end{cases}$$

while the boundary points are advanced by

$$\lambda_i \mathcal{F}_{i,j}^{n+1} = \begin{cases} \lambda_1 \mathcal{F}_{1,j}^{n+\frac{1}{2}} + \frac{2\gamma\Delta t}{v_j^2 \theta_{\frac{1}{2}}^2} \mathcal{X}_{\frac{1}{2},j}^{n+1} + \frac{\Delta t}{2} \lambda_1 (\hat{K}_{1,j}^n \mathcal{F}_{1,j}^{n+1} + S_{1,j}^n) & \text{for } \begin{cases} i = 1 \\ \forall j \end{cases} \\ \lambda_m \mathcal{F}_{m,j}^{n+\frac{1}{2}} - \frac{\gamma\Delta t}{v_j^2 \sin \theta_m \Delta \theta_m} \mathcal{X}_{m-\frac{1}{2},j}^{n+1} + \frac{\Delta t}{2} \lambda_m (\hat{K}_{m,j}^n \mathcal{F}_{m,j}^{n+1} + S_{m,j}^n) & \text{for } \begin{cases} i = m \\ \forall j \end{cases} \\ \lambda_I \mathcal{F}_{I,j}^{n+\frac{1}{2}} - \frac{2\gamma\Delta t}{v_j^2 (\pi - \theta_{I-\frac{1}{2}})^2} \mathcal{X}_{I-\frac{1}{2},j}^{n+1} + \frac{\Delta t}{2} \lambda_I (\hat{K}_{I,j}^n \mathcal{F}_{I,j}^{n+1} + S_{I,j}^n) & \text{for } \begin{cases} i = I \\ \forall j \end{cases} \\ \lambda_{iul} \mathcal{F}_{iul,j}^{n+\frac{1}{2}} - \frac{\gamma\Delta t}{2v_j^2 \sin \theta_{iul} \Delta \theta_{iul}} (\mathcal{X}_{iul-\frac{1}{2},j}^{n+1} - 2\mathcal{X}_{iul+\frac{1}{2},j}^{n+1} - \mathcal{X}_{iul+\frac{3}{2},j}^{n+1}) + \frac{\Delta t}{2} \lambda_{iul} (\hat{K}_{iul,j}^n \mathcal{F}_{iul,j}^{n+1} + S_{iul,j}^n) & \text{for } \begin{cases} i = iul \\ \forall j \end{cases} \end{cases} \quad (B7)$$

The value of $\mathcal{F}$ at the origin will be discussed presently. In the algorithm (B7) we have defined

$$\mathcal{X}_{i+\frac{1}{2},j}^{n+1} = (D^n \mathcal{F}^{n+1})_{i+\frac{1}{2},j} + E_{i+\frac{1}{2},j}^n \frac{\mathcal{F}_{i+\frac{1}{2},j+1}^{n+\frac{1}{2}} - \mathcal{F}_{i+\frac{1}{2},j-1}^{n+\frac{1}{2}}}{2\Delta v_j} + F_{i+\frac{1}{2},j}^n \frac{\mathcal{F}_{i+1,j}^{n+1} - \mathcal{F}_{i,j}^{n+1}}{\Delta \theta_{i+\frac{1}{2}}}. \quad (B8)$$

Density is conserved identically during the second half of the splitting algorithm as in the first half. This may be shown in an exercise similar to that in (B6).

There remains the matter of a unique value for $\mathcal{F}$ at the origin, $v_0 = 0$. This is accomplished by requiring that

$$\mathcal{F}_{i,1}^{n+1} = \frac{\sum_{i=1}^I a_i \lambda_i \mathcal{F}_{i,1}^{n+\frac{1}{2}}}{\sum_{i=1}^I a_i \lambda_i} \quad \forall i. \quad (B9)$$

The algorithm for solving the two sets of equations (B3) and (B7) representing three coupled tri-diagonal systems, is standard. However, the coupling of the three systems corresponding to the co- and counter-passing regions and the trapped region of velocity space, is somewhat uncommon and is presented here in the interest of completeness.

Equations (B3) may be cast in the following canonical form:

$$-\alpha_{i,j}^n \mathcal{F}_{i,j+1}^{n+\frac{1}{2}} + \beta_{i,j}^n \mathcal{F}_{i,j}^{n+\frac{1}{2}} - \gamma_{i,j}^n \mathcal{F}_{i,j-1}^{n+\frac{1}{2}} = \delta_{i,j}^n \quad (\text{B10})$$

where α , β , γ , and δ are known at time step n . The indices i, j range over all values on the mesh; vacuous terms such as $\gamma_{i,1}^n$ are ignored since $\mathcal{F}_{i,0}$ does not exist.

The procedure for solving (B10) involves two sweeps and is standard (Richtmyer²⁴). Recast (B10) as

$$\mathcal{F}_{i,j}^{n+\frac{1}{2}} = \sigma_{i,j} \mathcal{F}_{i,j+1}^{n+\frac{1}{2}} + \tau_{i,j} \quad (\text{B11})$$

where

$$\sigma_{i,j} = (\alpha_{i,j}^n - \gamma_{i,j}^n \sigma_{i,j-1})^{-1} \alpha_{i,j}^n \quad (\text{B12})$$

and

$$\tau_{i,j} = (\beta_{i,j}^n - \gamma_{i,j}^n \sigma_{i,j-1})^{-1} (\delta_{i,j}^n + \gamma_{i,j}^n \tau_{i,j-1}). \quad (\text{B13})$$

The quantities $\sigma_{i,1}$ and $\tau_{i,1}$ determined through the relevant boundary relation ($j = 1$) in (B3) allow initialization of the recurrence relation for $j > 1$. The condition for ($j = J$) in (B3) determines $\mathcal{F}_{i,J}$ thus initializing the back solve to evaluate $\mathcal{F}_{i,j}$ for $j < J$.

The velocity split thus involves only two sweeps (a forward recursion, then a back solve) over the index j for each value of index i . The pitchangle split, however, involves six sweeps over the index i for each value of index j . Each sweep-solve pair corresponds to one of the three sectors of velocity space joined at the boundary layer; namely $[0, \theta_{il}]$, $[\frac{1}{2}\pi, \theta_{il}]$, and $[\pi, \theta_{il}]$. The first three boundary relations in (B7) are used to initialize each of the three forward sweeps to determine the $\{\sigma\}$ and the $\{\tau\}$. Then the last condition of (B7), which links the regions at the boundary layer, is used to initialize each respective back solve.

Schematically, the procedure is as follows: With the exception of the cases $i \in \{itl, itu\}$ the difference equations (B7) are cast in the form

$$-\epsilon_{i,j}^n \mathcal{F}_{i+1,j}^{n+1} + \mu_{i,j}^n \mathcal{F}_{i,j}^{n+1} - \nu_{i,j}^n \mathcal{F}_{i-1,j}^{n+1} = \varrho_{i,j}^n \quad (\text{B14})$$

(again ignoring vacuous terms such as $\nu_{i,1}$). At $i \in \{itl, itu\}$ (B7) does not assume the form (B14) since the boundary layer involves values of $\mathcal{F}$ at four points at the advanced time rather than three. Expressed in recursive form in the index i , (B14) is treated in the same manner as (B10) on each sector separately; viz the forward recursion over $[\frac{1}{2}\pi, \theta_{il}]$ is represented by

$$\mathcal{F}_{i,j}^{n+1} = \sigma_{i,j} \mathcal{F}_{i-1,j}^{n+1} + \tau_{i,j}; \quad i = itl + 1, \dots, m.$$

The relevant $\{\sigma\}$ and $\{\tau\}$ are determined through recursive equations with $\sigma_{i,j}$, $\tau_{i,j}$ for $i \in \{1, m, I\}$ set by the boundary relations in (B7).

One further piece of information is necessary to complete the algorithm. The value of $\mathcal{F}$ in the boundary layer given by the last relation in (B7) is used to initialize the three back solve recursions as follows: Recasting that condition in the form

$$\mathcal{F}_{itl,j}^{n+1} = \epsilon_{itl,j}^n \mathcal{F}_{itl+1,j}^{n+1} + \mu_{itl,j}^n \mathcal{F}_{itu+1,j}^{n+1} + \nu_{itl,j}^n \mathcal{F}_{itl-1,j}^{n+1} + \varrho_{itl,j}^n \quad (\text{B15})$$

we substitute the recurrence formula for $\mathcal{F}_{iil,j}^{n+1}$ into (B15) to obtain

$$\begin{aligned} \mathcal{F}_{iil,j}^{n+1} = & \epsilon_{iil,j}^n \left(\sigma_{iil+1,j} \mathcal{F}_{iil,j}^{n+1} + \tau_{iil+1,j} \right) + \mu_{iil,j}^n \left(\sigma_{iit+1,j} \mathcal{F}_{iit,j}^{n+1} + \tau_{iit+1,j} \right) \\ & + \nu_{iil,j}^n \left(\sigma_{iil-1,j} \mathcal{F}_{iil,j}^{n+1} + \tau_{iil-1,j} \right) + \varrho_{iil,j}^n. \end{aligned} \quad (\text{B16})$$

This is the equation for $\mathcal{F}_{iil,j}^{n+1}$ whose solution initializes the back solve recursions.

The drawbacks of operator splitting are primarily associated with limitations on the size of the timestep in connection with stability constraints. Since the scheme is not fully implicit, and because it is a two-step algorithm, temporal evolution of the solution is the only practical way to the steady state. In cases where the steady state is of principal interest, the algorithm, while functional, is less than optimal.

APPENDIX C: TANGENT RESONANCE PHENOMENA

We wish to evaluate bounce-averages such as $\langle \langle S_n \sqrt{\psi} e^{i\psi} \rangle \rangle$ (cf. (1.35-1.38)) in the neighborhood of tangent resonance. We represent these trajectory integrals as

$$I = \langle \langle \Pi(\tau) e^{i\Psi(\tau)} \rangle \rangle \quad (\text{C1})$$

where

$$\Psi = - \int_0^\tau dr' (\omega - k_{\parallel} v_{\parallel} - m\Omega)(r') = - \int_0^\tau dr' \nu(r') \quad (\text{C2})$$

and Π is a slowly varying function of τ . Near the stationary phase points we expand the (rapidly oscillating phase) factor Ψ as:

$$\Psi = \Psi_R + \nu \Delta t + \dot{\nu} \frac{\Delta t^2}{2!} + \ddot{\nu} \frac{\Delta t^3}{3!} + \dots, \quad (\text{C3})$$

Δt being measured from the stationary phase point ($\nu = 0$). Placing (C3) in (C1) we can write

$$I = \sum_i \frac{1}{\tau_B} \int_{\tau_i - D\tau}^{\tau_i + D\tau} d\Delta\tau \Pi(\tau_i + \Delta\tau) \exp \left[i \left(\Psi_R + \nu \Delta\tau + \dot{\nu} \frac{\Delta\tau^2}{2!} + \ddot{\nu} \frac{\Delta\tau^3}{3!} \right) \right]. \quad (\text{C4})$$

Here $\Delta\tau = \tau - \tau_i$, $\nu(\tau_i) = 0$. Now we pass to the limit $D\tau \rightarrow \infty$:

$$\begin{aligned} \lim_{D\tau \rightarrow \infty} I &= \sum_i \frac{1}{\tau_B} \Pi(\tau_i) e^{i\Psi_R(\tau_i)} \int_{-\infty}^{+\infty} d\Delta\tau \exp \left[i \dot{\nu} \frac{\Delta\tau^2}{2!} \right] \\ &= \sum_i \frac{1}{\tau_B} \Pi(\tau_i) e^{i\Psi_R} e^{i\pi/4} \sqrt{\frac{2\pi}{\dot{\nu}}}. \end{aligned} \quad (\text{C5})$$

A simple modification to the result (C5) can be used to model the effects of gyrophase decorrelation. Let the n -bounce interaction history in the case of a single isolated resonance be represented in the form

$$I_n = \Pi(\tau_R) |\tau_c| \Sigma_n; \quad \nu(\tau_R) = 0 \quad (C6)$$

where Σ_n is defined recursively by

$$\Sigma_n = \Sigma_{n-1} \left\{ 1 - e^{-i4(\Psi_z + \delta\phi_n)} \right\} = S_n \Sigma_0 \quad (C7)$$

with

$$\Sigma_0 = e^{i(\Psi_R + \pi/4 + \delta\phi_0)} + e^{-i(\Psi_R + \pi/4 + \delta\phi_0)} \quad (C8)$$

Here we have introduced the phase integrals

$$\Psi_R = \int^{\tau_R} d\tau \nu(\tau); \quad \Psi_x = \int^{\tau_B} d\tau \nu(\tau)$$

and the random gyrophase increments $\delta\phi_m$. These gyrophase increments may be caused by collisions or random fluctuations in the RF field structure. For purposes of illustration, we choose the distribution of the $\delta\phi_m$ to be Gaussian:

$$P(\delta\phi_m) = \frac{1}{\sqrt{\pi} D_m} e^{-(\delta\phi_m/D_m)^2}$$

where D_m will depend on the collisionality and the orbit parameters.

The coefficients represented in (1.35-1.38) involve products of the single bounce interaction with the entire prior history of the interaction, viz.

$$B_{0q} \sim \lim_{n \rightarrow \infty} I_0^* I_n = \Pi^2(\tau_R) \tau_c^2 (\Sigma_0^* \Sigma_\infty + cc.) \quad (C9)$$

Within the context of certain simplifying assumptions, we may perform the ensemble average over the distribution of random gyrophase increments $\delta\phi$: Defining

$$\langle Q \rangle_m = \int_{-\infty}^{+\infty} d\delta\phi_m P(\delta\phi_m) Q$$

it is easily shown that

$$\langle \Sigma_0^* \Sigma_n \rangle_0 = 2S_n \left(1 + e^{-D_0^2} \cos 2(\Psi_R + \pi/4) \right)$$

For simplicity, let $P_m \equiv P \forall m > 0$ so that the same distribution of gyrophase increments is operant on any given bounce (Cohen, et al.²³ have shown the diffusion on subsequent bounces in cumulative; this, however, is incidental to the argument here). Iterating the procedure outlined and using the recursion (C7), the gyrophase decorrelated version of (C9) becomes

$$\langle B_{0q} \rangle_\infty = \Pi^2(\tau_R) \tau_c^2 2 \left(1 + e^{-D_0^2} \cos 2(\Psi_R + \pi/4) \right) \Upsilon \quad (C10)$$

where coherence carried through prior bounces is represented by

$$\Upsilon = \frac{2 \left(1 + e^{-4D^2} \cos 4\Psi_x \right)}{1 - 2e^{-4D^2} \cos 4\Psi_x + e^{-8D^2}}$$

In practice, D is sufficiently large to wash out any prior bounce coherence in modern tokamaks (though marginally so in mirror end cells). In particular, Υ can never become singular for $D > 0$. However, D_0 can quite generally be small for *single* bounce resonances which are not well separated, yet not close enough to ignore the gyrophase diffusion.

For cases in the proximity of $\nu(\tau_i) = \nu(\tau_i) = 0$, tangent resonance, $D_0 \rightarrow 0$, the quantity I as represented in (C5) diverges, gyrophase diffusion vanishes, and the procedure becomes a bit more elaborate. Expanding about the tangent resonance point we have for (C4) that

$$I = \frac{1}{\tau_B} \int_{\tau_i - D\tau}^{\tau_i + D\tau} d\Delta\tau \Pi(\tau_i + \Delta\tau) \exp \left[i \left(\Psi_R + \nu\Delta\tau + \ddot{\nu} \frac{\Delta\tau^3}{3!} \right) \right]. \quad (C11)$$

Π is a weak function of τ and we represent it in the interval $\tau_i - D\tau < \tau < \tau_i + D\tau$ as

$$\Pi(\tau) = \Pi_R(\tau_i) + \dot{\Pi}_R(\tau_i) \Delta\tau \quad (C12)$$

so that (C11) may be written as

$$I = \frac{1}{\tau_B} \left\{ \Pi_R(\tau_i) e^{i\Psi_R(\tau_i)} \int_{-\infty}^{+\infty} d\Delta\tau \exp \left[i \left(\nu\Delta\tau + \ddot{\nu} \frac{\Delta\tau^3}{3!} \right) \right] + \dot{\Pi}_R(\tau_i) e^{i\Psi_R(\tau_i)} \int_{-\infty}^{+\infty} d\Delta\tau \Delta\tau \exp \left[i \left(\nu\Delta\tau + \ddot{\nu} \frac{\Delta\tau^3}{3!} \right) \right] \right\}. \quad (C13)$$

In the following, we make use of the relation defining the Airy function:

$$\frac{2\pi}{\sqrt[3]{3a}} \text{Ai} \left(\pm \frac{x}{\sqrt[3]{3a}} \right) = \text{Re} \int_{-\infty}^{+\infty} d\zeta e^{i(a\zeta^3 \pm x\zeta)}. \quad (C14)$$

From (C14) there follows directly that

$$\begin{aligned} \frac{d}{dx} \frac{2\pi}{\sqrt[3]{3a}} \text{Ai} \left(\pm \frac{x}{\sqrt[3]{3a}} \right) &= \pm \frac{2\pi}{\sqrt[3]{3a}^2} \text{Ai}' \left(\pm \frac{x}{\sqrt[3]{3a}} \right) \\ &= \text{Re} \pm \int_{-\infty}^{+\infty} d\zeta i\zeta e^{i(a\zeta^3 \pm x\zeta)}. \end{aligned} \quad (C15)$$

Clearly, then,

$$\int_{-\infty}^{+\infty} d\Delta\tau \Delta\tau \exp \left[i \left(\nu\Delta\tau + \ddot{\nu} \frac{\Delta\tau^3}{3!} \right) \right] = -i2\pi \left(\sqrt[3]{\frac{2}{\ddot{\nu}}} \right)^2 \text{Ai}' \left(\nu \sqrt[3]{\frac{2}{\ddot{\nu}}} \right). \quad (C16)$$

Using (C16) in (C13) we can represent I as

$$I = \frac{1}{\tau_B} \left\{ \Pi_R(\tau_i) e^{i\Psi_R(\tau_i)} 2\pi \sqrt[3]{\frac{2}{\bar{\nu}}} Ai \left(\nu \sqrt[3]{\frac{2}{\bar{\nu}}} \right) - i \dot{\Pi}_R(\tau_i) e^{i\Psi_R(\tau_i)} 2\pi \left(\sqrt[3]{\frac{2}{\bar{\nu}}} \right)^2 Ai' \left(\nu \sqrt[3]{\frac{2}{\bar{\nu}}} \right) \right\} \quad (C17)$$

where $\dot{\nu}(\tau_i) = 0$ and $\nu / \sqrt[3]{2/\bar{\nu}} \leq \zeta_0 \approx 1$. The latter expression serves to define *proximity* as alluded to following (C5). We evaluate $\dot{\Pi}_R(\tau_i)$ as follows:

$$\dot{\Pi}_R(\tau_i) = v_{\parallel}(\tau_i) \frac{\partial \Pi_R}{\partial s} = v_{\parallel} \frac{\partial \Pi_R}{\partial \psi} \frac{\partial \psi}{\partial s}(\tau_i). \quad (C18)$$

From (C18) we notice immediately that

$$\dot{\Pi}_R = 0 \text{ for } \psi'(\tau_i) = 0; \quad \text{midplane and throat resonance} \quad (C19)$$

and

$$\dot{\Pi}_R = 0 \text{ for } v_{\parallel}(\tau_i) = 0 \Rightarrow k_{\parallel} = 0 \quad \text{tangent (turning point) resonance.} \quad (C20)$$

The resonant diffusion tensor elements $B_{0_{qi}} \rightarrow F_{0_{qi}}$ (cf. (1.35-1.38)) can be written in the form

$$\begin{aligned} B_{0_{qi}} &= B_0(b^*b + cc), & C_{0_{qi}} &= C_0(b^*f + cc) \\ F_{0_{qi}} &= F_0(f^*f + cc), & E_{0_{qi}} &= E_0(f^*b + cc) \end{aligned} \quad (C21)$$

so that

$$\begin{aligned} \mathbf{D}_{0_{qi}} &= B_{0_{qi}} \hat{\mathbf{e}}_{v_0} \hat{\mathbf{e}}_{v_0} + C_{0_{qi}} \hat{\mathbf{e}}_{v_0} \hat{\mathbf{e}}_{\theta_0} + E_{0_{qi}} \hat{\mathbf{e}}_{\theta_0} \hat{\mathbf{e}}_{v_0} + F_{0_{qi}} \hat{\mathbf{e}}_{\theta_0} \hat{\mathbf{e}}_{\theta_0} \\ &= \begin{pmatrix} B_{0_{qi}} & C_{0_{qi}} \\ E_{0_{qi}} & F_{0_{qi}} \end{pmatrix} \end{aligned} \quad (C22)$$

Now since the relation $B_0 F_0 = C_0 E_0$ holds everywhere, outside the proximity of tangent resonance it is easily shown that $B_{0_{qi}} F_{0_{qi}} = C_{0_{qi}} E_{0_{qi}}$, or, stated another way, $\det \mathbf{D}_{0_{qi}} = 0$ and the diffusion is one dimensional.

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Figure Captions

FIG. 1. Phaseflow corresponding to the set of velocity space meshpoints chosen to represent the distribution $\mathcal{F}$. The abscissa is the arclength along the magnetic field measured from the midplane normalized to the arclength from the midplane (minimum B) to the throat (maximum B). The ordinate is the cosine of the pitchangle along a gyrocenter trajectory. The orbits above the separatrix correspond to co-passing orbits, those below the separatrix correspond to counter-passing orbits, and the elliptical trajectories correspond to trapped orbits. Since kinetic energy is a constant of the motion, these orbits are independent of v_0 . The vertical curves represent the relation $\nu = \omega - k_{\parallel} v_{\parallel} - m\Omega = 0$ in this space for a set of values v_0 on the chosen velocity mesh. The solid curves are those for which $k_{\parallel} > 0$ and the dotted curves are those for which $k_{\parallel} < 0$. The intersections of the gyrocenter trajectories and the curves $\nu = 0$ are the resonances corresponding to positions of stationary wave-particle interaction phase. For $k_{\parallel} = 0$, the vertical curves all collapse into a single vertical line passing through the common intersection of the curves. That intersection corresponds to the turning point resonance. For $k_{\parallel} \neq 0$ the loci of points in the phase space at which wave-particle resonance occurs coincidentally with $\dot{\nu} = 0$, tangent resonance, is shown as the remaining curve. This curve, of course, also goes through the turning point resonance, since there is no parallel Doppler shift at that point.

FIG. 2. Resonance loci. Loci in the midplane velocity space of points corresponding to orbits all of which pass through resonance at the same arclength position (or equivalently, the same poloidal angle). Some velocity space points are covered by two resonance loci. These orbits pass through resonance twice, at two different arclength positions. Along the inner boundary of this region, the two resonance loci covering a given point are tangent to one another, and to a curve we will call the tangent resonance locus.

FIG. 3. Tangent resonance loci. Loci in the midplane velocity space of points corresponding to orbits which pass through resonance ($\nu = \omega - k_{\parallel} v_{\parallel} - m\Omega = 0$) with $\dot{\nu} = 0$. As $k_{\parallel} \rightarrow 0$ the tangent resonance curves approach straight lines, corresponding to turning point resonance. These straight lines coincide with the quasilinear resonant diffusion asymptotes. Wave-particle interaction causes no pitchangle diffusion along these lines.

FIG. 4. Diffusion characteristics of the quasilinear resonant diffusion operator. Resonant diffusion for a definite wave frequency is one dimensional. Phase space density diffuses along these curves under the action of waves of a definite frequency, driven by density gradients locally tangent to the curves. For small $k_{\parallel}$ the asymptotes correspond to orbits which bounce at resonance.

FIG. 5. The quasilinear coefficients of (1.43-1.46) as functions of $\cos \theta_0$ for various values of v_0 . The prominence in the energy diffusion coefficients occurs in the vicinity of tangent resonance. This particular case represents single frequency fundamental minority species excitation with rather small $k_{\parallel}$. The vertical lines coincide with the trapped/passing boundary. The energies represented are too small to see significant finite gyrophase interference effects, and the mesh spacing is too coarse to depict bouncephase interference.

FIG. 6. The ion distribution function of a minority species excited at its fundamental cyclotron harmonic. In (a) is shown the initial state prior to excitation, in (b) is shown the distribution at $t = 2$ msec.

FIG. 7. **Velocity space fluxes for a minority species excited at its fundamental cyclotron harmonic.** The three contour plots show the fluxes driven by (a) resonant diffusion, (b) collisional diffusion, and (c) the combination of these. The energy diffusion is maximal in the neighborhood of the tangent resonances (cf. Appendix C).

FIG. 8. **Gyrocenter orbits projected on a constant toroidal angle plane for various energies and pitchangles.**

FIG. 9. **Contour map of bounce-averaged charge-exchange Krook term on midplane velocity space** discussed in Section 1.4 and Section 3.3. Contours are level curves of ion loss by charge exchange with the ambient neutral background resulting from a neutral transport calculation. The solid bold contours show where the edge of the gas layer and the wall would be, in the sense that orbits on these contours have finite banana excursions which bring them into contact with those respective (minor) radii ($r/a = .9, 1.$). The position of the flux surface with which the (small banana width) ensemble is associated is $r/a = .85$.

FIG. 10. **Boundary layer geometry.** One of four volume elements adjacent to the boundary separating trapped and passing orbits is detailed. Orbit ensemble averages of fluxes in these volume elements are employed to define fluxes in and into the boundary layer (see Section 2.3 and Appendix B).

FIG. 11. **Ratio of resistivity (neoclassical value) to Spitzer resistivity (classical value), η/η_S as a function of inverse aspect ratio, ϵ :** (a) solid line - code result; (b) upper broad dashed - determined by (3.3.4); (c) fine dashed - determined by the analysis of Connor, et al.²; Dashed line beginning at $\epsilon = .65$ is the Connor, et al.² result for the asymptotic case $\epsilon \rightarrow 1$. (see Section 3.1).

FIG. 12. **Parallel velocity electron distribution function.** $\mathcal{F}_{\parallel} = \int dv_{\perp} v_{\perp} \mathcal{F}$ vs $v_{\parallel}$. The (parallel velocity) distribution describes electrons in a large parallel electric field ($E_{\parallel} = .07$ v/cm) after evolving from a Maxwellian for a period of $t \sim 50 \tau_{ei}$. Note the positive slope at small $v_{\parallel}$. (see Section 3.1)

FIG. 13. **Electron slideaway distribution function.** (a) Contours of distribution $\mathcal{F}$ for the same case as Figure 12 at $t \sim 50 \tau_{ei}$. (b) Contours of $\Delta t \partial \mathcal{F} / \partial t$ for the same case.

FIG. 14. **Velocity space fluxes for the same case as Figure 13:** (a) Electric field driven velocity flux, (b) collisional flux and (c) combined fluxes are shown. Note that (a) dominates (b) for nearly all positive $v_{\parallel}$.

FIG. 15. **Contours of the steady state ion distribution function $\mathcal{F}$ represented in midplane velocity space coordinates ($v_{\parallel 0}, v_{\perp 0}$).** Tail structure sustained by second harmonic RF excitation (see Section 3).

FIG. 16. **Equatorial plane crossings.** All gyrocenter orbits which pierce the midplane velocity space (viewed as a Poincare map of the orbits) along a given contour will cross the equatorial plane of the torus (a physical plane) at the same minor radius. In this sense, each contour can be viewed as the image in midplane velocity space of the locus of points in physical space all of a given minor radius at which detectable charge-exchange events can take place. Coincidentally, the contour corresponding to minor radius $r = a$ is also sometimes referred to as the "loss cone boundary". The two figures differ in the passing orbit region of velocity space: (a) shows the outboard equatorial crossings and (b) shows the inboard equatorial crossings. This distinction is unnecessary for trapped orbits (cf. Section 3.3).

FIG. 17. **Contours of equatorial R_T on the midplane velocity space.** Those orbits which underlie a given contour correspond to ions which upon charge exchange at equatorial plane crossing generate neutrals moving along lines of sight with equal R_T . There are no contours near $\theta_0 = 0, \pi$, since those orbits have insufficient $v_\perp$ to produce neutrals whose motion remains in the equatorial plane.

FIG. 18. **Scanning charge exchange analyzer geometry viewed in the equatorial plane of the Tokamak.**

FIG. 19. **Spectral neutral flux at the charge exchange analyzer port as a function of R_T for various energies.** These spectra are computed by the neutral transport package (Stockdale³³). That calculation presumes a cylindrically symmetric plasma; there are neither trapped particle effects nor finite banana width effects operant in the model. The spectra are the result of charge exchange events which occur in the equatorial plane between the ambient neutrals and the local Maxwellian ion background, attenuated along the line of sight to the charge exchange analyzer port, and (the result) integrated over the minor radius. At large R_T the neutrals entering the analyzer are neutrals which have been born at the plasma periphery, and therefore their spectra are characteristic of a lower average energy.

FIG. 20. **Thermal neutral spectra arising through charge exchange of ions all in the same (average) flux surface ensemble with ambient neutrals:** Finite banana width effects are included. Charge exchange events which occur on the (outer) co-moving leg of trapped orbits produce larger neutral fluxes at the analyzer than those which occur on the (inner) counter-moving leg. This is a consequence of both larger neutral density at larger minor radii and the relatively smaller attenuation integral associated with outboard events.

FIG. 21. **Neutral spectra: RF signature.** The same case as Figure 20 after the application of fundamental minority species heating RF excitation. The distributions mapped through this diagnostic procedure are shown respectively in Figures 6(a) and 6(b). The finite banana width equivalent to Figure 19 would be a sum of results like Figure 21, one for each flux surface.

Figure 1

Figure 2

Figure 3

Figure 4

Figure 6

Figure 6

Figure 7

Figure 8

Figure 9

Figure 10

Figure 11

Figure 12

Figure 13

Figure 14

Figure 15

Figure 16

Figure 17

Figure 19

Figure 20

Figure 21