ENERGY VARIATIONS IN DIFFUSIVE CAVITY GROWTH

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ABSTRACT

We discuss the assignment of boundary values for the chemical potential and the calculation of energy release rates for the growth of creep cavities along grain boundaries by self-diffusion. For simplicity it is assumed that the boundaries are flat and that surface and grain-boundary diffusion are the dominant transport mechanisms. As matter diffuses from the void surface into and along the grain boundary, misfit residual stresses are induced to alleviate the high stress concentration ahead of the cavity apex. As a result, it is shown that the contribution of strain energy terms to the chemical potential can be neglected in typical cases. Also, contrary to the Griffith crack extension model, the energy dissipation incurred by diffusive removal of material from the cavity surface and deposition in the grain boundary is a major term in the energy transfers associated with cavity growth. We show that the primary energy "sink" in diffusive cavity growth arises from the work done by the grain-boundary normal stress when matter is inserted in the near-tip region by diffusion, and not from the loss of strain energy of matter that is removed from the cavity at its tip or from a work of bond separation. We also comment on thermodynamic restrictions on the angle formed by the void surfaces at their apex, where they join the grain boundary. Further, our derivation of boundary values for the chemical potential is carried out in a manner appropriate for arbitrarily large but elastic distortions of material near the cavity tip and, by contrast to most previous work in the area, we include rigorously the effects of surface tension (i.e., of "surface stress", as distinct from surface energy).

I.   INTRODUCTION

When subject to creep conditions, polycrystalline materials tend
to develop cavities at grain interfaces. Their growth and final
coalescence leads to intergranular failure. The kinetics of cavity
growth by diffusion has become a topic of extensive recent interest,
and there is not full agreement in the literature on fundamentals of
the subject. The papers by Hull and Rimmer (1959) and others (Speight
and Harris, 1967; Weertman, 1973; Vitovec, 1972; Raj and Ashby, 1975)
considered the diffusive growth of an array of spherical-caps (or
lenticular] cavities in a planar grain interface. This assumption as
to cavity shape implies that surface diffusion is sufficiently rapid
that cavity growth is controlled only by grain boundary (g.b.) diffusion.
Further, the grains were assumed to be non-deforming (rigid), and the
result predicted is that the growth rate varies linearly with the applied
stress. However, conditions do not always allow this quasi-equilibrium
cavity shape and g.b. cavities sometimes have an elongated, crack-like
shape. Hence, Chuang and Rice (1973) and others (Vitek, 1978; Speight
et al., 1978; Beere and Speight, 1978; Chuang et al., 1979) considered the
other limiting case of a thin crack-like cavity growing in a g.b. and
established different relationships between applied stress and growth
speed.

In an overview of non-equilibrium models, Chuang et al. (1979)
examined the entire spectrum of interfacial void shapes in diffusive
cavitation and concluded that over a variety of conditions, a cavity can
grow from one extreme case of slow advance in a lenticular shape to the
other extreme case of rapid advance in a crack-like shape. The growth

mechanisms on which those analyses are based are such that under the action of the applied stress normal to the boundary where cavities are located, matter on the cavity surfaces is driven by surface diffusion toward the cavity apex and into and along the g.b.

Based on this kinetic model, it is worthwhile to investigate the thermodynamic forces driving a cavity. Indeed, the subject of chemical forces in a stressed body containing defects has been studied by several authors. Specifically, Stevens and Dutton (1971) considered Griffith crack propagation by mass transport and formulated the thermodynamic potentials for incremental advance of crack due to various diffusion paths. McCartney (1977) and Heald and Speight (1977) investigated the thermodynamic stability of a cavity and determined the shape at which subcritical growth can be maintained. Their studies are essentially limited to cases in which cavities are located in a perfect crystal. However, it is well recognized that grain boundaries are perfect matter sinks (or vacancy sources) and the fact that voids are often observed at the g.b. suggests that it plays an important role during the stages of nucleation and growth.

The objective of this paper is to re-examine the formulation of chemical potentials and the calculation of energy release rates in cavity growth. We place emphasis on the special features induced by the existence of grain boundaries. Their action as matter sinks can change the whole picture of the formulation. Specifically, in contrast to Griffith cracks, we show that the strain energy term in the chemical potential expression can typically be neglected even in cavity growth in a relatively narrow crack-like shape. We also show that the major sink

of energy in cavity growth is from the work of normal stresses on the effective opening of the grain boundary due to the addition of matter to it by diffusion.

While our considerations have modelled the material as an elastic solid, undergoing self-diffusion along interfaces, it is well to remember that the cavitation process is often accompanied by significant amounts of plastic creep flow through dislocation motion.

II. THE BOUNDARY VALUES OF CHEMICAL POTENTIAL

In phenomenological terms, diffusive fluxes of a substance are
driven by thermodynamic forces generated by chemical potential gradients.
Consider a stressed body containing defects in the form of voids and
grain boundaries, and suppose for simplicity that the body consists
of a single component which, again for simplicity, is taken as isotropic
with respect to its elastic and surface energy properties. Temperature
is held constant and we suppose that an unstressed matter reservoir,
without defects, is available at the same temperature as the body.
All thermodynamic potentials, including the chemical potential μ , can
arbitrarily be assigned the value zero in the reservoir. It is conven-
tional to define μ in units of energy per atom although the concepts
are continuum in character and treat matter as being indefinitely
divisible; atoms are merely a convenient unit for mass.

The classic work on establishing boundary values for μ is that
of Herring (1951,1953). We follow essentially his procedures, but three
comments are in order. First, since we shall assume that matter is
always added coherently to surfaces, at full composition equilibrium with
the immediately adjoining bulk, there is no need to distinguish separate
potentials for atoms and vacancies as he does. Indeed, Herring makes
similar assumptions about matter addition, so that in the end the distinc-
tion is unnecessary and only $\mu_a - \mu_v$ appears in his formulae ("a" denotes
atoms, "v" vacancies). Second, although Herring classifies carefully the
difference between surface energies and surface tensions (i.e., surface
stresses), in a context which assumes clearly that surface energies are
dependent on elastic strains of surface elements, in his evaluations of μ

he gives no discussion of energy changes that come about because matter
addition will, in general, alter the strains along a surface. We derive
expressions for μ in a manner appropriate to arbitrarily large elastic
distortions (important because some models for diffusive cavitation assume
large strains at the cavity tip) and show, by a simple verbal argument
in the following text and a more detailed analysis in the Appendix, that
Herring's expressions do in fact remain valid so long as the terms within
them are properly referenced to the current deformed state. Third,
although some of Herring's discussion of potentials is directed to
(global) non-equilibrium states, notably to diffusion, his methods of
evaluating μ are phrased mostly in terms of conditions for equilibrium.
Our methods are somewhat more in keeping with the spirit of irreversible
thermodynamics although, certainly, we adopt tacitly the notions of local
thermodynamic equilibrium which are inherent to the accepted, if thereby
limited, procedures of that subject.

To proceed, we observe that μ is defined at any location within
the stressed body such that $\mu\delta n$ is the reversible work of taking an
infinitesimal element of matter, equivalent to δn atoms, from the reser-
voir and placing it at that location, while the displacements are held
fixed on the loaded external boundary. Equivalently, $\mu\delta n = \delta F$ where
F is the total Helmholtz free energy of the body, and more generally,
when matter is transferred without restrictions on the external boundary
displacements,

$$\delta F = \delta W + \mu\delta n \qquad (1)$$

where δW is the work of external loads. Our concern here is with

matter addition to (or removal from) void surfaces and grain boundaries. If these, collectively, have an area in the current deformed configuration denoted by A , then μ satisfies

$$\delta F = \delta W + \int_A \mu \ \delta N \ dA \qquad (2)$$

for arbitrary additions of infinitesimal matter layers to A , consisting of δN atoms per unit (current) area. Expressions in the form of (2) arise in "internal variable" formulations (Rice, 1974) of inelastic processes in solids, due to structural rearrangements of constituent elements of material by diffusion or (with appropriate terms analogous to $\mu \ \delta N$) slip, phase changes, etc. In applying (2) we will generally write $\delta N = \delta t / \Omega$ where δt is the local thickness of the (coherently) added layer and Ω is the local volume per atom in the deformed configuration. Further, F will be represented as $F_e + F_s$, where F_e is the elastic strain energy, of local density w per unit volume of the current deformed configuration, and F_s is the energy of interfaces, of density γ per unit (current) area ($\gamma = \gamma_s$ on cavity surfaces and $\gamma = \gamma_b$ on grain boundaries).

Consider the addition of a layer of matter of (variable) local thickness δt over some portion of a cavity surface, under circumstances for which the external boundary is fixed ($\delta W = 0$) . To obtain μ we need to calculate δF , and this has contributions from the following sources: (i) the added layer must be given a strain energy w to make it fit coherently; (ii) the area of the surface changes such that, if additional

elastic strains $\delta\varepsilon_{\alpha\beta}$ of surface elements are neglected, $\delta(dA) =$ $-(\kappa_1 + \kappa_2)\delta t\, dA$ by an easy geometric construction, where κ_1 and κ_2 are the principal curvatures and the sign convention is such that both would be positive on a spherical cavity; (iii) surface elements strain by $\delta\varepsilon_{\alpha\beta}$ causing a change in γ and an additional change in dA ; (iv) the tractions acting on the bulk solid immediately below the surface, which are in general non-zero whenever surface tensions exist, are carried through some additional displacements δu_i when matter is added, thus causing a change in strain energy of material external to the surface. Herring's (1951,1953) discussion makes no reference to (iii) and (iv). It is carried out as if the product $\gamma\, dA$ is unaffected by the strains $\delta\varepsilon_{\alpha\beta}$; in such circumstances the contribution from (iii) would be zero and because (see Appendix) the surface tensions vanish in these circumstances, there would then be no contributions from (iv) either. (Rice and Drucker (1967) observed that when matter is added or removed from an unstressed surface the energy alteration analogous to (iv) is of second order in δt , hence zero for our present purposes).

We evaluate δF as if $\gamma\, dA$ was invariant to strain and then explain briefly (relegating a detailed analysis to the Appendix) why the result is valid in general. Hence the contributions to δF arise from (i) and (ii), and are

$$\delta F = \int_A w\, \delta t\, dA + \int_A \gamma_s [-(\kappa_1+\kappa_2)\delta t]dA \ . \tag{3}$$

If we now write $\delta t = \Omega\, \delta N$ and compare with (1), it is seen that

$$\mu = w\Omega - \gamma_s(\kappa_1+\kappa_2)\Omega \tag{4}$$

on a cavity surface. This is the same as the expression given by Herring, so long as it is understood that w, Ω, γ_s and the κ's are referenced to the current deformed state at the place of matter addition.

Now, let us drop the generally untenable assumption that γ dA is invariant to surface strain and that surface tensions vanish. We can calculate δF by first carrying out steps (i) and (ii), while applying to the system whatever system of (workless) forces are necessary to prevent occurrence of the surface strains $\delta\varepsilon_{\alpha\beta}$ and displacements δu_i. This gives a δF exactly as in eq. (3) and the net additional δF, associated with steps (iii) and (iv), is just minus the work of removing this additional set of forces. But the final state to which the system goes is one of elastic (as opposed to compositional, or surface shape) equilibrium, and at this state F is stationary with respect to elastic distortions. Hence the net work of removing the set of constraining forces is a quantity of second order in δt, and the net contribution of (iii) and (iv) to δF is therefore zero to the first order terms of interest. Hence (3) and, particularly, (4) for μ are valid generally. A fuller analysis, necessarily of some mathematical complexity, is given in the Appendix. Indeed, it is shown there that when surface tensions exist δF_e and δF_s both separately contain terms of first order in δt (in addition to those already listed in eq. 3), but these terms are of opposite sign and cancel when $\delta F = \delta F_e + \delta F_s$ is formed.

Often, throughout the body of the paper, we simplify derivations by neglecting surface tension and the variation of γ dA with surface strain. As we remark, the final results in each case are valid without these assumptions, provided that terms are properly interpreted, and details

of complete derivations are given in the Appendix.

Consider now the addition of a matter layer δt to a flat grain boundary subject to tensile stress, again under conditions for which $\delta W = 0$. The layer must be given strain energy appropriate for coherent fit, and the boundary must be separated by the distance δt to allow the new matter, resulting in a change $-\int \sigma \delta t \, dA$ of strain energy of the adjoining material, where σ is the normal stress ("true" stress, i.e., force per unit current area) acting on the grain boundary. Thus

$$\delta F = \int_A (w - \sigma) \delta t \, dA \qquad (5)$$

and writing $\delta t = \Omega \delta N$,

$$\mu = w\Omega - \sigma\Omega \qquad (6)$$

along a grain boundary. Again, this agrees with Herring's result so long as the terms are properly referenced to the current deformed state, and (see Appendix) the result remains valid when surface tension along the grain boundary and dependence of γ_b on strain is included.

Indeed, this last formula makes evident our reasons for care in defining quantities per unit volume or area of the deformed configuration. The difference between $\sigma\Omega$ and $\sigma\Omega_r$, where Ω_r is the value of Ω in the unstressed reservoir, is itself of the same order as the strain energy term $w\Omega$ in (6). If, instead $\hat{w}$ is the strain energy per unit volume as measured in the unstressed state, and $\hat{\sigma}$ and $\hat{\gamma}_s$ are force and surface energy per unit area of the unstressed state (so that $\hat{\sigma}$ is a

"nominal" stress), then it is straightforward to show that (4) and (6) are modified to

$$\mu = \hat{w}\Omega_r - \hat{\gamma}_s (1+\epsilon_n)(\kappa_1+\kappa_2)\Omega_r \qquad (4')$$

on the cavity surface and

$$\mu = \hat{w}\Omega_r - \hat{\sigma}(1+\epsilon_n)\Omega_r \ , \qquad (6')$$

on the grain boundary, where ϵ_n is the strain of adjoining material elements in the direction normal to the surface being considered. Obviously, $\hat{\sigma}\epsilon_n$ is of the same order as $\hat{w}$, and hence great care is required in discussing the effect of strain energy contributions to the chemical potentials (Stevens and Dutton, 1971; McCartney, 1977; Charles, 1976; Dutton and Puls, 1978). Similar remarks apply to (4') since $\kappa_1\gamma_s$ is generally of the same order as σ at a cavity tip (see below).

In a material capable of matter transport by diffusion, μ must be continuous at the cavity apex where it meets the grain boundary; otherwise an unbounded matter flux would result there. Hence, when the strain energy terms are negligible

$$\gamma_s(\kappa_{1_0}+\kappa_{2_0}) = \sigma_0 \qquad (7)$$

where κ_{1_0} , κ_{2_0} and σ_0 are, respectively, the cavity surface curvatures as the apex is approached along the cavity walls and the grain boundary normal stress as the apex is approached. Using this formula as an approximate

estimate of stress in the vicinity of the apex, we can estimate the relative importance of the strain energy terms. For simplicity, consider cylindrical voids under plane strain conditions $(\kappa_1 = \kappa$, $\kappa_2 = 0)$ and assume linear elasticity. Then

$$w \approx \frac{1-\nu^2}{2E} \; \sigma_o^2 \approx \frac{\gamma_s \kappa_o}{2E} \gamma_s \kappa_o \approx \frac{b}{50\rho_o} \gamma_s \kappa_o \tag{8}$$

where b is the lattice spacing, $\rho_o = 1/\kappa_o$ is the radius of curvature of the cavity wall at the apex, and we use the estimate $\gamma_s \approx Eb/25$. Since ρ_o typically has values of 50 to 500 b (i.e., 100 Å to 1 μm) , the strain energy term is quite negligible.

We emphasize, however, that this result occurs because of the mobility of matter in the circumstances considered. Surface diffusion allows the attainment of large values of ρ_o (compared, say, to that for a flat. Griffith crack) and misfit stresses induced by matter flow into the g.b. effectively thicken the boundary near the tip, alleviating the stress con-centration normally associated with cracks or cavities. Very different conclusions on the importance of w could arise when the cavity does not lie along, or itself constitute, a high diffusivity path.

These latter circumstances seem to coincide with those assumed by Stevens and Dutton (1971) and Dutton and Puls (1978) for the diffusive growth of Griffith cracks and, at this point, it is perhaps interesting to compare the directions of matter flow in the different models proposed. As Dutton and coworkers remarked, in the case of their Griffith crack extension model the chemical potential at the tip region is very high due to high strain energy density developed in this region, and they consider

the first term in (4) to dominate. Hence they assumed that the direction
of diffusion is away from the tip along the cavity surface. On the other
hand, as envisioned in our model, based on the Hull-Rimmer model and its
generalizations, misfit stress has been induced to reduce greatly the
stress concentration associated with a notch so that the second term of
(4) takes over. The chemical potential at the cavity surface far away
from the tip is higher than at the tip region and the flow is then reversed
toward the tip and into the g.b. Intuitively, this must be true since in
our model the g.b. which lies ahead of the cavity tip is capable of
accepting the matter previously diffused from the cavity surface, whereas
in the Griffith model, the cavity is regarded as being essentially the same
as if it were located in a perfect crystal so that no sinks are available
to accommodate the matter, and thus flow must carry atoms away from the tip.
However, it remains an open question as to whether a full solution to the
coupled equations of elasticity and diffusion would actually produce
cavity growth in the narrow crack-like mode assumed in the Griffith model
of Dutton and coworkers. Instead, e.g., it is possible that a full analysis
would merely predict a rounding out of the crack near its tip. By contrast,
a variety of complete solutions have been carried out for the model which
we envision for cavities along grain boundaries, and these do indeed pre-
dict growth, ultimately in a comparatively narrow crack-like mode (Chuang,
1974; Vitek, 1978; Chuang et al., 1979), but with negligible strain energy
terms at the tip so long as $\rho_o \gg b$.

III. ENERGY RELEASE RATE IN CAVITY GROWTH

In fracture mechanics terminology, there is an energy release rate associated with cavity growth. Since there is not full agreement in the published literature, we examine here the computation of this rate and answer the question as to where the energy goes for diffusive crack growth. This is done with reference to a plane strain mode (i.e., cylindrical cavities or cracks) so that all works and energies are on a unit thickness basis.

In the case of elastic-brittle crack growth without diffusion, as formulated within the Griffith context, an energy release rate G is defined by

$$G\dot{a} = \dot{W} - \dot{F}_e \qquad (9)$$

where a is crack length. Here it is customary to neglect surface tension effects (whether they can be included consistently within a model for a mathematically sharp tipped crack remains an open question), so that F_e is just the ordinary strain energy of the cracked body as computed from continuum elasticity theory and G is given by the well-known expression $(1-\nu^2)K^2/E$ for an isotropic linear elastic solid. Further, in the Griffith model dF_s is written as $(2\gamma_s - \gamma_b)da$ because in an increment da of growth a length $2da$ of crack surface is created at the expense of length da of grain boundary. Actually, this should be written as $(2\hat{\gamma}_s - \hat{\gamma}_b)d\hat{a}$, where $d\hat{a}$ is the growth increment as measured off relative to some reference configuration, and the $\hat{\gamma}$'s are based on unit area of that configuration (the $\hat{\gamma}$'s are independent of surface strain when

surface tensions vanish). Understanding the terms in this way one has, for the Griffith model of reversible crack growth

$$dW = dF = dF_e + dF_s = dF_e + (2\gamma_s - \gamma_b)da \tag{10}$$

during growth, and hence the required value of G is

$$G = 2\gamma_s - \gamma_b \tag{11}$$

It has been emphasized recently (Rice, 1978) however, that the actual requirement of thermodynamics for quasi-static, isothermal crack growth should be phrased as the inequality $\dot{W} \geqslant \dot{F}$ (i.e., the inequality assuring non-negative entropy production) which requires only that

$$\dot{W} - \dot{F} \equiv [G - (2\gamma_s - \gamma_b)]\dot{a} \geqslant 0 \tag{12}$$

for growth (or healing, $\dot{a} < 0$), and this is general enough to include lattice trapping with thermally activated growth at values of G different from (11). Here $2\gamma_s - \gamma_b$ can be interpreted as the reversible work of pulling the interface apart against cohesive forces.

The formulation is rather different for the diffusive growth process considered here, because there is no direct rupturing of cohesive bonds. We assume, consistently with procedures in the thermodynamics of diffusion, that μ is defined as in eqs. (4) and (6) during processes, and that diffusive fluxes J are established on the cavity surface and grain boundary, such that $J\partial\mu/\partial s < 0$ where s is arc length along the flow path.

We consider a half-void of length a as in fig. 1 and let Γ_b denote the grain interface and Γ_s the upper surface of the cavity. Then eq. (2), which is regarded as an identity given the definitions of μ , can be written with $F = F_e + F_s$, where

$$F_s = 2 \int_{\Gamma_s} \gamma_s \, ds + \int_{\Gamma_b} \gamma_b \, ds \qquad (13)$$

and ds denotes arc length. Hence if an energy release rate is defined as $\dot{W} - \dot{F}_e$ the rate is given by

$$\dot{W} - \dot{F}_e = \dot{F}_s - 2 \int_{\Gamma_s} \mu \dot{N} \, ds - \int_{\Gamma_b} \mu \dot{N} \, ds \quad . \qquad (14)$$

Some rearrangements of this equation are useful. We note, however, that contrary to the case for Griffith cracks, the quantity like G , namely $(\dot{W} - \dot{F}_e)/\dot{a}$, will not in general be simply a function of applied load and notch size but, as shown by examples later, will depend on details of the matter placement along the grain boundary. First, observing that J and $\dot{N}$ are related by mass conservation,

$$\partial J/\partial s + \dot{N} = 0 \ , \qquad (15)$$

that $\mu_s = \mu_b$, $2J_s = J_b$ at the cavity apex, and that J vanishes at the left and right boundaries (the latter if the right boundary is a half-spacing between cavities), integration by parts leads to

$$\dot{W} - \dot{F}_e = \dot{F}_s + \left[2 \int_{\Gamma_s} (-J \, \partial\mu/\partial s) ds + \int_{\Gamma_b} (-J \, \partial\mu/\partial s) ds \right] \quad . \qquad (16)$$

Hence the energy release rate $\dot{W}-\dot{F}_e$ supplies two terms, the first $(\dot{F}_s)$ being the energy which goes to cavity surface creation and grain boundary removal, and the second being diffusive dissipation in the form of necessarily positive products $(-J \partial\mu/\partial s)$.

Next, we observe that directly from elasticity theory, the rate of change in strain energy (calculated, e.g., as by Rice and Drucker (1967), Rice (1968a), and Eshelby (1970) when material is added or removed from surfaces) is

$$\dot{F}_e = 2 \int_{\Gamma_s} w\Omega\dot{N} \, ds + \int_{\Gamma_b} w\Omega\dot{N} \, ds + \dot{W} - \int_{\Gamma_b} \sigma\Omega\dot{N} \, ds \quad . \qquad (17)$$

Here $\Omega\dot{N}$ (generally positive on Γ_b and negative on Γ_s) is the volumetric rate of matter addition per unit area, or rate of thickening $\dot{t}$; the first two terms represent the strain energy of the added matter, and the last two the work of all applied tractions, both external and those $(-\sigma)$ acting along the g.b. Equation (17) is written for the case when no surface tensions exist, consistently with the references by Rice and others cited above. The full form of the equation is given in the Appendix. We can also compute $\dot{F}_s$, where F_s is given by (13). Consistently with the neglect of surface tension in (17), in evaluating $\dot{F}_s$ the effects of the surface strain rate $\dot{\epsilon}$ on products of the form γds are neglected (again, see the Appendix for the full expression), and hence

$$\dot{F}_s = -2 \int_{\Gamma_s} \gamma_s \kappa\Omega\dot{N} \, ds + (2\gamma_s \cos\phi_0 - \gamma_b)_0 \dot{a} \quad . \qquad (18)$$

The integral here represents the effect of adding matter at the rate $\dot{t} \equiv \Omega \dot{N}$ to the surface (of course, $\dot{t}$ will generally be negative along Γ_s when cavity is growing); as in our earlier discussion, $-\kappa \dot{t} \, ds = -\kappa \Omega \dot{N} \, ds$ is the rate of change of an arc ds, at a place where the curvature is κ, due to matter addition. The last term of eq. (18) arises because the integral only gives the rate of change of F_s from matter addition (or removal) from the existing arc Γ_s but, as illustrated by the inset diagram in fig. 1, it does not include the new surface element created during growth. As shown in the diagram, in cavity growth by da into the grain boundary, two arcs of length $da \cos\phi_o$ (ϕ_o is the angle between Γ_s and the negative x-axis at the cavity tip) are added to the cavity surface and the grain boundary diminishes by da, thus leading to the last term of (18) for $\dot{F}_s$. The subscript "o" on this last term means that the γ's refer to values at the cavity tip.

When eqs. (17) and (19) are added together there results

$$\dot{F} = \dot{F}_e + \dot{F}_s = \dot{W} + 2 \int_{\Gamma_s} (w - \gamma_s \kappa) \Omega \dot{N} \, ds$$

$$+ \int_{\Gamma_b} (w - \sigma) \Omega \dot{N} \, ds + (2\gamma_s \cos\phi_o - \gamma_b)_o \, \dot{a} \qquad (19)$$

and, as shown in the Appendix, this equation is correct generally, i.e., when surface tensions and associated dependences of products γds on surface strains are present, but note there that if surface tension τ_s exists at the tip, the distribution of σ must be regarded as including a concentrated tension $\tau_s \sin\phi_o$ at the tip in this formula and in eqs. (23,24) to follow. As noted in the Appendix, however, there are grounds for arguing that τ_s and τ_b must vanish at the tip. If we now recall

the definitions of μ in eqs. (4) and (6), recognizing that $\kappa_1 = \kappa$, $\kappa_2 = 0$ in the present case, and then compare eqs. (14) and (19), we conclude that ~

$$(2\gamma_s \cos\phi_o - \gamma_b)_o = 0 \text{ , or } \cos\phi_o = (\gamma_b/2\gamma_s)_o \qquad (20)$$

during cavity growth. This is the well-known expression for the equilibrium angle ϕ_o at the apex of a grain boundary void. The same expression has generally been assumed to apply during growth, and the present analysis provides a rationalization for this.

On the other hand, eq. (2), and hence eq. (14), were assumed to apply as equalities during growth. If instead they were assumed to apply as inequalities in the form (notation of eq. (2))

$$\dot{W} \geq \dot{F} - \int_A \mu\dot{N} \, dA \quad , \qquad (21)$$

to allow for some form of entropy production at the cavity tip in addition to the diffusive entropy production $(-J \, \partial\mu/\partial s)$ already exhibited, e.g., in eq. (16), then combination of eq. (14), with = replaced by $\geq$, and eq. (19) (which remains valid as an equality) leads to

$$(2\gamma_s \cos\phi_o - \gamma_b)_o \dot{a} \leq 0 \quad . \qquad (22)$$

This inequality will be satisfied during cavity growth (i.e., when $\dot{a} > 0$) only if $\cos\phi_o \leq (\gamma_b/2\gamma_s)_o$; that is, only if ϕ_o is greater than or equal to the angle given by the equilibrium expression. During cavity shrinkage (i.e., sintering) it will be satisfied only if ϕ_o is less than or equal to the result of the equilibrium expression. Consequences of the inequality version of the Second Law of Thermodynamics are perhaps not often surprising

but this seems to be an exception. One might well expect that because the cavity advances by the drawing-in of material to the grain boundary at the cavity apex, under action of applied stress, that ϕ_o might <u>decrease</u> from the value given by the equilibrium expression as $\dot{a}$ <u>increases</u> from zero, and conversely. But thermodynamics prohibits this. If there is to be <u>any</u> deviation from the equilibrium expression, it must be in the opposite sense. In the circumstances it seems reasonable to assume that ϕ_o always retains the value given by the relation for equilibrium and, indeed, as shown by eq. (20) this must necessarily be the case if the cavity apex does not constitute a "point source" of entropy production. We have been careful to use the phrasing "angle given by the equilibrium expression" rather than "equilibrium angle" in this discussion. For example, owing to the dependence of the γ's on surface strain it is possible that the angle ϕ_o given by the equilibrium expression in a <u>stressed</u> solid would differ from the corresponding angle ϕ_o in an unstressed solid.

As a final expression for the energy release rate, which is perhaps most revealing in comparing different models, we may rearrange eq. (17), which retains validity regardless of whether eq. (21) is regarded as an equality or inequality, to read

$$\dot{W}-\dot{F}_e = \left[2\int_{\Gamma_s} w(-\Omega\dot{N})ds - \int_{\Gamma_b} w\Omega\dot{N}\ ds \right] + \int_{\Gamma_b} \sigma\Omega\dot{N}\ ds \quad . \qquad (23)$$

(Since eq. (17) neglects surface tension effects, so also does this equation. However, we show in the Appendix that this equation and, hence, our subsequent discussion in this section remains valid if F_e is replaced by a new quantity F_e' which is, in fact, a more logical quantity to use in calculating an energy release rate, as $\dot{W}-\dot{F}_e'$, when surface tension

tension effects exist. Also, as noted already, σ then includes the concentrated tension $\tau_s \sin\phi_0$ at the tip). Equation (23) can be interpreted relative to fig. 2 where, in growing the void by some infinitesimal increment Δa, material of local thickness Δn is removed from the notch surface and the grain boundary effectively thickens by an amount $\Delta\delta$. Hence

$$\Delta W - \Delta F_e = 2 \int_{\Gamma_s} w\Delta n \ ds - \int_{\Gamma_b} w\Delta\delta \ dx + \int_{\Gamma_b} \sigma\Delta\delta \ dx , \qquad (24)$$

and from this the difference between the Griffith-like model envisioned by Stevens and Dutton (1971) and the misfit model is clear. In the former case there is assumed to be no or negligible opening $\Delta\delta$ ahead of the crack, and all of the energy release goes to the first term of (24), representing strain energy of material diffused away from the highly stressed notch surface. As remarked, it remains to be demonstrated that a model of this type will actually lead to cavity growth. On the other hand, in the Hull-Rimmer model and its generalizations, high stresses are alleviated by the misfitting material which effectively thickens the grain boundary at the notch tip. As we have demonstrated, in this case the w terms are negligible and all of the energy release goes to the last term of (24), which represents work done by local stresses in opening the grain boundary to accommodate the diffused matter.

It is tempting to regard the left side of (24) as being $G \Delta a$ where $G = (1-\nu^2)K^2/E$ is the usual energy release rate of linear elastic fracture mechanics. However, this is valid only in special cases, considered by Chuang (1974), of essentially steady state growth in which the creep

cavity is indeed thin (i.e., crack-like) in shape, and in which all
effective thickening of the grain boundary is limited to a small vicinity
of the void tip, with the diffused matter being accommodated by elastic
distortion of the adjoining grains. In such cases Chuang shows, by
developing numerically an explicit solution to the coupled (linearized)
equations of elasticity and diffusion, that the G level to drive the
void is always greater than the Griffith level of eq. (11). Indeed, this
is required by general considerations embodied in eq. (16) (in which
we may set $\dot{F}_s = (2\gamma_s - \gamma_b)\dot{a}$ for the present case), since $(-J \, \partial\mu/\partial s)$
is necessarily positive. In fact, Chuang's analysis suggests that K
can never fall below approximately 1.7 times the Griffith value, and
by an application of the J-integral (Rice, 1968a,b; Eshelby, 1970) he
was able to derive an equation analogous to (24), neglecting the w terms,
and demonstrate that his numerical results for $\sigma(x)$ and $\delta(x)$ closely
satisfied this equation.

IV. TOTAL ENERGY CHANGE DUE TO INTRODUCING A CAVITY

Stevens and Dutton (1971) have considered the total energy change ΔE due to introducing a cavity into a stressed body. In our notation $E = F - W$ where the external loads are assumed to be fixed so that W varies in proportion to the displacement of the loaded boundary. Following their classification, with slight modifications, ΔE includes contributions from the following four sources (here we neglect completely surface tension effects):

(1) change of energy due to void surface area introduced and lost grain boundary area (ΔF_s) ,

(2) elastic strain energy change $(\Delta \tilde{F}_e)$ due to removal of material from the body to form a void,

(3) work done by external loads $(\Delta \bar{W})$ during removal of material,

(4) work done in processes of local matter rearrangement, such as by local stress fields when the matter removed from the void is deposited at the g.b. $(\Delta W')$. (All other possible sources such as dislocation, slip, etc., are neglected).

It is important to note, as was not emphasized by Stevens and Dutton, that the $\Delta \tilde{F}_e$ and $\Delta \bar{W}$ terms calculated in step (2) and (3) are not the <u>total</u> terms ΔF_e and ΔW (which, for example, enter into the energy release rate expression). Both ΔF_e and ΔW include a contribution from the work W' in step (4), and typically this is a very significant term. Indeed, the work W' is analogous to the effect of σ term in eq. (6) for μ on a grain boundary; the part $-\sigma\Omega\delta N$ of $\mu\delta N$ in eq. (2) is essentially a work of type W' , and we have seen its importance in eq. (24).

With the preceding classification of terms, the total energy change is

$$\Delta E \equiv \Delta F_s + \Delta F_e - \Delta W = \Delta F_s + \Delta \bar{F}_e - \Delta \bar{W} + W' \; . \tag{25}$$

In fact, the specific terms $\Delta \bar{F}_e$, $\Delta \bar{W}$ and W' depend on the sequence of operations and are different if matter (say, from a reservoir) is first inserted into the grain boundary, and then an equivalent amount of matter is removed from the body to form the void, and placed in the reservoir. Of course, the total effect, $\Delta F_e - \Delta W$, which these terms represent, is independent of sequence.

Consider a small elliptical cylindrical void lying along a g.b. as illustrated in fig. 3 with its major axis normal to the direction of the uniaxial applied stress σ_{∞} . We proceed to compute the energy change ΔE of this configuration in the following manner: First, we cut out the "volume" V (actually, an area in the 2-D model), thus creating a void on the boundary, and temporarily deposit it in an unstressed matter reservoir. This operation produces the free energy changes ΔF_s , $\Delta \bar{F}_e$ and $\Delta \bar{W}$, where (Rice, 1968a)

$$\Delta \bar{F}_e - \Delta \bar{W} = - \frac{\pi (1-\nu^2)\sigma_{\infty}^2 a^2}{E} \; (1+d/2a) \quad . \tag{26}$$

Next we take the volume V out of the reservoir and insert it into the g.b., thus adding a non-uniform layer of material of local thickness δ , such that

$$\int_{g.b.} \delta(x)dx = V = \pi ad \; . \tag{27}$$

To get the W' term, we calculate the work done during the insertion process. This includes the work done on the surrounding material, plus the work necessary to bring the strain energy of the material being inserted to the proper level. Let $\sigma^a(x)$ be the normal tensile stress of an Inglis-type stress distribution induced by σ_∞, and let $\sigma^b(x)$ be the final total stress that results after the matter V has been stuffed into the g.b. To compute W', we first cut the g.b., hold it in place by applying σ^a, and quasi-statically alter these stresses until the cut has opened locally by $\delta(x)$ (at which point the g.b. stresses are $\sigma^b(x)$). W' is simply the (negative) work done on the surface of the cut during this operation and if we assume the material is linearly elastic, then

$$W' = -\frac{1}{2} \int_{g.b.} \left(\sigma^a + \sigma^b\right) \delta \, dx \ . \tag{28}$$

Of course, the material V that is inserted into the opened cut must be stressed and thus there will be an additional part of W' related to the strain energy increase in the inserted volume V at state (b), but this portion is insignificant when σ^b is small compared to E. Still, if σ^b is of the order of σ_∞ over the region where the matter is added, the term is of order $\pi\left(1-\nu^2\right)\sigma_\infty^2 \, ad/2E$, which is of the same order as $\Delta\bar{F}_e - \Delta\bar{W}$ of (26) unless $d/a \ll 1$.

There are two observations to make about W'. First, it is usually far greater in magnitude than $\Delta\bar{F}_e - \Delta\bar{W}$, and hence cannot be neglected. Suppose, for example, that the material is added uniformly over the entire

g.b., so that $\sigma^b = \sigma^a$ and both have average values of σ_∞ . Then

$$W' = -\sigma_\infty V = -\sigma_\infty \pi a d$$

$$= \left(\Delta \bar{F}_e - \Delta \bar{W}\right) (E/(1+\nu^2)\sigma_\infty)(d/a)/(1+d/2a) , \tag{29}$$

and the factor multiplying $(\Delta \bar{F}_e - \Delta \bar{W})$ will be much greater than unity unless d/a is the same order as σ_∞/E . Second, there is no unique value for W' ; it depends on how the matter is inserted. For example, concentration of the matter along a limited segment of the g.b. could make σ^b large and negative there, so much so that W' could become positive (and indefinitely large). Also, addition of matter just over the highly stressed portions of the g.b., so as to reduce σ^b to negligible values there, would produce a different expression for W' than (29). Since W' is not unique, neither are the terms ΔF_e and ΔW and, hence, neither is the net energy change ΔE . Unique values result only for a given $\delta(x)$. In this sense it is meaningless to talk about an energy release rate (except in the limiting case of steady state crack-like cavity growth discussed in the last section) unless it is understood that the distribution $\delta(x)$ is determined by the coupled equations of elasticity and diffusion as part of the analysis.

V.   DISCUSSION

Dutton and Puls (1978) in a comment on the formulation presented
by Charles (1976), equated the crack extension force to the elastic
fracture mechanics strain energy release rate $G = (1-\nu^2)K^2/E$. As we
have seen, the usual fracture mechanics expressions are valid only in
special circumstances. Essentially, their arguments are based on a
Griffith-like crack model and treat the strain energy term in (24) as
dominant. We have pointed out, however, that when a crack is located
in a g.b. rather than in a perfect matrix, the g.b. is capable of
accommodating the material being removed from the crack surface and thus
relax the high stress concentration ahead of the crack tip. Hence the
strain energy contribution appears to be minor in the energy release
expression and the major portion comes from the work done by the boundary
stress σ on the opening $\Delta\delta$ as it appears in the last term of (24).
Indeed, Charles (1978) noted in his reply that he believes that the
"PV" work of insertion of a diffusible species in a highly stressed volume
dominates in the stress rupture process, although he did not provide a
full analysis as we do here.

Speight, Beere and Roberts (1978) have recently presented an inter-
granular crack growth model based on the mechanisms discussed here. Rather
than relying on a full solution of relevant equations, as by Chuang (1974)
and Vitek (1978), they imposed an approximate distribution δ (see
eq. (1) of Speight et al., 1978) in order to solve for the maximum stress
in the g.b. Further, they attributed the energy release rate to the
"plastic work" done by σ_∞ on the crack volume $(W' = -\sigma_\infty V)$ as in
eq. (29). But we have shown that the work term W' is dependent on the

distribution of δ and their formulation does not appear to be fully consistent in this respect, although it is not clear as to how critically these assumptions affect their conclusions.

VI. CONCLUSIONS

We have presented precise expressions for chemical potentials on grain boundaries and free surfaces, and noted that in representative cases the strain energy contributions to these are negligible for the diffusive growth of grain boundary cavities. This is because surface diffusion tends to round-out a cavity near its tip, and grain boundary diffusion generates misfit stresses to alleviate its stress concentration, reducing local stresses to the order of γ_s / ρ_o. An evaluation of expressions for the energy release rate in diffusive cavity growth shows that the major energy sink is provided by the work of normal stresses on the grain boundary, as it is effectively thickened by matter addition, and not from the loss of strain energy of material diffused from the cavity tip. We discuss the computation of net energy changes associated with introduction of cavities, and point out that the dominant term is dependent on details of the matter placement along the grain boundary. In addition, we have commented on restrictions placed by thermodynamics on the angle formed by cavity walls where they join the grain boundary.

VII. APPENDIX: SURFACE TENSION EFFECTS

At various points throughout the paper surface tension (or surface stress) effects were ignored for ease of presentation, with the promise that all would be made well here. Let θ^1, θ^2 be curvilinear coordinates in a surface (cavity surface, grain boundary) in some arbitrary reference configuration, and let $g^o_{\alpha\beta}$ be the surface metric tensor $(\alpha,\beta=1,2)$ in this configuration. If the coordinate lines are regarded as being scribed onto the surface, in an elastic deformation the coordinates are convected and the metric tensor changes to $g_{\alpha\beta}$. The (covariant) components of strain are defined as

$$\epsilon_{\alpha\beta} = (g_{\alpha\beta} - g^o_{\alpha\beta})/2 \quad . \tag{A-1}$$

The contravariant components (on the convected coordinates) $\tau^{\alpha\beta}$ of the surface tension tensor are defined such that $\tau^{\alpha\beta}\delta\epsilon_{\alpha\beta}dA$ is the work of incremental elastic distortion $\delta\epsilon_{\alpha\beta}$ of an area dA . Since $\gamma\,dA$ is the surface energy,

$$\tau^{\alpha\beta}\delta\epsilon_{\alpha\beta}dA = \delta(\gamma\ dA) = \delta\gamma\ dA + \gamma g^{\alpha\beta}\delta\epsilon_{\alpha\beta}\ dA \ , \tag{A-2}$$

where $g^{\alpha\beta}$ are the contravariant components of the metric, and we obtain the well-known relation between surface tension and surface energy (e.g., Herring, 1951,1952; Murdoch, 1976)

$$\tau^{\alpha\beta} = \frac{\partial\gamma}{\partial\epsilon_{\alpha\beta}} + \gamma g^{\alpha\beta} \quad , \tag{A-3}$$

valid for arbitrary deformation. Note that if we define $\hat{\gamma}$ by $\hat{\gamma} \, dA_o = \gamma \, dA$, where dA_o is the area of the considered element in the reference configuration, then

$$\tau^{\alpha\beta} = \frac{dA_o}{dA} \frac{\partial \hat{\gamma}}{\partial \epsilon_{\alpha\beta}} \quad . \tag{A-4}$$

(For readers unfamiliar with curvilinear tensors: If the coordinates lines are locally orthogonal at the point considered, and normalized so that $d\theta^1$, $d\theta^2$ are equal to the associated changes in arc length along the surface, then $g^{\alpha\beta}$ of eqs. (A-2) and (A-3) is just the Kronecker tensor and $\tau^{\alpha\beta}$ is the force acting in the β direction per unit length of a cut along the surface with outward normal in the α direction. Also, when the reference configuration differs only by an infinitesimal amount from the current configuration, $\epsilon_{\alpha\beta}$ is just the classical strain tensor).

Mechanical equilibrium equations for $\tau^{\alpha\beta}$ are complicated (Gurtin and Murdoch, 1975), but it is known that the field equations of equilibrium are fully implied by the principle of virtual work, and the latter is most convenient for our purposes. Let A denote a region of cavity surface with bounding contour C , let n_i be components (in 3-D space) of the outward normal to A (i.e., pointing away from the adjacent material, let σ^{ij} be the stress tensor acting immediately beneath the surface (Latin indices i,j have the range 1 to 3), and let T^α be components of the surface tension acting normal to C . Hence the surface may be considered as a membrane loaded with forces per unit area $-n_i \sigma^{ij}$ on A and forces per unit length T^α along C . The principle of virtual work then requires that

$$\int_A \tau^{\alpha\beta} \delta\varepsilon_{\alpha\beta} \ dA = \int_C T^\alpha \delta u_\alpha \ dC - \int_A n_i \sigma^{ij} \delta u_j \ dA \qquad \text{(A-4)}$$

for <u>arbitrary</u> virtual displacement fields δu_i and associated virtual surface strains $\delta\varepsilon_{\alpha\beta}$. Some well-known consequences are, for example, that

$$T^\beta = m_\alpha \tau^{\alpha\beta} \quad , \quad \tau^{\alpha\beta} \kappa_{\alpha\beta} = n_i \sigma^{ij} n_j \qquad \text{(A-5)}$$

where m_α denotes the outward normal (in the tangent plane of the surface) to C, and $\kappa_{\alpha\beta}$ is the surface curvature tensor.

An equation identical to (A-4) can be written for an arbitrary area A of the grain boundary, where $\tau^{\alpha\beta}$ and T^α are tensions in the g.b., but now the integral involving σ^{ij} must be carried out over both sides of the grain boundary. Observe that when the grain boundary is flat, as considered here, the normal components of σ^{ij} are equal on both sides (and identical to what we have called σ). This means that for grain boundaries we can write $n_i \sigma^{ij} \delta u_j$ as $n_i \sigma^{i\alpha} \delta u_\alpha$ where δu_α are components of δu_i parallel to the boundary and $\sigma^{i\alpha}$ are shear stresses in the immediately adjacent material. Observe that for symmetrically loaded boundaries as in figs. 1 to 3, $\sigma^{i\alpha}$ will reverse sign on crossing the boundary, and $\sigma^{i\alpha}$ is generally non-zero whenever the surface tensions in the boundary are non-uniform.

To calculate μ (section II) we proceeded by calculating δF when a layer of local thickness δt was added to cavity surfaces and/or grain boundaries, under conditions for which $\delta W = 0$. We write $\delta F = \delta F_e + \delta F_s$, and observe that when matter is added, there are associated displacements

δu_i and strains $\delta \varepsilon_{\alpha \beta}$ along A_s (the cavity surfaces) and A_b (the grain boundaries). The change in elastic strain energy is

$$\delta F_e = \int_{A_s} w \, \delta t \, dA + \int_{A_b} w \, \delta t \, dA - \int_{A_b} \sigma \, \delta t \, dA$$

$$+ \int_{A_s} n_i \sigma^{ij} \delta u_j \, dA + \int_{A_b} n_i \sigma^{i\alpha} \delta u_\alpha \, dA \quad . \tag{A-6}$$

The first three terms have been discussed in the body of the paper, the fourth corresponds to item (iv) in the list enumerated shortly before eq. (3), and the last three terms together give the change in strain energy of the material lying outside the added layers. The last integral on A_b covers both sides of the g.b.'s and it is noted that the normal stresses σ and shear stresses $\sigma^{i\alpha}$ appear separately. Further, for symmetrically loaded grain boundaries as we consider, δu_α is the same on both sides of the boundary. The change in F_s is

$$\delta F_s = - \int_{A_s} \gamma_s (\kappa_1 + \kappa_2) \delta t \, dA + \int_{A_s} \left(\delta \gamma_s + \gamma_s g^{\alpha \beta} \delta \varepsilon_{\alpha \beta} \right) \, dA$$

$$+ \int_{A_b} \left(\delta \gamma_b + \gamma_b g^{\alpha \beta} \delta \varepsilon_{\alpha \beta} \right) \, dA \quad . \tag{A-7}$$

The first term has been discussed in the body of the paper. The second two are contributions, of type (iii) in the list shortly before eq. (3), due to effects of elastic strain in changing γ by $\delta \gamma$ and dA by $g^{\alpha \beta} \delta \varepsilon_{\alpha \beta} \, dA$. Using eq. (A-2),

$$\delta F_s = -\int_{A_s} \gamma_s (\kappa_1 + \kappa_2) \phi t \ dA + \int_{A_s} \tau^{\alpha\beta} \delta\epsilon_{\alpha\beta} \ dA$$

$$+ \int_{A_b} \tau^{\alpha\beta} \delta\epsilon_{\alpha\beta} \ dA \quad . \tag{A-8}$$

Mechanical equilibrium is expressed by the principle of virtual work in eq. (A-4). Let us write this equation separately for the grain boundaries and for the cavities, taking C to be the arc where the two make contact. Adding these equations together, we observe that contributions on C cancel (because of mechanical equilibrium between τ_s and τ_b at the cavity tip, $2\tau_s \cos\phi_o = \tau_b$) <u>if</u> there is a unique displacement δu_i at the tip. On the other hand, if there is matter addition δt $(=\Omega\delta N)$ to the grain boundary at the tip, so that the vertical components $\tau_s \sin\phi_o$ of τ_s separate by δt, the sum of the two terms involving C is

$$- \int_C \tau_s \sin\phi_o \ \delta t \ dC \ .$$

Hence the resulting "grand" form of the principle of virtual work is

$$\int_{A_s} \tau^{\alpha\beta} \delta\epsilon_{\alpha\beta} \ dA + \int_{A_b} \tau^{\alpha\beta} \delta\epsilon_{\alpha\beta} \ dA = \int_C \tau_s \sin\phi_o \ \delta t \ dC$$

$$- \int_{A_s} n_i \sigma^{ij} \delta u_j \ dA - \int_{A_b} n_i \sigma^{i\alpha} \delta u_\alpha \ dA \tag{A-9}$$

where the last integral covers both sides of A_b . Since the actual field δu_i , $\delta\epsilon_{\alpha\beta}$ associated with matter addition involves continuity of δu_α across A_b , it can be taken as the field to be entered in this form of

the virtual work equation. Now, when we add together eqs. (A-6) and (A-8), so as to write $\delta F = \delta F_e + \delta F_s$, we find that because of eq. (A-9), all the terms involving $\tau^{\alpha\beta}$, σ^{ij} and $\sigma^{i\alpha}$ cancel one another. Hence

$$\delta F = \delta F_e + \delta F_s = \int_{A_s} [w - \gamma_s(\kappa_1 + \kappa_2)]\delta t \, dA$$

$$+ \int_{A_b} (w - \sigma)\delta t \, dA \qquad (A-10)$$

where here, and in eq. (A-15) to follow, for simplicity in the presentation of formulae, the integral involving σ is understood to include a "delta function" contribution to σ at the cavity tip due to the vertical component of concentrated tension, $\tau_s \sin\phi_o$, acting there.

Thus, if we write $\delta t = \Omega \delta N$ and observe that μ is defined by

$$\delta F = \int_{A_s} \mu \, \delta N \, dA + \int_{A_b} \mu \, \delta N \, dA \qquad (A-11)$$

(in the case considered of matter addition under conditions for which $\delta W=0$) , then we obtain at once eqs. (4) and (6) in the body of the paper for μ . Thus we see that while the terms associated with δu_i and $\delta\epsilon_{\alpha\beta}$ make first order contributions to δF_e and δF_s separately, they make only higher order contributions to the total δF . There is, however, a curious result at the tip. Because σ has the delta-function contribution there, of integrated intensity $\tau_s \sin\phi_o$, we must conclude that μ has a similar negatively infinite delta-function form of integrated intensity $-\tau_s \sin\phi_o \Omega$, at the tip. We leave open the interpretation to be given to this. Certainly, to the extent that thermodynamics required continuity

of μ , such a condition can be met only if the material adjusts its state of surface strain at the tip in a manner such that τ_s (and hence τ_b too) vanishes there.

In the discussion of plane strain growth of a cylindrical cavity, surface tension effects were ignored in deriving eqs. (17) and (18). Now we can drop indices on $\tau^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$, writing τ and ϵ for the components in the plane of deformation. With the previous discussion as background, the full version of eq. (17) is readily seen to be

$$\dot{F}_e = 2 \int_{\Gamma_s} w \, \Omega\dot{N} \, ds + \int_{\Gamma_b} w \, \Omega\dot{N} \, ds + \dot{W} - \int_{\Gamma_b} \sigma \, \Omega\dot{N} \, ds$$

$$+ 2 \int_{\Gamma_s} n_i \sigma^{ij} \dot{u}_j \, ds + \int_{\Gamma_b} n_i \sigma^{ix} \dot{u}_x \, ds \qquad \text{(A-12)}$$

where the integral on Γ_b covers both sides of the g.b. and, as in figs. 1 to 3, the x direction is parallel to the grain boundary. Here i,j range over x,y . Similarly, the full version of eq. (18) is:

$$\dot{F}_s = -2 \int_{\Gamma_s} \gamma_s \kappa \Omega\dot{N} \, ds + (2\gamma_s \cos\phi_0 - \gamma_b)_0 \dot{a}$$

$$+ 2 \int_{\Gamma_s} \tau\dot{\epsilon} \, ds + \int_{\Gamma_b} \tau\dot{\epsilon} \, ds \quad . \qquad \text{(A-13)}$$

Again, a grand form of the principle of virtual work may be written, analogously to (A-9), and this is

$$2 \int_{\Gamma_s} \tau \; \delta\epsilon \; ds + \int_{\Gamma_b} \tau \; \delta\epsilon \; ds$$

$$= -(\tau_s \sin\phi_o \delta t)_o - 2 \int_{\Gamma_s} n_i \sigma^{ij} \delta u_j \; ds - \int_{\Gamma_b} n_i \sigma^{ix} \delta u_x \; ds \; . \quad \text{(A-14)}$$

Thus, when eq. (A-12) and (A-13) are added together the terms multiplying $\dot{\epsilon}$ and $\dot{u}_i$ cancel, and we obtain eq. (19) in the body of the paper (which we remarked there to be valid in the general case when surface tension effects exist, so long as σ is taken to include the concentrated tension at the tip--if, indeed, this is non-zero).

Finally, if eq. (A-14) is used in (A-12), and then (A-12) is re-arranged to the form of eq. (23) in the body of the paper, we obtain

$$\dot{W} - \dot{F}_e' = \left[2 \int_{\Gamma_s} w(-\Omega\dot{N})ds - \int_{\Gamma_b} w \; \Omega\dot{N} \; ds \right] + \int_{\Gamma_b} \sigma \; \Omega\dot{N} \; ds \quad \text{(A-15)}$$

(i.e., the same right side as eq. (23)) <u>if</u> we define

$$\dot{F}_e' = \dot{F}_e + 2 \int_{\Gamma_s} \tau\dot{\epsilon} \; ds + \int_{\Gamma_b} \tau\dot{\epsilon} \; ds \quad . \quad \text{(A-16)}$$

When we recall that $\dot{F}_e$ can be written as the integral over the volume of the body of stress components times corresponding components of strain rate, these added terms seem certainly to be appropriate, and $\dot{W} - \dot{F}_e'$ seems to be the obvious extension of the concept of an energy release rate to cases in which surface tensions exist during cavity growth.

VIII. ACKNOWLEDGEMENT

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FIGURE CAPTIONS

Figure 1.   Cylindrical void in a grain boundary.

Figure 2.   Illustration of matter transfer in diffusive cavity growth.
In growth Δa a layer of local thickness Δn is removed
from the cavity surface and the grain boundary, effectively,
thickens by a local amount $\Delta \delta$ due to matter addition.

Figure 3.   An elliptical cavity in a linear elastic material; σ^a is
the grain boundary tensile stress if there is no matter
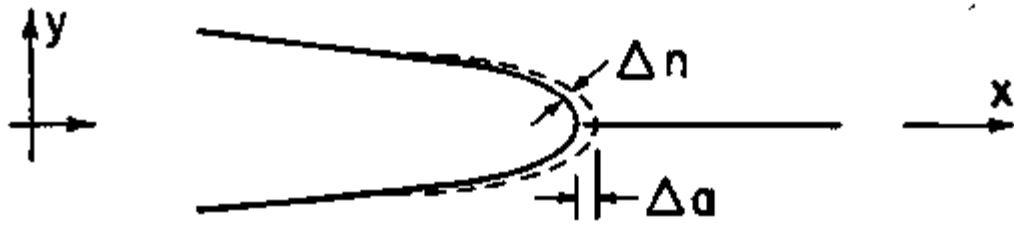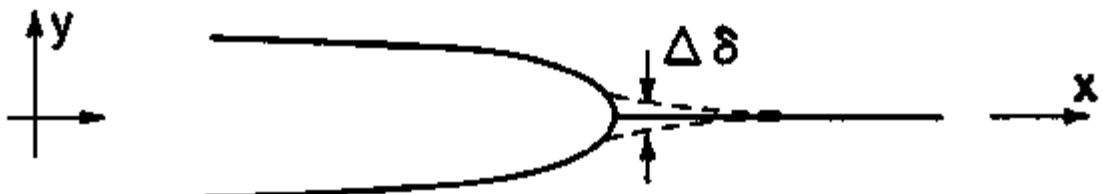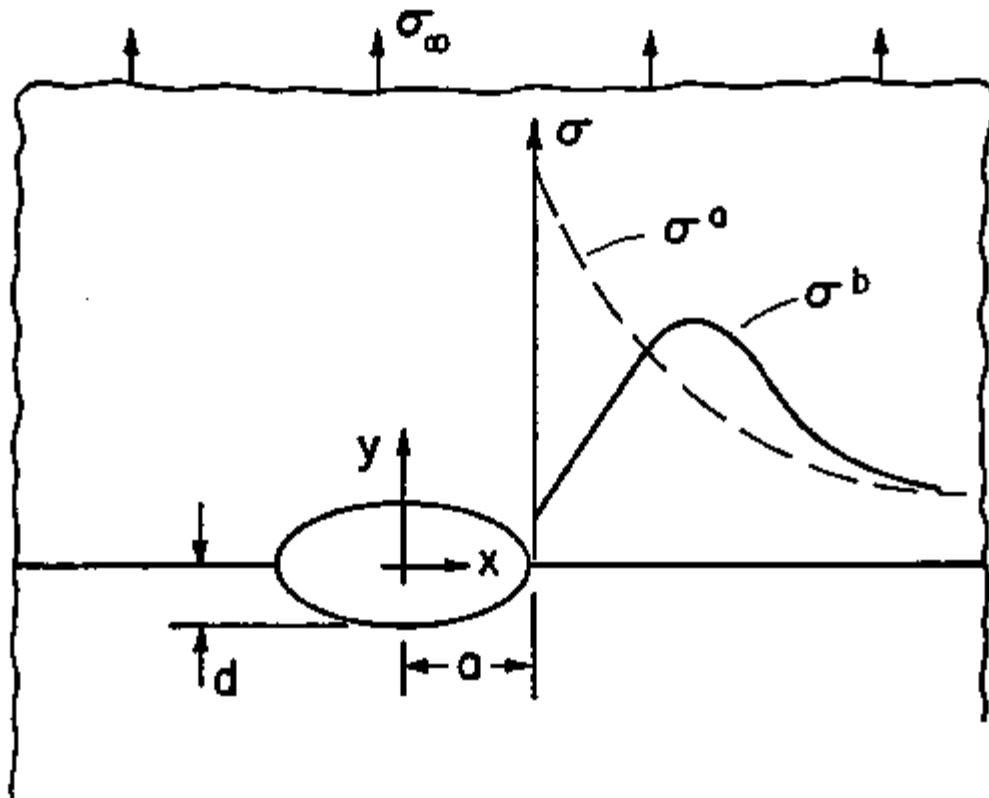placement in the grain boundary, and σ^b results after
placement.
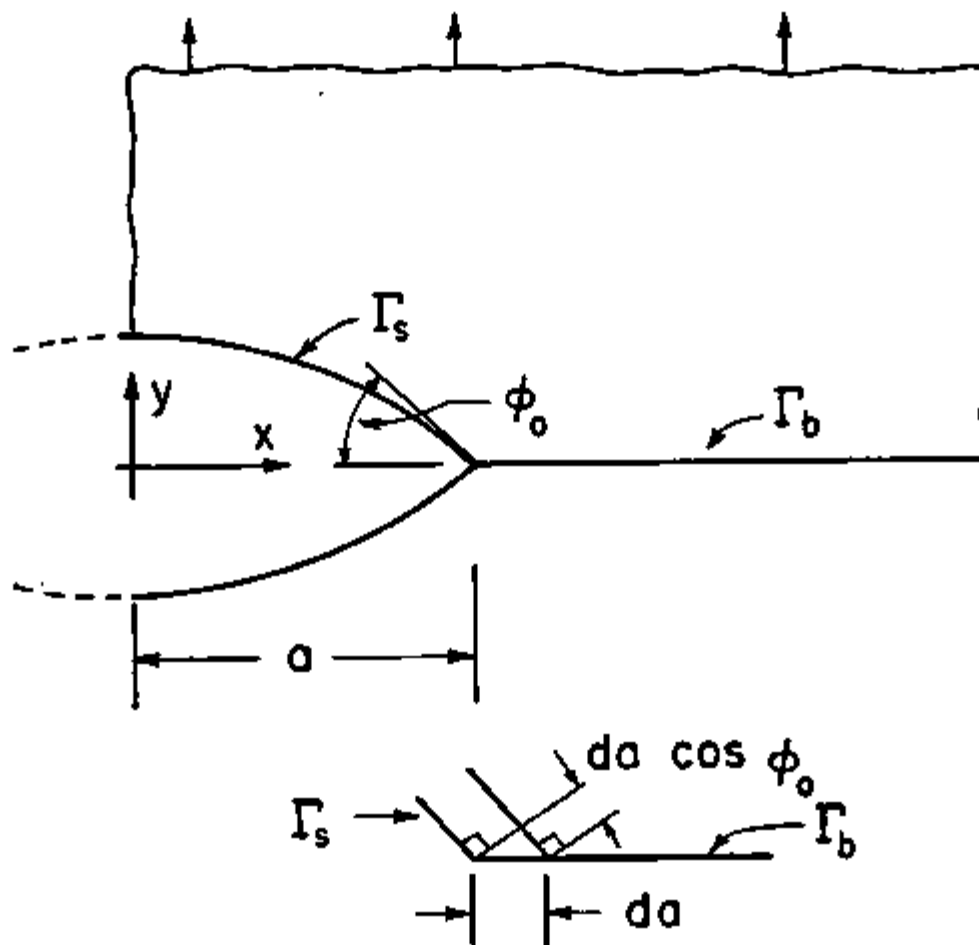
Figure 1

(a)

(b)

Figure 2

Figure 3