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Report No. UVA-ER-953-84U
(GCTG-28)

Topical Report

UNSYMMETRICAL FLOW IN A CENTRIFUGE (U)

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September 1984

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These notes discuss the differential equations for a centrifuge sector: a cylinder whose cross section is a slice of pie, presumably mud. Three boundary layers are considered: the cylindrical wall layer, Ekman layers at the ends, and the "buoyancy" layers on the $\theta = \text{constant}$ walls. This exercise was provoked by a Matsuda and Nakagawa opus (Ref. 1). They consider most of the time that the cylinder is infinitely long, and all of the time that the atmospheric scale height is of the order of the bowl radius. Both restrictions are relaxed herein.

As will be seen, much discomfort arises and the results need more scrutiny.

The linearized unsymmetric equations in bowl-fixed coordinates take the familiar form

$$\begin{aligned}
 (\rho_o w')_z + \frac{1}{r} (r \rho_o u')_r + \frac{\rho_o v'_{\theta}}{r} &= 0 \\
 P'_z &= \mu [w'_{zz} + \frac{1}{r} (r w'_r)_r + \frac{w'_{\theta\theta}}{r^2} + \frac{1}{3} (\nabla \cdot \bar{v}')_z] \\
 P'_r &= 2\Omega \rho_o v' + \Omega^2 r \rho' + \mu [u'_{zz} + \frac{1}{r} (r u'_r)_r - \frac{u'}{r^2} + \frac{u'_{\theta\theta}}{r^2} + \frac{1}{3} (\nabla \cdot \bar{v}')_r] \\
 \frac{P'_{\theta}}{r} + 2\rho_o \Omega u' &= \mu_o [v'_{zz} + \frac{1}{r^2} (r^2 v'_r - r v')_r + \frac{v'_{\theta\theta}}{r^2} + \frac{1}{3r} (\nabla \cdot \bar{v}')_{\theta}] \\
 -\rho_o \Omega^2 r u' &= k [T'_{zz} + \frac{1}{r} (r T'_r)_r + \frac{1}{r^2} T'_{\theta\theta}] \\
 \frac{P'}{P_o} &= \frac{P'}{\rho_o} + \frac{T'}{T_o} .
 \end{aligned} \tag{1}$$

From this set, the boundary layer equations appropriate to walls aligned in the three directions are readily found. Although with a lot of care one can derive the requisite forms, it will be done here using the time honored technique of arm-waving.

We consider that all dependent variables are of unit relative order except for the velocity normal to the local surface and that this is of order ϵ . The boundary layer thickness is of order ϵ , the size of which is to be found as we go on.

Finally, the scale height is assumed small enough that radial density variations are important, but curvature is otherwise ignored.

Consider in turn the radial (cylinder wall), axial (Ekman) and circumferential boundary layers.

Cylindrical Wall ($r=a$)

Nondimensionalize things as follows (primes are dimensional, unprime is undimensional).

$$\begin{aligned} P &= P'/P_0(r) & v &= v'/\Omega a & y &= z/a \\ T &= T'/T_0 & u &= u'/\Omega a & \eta &= r/a \\ \rho &= \rho'/\rho_0(r) & w &= w'/\Omega a \end{aligned} \quad (2)$$

and remember

$$A^2 = \frac{\Omega^2 a^2}{2RT_0} \approx M^2/2$$

$$Re = \frac{\rho_0(r) \Omega a^2}{\mu_0} \quad (3)$$

$$S = 1 + \frac{\gamma-1}{2\gamma} A^2 Pr, \quad (4)$$

where Pr is Prandtl Number.

Note well: Re as used here is not a constant but is really a nondimensional density.

Now, we can write the cylindrical wall boundary layer equations, simplifying (1) as described above. In nondimensional variables, using (2), (3), we get

$$\text{Re}(w_y + v_\theta) + (\text{Re } u)_\eta = 0 \quad (5)$$

$$P_y = \frac{2A^2}{\text{Re}} w_{\eta\eta} \quad (6)$$

$$P_\eta = 4A^2(v - T/2) \quad (7)$$

$$P_\theta + 4A^2u = \frac{2A^2}{\text{Re}} v_{\eta\eta} \quad (8)$$

$$-4(S-1)u = \frac{1}{\text{Re}} T_{\eta\eta} \quad (9)$$

$$P = \rho + T. \quad (10)$$

Observe that these equations differ from those of Matsuda (his Eq. 6.2-6.6) only in his omission of $u\text{Re}_\eta$ in (5). Eliminate P between (6) and (7), remembering (3)

$$\left(\frac{w_{\eta\eta}}{\text{Re}}\right)_\eta = 2\left(v - \frac{T}{2}\right)_y. \quad (11)$$

Then (8) and (9) yield

$$\frac{1}{\text{Re}} \left(v - \frac{T}{2}\right)_{\eta\eta} = 2Su + \frac{P_\theta}{2A^2} \quad (12)$$

and (8), (9) and (6) produce

$$\frac{1}{\text{Re}}[T + 2(S-1)v]_{\eta\eta y} = \frac{S-1}{A^2} P_{\theta y} = \frac{2(S-1)}{\text{Re}} w_{\eta\eta\theta}.$$

Integrating twice gives

$$[T + 2(S-1)v]_y - 2(S-1)w_\theta = \frac{\partial T_w}{\partial y}(\theta, y) \quad (13)$$

T_w is the wall temperature. Rearranging, from (13), (11)

$$2S v_y = (T_w)_y + 2(S-1) w_\theta + \left(\frac{w_{\eta\eta}}{\text{Re}}\right)_\eta.$$

Using this and (6), (5) we get from (12)

$$\begin{aligned}
(v - \frac{T}{2})_{\eta\eta y} &= (\frac{Re P_\theta}{2A^2})_{\eta y} + 2S(Re u)_{\eta y} \\
&= w_{\eta\eta\eta\theta} - 2S Re (v_{\theta y} + w_{yy}) \\
&= w_{\eta\eta\eta\theta} - 2S Re w_{yy} \\
&\quad - Re[(T_w)_{y\theta} + 2(S-1)w_{\theta\theta} + (\frac{w_{\eta\eta}}{Re})_{\eta\theta}] \\
&= -2S Re w_{yy} - 2(S-1) Re w_{\theta\theta} + 2A^2 w_{\eta\eta\theta} - Re(T_w)_{y\theta}
\end{aligned}$$

where we remember (?) that

$$\frac{\rho_{o\eta}}{\rho_o} = \frac{Re}{Re} \frac{\eta}{Re} = 2A^2.$$

At long last we can eliminate $(v-T/2)$ from this treasure using (11) to arrive at

$$\frac{1}{Re} (\frac{w_{\eta\eta}}{Re})_{\eta\eta\eta\eta} = -4S w_{yy} - 4(S-1)w_{\theta\theta} + \frac{4A^2}{Re} w_{\eta\eta\theta} - 2(T_w)_{y\theta}. \quad (14)$$

This can be put in the familiar pancake form. Define

$$\begin{aligned}
x &= A^2(1 - \eta^2) \\
dx &= -2A^2\eta d\eta \Rightarrow -2A^2 d\eta \\
Re &= \overline{Re} \left[\frac{\rho_o}{\rho_w} = e^{-x} \right]
\end{aligned}$$

where $\overline{Re}$ is the (constant) wall Reynolds number.

Split w into

$$w = \frac{1}{2} w_p(\theta, y) + [\bar{w} = e^x \chi_{xx}]. \quad (15)$$

This defines χ without involving the (nonexistent) stream function. Then

$$S w_{pyy} + (S-1) w_{p\theta\theta} = -(T_w)_{y\theta} \quad (16)$$

$$\begin{aligned}
& [e^x (e^x \chi_{xx})_{xx}]_{xx} + 4S \left(\frac{\overline{Re}}{8A^6} \right)^2 \chi_{yy} + 4(S-1) \left(\frac{Re}{8A^6} \right)^2 \chi_{\theta\theta} \\
& - 2 \left(\frac{\overline{Re}}{8A^6} \right) e^x \chi_{xx\theta} = 0 .
\end{aligned} \tag{17}$$

This agrees with Babarsky's unpublished result (Ref. 2).

The first three terms of (17) consistently describe a layer whose thickness in scale height units is of order

$$\varepsilon_x = 2A^2/Re^{1/3} = M^2/Re^{1/3}$$

while the last term is smaller by order ε_x . It is easy to imagine circumstances when M is large enough and the wall pressure moderate enough that ε can approach unity so that the boundary layer thickness approaches the scale height. Observe however that this is not a statement about a lack of boundary layer character. In physical variables (r/a), the layer thickness is of order

$$\varepsilon_r = Re^{-1/3}.$$

The fact does remain however that the $\chi_{xx\theta}$ term is only of order ε_x smaller than the remaining terms in the equation. If this is retained, then its contribution of an odd derivative in θ makes solution by separation of variables inconvenient, to put it gently.

Ekman Layers (End Wall, $z=0$)

Proceeding as before, the end wall boundary layers are defined, from (1), by

$$Re(w_y + v_\theta) + (Re u)_\eta = 0 \tag{18}$$

$$P_y = 0 \tag{19}$$

$$P_\eta = 4A^2 (v - T/2) + \frac{2A^2}{Re} U_{yy} \tag{20}$$

$$P_\theta + 4A^2 u = \frac{2A^2}{Re} v_{yy} \tag{21}$$

$$4(S-1)u = \frac{1}{\text{Re}} T_{yy}. \quad (22)$$

Beginning with (20) and using successively (19), (22), (21)

$$\begin{aligned} \frac{2A^2}{\text{Re}} u_{yyyy} &= 4A^2 \left(\frac{T}{2} - v \right)_{yy} \\ &= 2A^2 \text{Re}[4(S-1)u] - 2\text{Re}[P_\theta + 4A^2 u] \end{aligned}$$

or

$$u_{yyyy} = 4\text{Re}^2 S u - \frac{\text{Re}^2}{A^2} P_\theta$$

so

$$u = i[A_1 e^{-(1+i)\sqrt{\beta}y} - B_1 e^{-(1-i)\sqrt{\beta}y}] + \frac{P_\theta}{4A^2 S} \quad (23)$$

where $\beta = \text{Re} \sqrt{S}$.

From (20)

$$\phi = v - \frac{T}{2} = \frac{\beta}{\text{Re}} [A_1 e^{-(1+i)\sqrt{\beta}y} + B_1 e^{-(1-i)\sqrt{\beta}y}] + [C(\eta, \theta) = \frac{P_\eta}{4A^2}]. \quad (24)$$

Next, (21) and (22) yield

$$[T + 2(S-1)v]_{yy} = \frac{\text{Re}}{A^2} (S-1)P_\theta \quad (25)$$

and, from (18), observing that

$$v = \frac{(v - \frac{T}{2}) + \frac{1}{2}(T + 2(S-1)v)}{S}$$

we get

$$\begin{aligned} w_y + i[A_{1\eta} + 2A^2 A_1] e^{-(1+i)\sqrt{\beta}y} - (B_{1\eta} + 2A^2 B_1) e^{-(1-i)\sqrt{\beta}y} \\ + \frac{1}{\sqrt{S}} [A_{1\theta} e^{-(1+i)\sqrt{\beta}y} + B_{1\theta} e^{-(1-i)\sqrt{\beta}y}] \\ = \frac{P_\theta}{2S} - \frac{1}{2S} [T + 2(S-1)v]_\theta. \end{aligned} \quad (26)$$

If $P_\theta = 0$ these are the familiar results found for the axisymmetric case. To the extent that $P_\theta \neq 0$ terms must be retained, (25) and (26) yield quadratic and cubic terms.

If P_θ and $(T_w)_\theta$ can be set to zero, and u_w, ϕ_w, w_w are wall values, then on integrating (26) and evaluating at the wall along with (23) and (24), there follows

$$\begin{aligned} u_w &= i(A_1 - B_1) \\ \phi_w &= \frac{1}{\sqrt{S}} (A_1 + B_1) + \phi_i \\ w_w &= \frac{1}{2\sqrt{\beta}} [(1+i)(A_{1\eta} + 2A^2 A_1) + (1-i)(B_{1\eta} + 2A^2 B_1)] \\ &\quad + \frac{1}{2\sqrt{\beta S}} [(1-i)A_{1\theta} + (1+i)B_{1\theta}] + w_i. \end{aligned} \quad (27)$$

ϕ_i and w_i are interior values.

Elimination of A_1, B_1 gives

$$\begin{aligned} 2\sqrt{\beta}(w_w = w_i) &= u_{w\eta} + 2A^2 u_w - \frac{u_{w\theta}}{\sqrt{S}} + \frac{1}{\sqrt{S}} [\phi_w - \phi_i]_\eta \\ &\quad + 2A^2 (\phi_w - \phi_i) + \frac{1}{S} (\phi_w - \phi_i)_\theta \end{aligned}$$

which is the extension of the Carrier-Maslen condition.

Remember that the P_θ terms have been summarily dismissed for no very defensible reason.

Buoyancy Layers ($\theta = \text{constant}$)

For layers associated with a wall defined by constant θ , the same recipe as before produces

$$\text{Re}(w_y + v_\theta) + (\text{Re } u)_\eta = 0 \quad (29)$$

$$P_z = \frac{2A^2}{\text{Re}} w_{\theta\theta} \quad (30)$$

$$P_\eta = -2A^2 T + \frac{2A^2}{\text{Re}} u_{\theta\theta} \quad (31)$$

$$P_\theta = 0 \quad (32)$$

$$\frac{1}{\text{Re}} T_{\theta\theta} = 4(S-1)u. \quad (33)$$

Then (31) gives, using (32), (33)

$$u_{\theta\theta\theta\theta} = -\frac{\text{Re}}{2A^2} P_{\eta\theta\theta} + \text{Re } T_{\theta\theta} = 4(S-1)\text{Re}^2 u$$

so that

$$u = i[A_2 e^{-(1+i)\sqrt{\alpha}\theta} - B_2 e^{-(1-i)\sqrt{\alpha}\theta}] \quad (34)$$

where

$$\alpha = \text{Re}\sqrt{S-1}. \quad (35)$$

Then (31) allows

$$T = -2\sqrt{S-1}[A_2 e^{-(1+i)\sqrt{\alpha}\theta} + B_2 e^{-(1-i)\sqrt{\alpha}\theta}] - [D = \frac{P_{\eta}}{2A^2}]. \quad (36)$$

Then, by (29),

$$\begin{aligned} v_{\theta} &= -w_y - u_{\eta} - 2A^2 u \\ &= -i[A_{2\eta} + 2A^2 A_2] e^{-(1+i)\sqrt{\alpha}\theta} - [B_{2\eta} + 2A^2 B_2] e^{-(1-i)\sqrt{\alpha}\theta} - w_y \end{aligned} \quad (37)$$

where w is given by (30).

Analogously as in the case of the Ekman layers, the presence of P_z makes w quadratic in the boundary layer dimension θ and v cubic in θ .

Clearly, from (36), (37) the layer thickness is of order $1/\sqrt{\alpha}$. Set $\lambda = \theta\sqrt{\alpha}$. Then (30) is

$$w_{\lambda\lambda} = \frac{P_z}{2A^2(S-1)}.$$

If we argue that this is small because $A^2 \gg 1$, then

$$w = 0 \quad (38)$$

and

$$\begin{aligned} v &= \frac{1}{2\sqrt{\alpha}}[(A_{2\eta} + 2A^2 A_2)(1+i)e^{-(1+i)\sqrt{\alpha}\theta} \\ &\quad + (B_{2\eta} + 2A^2 B_2)(1-i)e^{-(1-i)\sqrt{\alpha}\theta}] + v_i \end{aligned} \quad (39)$$

$$T = -2\sqrt{S-1}[A_2 e^{-(1+i)\sqrt{\alpha}\theta} + B_2 e^{-(1-i)\sqrt{\alpha}\theta}] + T_i. \quad (40)$$

Then, proceeding as for the Ekman case, we are led to

$$2\sqrt{\alpha}(v_w - v_i) = u_{w\eta} + 2A^2 u_w - \frac{1}{2\sqrt{S-1}} [(T_w - T_i)_\eta + 2A^2(T_w - T_i)]. \quad (41)$$

This is the appropriate end (in θ) condition consistent with the wall differential equation (17) which is second order in θ . Remember again that P_z has been thrown out.

References

1. Matsuda, T. and Nakagawa, K., "A New Type of Boundary Layer in a Rapidly Rotating Gas," J. Fluid Mech., 126, 431-442, 1983.
2. Babarsky, R. J., "Compressible Flow in a Pie-Shaped Cylinder Rotating About the Apex," 1984, unpublished.