Electron Multiple, Plural, and Single Scattering

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by

Joseph J. Devaney

ABSTRACT

The angular distribution of a beam of electrons penetrating an infinite slab is studied with a view to providing simple accurate formulae for use in electron transport codes. We report success, but a highly disappointing success, for although the literature is vast, (over 150 papers), the experiments are limited, sparse, and often poorly described. The theory suffers from complexity, from incomplete, erroneous, or even absent comparisons with other theories and with experiment. Consequently, we cannot say with any assurance what the best theory is, nor to any great accuracy, especially in limiting cases, how good the theory we recommend is. We do give comparisons with the experiments known to us.

With that caveat, we recommend the theory of Molière as given by Bethe and by Hanson et al. Formulas are given. We expect accuracies of a few percent for multiple scattering. For the (in the cases studied) less dominant concomitant plural and single scattering the theory cannot do better than 10 to 30%. Because of the complexity of the functions, either series or tables must be employed.

A bright spot in this somewhat gloomy state of affairs is the calculation of the $1/e$ width which can be done accurately ($\lesssim 2\%$) by a simple approximation due to Hanson et al.

Using theories of Hanson et al and variants thereof, an approximate "Gaussian" gives a good (7-20% error) simple description of the multiple scattering. Approximate single scattering formulas are also offered.

*For a brief overview of this paper, we suggest reading the Introduction, Section I, and then the Conclusions and Recommendations, Section VII.

I INTRODUCTION

In attacking the problem of electron transport, it is of fundamental advantage* to know the angular dispersion of a beam of electrons as it passes through matter. The problem is difficult for a number of reasons. To the ordinary difficulty of calculating scattering from Coulomb fields is added the difficulty and uncertainty of atomic electron shielding of such fields. Unless the matter traversed is both very thin and not wide, the scattering is greater than single so that the probabilistic treatment of many scatterings is involved. Although large angle scattering is rare and (multiple) small angle scattering probable, for a full description one must treat simultaneously: single, a few (plural), and many (multiple) scattering of a beam of electrons. Last, the algebra involved is complicated and the experiments are difficult.

Usually, in fact nearly universally, the theories treat the scattering in the small angle approximation, i.e. $\sin \theta$, $\tan \theta \rightarrow \theta$. The Thomas-Fermi statistical atom is used to estimate shielding. In order to make the formidable algebra tractable, further mathematical approximations are usually introduced. Enormous effort has been expended on the problem. Some 150 papers are referenced by W. T. Scott in his 1963 review article.¹ The bench-mark theory is without doubt that of Molière.² A clearer, simpler exposition of Molière's theory is to be found in Bethe's work.³ A key advantage to the Molière theory is that it gives not only the "Gaussian" central multiple scattering but also the transition through plural to single scattering as the angle of scattering increases. As a consequence, experimentalists universally test their results against the Molière theory. "Good agreement has usually been obtained."⁴ We substantiate that remark. See Tables III to VI. For those reasons we recommend and will provide formulas below for the Molière theory. However, because authors have concentrated on the 1/e width of the distributions which have frequently been claimed to be accurate to about a percent, some have missed the fact that the rest of the distribution can have even an order of magnitude greater error as we will show below. Moreover, there is contention among theories and uncertainty in the experiments that lead to considerable lack of confidence in any results. Thus, in addition to

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the inconvenience of the complex algebra of full theories, like Molière's, the accuracy thereof may only be of modest improvement over a simpler theory in the dominant central scattering region. For the central multiple scattering, the Hanson et al⁵ Gaussian approximation does as well as the Molière theory in giving the $1/e$ width^{5,6} (Table III), and for the distribution function is possibly as good as at least some experimental error for angles less than the $1/e$ angle. However, the approximation does not give the less probable plural scattering at all. In Hanson's paper⁵, Molière matches experiment well except at about 3 times the $1/e$ angle where the theory is as much as 30% low. However, this large angle scattering is at a scattered intensity of only $\sim 1/1000$ of the peak at 0° scattering angle and is nearly in the single scattering region⁵.

In the next section we give a brief critique of the literature. Then we give formulas for the Molière-Bethe-Hanson theory which is recommended as the practical best, but with reservations. Then we give the Hanson approximate theory with both a central "Gaussian" and a single scattering part. This approximation, or our variant thereof, may well be the optimum presently for calculations. It is far and away the simplest. We give modifications to the Hanson et al approximation. We give what experimental comparisons we can find. We summarize and give recommendations and finish with a plea for further research.

II. LITERATURE CRITIQUE

In the opinion of this writer the literature on (multiple) electron scattering is disappointing in that much difficult, innovative work has been done, but very frequently indeed: results are not stated; results, if stated, are not rendered in readily useable form; previous work is not compared, nor is theoretical work compared with experiment. For example, a new theory may be proposed but no evidence is given that it is in fact better than previous theories. Often, when comparisons are made, they are made only among $1/e$ widths. Some experimental results are only relative, leaving the reader to

attempt numerical integration with only small graphs available. Sometimes only projected (on a plane containing the initial electron beam) distributions are given. Particular comments follow.

Scott¹ repeats the derivations of many theories up to about 1961 or so, explaining and improving the methods, but recommends no best theory, nor any best approximate theory. A great pity, because Scott has, at great effort, ploughed through the different theories with great care and in enormous detail. He is quite likely the most knowledgeable of their comparative merits. He makes some comparison with experiment, e.g. with Hanson et al,⁵ but only compares the widths.

The early work of Goudsmit and Saunderson⁷ (1940) alleges an exact theory valid to all angles, but the theory is given in terms of particle path length which is effectively of little accurate use except for small angles. Further, E. Breitenberger argues⁸ that one of their hidden assumptions is good only for small angles. See also Molière.²

Nigam, Sundaresan, and Wu allege⁹ that Molière uses an inconsistent approximation to all orders in the parameter $\alpha \equiv \frac{Z}{137\beta}$. (z 's are charge numbers, β is electron velocity divided by light velocity, v/c). However, Molière generally agrees with experiment, see Section VI, but Nigam et al require an adjustable parameter (for atomic screening differences) which for Au and Be amounts to 1.8, (Ref. 9). Because this parameter is not derived by theory (although it can be) and checked by experiment, it must be regarded as somewhat ad hoc and is a whopping 80% correction.

Experimental error may be larger than the error bars given. For example in Kulchitsky and Latyshev¹⁰, left to right scattering cross-sections or intensities differ by 14% at 10.9° and 50% at 32.6° for 0.0174 g/cm² of Sn, and by 8% at 8° and 35% at 16.2° for 0.0079 g/cm² of Pb, both for 2.25 MeV incident electrons. In contrast their error bars are of the order of $\pm 3\%$ or less. Kulchitsky and Latyshev give relative distributions only. Their $1/e$ values are 10.65° for Sn and 9.70° for Pb.

Nearly all the theories restrict themselves to small angles, i.e. such that $\sin\theta \sim \theta$ or say for $\theta \leq 30^\circ$. However, the naive view that large angle scattering is single and thus adequately given by Rutherford's (or Mott's = relativistic) formula is only correct on the smaller angle end when plural scattering is negligible, and on the larger angle end when the electron energy and scattering angle are low enough that the electron wave function does not sample the nuclear fields significantly and also when the large angle is not so large that the probability of multiple scattering in the target slab after the single-large-angle-scattering is not large. This last restriction applies to the thinnest target. See the interesting work of J. H. Jacob.¹¹

J. H. Jacob¹¹ alleges an exact electron scattering theory (for constant electron energy) and for any angles. The results require truncation of an infinite matrix expansion and so are likely beyond the present anticipated practical use of X-Division. In addition there is nearly zero comparison with experiment, primarily because there is no experiment at the large angles. Indeed we need high quality experiment for all angles. The theory does match earlier theories which in turn match Hanson et al. However, Jacob's interesting work certainly deserves further study.

III. HANSON, LANZL, LYMAN, SCOTT, BETHE, AND MOLIERE THEORY

This section gives the Molière Theory² as presented by Bethe³ and by Hanson et al.⁵ It is, as given here, supposed to be good to a percent or better,³ but I cannot demonstrate that accuracy against experiment^{5,6,10} (see Tables IV, V, and VI), except for the $1/e$ widths, (Table III). The 1% accuracy quoted³ apparently refers also to the accuracy of the Bethe-Molière Theory's first three terms relative to the full sum, but other errors arising from the approximate treatment of Bethe-Molière could be significant. Use of the Thomas-Fermi Statistical Atom comes to mind^{3,5,9} But see Ref. 14 for a contrary finding. We will compare the three terms Hanson et al, Bethe, Molière theory, which we call HBM for convenience, with experiment below (Tables III to VI). Unfortunately the number of experimental studies is low, and their utility lower yet, as described in the preceding section. Nonetheless the

Molière theory appears by far to be the most tested against experiment with good agreement generally found.^{5,6,10,12-14} The tests do not include very thin foils nor very thick foils. In the former, however, the theory does possess the proper limit, to wit a δ -function for the distribution, (and Rutherford single scattering for thin slabs). For the latter the limit of the theory to small angle scattering effectively restricts the slab thickness relative to the electron velocities. For electron energies and slab thicknesses that limit the beam broadening to an experimentally convenient 10 degrees or so, experiment has tested the theory from 1.66 MeV¹² through 2.25 MeV¹⁰, 4.5 MeV⁶, and 15.7 MeV⁵, to 600 MeV¹⁴ (at 600 MeV the scattering angles are only $\leq \sim 0.01^\circ$).

The HBM theory presented below has the following limitations:

1. To small scattering angles, i.e. such that $\sin \theta \sim \theta$, or say for angles $\theta \leq 30^\circ$, (difference of θ to $\sin \theta$ at 30° is + 4.7%).
2. No spin corrections. These and relativistic corrections are only appreciable for large angle single scattering.³
3. The sole relativistic correction is found in the formula:

$$p^2 v^2 = (2 m_0 c^2 K + K^2) \beta^2, \quad (1)$$

where p is momentum, v velocity, $\beta \equiv v/c$, c is the velocity of light, m_0 the rest mass of the electron, and K the electron kinetic energy.

4. The Thomas-Fermi statistical atom is used. Some erratic results may therefore be expected for low Z , (Ref. 5), but see the results of Mohr and Tassie given in Ref. 14 for a contrary view.
5. No effects of electron energy loss are included. For slabs not too thick, the mean energy in the foil should be used. For very thick slabs, but not violating limitation 1 above, integration is required.

6. No nuclear effects are included; i.e. we assume a point nucleus. (Nuclear size and reactivity affect most the large angle, low Z, high electron energy scattering).
7. Atom location is random, i.e., no collective effects.
8. Electron beam intensity is sufficiently low (and energy sufficiently high) that self repulsion of the electrons is negligible in angle.
9. The additional effect of the atomic electrons on the electron scattering is adequately approximated by replacing one Z by (Z + 1) in the scattering term.
10. Neither the density of atoms nor the electron beam density is so great that the Pauli Exclusion Principle must be invoked.
11. Some integrals are evaluated approximately.^{2,3}

This list of limitations is not by any means impossibly restrictive. Of the few experiments we possess, the theory fits the multiple, plural, and single scattering from well to fair, possibly to within realistic experimental error, or at least close to that. It is impossible to say precisely how good the fits are because the data itself are often given without errors.

Because no polarizations or other azimuthal asymmetries are treated there is no manifest azimuthal angular dependence. The normalized differential cross-section for a slab is then written:³

$$2\pi \frac{d\Sigma}{d\Omega} \sin\theta d\theta = w^{-2} \sin\theta [f^{(0)}(\theta^2/w^2) + B^{-1} f^{(1)}(\theta^2/w^2) + B^{-2} f^{(2)}(\theta^2/w^2)] \quad (2)$$

where $f(\theta)$ is the angular distribution function in the scattering angle θ , (no ϕ dependence), with the normalization,³

$$\int_0^\infty f(\theta) \theta d\theta = 1 \quad . \quad (3)$$

Note that our solid angle $d\Omega = 2\pi \theta d\theta$ since $\sin \theta \rightarrow \theta$ and we have azimuthal symmetry. Σ is the psuedo cross-section of the entire slab. w is a width, the "Gaussian" width of the first term, $f(0)$.

$$w \equiv \chi_c B^{1/2} \quad . \quad (4)$$

We define a unit probability scattering angle, χ_c , as

$$\chi_c^2 \equiv \frac{4\pi\sigma N_a Z(Z+1)e^4 z^2}{A (pc)^2 \beta^2} \quad (5)$$

where

$$(pc)^2 = K^2 + 2m_Q c^2 K \quad (6)$$

and

$$\beta^2 = 1 - \left(1 + \frac{K}{m_Q c^2}\right)^{-2} \quad . \quad (7)$$

(It turns out that differing values for the constants in Eq. (5) can lead to ~ 1% differences in χ_c which in turn leads to yet greater differences in the distribution functions, f .) σ is the slab thickness in g/cm², N_a is Avogadro's number, Z the charge number of the target atoms, e the electron charge, z the charge number of the projectiles. For electrons, $z = -1$. A is the atomic weight of the target, pc the projectile momentum in energy units, $\beta = v/c$ the ratio of projectile velocity to light velocity, K is the projectile kinetic energy, and m_0c^2 the projectile rest mass. The quantity χ_c^2/θ^2 is the relative probability of having a single scattering of angle greater than θ in the foil; thus the relative probability of having a scattering greater than the angle χ_c is exactly 1 (Ref. 3). χ_c is a measure of the effective foil thickness. The Molière characteristic scattering angle χ_a is given by:

$$\chi_a^2 = \left[\frac{\hbar Z^{1/3} c}{0.885 pc a_0} \right]^2 \cdot \left[1.13 + 3.76 \left(\frac{Z}{137\beta} \right)^2 \right] \quad (8)$$

$\hbar$ is Planck's constant divided by 2π , $a_0 = \hbar^2/m_0c^2$ is Bohr's atomic radius. B is given by:

$$B = \ln B + \ln [6\chi_c^2/7\chi_a^2] \quad (9)$$

which is readily solved by a few iterations starting with a guess for B_0 (say try 8 for 1/e scattering of $\sim 10^\circ$):

$$B_n = \ln B_{n-1} + \ln [6\chi_c^2/7\chi_a^2] \quad (10)$$

The characteristic scattering angle, χ_a , is the prime descriptor in the theory of the details of the scattering atom, i.e. the target atom. It, together with χ_c , completely gives the whole scattering theory. Defining:

$$x \equiv (\theta/w)^2 = \theta^2/\chi_c^2 B \quad (11)$$

the functions $f^{(i)}$ of Eq. (2) are:³ the "Gaussian" (because $x = \theta^2/w^2$):

$$f^{(0)} = 2 e^{-x}, \quad (12)$$

$$f^{(1)} = 2e^{-x}(x-1) \left[\gamma + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!} \right] - 2(1 - 2e^{-x}) \quad (13)$$

where $\gamma = 0.5772156649$ is Euler's constant, and

$$f^{(2)} = e^{-x} \left\{ 1.246465014(x^2 - 4x + 2) + 4 \sum_{n=0}^{\infty} \frac{1}{n+1} \left[0 + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - 0.922784345 \right] \cdot \left[\frac{x^{n+3}}{(n+3)!} - \frac{2x^{n+2}}{(n+2)!} + \frac{x^{n+1}}{(n+1)!} \right] \right\} \quad (14)$$

Note: a factor of $1/4$ is missing from the first term of Bethe's expression (29), Ref. 3., for $f^{(2)}$. Bethe's $f^{(2)}$ in Table II, Ref. 3, is however correct, but not his $(1/2) f^{(2)}$ first two entries. See my Table I. See Molière, second paper Ref. 2, D, 9, p. 96.

For small angles, $f^{(0)}$ is the dominant term; it describes multiple scattering. $f^{(1)}$ and $f^{(2)}$ give plural and single scattering. In fact for $n \geq 1$ (Ref. 3).

$$f^{(n)} \xrightarrow{\theta \rightarrow \infty} \text{constant} \cdot \theta^{-2n-2} \quad (15)$$

so that $f^{(1)}$ goes over into the single scattering (Rutherford) law

$$f_{\theta \rightarrow \infty}^{(1)} \propto \theta^{-4}, \text{ (in fact } f^{(1)} = 2w^4/\theta^4 \text{ in the limit) } . \quad (16)$$

$f^{(0)}$ by itself is a fair approximation, but a (usually) better "Gaussian" approximation is found in Section IV and V next. $f^{(0)} + B^{-1}f^{(1)}$ is a good approximation and the three term $f^{(0)} + B^{-1}f^{(1)} + B^{-2}f^{(2)}$ is expected to be a very good approximation within the limitations above to the angular distribution function. However, as I noted above, I cannot demonstrate the high accuracies claimed³ against experiment. See below, Section VI.

Table I is derived from Bethe's Table II (Ref. 3) giving values for the functions $f^{(0)}$, $f^{(1)}$, $f^{(2)}$. His

$$\mathcal{J} \equiv \theta/w \quad (17)$$

in the Table. The last two columns are more suitable for interpolation (for $(\theta/w) \geq 1$) than columns 3 and 4. Entries two and three in Col. 5 were in error, as noted.

IV. HANSON, LANZL, LYMAN, AND SCOTT APPROXIMATE THEORY

This section gives the Hanson et al⁵ approximate theory which we shall for brevity call the Hanson or H theory. The HBM theory given above treats not only multiple scattering, but also plural and single scattering. The

TABLE I
Values of the $f^{(1)}$ Functions
(after Bethe³ Table II)

y	$f(0)$	$f(1)$	$f(2)$	$(1/2) y^2 f(1)$	$(1/2) y^2 f(2)$
0	2	0.8456	2.4929	0	0
0.2	1.9216	0.7038	2.0694	0.0006	0.00166(Bethe: 0.0000)
0.4	1.7214	0.3437	1.0488	0.0044	0.01343(Bethe: 0.0034)
0.6	1.4094	-0.0777	-0.0044	-0.0050	-0.00285(Bethe: -0.003)
0.8	1.0546	-0.3981	-0.6068	-0.0815	-0.1246
1	0.7338	-0.5285	-0.6359	-0.2642	-0.318
1.2	0.4738	-0.4770	-0.3086	-0.4946	-0.320
1.4	0.2817	-0.3183	0.0525	-0.6113	0.101
1.6	0.1546	-0.1396	0.2423	-0.4573	0.669
1.8	0.0783	-0.0006	0.2386	-0.0032	1.251
2	0.0366	+0.0782	0.1316	0.6258	1.053
2.2	0.01581	0.1054	0.0196	1.2347	0.475
2.4	0.00630	0.1008	-0.0467	1.6718	-0.775
2.6	0.00232	0.08262	-0.0649	1.8877	-1.483
2.8	0.00079	0.06247	-0.0546	1.9200	-1.676
3	0.000250	0.04550	-0.03568	1.9429	-1.448
3.2	7.3×10^{-5}	0.03288	-0.01923	1.7240	-1.008
3.4	1.9×10^{-5}	0.02402	-0.00847	1.6050	-0.566
3.6	4.7×10^{-6}	0.01791	-0.00264	0.5038	-0.222
3.8	1.1×10^{-6}	0.01366	0.00005	1.4237	0.005
	1000 $f(0)$	1000 $f(1)$	1000 $f(2)$		
4	2.3×10^{-4}	10.638	1.0741	1.3617	0.1375
4.5	3×10^{-6}	6.140	1.2294	1.2588	0.2521
5	2×10^{-7}	3.831	0.8326	1.1972	0.2602
5.5	2×10^{-10}	2.527	0.5368	1.1563	0.2456
6	5×10^{-13}	1.739	0.3495	1.1275	0.2264
7	1×10^{-18}	0.9080	0.1584	1.0901	0.1901
8	3×10^{-25}	0.5211	0.0783	1.0679	0.1604
9	1×10^{-32}	0.3208	0.0417	1.0523	0.1369
10	1×10^{-40}	0.2084	0.0237	1.0419	0.1186

"Gaussian," $f^{(0)}$, of that theory gives the dominant multiple scattering term, except for slabs so thick that the $\sin \theta \approx \theta$ approximation is itself violated so invalidating the HEM theory. This Gaussian alone is sufficient to describe well the dominant multiple scattering part. However, by a small correction, Hanson et al⁵ made the Gaussian generally better, in fact just as good as the full theory as far as the $1/e$ width is concerned. So, unless one is interested also in the low probability, wider angle scattering in the wings of the distribution, the Hanson theory is a relatively good approximation. Its simplicity is great, for it omits entirely the complicated expressions for $f^{(1)}$ (Eq. (13)) and $f^{(2)}$ (Eq. (14)).

The Hanson distribution, f_H , is

$$f_H = 2\pi \, d\Sigma_H/d\Omega = (2/w_H^2) \cdot e^{-(\theta^2/w_H^2)} \quad (18)$$

with usual normalization:

$$\int_0^\infty f_H \, \theta d\theta = 1 \quad (19)$$

where

$$w_H \equiv \chi_c (B - 1.2)^{1/2} \quad (20)$$

Eqs. (18) and (20) give the central, dominant, multiple scattering part of a beam divergence. χ_c is obtained from Eq. (15), and B from Eq. (9) or (10). For large angles (but $< 30^\circ$ so that $\sin \theta \approx \theta$) where single scattering will dominate, (i.e. $2w_H$ also $< 25^\circ$ so that multiple and plural scattering do

not dominate the scattering), Hanson et al⁵ give a good rule of thumb for the scattering distribution:

$$f_H \xrightarrow[2w_H < 25^\circ]{\text{large, but } < 30^\circ} 2 \chi_c^2 (1 + 0.012\theta)/\theta^4 . \quad (21)$$

Eqs. (18) and (21) do a fair job of the entire distribution if one uses Eq. (18) up to $\theta = 1.8 w_H$ and then Eq. (21) beyond. Plural scattering is not included so the scattering at about $2w_H$ to $4w_H$ cannot be expected to be well represented and so it proves. Maximum errors of $\sim 65\%$ are found comparing Eq. (21) to Hanson et al⁵ experiments. Note that in Hanson et al⁵ the Gaussian displayed is not that of Hanson's width w_H (our Eq. (20)), but rather of Molière's width w (our Eq. (4)). The Molière Gaussian being wider, matches experiment better in the plural range and so should be used optimally out to $\theta = 2.5 w$ before using Eq. (21). This procedure leads to lower maximum errors compared with the Hanson et al data in the plural range, but usually to higher errors in the dominant, small angle, multiple scattering regime than the Hanson approximation. Note also that plural scattering typically⁵ begins to be important when the multiple scattering has dropped in intensity by more than an order of magnitude. So that the foregoing errors occur in a very small, usually negligible (if larger angle scattering per se is not important) part of the total distribution. In the next section we discuss possible improvements to Hanson's approximation.

V. MODIFICATIONS OF THE HANSON, LANZL, LYMAN, AND SCOTT APPROXIMATE THEORY

Some simple modifications of the Hanson et al theory⁵ give better agreement with selected theory in selected energy regions. This section gives two such modifications. They are specialized improvements and are only slightly, if at all, superior in general to the Hanson et al approximation. They are inferior in some aspects, for example the $1/e$ width. Were there a

good and more complete "corpus experiment," one might expect to make general improvements. But there isn't. So further effort cannot be proved to be superior. We will restrict ourselves to the Hanson et al data to calculate the modifications given in this section and then compare with other experiments as well in the next section.

Modification I

This is a normalized "Gaussian" matched to experiment at zero scattering angle, $\theta = 0$. Because the multiple scattering probability distribution peaks at zero scattering angle, it can be argued, on the basis that any approximation should fit the highest probability best, that one should match experiment at zero scattering angle. (In contrast the H approximation fits the angular spread best, i.e. at $1/e$).

The Mod. I distribution is:

$$2\pi \frac{d\Sigma_I}{d\Omega} = f_I = (2/w_I^2) e^{-(\theta^2/w_I^2)} \quad (22)$$

with usual normalization:

$$\int_0^\infty f_I \theta d\theta = 1 \quad (23)$$

w_I is now determined from matching the normalized experiment, $f^{(0)}$ experiment at $\theta = 0$.

$$w_I = \sqrt{2/f_{\text{experiment}}^{(0)}} = 1 / \sqrt{\pi \left. \frac{d\Sigma}{d\Omega} \right|_{\theta=0} \text{experiment}} \quad (24)$$

At 15.7 MeV electrons incident, we get the values given in Table II. We will compare these widths with others in Table III.

Where x is defined:

$$w_I \equiv \chi_c \sqrt{B - x} \quad (25)$$

with the unit scattering angle χ_c given by Eq. (5) and B by Eq. (9) or (10). For the Hanson et al approximation of Section IV, $x = 1.2$. The departure of x in Table II from 1.2 is an indication of the inexactness of the Hanson Approximation in fitting $f(\theta = 0)$. Moreover, (Tables IV, V, and VI). The parameter x of Table II is precisely fit by:

$$x = 0.579 + 1.13\sigma + (-0.386 + 9.08\sigma) \cdot \left(\frac{Z-4}{75}\right) \quad (26)$$

where σ is the target thickness in g/cm^2 and Z is the target atomic number. Mod. I approximation is given by Eqs. (24), (25), and (22). Its accuracy will be tested in the next section. Of course, by construction, x determined by Eq. (26) gives a perfect fit to the four Hanson et al⁵ experimental distributions at $\theta = 0^\circ$. In the absence of experiment at $\theta = 0^\circ$, Eq. (24) cannot be used. Mod. I* approximation, now designated by an asterisk, will then be given by Eqs. (26), (25), and (22).

Modification II

One can improve Hanson's large angle approximation, Eq. (21), both to fit his Au data better, (he does not give large angle data for Be, so we cannot be sure that Mod. II is better also for Be), and to be more accurate for $\theta \geq 30^\circ$. We remind the reader that for net scattering angles near 90° the

TABLE II
HANSON, LANZL, LYMAN, AND SCOTT MODIFICATION I PARAMETERS
(Incident Electron Energy = 15.7 MeV)

Element	Be	Be	Au	Au
Target Thickness (g/cm ²)	0.257	0.4913	0.01866	0.03728
$\frac{d\Sigma}{d\Omega}$ experiment, $\theta = 0$ deg ⁻² (Ref. 5)	0.0317	0.0153	0.0441	0.0200
w_I	3.17°	4.56°	2.69°	3.99°
x	0.868	1.132	0.383	0.574

scattering is not Rutherford even for thin slabs.¹¹ The reason is that a scattering by about 90° forces the particle then to travel through a much greater amount of material than the slab thickness with the consequent near certainty of subsequent (multiple) scattering leading to large departures from the Rutherford (or Mott) law.¹¹

Each of Hanson's Au curves can be fitted separately and a thickness dependent parameter set developed, but, for simplicity, we leave all the thickness dependence in the parameter χ_c of Eq. (5) to give:

$$2\pi \frac{d\Sigma_{II}}{\sin\theta d\theta d\phi} = f_{II}(\theta) = 0.9028 \cdot 2\pi \cdot \chi_c^2 \cdot (1 - 1.0093 \sin \theta) / (\sin \theta)^4 \quad (27)$$

which does a little better than Hanson's Approximation. Eq. (21), to wit a maximum error of ~35% (for Au) versus ~ 65%, plus lower errors (0 to 17%) near $2w$ (where the "Gaussian" fails) and lower errors (~0%) in the large angle limit, except near 90° as noted above.

VI. COMPARISONS WITH EXPERIMENT

In this section we compare the Hanson-Bethe-Molière (HBM) theory with experiment. We also compare the Hanson et al approximation, as well as our modifications of it, with experiment. As we noted in our critique of the literature in Section II, authors often give only a comparison of $1/e$ widths and not of the whole scattering distribution. However, it would be wrong to conclude that such comparisons are totally valueless, they are rather the very concise reduction of an entire curve to but a single number. That number is, however, the most representative single number describing the distributions since they are, for much of the parameter space, dominated by a "Gaussian," i.e. an exponential in θ^2 .

Table III gives comparisons of $1/e$ widths, theoretical to experimental for all cases where I could find the parameters given. W_{HBM} is of course the $1/e$ width from the Hanson et al, Bethe, Molière theory of Section III. W_H is

TABLE III

Comparisons of $1/e$ widths, w , of Electron Beam Broadening Following Slab Penetration

Author and Reference	(Mean) Electron Energy (MeV)	Element	Slab Thickness (g/cm^3)	w Experiment	w_{HBM}	% difference with Experiment	w_{H}	% difference with Experiment	w_{I}	% difference with Experiment
Hanson ⁵	(15.47)	Be	0.257	3.06°	3.13°	+2.3%	3.12°	+2%	3.17°	+3.6%
	(15.24)	Be	0.4913	4.25°	4.57°	+7.5%	4.58°	+7.8%	4.56°	+7.3%
	(15.69)	Au	0.01866	2.58°	2.52°	-2.3%	2.53°	-1.9%	2.69°	+4.1%
	(15.67)	Au	0.03728	3.76°	3.83°	+1.9%	3.83°	+1.8%	3.99°	+6.1%
Heymann ⁶	4.5	Al	0.0652	8.23°	8.16°	-0.9%	8.12°	-1.3%	8.41°	+2.2%
		Cu	0.0622	9.14°	9.30°	+1.8%	9.32°	+2.7%	9.68°	+5.9%
		Mo	0.0532	11.98°	12.02°	+0.3%	12.05°	+0.6%	12.49°	+4.3%
		Ag	0.0484	11.96°	11.92°	-0.3%	12.00°	+0.3%	12.46°	+4.2%
		Pt	0.0268	9.89°	9.97°	+0.8%	10.03°	+1.9%	10.70°	+8.2%
			0.0544	15.76°	15.30°	-2.9%	15.35°	-2.6%	15.87°	-0.7%
Kageyama ¹²	1.66	Al	0.0035	3.4±.1°			3.4°	0		
			0.014	8.4±.3°			8.4°	0		
		Cu	0.0063	8.0±.3°			8.2°	+2.5%		
			0.0126	12.2±.4°			12.6°	+3.3%		
		Pb	0.0062	11.2±.4°			11.5°	+2.7%		

TABLE III (cont)

Author and Reference	Electron Energy (MeV)	Element	Slab Thickness (g/cm ³)	w Experiment	% difference with Experiment		% difference with Experiment	
					w _H		w _I	
Mozley ⁺¹⁴	600	Be	0.0123	0.01371° ±2.5%	0.01376°	+0.3%		
		Al	0.00244	0.00849° ±3%	0.00859°	+1.1%		
			0.00706	0.01708° ±3%	0.01752°	+2.6%		
			0.0205	0.03343° ±2.5%	0.03411°	+2.0%		
		Au	0.0103	0.04636° ±3%	0.04645°	+0.2%		

TABLE IV, A

Comparisons with Experiment: Hanson et al⁵ for Au, 0.01886 g/cm², Mean Electron Energy 15.69 MeV
 $(w = 2.768^\circ$ Eq.(4), $B = 6.5077$ Eq.(9), $\chi_C = 1.085^\circ$ Eq.(5), $w_H = 2.53^\circ$ Eq.(20)
 (we use Hanson's $\chi_C = 1.10^\circ$), $\chi = 0.383$ Eq.(26), $w_I = 2.69^\circ$ Eq.(24).

Parameter	<div style="display: flex; justify-content: space-between; align-items: center;"> <div style="text-align: center;"> Dominant Central ← "Gaussian" Region → </div> <div style="text-align: center;"> Wings → </div> </div>							
	← Multiple Scattering ← Plural Scattering				← Single Scattering Dominant →			
	0	1	2	3	4	6	8	10
$\chi = \theta/w$ Eq. (17)	0	1	2	3	4	6	8	10
Scattering Angle, θ	0	2.768°	5.536°	8.30°	11.07°	16.61°	22.14°	27.7°
$\frac{f(\theta)}{2\pi} = \frac{d\Sigma}{d\Omega}$ Eq.(2) experiment	0.0441	0.0135	0.00130	1.74(-4)	4.86(-5)	7.32(-6)	2.38(-6)	9.6(-7)
$f_{HBM}/2\pi$, Eq. (2)	0.0456	0.0133	0.00108	1.33(-4)	3.45(-5)	5.72(-6)	1.70(-6)	6.8(-7)
% difference HBM: expt.	+3%	-1.6%	-16.9%	-24%	-29%	-22%	-29%	-29%
$f_H/2\pi$, Eq. (18), <small>small angle</small> (Gaussian) <small>plural</small>	0.0497	0.0150	4.03(-4)	9.17(-7)	1.74(-10)	-	-	-
$f_H/2\pi$, Eq. (21), <small>(large angle)</small> <small>single</small>	∞	6.84(-3)	4.57(-4)	9.78(-5)	3.23(-5)	6.02(-6)	2.03(-6)	8.8(-7)
% difference, H: expt.	+12.7%	+11.3%	-65.%	-44.%	-33.%	-17.7%	-14.8%	-7.9%
$f_I/2\pi$, Eq. (22), <small>Plural</small> (small angle) Gaussian	0.0441	0.0153	6.23(-4)	2.84(-6)	1.42(-9)	-	-	-
$f_{II}/2\pi$, Eq. (27), <small>single</small> (large angle) Rutherford	∞	0.0197	0.00107	2.12(-4)	6.49(-5)	1.04(-5)	3.01(-6)	1.13(-6)
% difference, $\frac{I}{II}$: expt.	0	+13.0%	-17.6%	+22.%	+33.%	+42.%	+26.%	+17.4%

TABLE IV (cont.), B

Comparisons with Experiment: Hanson et al⁵ for Au, 0.01866 g/cm², Mean Electron Energy 15.69 MeV
Dominant Small Angle "Gaussian Part"

Parameter							
θ/ω Eq. (17)	0	0.2	0.4	0.6	0.8	1.0	1.2
Scattering Angle, θ	0	0.554°	1.107°	1.661°	2.214°	2.768°	3.322°
$f(\theta)/2\pi$ experiment, Eq. (2)	0.0441	0.0422	0.0357	0.0277	0.0198	0.0135	0.00832
$f_{\text{HBM}}/2\pi$, Eq. (2)	0.0455	0.0432	0.0374	0.0290	0.0203	0.0133	0.00817
% difference, HBM:expt.	+3.1%	+2.4%	+4.6%	+4.7%	+2.6%	-1.6%	-1.9%
$f_{\text{H}}/2\pi$, Eq. (18) (Gaussian)	0.0497	0.0474	0.0411	0.0323	0.0231	0.0150	0.00887
% difference, H:expt.	+12.7%	+12.4%	+14.9%	+16.0%	+16.6%	+11.3%	+6.6%
$f_{\text{I}}/2\pi$, Eq. (2') (Gaussian)	0.0441	0.0423	0.0372	0.0301	0.0224	0.0153	0.00956
% difference, I:expt.	0	+0.2%	+4.1%	+8.5%	+12.7%	+13.0%	+14.9%

the $1/e$ width from the Hanson et al approximation discussed in Section IV, and w_I is a variant of w_H , adjusted to match experiment at $\theta = 0^\circ$, the most probable event, i.e. $f(0) \geq f(\theta)$, except possibly for extreme thicknesses where $\sin\theta \approx \theta$ is no longer even a fair approximation. w_I is discussed in Section V. It can be seen that the HBM theory widths agree to within 1 to 3%, usually, with the experimental widths, and the far simpler Hanson approximation widths do just as well. Unfortunately, as we've noted, width matching does not guarantee curve matching, as also we will show below. A measure of the discrepancy is given by the difference of the widths, w_I , which were derived by setting a normalized "Gaussian" equal to $f(0)$. We see that the width error approaches 7-8% for these widths, whereas the HBM and H widths generally run to $< 3\%$ different than experiment.

Table IV gives a detailed comparison of the theories for one Hanson et al⁵ target and energy, to wit Au of 0.01866 g/cm^2 thickness with 15.7 MeV electrons incident. The first part of Table IV, A, gives an overview far out into the wings to where the distribution has dropped to $2/100\,000$ of the central value. For this case and single scattering, Hanson's (Eq. (21)) simple, large angle expression is best. For multiple and plural scattering the Hanson-Bethe-Molière formulae are clearly superior. The important central region is examined in detail in the second part of Table IV, B. The superior performance of the HBM-theory is evident. The H-theory is quite good and vastly simpler than the HBM-theory. The Mod. I theory is a bit spuriously good here because it was normalized to the Hanson et al experiments at $\theta=0$. Use of Eq. (20) of course gives identical results because Eq. (26) was derived to precisely fit the x that gave the 0° scattering probability for the Hanson et al experiments.

Table V compares the theories with the experiments of Heymann and Jennings,⁶ at 4.5 MeV electrons incident. The Table concentrates attention to the high probability region of $\mathcal{N} \leq 2$, which region contains roughly 98% of the scattered electrons, (for a pure Gaussian, the region contains exactly $(1 - e^{-4}) = 98.168\%$). Out to about $\mathcal{N} = 1$ (-63% of all scattered electrons) the

Hanson-Bethe-Molière theory is indistinguishable from experiment, to the accuracy with which I can read Heymann and Jennings graphs (~ 3% ?). In the region $\gamma=1$ to $\gamma=2$, (~ 35% of electrons), the HBM theory is good to 10 or 20%. The Hanson et al approximation is good to about 10% over the entire range $\gamma < 1.4$. Amazingly, just the first term of HBM, $f^{(0)}$ (see Eqs. (2) and (12)) is good to 10% or less for $\gamma < 2$, clearly superior here to the Hanson Approximation. Matching a normalized "Gaussian" at $\theta=0$ does very well up to $\gamma = 0.6$ but remains superior to the Hanson Approximation throughout. Because the data of Heymann and Jennings were given in terms of $w^2 f$, we have presented the comparisons that way. From Eq. (2)

$$w^2 f = 2\pi w^2 d\Sigma/d\Omega. \quad (28)$$

$$\% \text{ HBM:expt} \equiv 100 \frac{w^2 f_{\text{HBM}} - w^2 f_{\text{expt}}}{w^2 f_{\text{expt}}} \quad (29)$$

for example. Again f_I is a bit spuriously good because it is normalized to experiment at $\theta=0$. A better test of the general or predictive usefulness of Mod. I is to use Eq. (26) for x rather than Eq. (24). The results are given in Table V, labeled $w^2 f_I^*$. The results are superior to Hanson but not as good as a simple $x = 0.6$, for example.

Table VI compares the theories with the 2.25 MeV incident measurements of Kulchitsky and Latyshev.¹⁰ Unfortunately, they did not give absolute numbers and numerical integration of their small graphs appeared to be likely inaccurate for reasonable effort. So their results are normalized to the theories, which of course gives a better agreement than generally warranted. They also give left and right scattering of the electrons. The left and right scattering differences usually exceed their graphed error bars. For Sn, for example, in the angles 16 to 30° the left right differences are 13 to 50% where the bars are $\leq \pm 3\%$. Nonetheless, we cannot find agreement within

TABLE V

Comparisons with Experiment: Heymann and Jennings⁶ for Al and Pt, 4.5 MeV Electrons IncidentAl, 0.0652 g/cm³, $\chi_c = 0.0532$, B = 8.44, w = 8.855°, w_H = 8.20°, w_I = 8.571°, x = 0.5337

$\mathcal{J} \equiv \theta/w$ Eq. (17)	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
θ , scatt. angle	0	1.771°	3.542°	5.313°	7.084°	8.855°	10.63°	12.40°	14.17°	15.94°	17.71°
w ² f _{expt} Eq.(2)	2.137	2.047	1.759	1.387	1.00	0.6518	0.440	0.287	0.178	0.087	0.053
w ² f _{HBM} , Eq. (2)	2.135	2.034	1.777	1.400	0.999	0.6623	0.413	0.245	0.135	0.0816	0.0477
% HBM:expt	0	-0.6%	+1.%	+0.9%	-0.1%	+1.6%	-6.1%	-14.6%	-24.2%	-6.2%	-10.%
w ² f ⁽⁰⁾ _{HBM} , Eq.(2)	2.	1.922	1.704	1.395	1.055	0.7358	0.4739	0.2817	0.1546	0.07833	0.03663
% HBM ⁽⁰⁾ :expt	-6.4%	-6.1%	-3.1%	+0.6%	+5.5%	+12.9%	+7.7%	-1.8%	-13.1%	-10.0%	-23.2%
w ² f _H , Eq. (18)	2.332	2.226	1.935	1.533	1.106	0.7267	0.4345	0.2370	0.1177	0.0533	0.02198
% H:expt	+9.1%	+8.7%	+10.0%	+10.5%	+10.6%	+11.6%	-1.3%	-17.4%	-34.%	-39.%	-59.%
w ² f _I , Eq. (22) x = 0.5337	2.137	2.048	1.801	1.454	1.078	0.7342	0.4585	0.2632	0.1388	0.06718	0.02986
% I:expt	0	0	+2.4%	+4.8%	+7.8%	+12.6%	+4.2%	-8.3%	-22.%	-23.%	-44.%
w ² f _I [*] , Eqs. (22,26) x = 0.6774	2.174	2.082	1.827	1.470	1.084	0.7331	0.4539	0.2579	0.1344	0.0642	0.0281
% I [*] :expt	+1.8%	+1.7%	+3.9%	+6.0%	+8.4%	+12.5%	+3.2%	-10.1%	-24.5%	-26%	-47%

TABLE V (cont.)

Comparisons with Experiment: Heymann and Jennings⁶, 4.5 MeV Electrons IncidentPt, 0.0544 g/cm², $x_c = 0.1049$, $B = 7.76$, $w = 16.743^\circ$, $w_H = 15.37^\circ$, $w_I = 16.148^\circ$, $x = 0.5413$

$\theta = \theta/w$ Eq. (17)	0	0.4	0.6	0.8	1.0	1.2	1.6	2.0
θ , scattering angle	0	6.697°	10.05°	13.39°	16.743°	20.09°	26.79°	33.49°
$w^2 f_{\text{expt}}$ Eq. (2)	2.149	1.779	1.396	1.001	0.6978	0.500	0.1439	not meas
$w^2 f_{\text{HBM}}$, Eq. (2)	2.150	1.783	1.399	0.9932	0.6551	0.4072	0.1406	0.04886
% HBM:expt.	0	+ 0.2%	+ 0.2%	- 0.1%	- 6.1%	- 18.6%	-2.3%	-
$w^2 f^{(0)}_{\text{HBM}}$, Eq. (2)	2	1.704	1.395	1.055	0.7358	0.4739	0.1546	
% HBM ⁽⁰⁾ :expt	-7.0%	-4.2%	-0.1%	+5.4%	+5.5%	-5.2%	+ 7.4%	-
$w^2 f_H$, Eq. (18)	2.373	1.963	1.548	1.111	0.7247	0.4299	0.1137	0.02058
% H:expt	+10.4%	+ 10.3%	+ 10.9%	+11.0%	+ 3.9%	- 14.0%	- 21.%	-
$w^2 f_I$, Eq. (22), $x = 0.5413$	2.149	1.810	1.460	1.081	0.7341	0.4574	0.1371	-
% I:expt	0	+ 1.7%	+ 4.6%	+ 8.0%	+ 5.2%	- 8.5%	- 4.7%	-
$w^2 f_I^*$, Eqs. (26), (22), $x = 0.747$	2.213	1.854	1.485	1.091	0.7319	0.450	0.1302	-
% I [*] :expt	+ 3.0%	+ 4.2%	+ 6.4%	+ 9.0%	+ 4.9%	- 10.0%	- 9.5%	-

TABLE VI

Comparison with Experiment: Kulchitsky and Latyshev¹⁰ for Al, Sn, and Pb; 2.25 MeV Incident Electron Energy (Normalized to HBM at $\gamma = 0.8$ or $\gamma = 0$).

Al, 0.0266 g/cm², $\chi_c = 0.06294$, $B = 7.387$, $w = 9.802^\circ$, $w_H = 8.97^\circ$

$\gamma \equiv \theta/w$, Eq. (17) θ , scattering Angle	0	0.2	0.4	0.8	1.2	1.6	2.0	2.4	3.0
	0	1.96°	3.921°	7.841°	11.76°	15.68°	19.60°	23.52°	29.4°
$f_{HBM} = 2\pi d\Sigma/d\Omega$ Eq. (2)	2.248(-2)	2.139(-2)	1.86(-2)	1.03(-2)	4.201(-3)	1.459(-3)	5.163(-4)	1.987(-4)	5.731(-7)
f_{expt} (normalized to f_{HBM} @ $\gamma = 0.8$)	2.157(-2)	2.089(-2)	1.73(-2)	1.03(-2) normalized	5.053(-3)	1.94(-3) to 1.64(-3)	4.28(-4)	2.55(-4)	8.5(-5)
% HBM:expt.	+4.2%	+2.4%	+7.5%	0 (nor- malized)	-16.9%	-25 to -14%	+21%	-22%	-33%
f_H Eq.(18)	2.482(-2)	2.37(-2)	2.053(-2)	1.158(-2)	4.456(-3)	1.17(-3)	2.098(-4)	2.568(-5)	5.37(-7)
% H (normalized to H_{HBM} $\gamma = 0.80$):expt.	+2.5%	+0.9%	+5.6%	0 (nor- malized)	-22%	-47%	-56%	-91%	-99.44%

Sn, 0.0174 g/cm², $\chi_c = 0.09083$, $B = 6.791$, $w = 13.561^\circ$, $w_H = 12.305^\circ$

θ , Scattering Angle	0	2.712°	5.424°	10.85°	16.27°	21.70°	27.10°	32.55°
f_{HBM}	1.183(-2)	1.126(-2)	9.759(-3)	5.344(-3)	2.158(-3)	7.575(-4)	2.772(-4)	1.095(-4)
f_{expt} (normalized to HBM @ 0)	1.183(-2) normalized	1.092(-2)	9.024(-3)	3.85 to 4.4 (-3)	1.26-1.43 (-3)	4.3-4.88 (-4)	1.38 - 2. (-4)	6.88(-5) to 1.032(-4)
% HBM:expt	0 (nor- malized)	+ 3.1%	+ 8.1%	+ 39 to 21%	+71 to 51%	+76 to 55%	+100 to 39%	+59 to 6.1%
f_H	1.3209(-2)	1.258(-2)	1.088(-2)	6.07(-3)	2.299(-3)	5.89(-4)	1.026(-4)	1.208(-5)
%H(normalized to HBM @ 0):expt.	0 (nor- malized)	+3.2%	+7.9%	+ 41 to 24%	+63 to 44%	+23 to 8.1%	-33 to -54%	-84 to -90%

Table VI (cont.)

Comparisons with Experiment: Kulchitsky and Latyshev,¹⁰ 2.25 MeV ElectronsPb, 0.0079 g/cm², $\chi_c = 0.075676$, $B = 5.450$, $w = 10.122^\circ$, $w_H = 8.939^\circ$

$\chi = \theta/w$, Eq. (17)	0	0.2	0.4	0.8	1.2	1.6	2.0	2.4	3.0
θ , Scattering Angle	0	2.024°	4.049°	8.098°	12.15°	16.20°	20.24°	24.29°	30.37°
$f_{HBM} = 2\pi d\Sigma/d\Omega$, Eq.(2)	2.185(-2)	2.069(-2)	1.776(-2)	9.38(-3)	3.668(-3)	1.338(-3)	5.405(-4)	2.266(-4)	7.22(-5)
f_{expt} (normalized to f_{HBM} @ $\chi = 0.8$)	1.861(-2)	1.815(-2)	1.54(-2)	9.38(-3) normalized	4.12(-3)	1.754(-3)	7.63(-4)	3.05(-4)	9.08(-5)
%HBM:expt.	+17.4%	+14.1%	+15.3%	0 normalized	-9.7%	-24.1%	-29.1%	-26.1%	-21.1%
f_H , Eq. (18)	2.503(-2)	2.378(-2)	2.039(-2)	1.102(-2)	3.946(-3)	9.38(-4)	1.49(-4)	1.56(-5)	2.43(-7)
%H (normalized to HBM @ $\chi = 0.8$):expt.	+14.5%	+11.5%	+12.7%	0 normalized	-18.5%	-54.1%	-83.1%	-95.7%	-99.77%

experimental error with either the HBM or the H theories above $\mathcal{J} = 1$ for Sn. For Al and Pb and for Sn $\mathcal{J} < 1$ the theories agree with experiment as far as we can tell from the relative measurements, i.e. the shape is proper. But our determinations are no better than say 4 to 17% to 26% in these regions. Thus Kulchitsky and Latyshev gives us a very weak check indeed on the theory. As before

$$\%H:expt \equiv 100 \cdot \frac{f_H - f_{expt}}{f_{expt}} . \quad (30)$$

VII. CONCLUSIONS, RECOMMENDATIONS

The scope of the experiments known to this writer is limited to thicknesses relative to energy where multiple scattering clearly dominates. The experiments are also for the most part restricted to small angle scattering. In the one case¹⁰ where I can independently check experimental error, the error is much worse than the author's error bars indicate. As a consequence: 1) we have no way of testing the limits of the theory; and 2) the accuracy of the experiments is suspect. One case, certainly, does not establish the latter, but we are also disturbed to find, upon calculation, that often the claimed theory to experiment agreement is sometimes not substantiated by the numbers. A frequent annoyance is the misrepresentation of the theories used, e.g. only the first term of the Molière theory represented as the full Molière theory. I estimate that experimental accuracy could be better than 5%, but could also be (and is) worse (especially at large but not very probable angles) than 50% in accuracy. One should bear these numbers in mind and our uncertainty in knowing them when deciding upon the merits of the theories.

Exceptions to this gloomy picture are the $1/e$ widths. These, it would appear, are accurate to a few percent. See, for example, Table III.

For the best theory of the electron angular distributions, we recommend the Hanson-Bethe-Molière Theory, (HBM Theory), of Eqs. (2), (4), (5), (6), (7), (8), (9) or (10), (11), (12), (13), and (14); or Table I. Its

limitations are described in the beginning of Section III. Its drawback is the complication of the expressions (13) and (14). It, itself, is an approximation, alleged improper by Nigam, Sundaresan, and Wu.⁹ Its advantage is that it is the most tested against what experiment we have, and it fits experiment very well indeed in the high probability central region, but is only fair at the large angles (but which have low probability). See Tables IV, V, and VI. Also it possesses the proper limits at large (but $\leq 30^\circ$) angles and at small thicknesses and has a theoretical basis, if only approximate, so that one has more confidence in it outside of the tested regimes.

If only the $1/e$ width in the multiple scattering regime is desired, the Hanson et al⁵ suggestion Eqs. (20), (5), (6), (7), (8), and (9) or (10) is fully as good as HBM, sometimes better, and vastly simpler. For the best widths of multiple scattering it is recommended.

Use of the Hanson et al width, w_H , in a normalized "Gaussian," Eq. (18), is a simple and fairly accurate ($\sim 10\%$ error) approximation good up to about $\mathcal{N} = 1.2$ which includes $\sim 75\%$ of the probability. Above $\mathcal{N} = 1.2$ the approximation is much too low. For plural and single scattering, therefore, one must add terms, i.e. for single scattering, Eq. (21) or (27). We have no plural scattering correction to the Hanson approximation. The Hanson Gaussian approximation by itself contains all the probability, i.e. 100%, but concentrates it somewhat in the small angles, $\mathcal{N} \leq 1$, in order to give an accurate $1/e$ width. Consequently, the approximation is low at higher angles, very low in the single scattering regime.

The comparisons of Tables IV, V, and VI show that the Hanson choice of $x = 1.2$ in Eq. (20) or (25) does not do as well as a smaller value for x in representing f or $dZ/d\sigma$ over a range of angles, θ . Accordingly, a simple approximation, more accurate than Hanson except for the $1/e$ widths, is to put $x = 0.6$ in Eq. (25) and use Eq. (22) which will give the dominant multiple scattering. χ_C and B are given by Eqs. (5), (6), (7), (8), and (9) or (10). We recommend this procedure for its simplicity and good accuracy, ($\leq \sim 10\%$ for $\theta \leq \chi_C \sqrt{B}$). We find, for the data known to us, that using Eq. (26) for x in place of $x = 0.6$ is generally a little better yet, especially for small angles

for which it was designed. It adds the evaluation of the expression (26), however. It, that is Eqs. (26), (25), and (22), is recommended as the best simple multiple scattering formula.

To the foregoing, one may add either the single scattering of Eq. (21) or that of Eq. (27). Neither is satisfactory; both omit plural scattering largely. Eq. (21) is simpler. Eq. (27) can be carried out to angles $\theta \geq 30^\circ$, but not to 90° for a wide slab. See the discussion in Modification II, Section V. One could adjust these equations ((20) and (27)) to fit better any given data. To do so with any confidence of giving accurately an unmeasured distribution would require research not yet in evidence in the literature. These additions slightly increase the total probability over 1, (provided by the Gaussian normalization). The HBM theory is strictly normalized in its entirety, term by term, (the first being 1 the rest 0).

We remind the user that no correction for electron energy loss has been included in this paper. Small loss can be compensated for by using the mean electron energy in the slab, rather than the incident energy. The mean energy may be determined from electron energy loss tables. Considerable loss requires integration of the scattering over the slab thickness.

VIII. FUTURE NEEDS

The field of electron scattering needs wide ranging experiments determined to accuracies at least approaching 1% although even up to 10% is still useful. The angular distribution results should be absolute, i.e. normalized to unit probability, and clearly presented to facilitate numerical comparisons. With a decent sampling: of thick to thin targets, of the energy range, and of the periodic table, one is then in the position to check the various theories to find the best and simplest, and also to provide simple empirical formulas that can give good accuracy without undue algebraic labor.

But without experiment, nothing is sure.

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