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PLASMAS IN PARTICLE ACCELERATORS:
ADIABATIC THEORIES FOR BUNCHED BEAMS

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MARCH 3, 1981

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Abstract

Three different formalisms for discussing Vlasov's equation for bunched beam problems with anharmonic space charge forces are outlined. These correspond to the use of a drift kinetic equation averaged over random betatron motions; a fluidkinetic adiabatic regime analogous to the theory of Chew, Goldberger, and Low; and an adiabatic hydrodynamic theory.

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I. Introduction

Induction linear particle accelerators are being designed for application as drivers for heavy-ion inertial fusion energy power plants [1-6]. As part of the process, it is necessary to identify and avoid regimes of longitudinal instability in bunched beams. Discussions of these problems [1-4,6] have generally been pursued using infinitely thin beams. The question of transverse structure in these instabilities has not been considered. Even though these longitudinal instabilities are usually axisymmetric, introduction of the transverse dimensions results in rather complicated three dimensional orbits for the individual particles. Especially in the presence of anharmonic transverse forces, the transverse motion are rosette orbits in a central force. The solution of Vlasov's equations (cf. Ref. [7]) involves the three dimensional effects of the betatron transverse orbit oscillations. A treatment including two spatial dimensions, three velocity ones and the time parameter might seem necessary. One purpose of this paper is to provide mathematical formalisms which show that the solution of Vlasov's equation for this problem can be reduced to two spatial dimensions and one in velocity (Sections II and III), or even to just two spatial dimensions (Section IV), within physical regimes which allow appropriate asymptotic ordering of parameters.

The discussion is based on the "warm-beam model" recently described by Mark and Smith (Ref. [8]; hereinafter referred to as Paper I). Namely, we take the point of view that the forces on the particle are governed by an axisymmetric averaged component $F_{av}(r, z, t)$ perturbed by a

smaller fluctuating part $F_{fl}(r, \theta, z, t, \epsilon t, \dots)$, where (r, θ, z) are cylindrical polar coordinates with z axis along beam axis and t is the time. The quantity ϵ is a small parameter which indicates, for example, that the force F_{av} varies more rapidly in the radial direction than the axial one, being directly related to the fact that beam bunches are much longer than they are wide. The time variation in F_{av} is also expected to be slower than that of transverse betatron time ($2\pi/\nu$) of the particles, thus the (ϵt) dependences.

As a first approximation, we already discussed in Paper I the problem of the "coasting-beam" equilibria in the warm beam limit with $F_{av}(r)$ only. The study of the orbits in the presence of strong anharmonic space charge forces as outlined there is now the starting point of our discussion of bunched beams. We allow in principle not just the external bunching forces, but also internal ones (eg. waves) but stipulate that they fit into the reasonable scheme where forces like $F_{av}(r, \epsilon z, \epsilon t)$ prevail, while those involving $F_{fl}(r, \theta, z, t, \epsilon t, \dots)$ are at least $O(\epsilon)$ smaller. Under these expected conditions where the dominant forces vary little over the betatron orbit time, we derive a "drift-kinetic" version of Vlasov's equation (Eq. II.23) in Section II where some effects of betatron orbits are averaged out. For axisymmetric problems, this is a "1-1/2 dimension" equation describing exact dynamics in the axial direction, but providing an approximate description of the transverse directions to determine self-consistent forces. In this equation (II.23), all that remains of the transverse motions (for axisymmetric problems) is an averaged radial (hydrodynamic) drift (one reason for the name "drift-kinetic equation").

A more accurate set of equations can be obtained (cf. Section III) under conditions similar to the so-called "adiabatic fluid-kinetic theory" of Chew, Goldberger and Low [9]. It is important to emphasize that we do not have axial magnetic fields in our problem, so that only careful analyses of particle orbits (such as those of Paper I) exhibits the analogy to the latter theory. The betatron vortex oscillations of Paper I are the analogue of the Larmor gyrations in an axial magnetic field. Within this adiabatic theory, the exact continuity and momentum moments (Eqs. III.12-III.15) of Vlasov's equations are used, while the longitudinal heat flow is governed by the above mentioned drift-kinetic equation (II.23).

Sometimes the adiabatic theory is not trivial to apply in practice, thus it is useful to note (Section IV) that an additional "adiabatic (fully) hydrodynamic theory" obtains if we further assume that part of the longitudinal forces are somewhat weaker than the ordering allowed in the earlier adiabatic fluid-kinetic regime. This approximation is equivalent to an additional requirement on the longitudinal length scale of beam bunches (or disturbances thereof) being much larger than the beam radius. (We note that the beam radius to length $\sim 1/50$, and in any case, the majority of particles barely make it across the beam length). Within this fully hydrodynamic regime, the moment equations for continuity and momentum are now closed by equations relating the stresses to density and velocity so that the description is by equations (III.12)-(III.15) with closure relations (IV.1) - (IV.3). An alternate justification for similar approximations following the "double-adiabatic" arguments of Chew et al. [10] might be feasible. In this case we presume (detailed justification has yet to be made) that the

fluctuating force component F_{fk} provides a field which "collides" sufficiently strongly with the longitudinal particle motion so as to sufficiently inhibit longitudinal heat flow in the beam bunch. Thus heat flow terms in the equations for stresses can be omitted.

Since we have retained full three dimensionality in the derivation of all our equations, the theory can be used to study three-dimensional instabilities. For example, the hydrodynamic equations (III.12)-(III.15) with closure relations (IV.1) - (IV.3) could be a justification for the fluid model used by Mark, Krafft and Wang [11].

II. The Drift-Kinetic Equation

For the present time dependent bunched beam problem we assume, for simplicity, that the averaged accelerating, focusing, bunching and space charge forces are represented by

$$\frac{1}{m} F_{av} = -\hat{z} \frac{\partial V_a(Z)}{\partial Z} - \nabla V(r, \theta, z, t), \quad (II.1)$$

where we now distinguish the z coordinate to be relative to the beam bunch which has an average position $Z(t)$ accelerating according to $Z = -\partial V_a / \partial Z$. (m is the mass of the ions). Rather than explicitly displaying the small parameter ϵ as in Section I, we now simply understand that relative to this average beam frame

$$\epsilon = O\left(\frac{\partial}{\partial t}\right) = O\left(\frac{\partial}{\partial \theta}\right) = O\left(\frac{\partial}{\partial z}\right). \quad (II.2)$$

For simplicity of presentation in Sections III-IV, we omit for the time being any discussions of magnetic fields and electromagnetic problems.

Using the corresponding momenta per unit mass (p_r, p_θ, p_z), we can write the equations of motion (relative to the beam bunch) as

$$\frac{dr}{dt} \equiv \dot{r} = p_r, \quad \dot{p}_r = \frac{1}{r^3} p_\theta^2 - \frac{\partial V}{\partial r}, \quad (11.3)$$

$$\dot{\theta} = \frac{1}{r^2} p_\theta, \quad \dot{p}_\theta = - \frac{\partial V}{\partial \theta}, \quad (11.4)$$

$$\ddot{z} = - \frac{\partial V}{\partial z}, \quad (11.5)$$

where the dots above the symbols indicate time derivatives. Vlasov's equation in these coordinates is well known.

In addition to the adiabatic assumptions (11.1)-(11.2), we incorporate also the ordering scheme corresponding to the warm-beam equilibria of Paper I. Namely, we assume that the presence of nonlinear space charge forces precludes the separate discussion of orbits in two orthogonal transverse directions (thus, our use of r, θ, z coordinates). And, in particular, we specify that typical orbits have

$$\left| \frac{r p_r}{p_\theta} \right| \sim \varepsilon \sim \left| \frac{\delta r}{r} \right| < 1, \quad (11.6)$$

which describe rosette orbits in the anharmonic averaged focusing forces. These orbits have radial excursions δr smaller than their typical radius (cf. Figs. 1-2 of Paper I).

Following Paper I, we may presume that there are two superposed counter-rotating beam components with averaged circular frequencies $\pm |\omega(r, \theta, z, t)|$. For each component, we transform Vlasov's equation into "drift-kinetic" form by isolating quantities that are our present analogues of magnetic moment, etc.. This transformation is achieved first through the momentum variables. We introduce the amplitudes α, β and phase ψ of the betatron-vortex oscillations (cf Paper I, Eqs. 14-21) through

$$p_r = U(r, \theta, z, t) + \alpha \sin \psi, \quad (11.7)$$

$$p_\theta = r^2 \Omega(r, \theta, z, t) + \beta \cos \psi, \quad (11.8)$$

$$\alpha \sim \beta \sim \frac{\delta r}{r} \sim \epsilon. \quad (11.9)$$

Differentiating these, substituting into equations (II.3)-(II.4) give,

$$\begin{aligned} \dot{p}_r &\equiv \frac{\partial U}{\partial t} + (U + \alpha \sin \psi) \frac{\partial U}{\partial r} + \left(\Omega + \frac{\beta}{r^2} \cos \psi \right) \frac{\partial U}{\partial \theta} + \\ &\quad + \dot{z} \frac{\partial U}{\partial z} + \dot{\alpha} \sin \psi + \alpha \dot{\psi} \cos \psi \\ &= - \frac{\partial V}{\partial r} + \Omega^2 r + \frac{2\beta \Omega}{r} \cos \psi + \frac{\beta}{r^3} \cos^2 \psi, \end{aligned} \quad (11.10)$$

$$\begin{aligned} - \frac{\partial V}{\partial \theta} = \dot{p}_\theta &\equiv \frac{\partial r^2 \Omega}{\partial t} + (U + \alpha \sin \psi) \frac{r v^2}{2 \Omega} + \left(\Omega + \frac{\beta}{r^2} \cos \psi \right) \frac{\partial r^2 \Omega}{\partial \theta} \\ &\quad + \dot{z} \frac{\partial r^2 \Omega}{\partial z} + \dot{\beta} \cos \psi - \beta \dot{\psi} \sin \psi, \end{aligned} \quad (11.11)$$

where (r, θ, z, t) is the generalized betatron frequency

$$\nu^2(r, \theta, z, t) \equiv \frac{1}{r^3} \frac{d}{dt} \left[r^4 \Omega^2(r, \theta, z, t) \right] \quad (11.12)$$

Solving these for $\dot{\psi}$ gives

$$\begin{aligned} \dot{\psi} = & -\frac{\cos \psi}{\alpha} \left[\frac{\partial V}{\partial r} - \Omega^2 r + \frac{\partial U}{\partial t} \right] + \frac{\sin \psi}{\beta} \left[\frac{\partial V}{\partial \theta} + \frac{\partial r^2 \Omega}{\partial t} \right] \\ & + \cos^2 \psi \left[\frac{\partial \beta \Omega}{\alpha r} - \frac{\beta}{\alpha r} \frac{\partial U}{\partial \theta} \right] + \frac{\alpha}{\beta} \frac{r \nu^2}{2 \Omega} \sin^2 \psi \\ & + \frac{1}{2} \left[\frac{d}{dt} \ln \frac{\beta}{\alpha} + \frac{\partial \Omega}{\partial \theta} - \frac{\partial U}{\partial \theta} \right] \sin 2\psi + \frac{\beta^2}{\alpha r^3} \cos^3 \psi, \end{aligned} \quad (11.13)$$

where $D/Dt = (\partial/\partial t + U\partial/\partial r + \Omega\partial/\partial \theta + z\partial/\partial z)$. In this equation, U and Ω are not yet specified, although we can view them as representing a radial drift and an averaged azimuthal rotation frequency. We now chose

$$\Omega^2(r, \theta, z, t) = \frac{1}{r} \frac{\partial V}{\partial r}(r, \theta, z, t), \quad (11.14)$$

$$U(r, \theta, z, t) = -\frac{\partial \Omega}{r \nu^2} \left[\frac{\partial V}{\partial \theta} + \frac{\partial r^2 \Omega}{\partial t} + \Omega \frac{\partial r^2 \Omega}{\partial \theta} + z \frac{\partial r^2 \Omega}{\partial z} \right], \quad (11.15)$$

so that equation (11.13) begins with a term of $O(1)$ rather than order $(1/\alpha) \sim (1/\beta) \sim (1/\epsilon)$. (Note that $U \sim \partial/\partial t \sim \partial/\partial \theta \sim \partial/\partial z \sim \epsilon$ and that $\partial U/\partial t \sim \partial U/\partial \theta \sim \partial U/\partial z \sim \epsilon^2$). This $O(1)$ coefficient of $\dot{\psi}$ becomes independent of ψ if we choose

$$\frac{\beta}{\alpha} = \frac{r \nu}{2 \Omega} \quad (11.16)$$

As we will immediately find, these choices are crucial in making a simple drift-kinetic equation with the properties stated above in Section I. Now

$$\dot{\psi} = v - \frac{v}{4\Omega r} \frac{\partial U}{\partial \theta} - \frac{\cos \psi}{\alpha} \left[\frac{\partial U}{\partial t} - \frac{3\alpha^2 v^2}{16\Omega^2 r} \right] - \frac{v \cos 2\psi}{4\Omega r} \frac{\partial U}{\partial \theta} \quad (11.17)$$

$$+ \frac{1}{2} \left[\frac{d}{dt} \ln \frac{\beta}{\alpha} + \frac{\partial \Omega}{\partial \theta} - \frac{\partial U}{\partial r} \right] \sin 2\psi + \frac{\alpha v^2}{16\Omega^2 r} \cos 3\psi$$

exhibiting the $\dot{\psi} = v$ behavior we expect for the phase of the betatron-vortex oscillations. From equations (II.10), (II.11), (II.14)-(II.16), we find

$$\frac{d \ln \alpha}{dt} = \frac{1}{2} \frac{d}{dt} \ln v - \frac{1}{2} \left[\frac{\partial U}{\partial t} - \frac{\alpha^2 v^2}{16\Omega^2 r} \right] \sin \psi$$

$$- \frac{v \sin 2\psi}{8 r \Omega} \frac{\partial U}{\partial \theta} + \frac{1}{4} \left[\frac{\partial U}{\partial r} - \frac{\partial \Omega}{\partial \theta} \right] \cos 2\psi + \frac{\alpha v^2}{32 r \Omega^2} \sin 3\psi. \quad (11.18)$$

Vlasov's equation for the distribution function $F(r, \theta, \alpha, \psi, z, \dot{z}, t)$

can be written as

$$\frac{\partial F}{\partial t} + (U + \alpha \sin \psi) \frac{\partial F}{\partial r} + (\Omega + \frac{v\alpha}{2\Omega r} \cos \psi) \frac{\partial F}{\partial \theta} + \dot{\alpha} \frac{\partial F}{\partial \alpha}$$

$$+ \dot{\psi} \frac{\partial F}{\partial \psi} + \dot{z} \frac{\partial F}{\partial z} - \frac{\partial V}{\partial z} \frac{\partial F}{\partial \dot{z}} = 0, \quad (11.19)$$

where the forms of $\dot{\alpha}$ and $\dot{\psi}$ were already recorded. Following Chew, et al. [9], we expand

$$F = \sum_{j=0}^{\infty} \epsilon^{j-2} F_j, \quad (11.20)$$

which begins with a term of $O(\epsilon^{-2})$ because the number density

$$n = \int F_0 \frac{1}{r} d p_r d p_\theta d \dot{z} = \int F_0 \alpha d \alpha d \psi d \dot{z} \quad (11.21)$$

is $O(1)$. (Note that $\alpha \ll 2\pi \ll \psi$ and $\alpha \ll \omega$, $\omega \ll \dot{z} \ll \omega$) and that F_j have zero average over ψ if $j \neq 0$). The lowest order contribution of equation (II.19) is

$$\frac{\partial \bar{F}_0}{\partial \psi} = 0 \quad , \quad (II.22)$$

or $F_0(r, \theta, \alpha, z, \dot{z}, t)$ only and is independent of ψ . The next order equation, after averaging over ψ , gives the drift kinetic equation

$$\frac{\partial \bar{F}_0}{\partial t} + U \frac{\partial \bar{F}_0}{\partial r} + \Omega \frac{\partial \bar{F}_0}{\partial \theta} + \dot{z} \frac{\partial \bar{F}_0}{\partial z} - \frac{\partial V}{\partial z} \frac{\partial \bar{F}_0}{\partial \dot{z}} = 0 \quad , \quad (II.23)$$

where we have taken advantage of the fact that to this order, we have an adiabatic invariant,

$$\frac{d\mu}{dt} = 0 \quad , \quad \mu \equiv \frac{\alpha^2}{\nu} \quad (II.24)$$

and defined $F_0(r, \theta, \nu, z, \dot{z}, t)$ in equation (II.23).

For axisymmetric problems, this drift kinetic equation now involves four parameters $(r, z, \dot{z}, t)$ while $U(r, z, \dot{z}, t)$ is a known quantity according to equations (II.12), (II.14)-(II.15). Equation (II.23) is substantially simpler than the original Vlasov equation because the rapid betatron-vortex oscillations are averaged out. This is advantageous for numerical simulations because the time-step now follows the macroscopic time relevant to the bunching process rather than the microscopic one involving detailed betatron oscillations. If the full particle orbits are followed in particle code simulations, there is always the danger that the relevant smooth slow drifts on the bunching time may be coarsely calculated because it represents only the residual of the rapid betatron and circular oscillations.

II. The Adiabatic Regime

The drift kinetic equation (II.23) is useful on its own merits when one desires more accurate kinetic information in the longitudinal direction but is satisfied with a first approximation in the transverse direction in order to determine the dependence of self-consistent force fields from bulk motion. The nature of the approximation of transverse motions can be seen by calculating averaged "hydrodynamic velocities" (v_r, v_θ, v_z) from the F_0 as determined by equation (II.23). Thus,

$$v_r \approx \langle U + (\mu v)^{\frac{1}{2}} \sin \psi \rangle_F = U(r, \theta, z, v_z, t), \quad (\text{III.1})$$

$$v_\theta \approx \langle \Omega r + \frac{\mu^{\frac{1}{2}} v^{\frac{3}{2}}}{2\Omega} \cos \psi \rangle_{F_0} = \Omega r, \quad (\text{III.2})$$

$$v_z \equiv \langle \dot{z} \rangle_F = \langle \dot{z} \rangle_{F_0}, \quad (\text{III.3})$$

where

$$\langle h \rangle_F \equiv \frac{v^2}{4\Omega n} \int h F d\mu d\psi d\dot{z}, \quad (\text{III.4})$$

and the number density n of particles is,

$$n \equiv \int F \frac{1}{r} dp_r dp_\theta d\dot{z} = \frac{v^2}{4\Omega n} \int F_0 d\mu d\psi d\dot{z}, \quad (\text{III.5})$$

From this calculation, we see that while no particular approximation has been made to $\langle z \rangle = v_z$ the transverse hydrodynamic velocities are those corresponding to $v_r = U$, $v_\theta = \Omega r$ which in fact is correct to the lowest order in ϵ . However, if we examine these velocities, they correspond to determining v_θ by force balance between centrifugal force and $(-\partial V/\partial r)$ while v_r is determined from v_θ by $p_\theta = d(\Omega r^2)/dt$. Thus, effects of fluid stresses and radial inertia term are omitted (because they are higher order in ϵ).

In the analogous plasma problem with axial magnetic fields, Chew, Goldberger and Low [9] have cleverly combined equation (II.23) with the exact continuity and momentum moments of Vlasov's equation to obtain an even better approximation. Namely, in their viewpoint, equation (II.23) is used only to provide information to evolve the stress tensor $\underline{P}$ in time, thus providing closure for the two exact continuity and momentum equations. Within this viewpoint, we calculate the components P_{rr} , $P_{r\theta}$, $P_{\theta\theta}$, etc., of the stress tensor $\underline{P}$,

$$P_{rr} = n \langle \mu v \sin^2 \psi \rangle_{F_0} = \frac{1}{2} n \langle \mu v \rangle_{F_0} \quad (\text{III.6})$$

$$P_{\theta\theta} = n \langle \mu \frac{v^3}{4\Omega^2} \cos^2 \psi \rangle_{F_0} = \frac{v^2}{4\Omega^2} P_{rr} \quad (\text{III.7})$$

$$P_{r\theta} = P_{\theta r} = n \langle \frac{\mu v^2}{4\Omega} \sin 2\psi \rangle_{F_0} = 0 \quad (\text{III.8})$$

$$P_{rz} = P_{zr} = n \langle (\mu v)^{\frac{1}{2}} \sin \psi (\dot{z} - u_z) \rangle_{F_0} = 0 \quad (\text{III.9})$$

$$P_{\theta z} = P_{ze} = n \left\langle \frac{\mu_z^{\frac{1}{2}} v_z^{\frac{3}{2}} \cos \psi (\dot{z} - v_z)}{2 \Omega} \right\rangle_{F_0} = 0 \quad (III.10)$$

$$P_{zz} = n \left\langle (\dot{z} - v_z)^2 \right\rangle_{F_0} \quad (III.11)$$

which could all be determined once F_0 is known.

The complete system of equations of the adiabatic approximation now comprises of the exact continuity and momentum moment equations

$$\frac{\partial n}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r n v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (n v_\theta) + \frac{\partial}{\partial z} (n v_z) = 0, \quad (III.12)$$

$$\begin{aligned} \frac{d v_r}{d t} &\equiv \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} \\ &= \frac{v_\theta^2}{r} - \frac{\partial V}{\partial r} - \frac{1}{nm} \left[\frac{\partial P_{rr}}{\partial r} + \frac{1}{r} (P_{rr} - P_{\theta\theta}) \right], \end{aligned} \quad (III.13)$$

$$\frac{1}{r} \frac{d r v_\theta}{d t} \equiv \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{1}{r} v_r v_\theta = - \frac{\partial V}{\partial \theta}, \quad (III.14)$$

$$\frac{d v_z}{d t} \equiv \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} = - \frac{\partial V}{\partial z} - \frac{1}{nm} \frac{\partial P_{zz}}{\partial z}, \quad (III.15)$$

together with the adiabatic drift equation (II.23) which determines the relevant stresses through (III.6), (III.7), (III.11) with subsidiary relations (II.12), (II.14), (II.15). (Instead of equation (II.12) for v we could instead use (IV.8) derived below).

In these equations of the adiabatic regime, the rapid radial and corresponding azimuthal oscillations of individual particle orbits have been averaged to give the stress contributions P_{rr} and $P_{\theta\theta}$, but some motions of the betatron timescales are retained in the smooth circulation of fluid motion. As we discussed in Paper I, typical accelerator beam bunches with anharmonic space charge forces could be represented by two superposed but counter-rotating streams of roughly equal densities. Finally, we note that in addition to our obvious absence of axial magnetic fields, our results differ from Chew, et al.[9] also in that $P_{\theta\theta}/P_{rr} = v^2/4\omega^2$ may be nearly unity but deviates more and more from unity when anharmonic space charge effects become important.

IV. The Adiabatic Hydrodynamic Approximation

In the induction linac for heavy-ion fusion, the particles in a beam bunch traverses the length of the bunch in a time comparable to or slightly less than the total time spent in the accelerator; but there are many betatron-vortex oscillations in this same period. The diversity of time scales is one consequence of the fact that the ratio of beam radius over length is $R_p/L \sim 10 \text{ cm}/5 \text{ m} \sim 1/50$. On the other hand, we intend to use the "warm-beam" approximation [8] even when $\epsilon \lesssim 1/3$. Thus, we can in fact justifiably say that $(R_p/L) \leq O(\epsilon^2)$. If the longitudinal disturbances also have length scales λ such that $(R_p/\lambda) = O(\epsilon^2)$ then relative to our implicit order of $(\partial/\partial r) = O(1)$ we can order all $(\partial/\partial z) = O(\epsilon^2)$

rather than $O(\epsilon)$ as used in previous sections. Under this latter ordering, we effectively diminish the "heat flow" terms such as $\langle (\dot{z} - v_z) \mu \rangle$ and $\langle (\dot{z} - v_z)^3 \rangle$ by as much as one order in ϵ . They can then be neglected, and we have an adiabatic (fully) hydrodynamic approximation when the stress components P_{rr} , P_{zz} are determined in terms of the lower moments. For example, we find [cf. also Ref. 12],

$$\frac{d}{dt} \left(\frac{P_{rr}}{n\nu} \right) \equiv \left(\frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z} \right) \left(\frac{P_{rr}}{n\nu} \right) = 0, \quad (\text{IV.1})$$

$$\frac{d}{dt} \left(\frac{P_{zz}}{n} \right) \equiv \left(\frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z} \right) \left(\frac{P_{zz}}{n} \right) = 0, \quad (\text{IV.2})$$

$$P_{\theta\theta} = \frac{\nu^2}{4\Omega^2} P_{rr}. \quad (\text{IV.3})$$

Actually, the $\partial/\partial z$ terms are smaller in this ordering but are kept for completeness of correspondence to equations (III.12)-(III.15) which are exact. Thus, we do not make any approximations in the continuity and momentum equations, but the only approximations are in those closure equations (IV.1)-(IV.3) which determine the time evolution of the stresses.

Equation (IV.3) is the same as (III.7) and follows from the discussions of Section III. To derive equation (IV.1), we multiply the drift-kinetic equation (II.23) by μ and integrate over μ , ψ and $\hat{z}$ (i.e. over velocity space). Term by term we get

$$\frac{\partial}{\partial t} \int F_0 \mu d\mu d\psi d\hat{z} = \frac{\partial}{\partial t} \left(\frac{\partial \Omega}{\nu^3} P_{rr} \right), \quad (IV.4)$$

while,

$$\begin{aligned} & \int U(r, \theta, z, \hat{z}, t) \frac{\partial F_0}{\partial r} \mu d\mu d\psi d\hat{z} \\ &= U(r, \theta, z, v_z, t) \frac{\partial}{\partial r} \int F_0 \mu d\mu d\psi d\hat{z} - \frac{\partial \Omega r}{\nu^2} \frac{\partial \Omega}{\partial \hat{z}} \int (\hat{z} - v_z) \frac{\partial F_0}{\partial r} \mu d\mu d\psi d\hat{z} \\ &\approx v_r \frac{\partial}{\partial r} \left[\frac{\partial \Omega}{\nu^3} P_{rr} \right] \end{aligned} \quad (IV.5)$$

according to relation (III.1), (III.4), (III.6) and the fact that the $\partial \Omega / \partial \hat{z}$ term is one order smaller in ϵ (note that a heat flux contribution is involved in this higher order term). Also, we have

$$\int \Omega \frac{\partial F_0}{\partial \theta} \mu d\mu d\psi d\hat{z} = \Omega \frac{\partial}{\partial \theta} \left(\frac{\partial \Omega}{\nu^3} P_{rr} \right) \approx \frac{v_\theta}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial \Omega}{\nu^3} P_{rr} \right), \quad (IV.6)$$

because of equations (III.2), (III.4) and (III.6). Together, equations (II.23) with $(\partial/\partial z) = O(\epsilon^2)$ and equations (IV.4)-(IV.6) imply that

$$\frac{d}{dt} \left(\frac{n P_{rr}}{\nu^3} \right) \equiv \left(\frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + v_\theta \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z} \right) \left(\frac{n P_{rr}}{\nu^3} \right) = 0, \quad (IV.7)$$

The continuity equation (III.12) with equations (II.15), (III.1)-(III.3) implies that

$$\frac{d}{dt} \left(\frac{\nu^2}{n R} \right) = 0, \quad (IV.8)$$

so that equations (IV.1) follows. It is interesting to note that if in the above derivation we had only omitted the heat flux terms but not all a/a_z terms, we would have had instead of equations (IV.7)-(IV.8)

$$\frac{d \ln \left(\frac{n P_{rr}}{\nu^3} \right)}{dt} = \left[\frac{2 n r}{\nu^2} \frac{\partial n}{\partial z} \frac{\partial v_z}{\partial r} - \frac{\partial v_z}{\partial z} \right] = \frac{d \ln \left(\frac{n R}{\nu^2} \right)}{dt}, \quad (IV.9)$$

which again implies (IV.1). This raises the possibility that equation (IV.1) might be alternately justifiable by assuming that the fluctuating force $F_{\text{an} f_k}$ mentioned in Section I is sufficient to inhibit longitudinal heat flux, without being so strong as to imply isotropic stresses.

The adiabatic relation (IV.2) follows in similar fashion by multiplying equation (II.23) with $\frac{1}{2} \epsilon^2$ and integrate over μ , ψ and $\hat{z}$, noting also equation (IV.8) and the fact that relation (III.15) is $dv_z dt \simeq 0(\epsilon^2) \simeq 0$.

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