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SPACE-CHARGE LIMITS FOR LINEAR ACCELERATORS

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ABSTRACT

Equations are developed which determine the space-charge limits for a generalized linear accelerator. A single parameter k , the ratio of space charge to restoring forces, is the only unknown parameter. Experience and computer simulations indicate that $k \approx 0.5$.

I. THE GENERAL FORMULATION

The field in a sphere of charge is given by*

$$\vec{E}_{sc} = \frac{Nze}{4\pi\epsilon_0 r_0^3} \vec{r}, \quad (1)$$

where we assume a uniform density distribution of N particles of charge ze .
 r_0 is the radius of the sphere.

We assume that there are linear restoring (focusing) forces $\vec{E} = \vec{E}' \vec{r}$,
 which keep the ball of charge together.

The equations of motion for each of the three dimensions give the results:

$$\ddot{x} = -\omega^2 x_0 \sin \omega t,$$

$$\dot{x} = \omega x_0 \cos \omega t,$$

where

$$\omega = \sqrt{\frac{ez \vec{E}' (1-k)}{Am_p}},$$

and

The maximum normalized emittance is given by

$$\epsilon_N = r_0^2 \frac{\omega}{c}. \quad (2)$$

Now the maximum electric field is $\vec{E}_{max} = \vec{E}' r_0$, so we can write

$$\epsilon_N = r_0^{3/2} \sqrt{\frac{ez \vec{E}_{max} (1-k)}{Am_p c^2}}. \quad (3)$$

*See Appendix for notation and definitions.

If we keep ϵ_N and $\bar{E}_{\max}$ fixed, there will be a limiting number of particles in the ball for some maximum value of k . Therefore, we obtain the following expression for the space-charge limit:

$$\frac{ez N_{sc}}{r_o} = \frac{4\pi\epsilon_o k}{(1-k)^{1/3}} \left(\frac{A}{z}\right)^{1/3} \epsilon_N^{2/3} \bar{E}_{\max}^{2/3} \cdot \left(\frac{m_p c^2}{e}\right)^{1/3} \quad (4)$$

If we were to view this ball of charge moving by with velocity βc , the space-charge-limited maximum current would be given by

$$i_{\max} = \frac{k 3\pi\epsilon_o c}{(1-k)^{1/3}} \left(\frac{m_p c^2}{e}\right)^{1/3} \left(\frac{A}{z}\right)^{1/3} \epsilon_N^{2/3} \bar{E}_{\max}^{2/3} \beta \quad (5)$$

II. LONGITUDINAL SPACE-CHARGE LIMITS

Now it is appropriate to consider the longitudinal motion separately.

If we suppose a linac with stable phase angle ϕ_s , frequency f , and electric field $\bar{E}_p$, the average accelerating field is $\bar{E} = E_p \sin \phi_s$. The longitudinal or "synchrotron" frequency is given by

$$\omega_s = \sqrt{\frac{\bar{E} \cot \phi_s 2\pi f ez (1-k)}{A m_p \beta c}} \quad (6)$$

The "phase" radius of the bunch is given by

$$r_o = \frac{2\pi f r_o}{\beta c}$$

Then

$$\epsilon_{NL} = \left(\frac{\beta c}{2\pi f} \right)^{3/2} \phi_0^2 \sqrt{\frac{\bar{E} \cot \phi_s (1-k) e z}{A_{mp} c^2}} \quad (7)$$

Using $\bar{E}_{max} = \bar{E} \cot \phi_s \phi_0$, and substituting into Eq. (5), we obtain

$$i_{max_L} = 1.2 \times 10^6 k \phi_0^2 \frac{\beta^2 \bar{E}}{f} \cot \phi_s \quad (8)$$

Typical values are $\phi_0 \sim 2\pi/10$ and $\cot \phi_s \sim 1$,

$$i_{max_L} = 4.73 \times 10^5 k \frac{\beta^2 \bar{E}}{f} \quad (9)$$

The average linac current, assuming all the buckets are filled, is given by the relation:

$$\bar{i}_{m_L} = \frac{4}{3} \frac{\phi_0}{2\pi} i_{max} \approx 0.1333 i_{max_L}$$

III. TRANSVERSE SPACE-CHARGE LIMITS

Now we can consider the transverse current limit by considering a quadrupole-focusing system. For our purposes here, we will consider electrostatic quadrupoles. We will formulate the space-charge limit in terms of the pole-trip field $\bar{E}_{0max}$. This can be converted to an equivalent magnetic channel by the relationship:

$$B_{0max} = \frac{E_{0max}}{\beta c}$$

Now we define a focusing cell as two drift tubes, each containing one quad.

We can write the acceptance as a function of the phase shift by noting

$$\mu_{\text{cell}} = \frac{\omega n}{f},$$

where f is the linac frequency and n is the number of half wavelengths per quad, i.e., $n = 1$ for $\beta\lambda/2$, $n = 2$ for $\beta\lambda$, etc. Then we can write

$$\epsilon_{\text{NT}} = r_0^2 \frac{f \mu_{\text{cell}}}{cn}. \quad (10)$$

The phase advance/cell is related to the zero space-charge advance by the following relation:

$$\mu_{\text{cell}} = \mu_0 (1-k)^{1/2}.$$

Now if we describe the channel in thin-lens approximation, we have a simple relationship for the phase advance.

We use two parameters to describe the lattice. The length of each quad is $l_Q = k_4 \beta c/f$. The radius of the quads is given by $r_0 = k_3 \beta c/f$.

Since the quad must fit in the drift tube, we have typically $k_4 \sim 0.25n$. k_3 cannot be very large or the transit time factor will be too small. Typical values for k_3 might be ~ 0.1 .

Using the thin-lens approximation for the transport matrices gives a simple expression for the phase shift/cell in terms of the linac parameters.

We obtain

$$E_{0_{\text{max}}} = \left\{ \mu_0 \frac{k_3}{k_4} \cdot \frac{6.26}{n} \right\} \frac{f A \beta}{z}. \quad (11)$$

Similarly, the acceptance can be determined in terms of these parameters.

$$\epsilon_{NT} = \left\{ \eta^2 k_3^2 c \frac{\mu_0}{n} (1-k)^{1/2} \right\} \frac{\beta^2}{f}, \quad (12)$$

where $\eta = r_{\max}/\bar{r}$ to account for the fluctuation of the β function inside the transport system. η is a weak function of μ_0 and for typical values of $\mu_0 \sim 1$, $\eta \sim 0.707$.

Now we can obtain an expression for the maximum current that can be transported due to transverse considerations*:

$$i_{\max T} = 2.34 \times 10^7 \left\{ \frac{k \mu_0^2 k_3^2 \eta^{4/3}}{n^2} \right\} \frac{A}{Z} (\beta\gamma)^3. \quad (13)$$

For typical values, $\mu_0 \sim 1.5$, $k_3 \sim 0.1$, $\eta = 0.707$, we get

$$i_{\max T} = 3.3 \times 10^5 \frac{k}{n^2} \frac{A}{Z} (\beta\gamma)^3. \quad (14)$$

Note that the current limit given by Eq. (13) does not depend upon the linac parameters. If we interpret k_3 in terms of the ratio of quadrupole radius to focusing cell length we get a maximum current limit for any transport system, since k_3 cannot become greater than a few tenths, and μ_0 will not be greater than ~ 1.5 . The correction for a cylinder of beam, rather than a ball, is a factor of $2/3$.

*The γ^3 term has been added to make the expression relativistically correct.

The current limit for any quadrupole-focusing system, then, is

$$i_{\max} \sim 1.1 \times 10^5 \frac{A}{Z} (\beta_Y)^3. \quad (14a)$$

IV. OPTIMIZING SIX-DIMENSIONAL PHASE-SPACE DENSITY

Now comparing the transverse and longitudinal current limits, we can equate them to arrive at the maximum brightness condition. We have

$$1.2 \times 10^6 k_{\phi_0}^2 \beta^2 \frac{E \cot \phi_s}{f} = 2.34 \times 10^7 \frac{k^2 \mu_0^2 k_3^2 \eta^{4/3}}{n^2} \frac{A}{Z} \beta^3$$

Strictly speaking, the space-charge factors k for the longitudinal and transverse cases need not be equal. Putting $\phi_0^2 = k_3^2 \times (2\pi)^2$, we get

$$\bar{E} \cot \phi_s = \frac{0.5 \mu_0^2 \eta^{4/3}}{n^2} \frac{fA}{Z} \beta. \quad (15)$$

Typical values for μ_0 are ~ 1.5 , $\eta \sim 0.707$. For a Wideroe, typically $n = 1$.

Then,

$$\bar{E} \cot \phi_s = \frac{0.709}{n^2} \frac{fA\beta}{Z}.$$

Recalling (11) and putting in typical values:

$$k_3 = 0.1, \quad k_4 = 0.25 n,$$

we have

$$E_{0\max} = \frac{3.75}{n^2} \frac{fA\beta}{Z}.$$

Therefore $\bar{E} \sim 0.2 E_{0\max}$ is the condition for getting the maximum density in 6-D phase space. Also, $\epsilon_{NL} \sim \epsilon_{NT}$.

The expression for the 6-D phase-space density is given by the following expression:

$$\rho_{6D} = \frac{\bar{I}/fez}{\epsilon_{NL}^2 \epsilon_{NT}} \quad (16)$$

where

$$\bar{I} = \frac{2}{3} \frac{\phi_0}{\pi} i_{\max}.$$

Using the previously derived relation for $\bar{I}$ and ϵ_{NL} , ϵ_{NT} , we obtain

$$\rho_{6D} \sim \frac{f^2}{\beta^3} \frac{A}{z}. \quad (17)$$

This indicates to us that if we want a super-bright beam we should start with an injection energy as low as is practicable, into a linac with as high a frequency as possible.

V. SINGLE-BUNCH SPACE-CHARGE LIMITS

These same methods can be used to obtain the space-charge limit of an induction accelerator. If we assume a parabolic distribution of charge, the space charge electric field gradient is given by

$$E'_{sc} = z \left(\frac{e}{4\pi\epsilon_0} \right) \cdot \frac{3}{2} \left(1 + 2 \ln \frac{b}{a} \right) \cdot \frac{N_{sc}}{l_0}. \quad (18)$$

The restoring force applied by the induction linac voltage slope can be described in terms of a tilt coefficient, k_T , where the restoring field gradient is

$$E' = \frac{k_T \bar{E}}{I_0}$$

Here, $\bar{E} \equiv$ average accelerating field. Then we obtain

$$\epsilon_{NL} = I_0^{3/2} \sqrt{\frac{e^2 z k_T \bar{E} (1-k)}{m_p c^2 A}} \quad (19)$$

Substituting into (18), we obtain

$$N_{sc} = \frac{4.44 \times 10^{14}}{(1 + 2 \ln \frac{b}{a}) z} \cdot \frac{k (k_T \bar{E})^{1/3}}{(1-k)^{2/3}} \cdot \frac{A^{2/3}}{z^{5/3}} \epsilon_{NL}^{4/3} \quad (20)$$

If we apply the same methods to a single bunch (nonrelativistic) circulating in an accumulator ring, we can derive an expression for the space-charge limit of a bunch. We obtain

$$N_{sc} = \frac{1.88 \times 10^{13}}{(1 + 2 \ln \frac{b}{a}) z} \cdot \frac{k}{(1-k)^{3/4}} \cdot \left(\frac{A}{z}\right)^{3/4} \epsilon_{NL}^{3/2} \left(\frac{Vf}{B_{YR}}\right)^{1/4}$$

Caution must be taken in applying this formula. That is because $\epsilon_{NL} = [(\Delta p/p) I_0]_{BY}$. For fixed ϵ_{NL} , one might be tempted to increase the space-charge limit by increasing V , or f , say. However, this will generally increase $\Delta p/p$. A value of $\Delta p/p$ much greater than about 1% is usually a limit for practical reasons associated with the accumulator.

For a single bunch in a linac, the longitudinal charge limit for a single bunch is

$$N_{sc} = \frac{4.5 \times 10^{13}}{z} \frac{k}{(1-k)^{3/4}} \left(\frac{A}{z} \right)^{3/4} \left(\frac{\bar{E} \cot \phi_s f}{\beta_Y} \right)^{1/4} \epsilon_{NL}^{3/2} \quad (21)$$

It is interesting to note that if $\bar{E}$ and f remain fixed, the space-charge limit decreases with increasing energy. If the linac was limited at injection, then we will get an increase of ϵ_{NL} such as to maintain N_{sc} as constant.

Another caveat. In attempting to design for a certain space-charge limit, one is not free to increase ϵ_{NL} . For a given $\bar{E}$, f , and β , ϵ_{NLmax} is readily computed. This can easily be seen by substituting (7) into (22).

$$N_{sc} = \frac{1.58 \times 10^{24}}{z} k \phi_0^3 \bar{E} \cot \phi_s \frac{\beta^2}{f^2}$$

The limits of ϕ_0 typically are $\sim \pi/5$.

APPENDIX

The normalized emittances, ϵ_{NL}^0 and ϵ_{NT}^0 , are defined as follows:

$$(\Delta X \Delta p)_0 = \epsilon_{NT}^0 A m_p c = (X X') \beta_Y A m_p c$$

$$(\Delta E \Delta t)_0 = \epsilon_{NL}^0 A m_p c = \left(\frac{\Delta p}{p} \frac{1}{\beta_0} \right) \beta_Y A m_p c$$

The units used throughout are mks, with E in volts/meter and B in tesla.