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Non-Linear Chromatic Effects in the Beta-Function.

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Summary

A theory is presented for computing the non-linear dependence of the β -functions on momentum. Results are found for the quadratic term. The results of the theory are compared with computed results. A procedure is proposed for computing the strengths of the sextupole correctors to correct the dependence of the β -function on momentum.

1. Introduction

The dependence of the β -function on particle momentum, $\Delta p/p$, may depart appreciably from being linear in $\Delta p/p$ for some lattices. This paper finds results for the quadratic term in the β -function dependence on $\Delta p/p$.

For the case where the operating v -value is close to a half-integer value, and the momentum dependence of the β -function is dominated by the corresponding half-integer stop band, it has been found that the maximum variation in the β -function with $\Delta p/p$ can be written as

$$\frac{\Delta\beta}{\beta} = \left(\frac{\Delta\beta}{\beta}\right)^{(1)} + \left(\frac{\Delta\beta}{\beta}\right)^{(2)} \quad (1.1a)$$

$(\Delta\beta/\beta)^{(1)}$ is the term which is linear in $\Delta p/p$,

$$\left(\frac{\Delta\beta}{\beta}\right)^{(1)} = \frac{2}{2(n-2v)} |\Delta v|, \quad (1.1b)$$

where n is the integer closest to $2v$, and Δv_2 is the half-width of the half-integer stop band at $v = n/2$. $(\Delta\beta/\beta)^{(2)}$ is the term which is quadratic in $\Delta p/p$,

$$\left(\frac{\Delta\beta}{\beta}\right)^{(2)} = \frac{3}{(n-2v)^2} C \frac{\Delta p}{p} |\Delta v_2| + \frac{1}{(n-2v)^2} |\Delta v_2|^2, \quad (1.1c)$$

where C is the chromaticity, $C = p dv/dp$. The above results are derived in Section 2.

The above results of the theory were compared with computed results using the SYNCH program for a lattice with a rapid dependence of the β -function on momentum. Using the above theoretical results, a least squares procedure is proposed for choosing the sextupole strengths that will minimize the variation of the β -function with momentum.

2. Theory of Non-Linear $\beta_A(p)$, $\beta_B(p)$

Courant and Snyder¹ presented the linear theory for the change in the β -function due to a change in the gradient function. They showed that the change in β -function $\Delta\beta$ caused by a change in the gradient function ΔK , obeys the equation

$$\frac{d^2}{d\theta^2} \left(\frac{\Delta\beta}{\beta}\right) + 4v^2 \left(\frac{\Delta\beta}{\beta}\right) = -v^2 B_2^2 \Delta K, \quad (2.1)$$

$$\Delta K = \Delta G/B\rho,$$

$$d\theta = ds/v\rho.$$

Equation (2.1) can be used to compute the linear variation of β with momentum by putting

$$\Delta K = (-G \Delta p/p + 2B_2 X_p \Delta p/p)/B\rho \quad (2.2)$$

where $B_2(s)x^2$ is the sextupole field and $X_p(s)$ is the horizontal closed orbit dispersion function.

To obtain an equation for $\Delta\beta$ which includes higher order terms in ΔK , we start from the equation for $b = \beta^{1/2}$,

$$\frac{d^2}{ds^2} b + Kb - \frac{1}{b^3} = 0. \quad (2.3)$$

Letting $K \rightarrow K + \Delta K$, and $b \rightarrow b + \Delta b$, and keeping up to second order terms in Δb and ΔK , one finds the equation for Δb

$$\frac{d^2}{ds^2} \Delta b + \left(K + \frac{3}{b^4}\right) \Delta b = -b \Delta K - \Delta K \Delta b + \frac{6}{b^3} \left(\frac{\Delta b}{b}\right)^2 \quad (2.4)$$

Now we carry out a transformation similar to the usual orbit theory transformation $\eta = x/\beta^{1/2}$. We put

$$\zeta = \Delta b/\beta^{1/2} = \Delta b/b \quad (2.5)$$

Let ψ , the betatron phase function, obey $d\psi/ds = 1/\beta = 1/b^2$, then

$$\begin{aligned} \frac{d}{ds} \Delta b &= \frac{db}{ds} \zeta + b \frac{d\zeta}{ds} \\ &= \frac{db}{ds} \zeta + \frac{1}{b} \frac{d\zeta}{d\psi} \end{aligned} \quad (2.6)$$

$$\begin{aligned} \frac{d^2}{ds^2} \Delta b &= \frac{d^2 b}{ds^2} \zeta + \frac{db}{ds} \frac{d\zeta}{d\psi} \frac{1}{b^2} + \frac{d^2 \zeta}{d\psi^2} \frac{1}{b^3} - \frac{d\zeta}{d\psi} \frac{1}{b^2} \frac{db}{ds} \\ &= \left(-Kb + \frac{1}{b^3}\right) \zeta + \frac{d^2 \zeta}{d\psi^2} \frac{1}{b^3}. \end{aligned}$$

Putting this result into Eq. (2.4) for Δb , one finds the equation for $\zeta = \Delta b/b$

$$\begin{aligned} \frac{1}{b^3} \frac{d^2 \zeta}{d\psi^2} + \left(-Kb + \frac{1}{b^3}\right) \zeta + \left(K + \frac{3}{b^4}\right) b \zeta &= \\ -b \Delta K - b \Delta K \zeta + \frac{6}{b^3} \zeta^2, \end{aligned}$$

or

$$\frac{d^2 \zeta}{d\theta^2} + 4v^2 \zeta = -v^2 b^4 \Delta K - v^2 \Delta K b^4 \zeta + 6v^2 \zeta^2 \quad (2.7)$$

Eq. (2.7) is correct to second order in ΔK . It may be interesting to note that one can also obtain the exact equation by replacing $6\zeta^2$ by $1/(1+\zeta)^3 - 1 + 3\zeta$, which gives

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$$\frac{d^2 \zeta}{d\theta^2} + 4v^2 \zeta = -v^2 b^4 \Delta K - v^2 \Delta E \zeta + v^2 \left\{ \frac{1}{(1-\zeta)^3} - 1 + 3\zeta \right\} \quad (2.8)$$

Eq. (2.7) reduces to the linear Courant-Snyder equation, Eq. (2.1), if one drops the second order terms on the right hand side.

Eq. (2.7) can be solved by successive approximations to find $\Delta E/\delta$ up to second order terms. The linear solution $\zeta^{(1)}$ can be found by neglecting the second order terms. This gives the solution found by Courant and Snyder.

$$\begin{aligned} \zeta^{(1)} &= -\frac{1}{4\sin(2\pi\nu)} \int ds_0 \cos(2\pi\nu - |\psi(s)| \\ &\quad - \psi(s_0)) |B(s_0)| \Delta K(s_0), \\ \Delta K &= (-G \Delta p/p + 2 B_2 X_p \Delta p/p)/B_0. \end{aligned} \quad (2.9)$$

Note that $\zeta^{(1)} = \Delta E/\delta$ to first order.

An alternative solution for $\zeta^{(1)}$ is

$$\zeta^{(1)} = \sum_n \zeta_n^{(1)} e^{in\theta}, \quad (2.10)$$

$$\zeta_n^{(1)} = \frac{(v^2 b^4 \Delta K)_n}{n^2 - 4v^2},$$

$$(v^2 b^4 \Delta K)_n = 2\nu \frac{1}{4\pi} \int ds \beta \Delta K e^{-in\theta},$$

$$\Delta K = (-G \Delta p/p + 2B_2 X_p \Delta p/p)/B_0.$$

If 2ν is close to the integer n , then the dominant term is

$$\begin{aligned} \zeta_n^{(1)} &= \frac{(v^2 b^4 \Delta K)_n}{n^2 - 4v^2}, \\ &= \frac{2\nu}{n^2 - 4v^2} \Delta v_2, \end{aligned} \quad (2.11)$$

$$\Delta v_2 = \frac{1}{4\pi} \int ds \beta \Delta K e^{-in\theta}.$$

Δv_2 is the half-width of the stopband at $\nu = n/2$, driven by the off momentum gradient $\Delta K = (-G \Delta p/p + 2B_2 X_p \Delta p/p)/B_0$.

The second order term in ζ may be found by putting $\zeta = \zeta^{(1)} + \zeta^{(2)}$, and solving the following equation for $\zeta^{(2)}$.

$$\frac{d^2}{d\theta^2} \zeta^{(2)} + (4v^2 + v^2 \Delta K b^4) \zeta^{(2)} = 6v^2 \zeta^{(1)2} \quad (2.12)$$

The $v^2 b^4 \Delta K$ term will shift the ν value by $\Delta\nu$ given by

$$\begin{aligned} 8v\Delta\nu &= (v^2 b^4 \Delta K)_0 \\ &= 2\nu C \Delta p/p, \end{aligned} \quad (2.13)$$

or

$$\Delta\nu = \frac{1}{4} C \Delta p/p,$$

$$C = \frac{1}{4\pi} \int ds \beta \Delta K,$$

where C is the chromaticity $C = p dv/dp$. We will write Eq. (2.12) as

$$\frac{d^2}{d\theta^2} \zeta^{(2)} + 4v^2 \zeta^{(2)} = 6v^2 \zeta^{(1)2}, \quad (2.14)$$

with the understanding that ν is shifted by the amount given by Eq. (2.13). This amount is often small and can be neglected. For example, if $C = 2$, and $\Delta p/p = .01$, $\Delta\nu = .005$ which is not appreciable unless 2ν is very close to the integer, n .

Eq. (2.14) for $\zeta^{(2)}$ can be solved by putting

$$\zeta^{(2)} = \sum_n \zeta_n^{(2)} e^{in\theta} \quad (2.15)$$

$$\begin{aligned} \zeta_n^{(2)} &= \frac{6v^2 (\zeta^{(1)2})_n}{4v^2 - n^2} \\ &= \frac{6v^2}{4v^2 - n^2} \sum_{n_1+n_2=n} \zeta_{n_1}^{(1)} \zeta_{n_2}^{(1)} \end{aligned}$$

An approximate more usable result can be found in the case where 2ν is close to an integer n . In this case $\zeta_n^{(1)}$ given by Eq. (2.11) dominates, and

$$\begin{aligned} \zeta_n^{(2)} &= \frac{6v^2}{4v^2 - n^2} 2 \zeta_n^{(1)} \zeta_0^{(1)}, \\ \zeta_0^{(1)} &= \frac{1}{4v^2} (-v^2 b^4 \Delta K)_0, \\ &= -\frac{1}{2\nu} C \Delta p/p. \end{aligned} \quad (2.16)$$

C is the chromaticity $C = p dv/dp$ and $C = \int ds \beta \Delta K/4\pi$. Terms like $\zeta_n^{(1)} \zeta_{2n}^{(1)}$ have been dropped. Thus one finds for $\zeta_n^{(2)}$,

$$\begin{aligned} \zeta_n^{(2)} &= \frac{12v^2}{4v^2 - n^2} \frac{2\nu \Delta v_2}{n^2 - 4v^2} \left(-\frac{1}{2\nu} C \Delta p/p \right), \\ \zeta_n^{(2)} &= \frac{12v^2}{(4v^2 - n^2)^2} C \frac{\Delta p}{p} \Delta v_2, \end{aligned} \quad (2.17)$$

$$\zeta_n^{(2)} = \frac{3}{4} \frac{1}{(2\nu - n)^2} C \frac{\Delta p}{p} \Delta v_2. \quad (2.18a)$$

Note that $\zeta_n^{(1)}$ is given by

$$\zeta_n^{(1)} = \frac{1}{2} \frac{1}{(n - 2\nu)} \Delta v_2 \quad (2.18b)$$

It may be noted that $\zeta_n^{(2)}$ is quadratic in $\Delta p/p$ as Δv_2 is linear in $\Delta p/p$. Eq. (2.17) shows that the second order term varies inversely as the square of $(2\nu - n)$, the distance from the resonance line.

The variation of β with momentum can be found from the exact relationship

$$\frac{\Delta \beta}{\beta} = 2 \frac{\Delta b}{b} + \left(\frac{\Delta b}{b} \right)^2, \quad (2.19)$$

$$\frac{\Delta\beta}{\beta} = 2\zeta + \zeta^2.$$

ζ is given by

$$\zeta = 2|\zeta_n^{(1)} + \zeta_n^{(2)}| \cos(n\theta + \delta_2),$$

since $\zeta_n^{(1)}$ and $\zeta_n^{(2)}$ have the same phase, the phase of $\Delta\beta_2$ which is denoted by δ_2 . Thus $\Delta\beta/\beta$ is given by

$$\begin{aligned} \frac{\Delta\beta}{\beta} &= 4|\zeta_n^{(1)} + \zeta_n^{(2)}| \cos(n\theta + \delta_2) \\ &+ 4|\zeta_n^{(1)}|^2 \cos^2(n\theta + \delta_2). \end{aligned} \quad (2.20)$$

$$\zeta_n^{(1)} = \frac{1}{2} \frac{1}{(n-2\nu)} \Delta\nu_2,$$

$$\zeta_n^{(2)} = \frac{3}{4} \frac{1}{(n-2\nu)^2} C \frac{\Delta p}{p} \Delta\nu_2.$$

The maximum variation of β with $\Delta p/p$ can then be written as

$$\left(\frac{\Delta\beta}{\beta}\right)_{\max} = \left(\frac{\Delta\beta}{\beta}\right)_{\max}^{(1)} + \left(\frac{\Delta\beta}{\beta}\right)_{\max}^{(2)} \quad (2.21)$$

$$\left(\frac{\Delta\beta}{\beta}\right)_{\max}^{(1)} = 4|\zeta_n^{(1)}| = \frac{2}{|n-2\nu|} |\Delta\nu_2|$$

$$\begin{aligned} \left(\frac{\Delta\beta}{\beta}\right)_{\max}^{(2)} &= 4|\zeta_n^{(2)}| + 4|\zeta_n^{(1)}|^2 \\ &= \frac{3}{(n-2\nu)^2} C \frac{\Delta p}{p} |\Delta\nu_2| + \frac{1}{(n-2\nu)^2} |\Delta\nu_2|^2 \end{aligned}$$

3. Comparison of Theory with Computed Results

During the design study of the CBA, a lattice was considered which has a strong half-integer stop-band at $\nu = 22.5 = 45/2$. The operating ν -value is $\nu = 22.65$. This lattice exhibited a strong dependence of the β -function on $\Delta p/p$. β_x varied from $\beta_x = 67$ at $\Delta p/p = 0$ to $\beta_x = 150$ at $\Delta p/p = .01$. The dependence of β_x on $\Delta p/p$ and on azimuth was computed using the SYNCH program², and the results compared with the theoretical results of the previous section.

For this lattice, the stop band half-width at $\nu = 22.5$ was computed to be $\Delta\nu_2 = .13$. In order to compute $\Delta\beta/\beta$ from Eq. (2.20), one needs the chromaticity C . A difficulty arises here, because accelerators tend to operate with low value of C . The lattice has a $C_x = -40$ before the sextupoles are excited which is then reduced to $C_x \approx 2$ using the sextupoles.

In the SYNCH run, the sextupoles were adjusted to give $C_x = 1.2$. If one computes C_x from Eq. (2.13) using the same sextupoles as used by SYNCH, one finds $C_x = -5.7$. These two results for C_x give considerably different results for $\zeta_n^{(2)}$ in Eq. (2.20) which is proportional to C_x . In what follows, the value $C_x = 1.2$ found by the SYNCH program was used. One then finds from Eq. (2.20)

$$\begin{aligned} \zeta_n^{(1)} &= -.22 \\ \zeta_n^{(2)} &= .013 \end{aligned}$$

The $\zeta_n^{(1)}$ term gives a linear variation in β of 4 $\zeta_n^{(1)}$ or $\Delta\beta = 59$ for $\Delta p/p = .01$, and a quadratic variation in $\Delta\beta/\beta$ of $4|\zeta_n^{(1)}|^2$, or $\Delta\beta = 13$. The $\zeta_n^{(2)}$ term gives a quadratic variation in $\Delta\beta/\beta$ of $4\zeta_n^{(2)}$ or $\Delta\beta = 3.5$.

The linear result predicts that $\Delta\beta/\beta$ will vary azimuthally like $\cos 2\psi$ where ψ is the betatron phase. In the lattice considered here, $\Delta\psi$ for one cell is $\pi/2$ and the variation of $\Delta\beta/\beta$ with θ has a period of 2 cells. The non-linear term in Eq. (2.2), $4|\zeta_n^{(1)}|^2 \cos^2(n\theta + \delta_2)$ changes this θ -dependence of $\Delta\beta/\beta$ adding a term that goes like $\cos^2 2\psi$. The following table shows the θ variation of β_x as given by the theory and by the SYNCH program.

	0	$\pi/2$	π
β_x -Theory	143	18	143
β_x -SYNCH	147	30	147

The agreement between theory and SYNCH at $\psi = \pi/2$ is only fair, but β_x is small there and large errors may be expected. There appears to be some evidence for the existence of the $\cos^2 2\psi$ term.

The linear result predicts that $\Delta\beta/\beta$ will vary linearly with $\Delta p/p$. The SYNCH results indicate an appreciable $(\Delta p/p)^2$ term. This is illustrated by the following table.

$\Delta p/p$	.01	0	-.01
β_x -Theory	143	67	25
β_x -SYNCH	147	67	30

References

1. E.D. Courant and H. Snyder, Annals of Physics, **3**, No. 1, (1958).
2. A.A. Garren and A.S. Kenney (unpublished), (1968).

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