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34

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Numerical Models for High Beta Magnetohydrodynamic Flow

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Abstract

The fundamentals of numerical magnetohydrodynamics for highly conducting, high-beta plasmas are outlined. The discussions emphasize the physical properties of the flow, and how elementary concepts in numerical analysis can be applied to the construction of finite difference approximations that capture these features. The linear and nonlinear stability of explicit and implicit differencing in time is examined, the origin and effect of numerical diffusion in the calculation of convective transport is described, and a technique for maintaining solenoidality in the magnetic field is developed. Many of the points are illustrated by numerical examples. The techniques described are applicable to the time-dependent, high-beta flows normally encountered in magnetically confined plasmas, plasma switches, and space and astrophysical plasmas.

Introduction

Magnetohydrodynamics (MHD) is a fluid model for the motion of an ionized gas in a magnetic field. It best describes collisional

plasmas that are in local thermodynamic equilibrium. It is often used to describe plasmas that are only weakly collisional. In such cases, the finite sized gyro-orbits of individual particles in a magnetic field replace the mean free path. When, and in what form, the MHD equations are applicable to plasma problems is discussed by Freidberg (1982).

MHD is used to model a very wide range of physical phenomena. In space plasmas, MHD is used to model the interaction of the solar wind with the earth's magnetosphere, [Nishida, 1982; Hones, 1984; Abdallah and Dutton, 1985; and Palmadesso et al, 1987], the dynamics of the solar corona [Priest, 1985] and the properties of astrophysical plasmas. [Kundu and Holman, 1985] MHD is used because "... the one fluid equations basically express the conservation of mass, momentum and energy, and it [is] perhaps not so surprising that the overall existence [i.e. behavior] of the solar wind is accessible via this approach." [Schwartz, 1985]

MHD is also used to model laboratory plasmas on a wide range of time and space scales. One typical application is the calculations of the dynamics of a field-reversed, theta-pinch plasma described in Section 5 [Milroy and Brackbill, 1982]. A completely different application with even more complex physics is the description of self-generated magnetic fields in laser target plasmas [Colombant and Winsor, 1977]. Such problems begin to test the boundary between fully kinetic models [Forslund and Brackbill, 1982] and MHD transport models [Brackbill and Goldman, 1983].

There are many MHD models, ranging in complexity from the derived, asymptotic equations [Strauss, 1976], through the ideal equations, which are discussed here, to the Braginskii transport models, [Braginskii, 1964]. For space plasmas, usually only dissipation may be assumed small. Thus, one must solve the full system of ideal equations, or something close to ideal, rather than a simpler set of derived or asymptotic equations as is often the case in other applications. One may, of course, add other physical effects to investigate details of the flow. For example, hybrid models with particle ions and fluid electrons are used [Brecht and Thomas, 1987].

Even in their ideal form, the MHD equations are more complex than the equations for ordinary fluid flow. However, they comprise a system of coupled, nonlinear, partial differential equations that have the same mathematical character as the Euler equations, the model for gas dynamics. Thus, many of the methods that have been developed for fluid flows also work for MHD.

Mathematically, the Lundquist equations are a symmetric hyperbolic system of equations in conservation form [Lax and Wendroff, 1960]. Theorems of existence and uniqueness of solutions for initial and boundary-value problems have been proven. Waves and shocks are among the solutions to the equations. [Friedrichs and Kranzer, 1958; Roberts, 1985] In these respects, the Lundquist equations are completely similar to the gas dynamic equations.

However, because of the greater complexity of the physics that the Lundquist equations describe, some aspects of the solutions are quite different. The magnetic field introduces a strong anisotropic character to the medium which causes wave propagation to depend on the direction of propagation with respect to the magnetic field. [Roberts, 1985] In addition, there are several distinct signal speeds so that, in general, the responses to disturbances are quite complex.

To capture the principal features of the solutions in numerical calculations, several problems must be addressed. Some of these problems are unique to MHD: for example, preserving the solenoidality of the magnetic field. Other problems, such as energy conservation, numerical stability, and computational diffusion, are similar to ordinary gas dynamics but are more complex or have different consequences for MHD than for ordinary fluid flow.

In the discussion that follows, elementary numerical analysis is applied to understanding how the MHD equations may be solved numerically. Numerical examples from calculations of MHD flows in two dimensions are included to illustrate the properties of the solutions.

1. Ideal Magnetohydrodynamics

When the plasma is collisional and both electrons and ions are magnetized, the ideal model for magnetohydrodynamics may be used. Ideal MHD is described by the Lundquist equations, which comprise a coupled system of first order, partial differential equations including Ampere's law,

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}, \quad (1.1)$$

Faraday's law,

$$\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \quad (1.2)$$

the solenoidal condition,

$$\nabla \cdot \mathbf{B} = 0, \quad (1.3)$$

and Ohm's law,

$$\mathbf{E} + \frac{\mathbf{u} \times \mathbf{B}}{c} = 0. \quad (1.4)$$

In general, when ions and electrons are sufficiently magnetized the Hall, diamagnetic, and collisional terms may be neglected. The evolution of the sources of the magnetic field is described by a mass continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0, \quad (1.5)$$

a momentum equation,

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \rho \mathbf{u} \mathbf{u} = -\nabla p + \frac{\mathbf{J} \times \mathbf{B}}{c} \quad (1.6)$$

and an energy equation,

$$\rho \frac{\partial i}{\partial t} + \nabla \cdot \rho i \mathbf{u} = -p (\nabla \cdot \mathbf{u}), \quad (1.7)$$

where i is the internal energy per unit mass.

Conservation Laws

Conservation laws play the same role in MHD as they do in ordinary gas dynamics. From these laws, not only can one identify the constants of the motion, such as total mass, momentum, energy, and magnetic flux which are neither created nor destroyed by the flow, but also one can derive Hugoniot relations which connect the states of the flow across shock discontinuities.

The conservation laws of MHD are easily derived by writing the equations of motion in conservation form. For example, the integral of the mass continuity equation, Eq. (1.5), over the domain is,

$$\frac{\partial}{\partial t_D} \int_D \rho dV = - \oint_S \mathbf{n} \cdot \mathbf{u} \rho dS \quad (1.8)$$

where S is surface enclosing the domain D , and $\mathbf{n}$ is it's unit outward normal. This equation, which states that the change in mass of D is equal to the net flux of mass across the surface S , is in conservation form. When $\mathbf{u}$ is zero on S , the mass enclosed by S is a constant of the motion.

Similar conservation laws can be derived for momentum and magnetic flux. The momentum integral is,

$$\frac{\partial}{\partial t_D} \int_D \rho \mathbf{u} dV = - \oint_S \left\{ \mathbf{n} \cdot \mathbf{u} \rho \mathbf{u} + \mathbf{n} p + \mathbf{n} \cdot \mathbf{M} \right\} dS \quad (1.9)$$

$\mathbf{M}$ is the Maxwell stress tensor, whose divergence is equal to the Lorentz force,

$$\frac{\mathbf{J} \times \mathbf{B}}{c} = \frac{1}{4\pi} \nabla \cdot \left(\frac{1}{2} \mathbf{B}^2 - \mathbf{B} \mathbf{B} \right) \quad (1.10)$$

when the solenoidal condition, Eq. (1.3), is satisfied.

The integral of the magnetic intensity is also in conservation form,

$$\frac{\partial}{\partial t_D} \int_D \mathbf{B} dV = - \oint_S \mathbf{n} \times \frac{\mathbf{u} \times \mathbf{B}}{c} dS \quad (1.11)$$

When the electric field is given by the ideal Ohm's law, Eq. (1.4), the magnetic field is frozen in to the plasma so that the magnetic flux threading S is a constant of the motion,

$$\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} = 0. \quad (1.12)$$

When S is a closed surface, Eq. (1.12) is equivalent to the solenoidal condition, Eq. (1.3), and is true no matter how the surface S moves relative to the fluid.

The total energy, comprising internal, magnetic, and kinetic energy, is also "conserved". The total energy is the sum of the kinetic,

$$\frac{\partial}{\partial t_D} \int_D \rho \frac{u^2}{2} dV = - \oint_S \mathbf{n} \cdot \mathbf{u} \left\{ \rho \frac{u^2}{2} + p \right\} dS + \int_D \left\{ \mathbf{E} \cdot \mathbf{J} + p \nabla \cdot \mathbf{u} \right\} dV \quad (1.15)$$

magnetic,

$$\frac{1}{4\pi} \frac{\partial}{\partial t_D} \int_D \frac{B^2}{2} dV = -c \oint_S \mathbf{n} \cdot (\mathbf{E} \times \mathbf{B}) dS - \int_D \mathbf{E} \cdot \mathbf{J} dV. \quad (1.16)$$

and internal energy,

$$\frac{\partial}{\partial t_D} \int_D \rho i dV = - \oint_S \mathbf{n} \cdot \mathbf{u} p i dS - \int_D p \nabla \cdot \mathbf{u} dV. \quad (1.17)$$

The last term is the work done by pressure. In the resulting total energy integral, the energy exchange terms cancel, yielding an equation in conservation form,

$$\frac{\partial}{\partial t_D} \int_D \rho \left\{ i + \frac{u^2}{2} + \frac{B^2}{8\pi\rho} \right\} dV = - \oint_S \mathbf{n} \cdot \mathbf{u} \left\{ \rho \left(\frac{u^2}{2} + i \right) + p \right\} dS - \oint_S \mathbf{n} \cdot (\mathbf{E} \times \mathbf{J}) dS \quad (1.18) \quad (1.18)$$

Any change in the energy in D is due to a flux of energy across the enclosing surface.

2. Waves and Stability

Magnetohydrodynamic Waves

Consider an infinite, homogeneous fluid in static equilibrium, with density ρ_0 , pressure p_0 , and magnetic field B_0 . If small amplitude perturbations about this state are imposed, they will evolve as described by the linearized equations of motion,

$$\frac{\partial \rho_1}{\partial t} + \rho_0 (\nabla \cdot \mathbf{u}_1) = 0 \quad (2.1)$$

$$\frac{\partial p_1}{\partial t} + \gamma p_0 (\nabla \cdot \mathbf{u}_1) = 0 \quad (2.2)$$

$$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} + \nabla (p_1 + \mathbf{B}_0 \cdot \mathbf{B}_1) - (\mathbf{B}_0 \cdot \nabla) \mathbf{B}_1 = 0 \quad (2.3)$$

$$\frac{\partial \mathbf{B}_1}{\partial t} + \mathbf{B}_0 (\nabla \cdot \mathbf{u}_1) - (\mathbf{B}_0 \cdot \nabla) \mathbf{u}_1 = 0 \quad (2.4)$$

If Eqs. (2.1-2.4) are differenced in time and Fourier transformed in space, they may be written,

$$\rho_1^1 - \rho_1^0 + \rho_0 i(\mathbf{k} \cdot \mathbf{u}_1^\theta) \Delta t = 0 \quad (2.5)$$

$$p_1^1 - p_1^0 + \gamma p_0 i(\mathbf{k} \cdot \mathbf{u}_1^\theta) \Delta t = 0 \quad (2.6)$$

$$\rho_0 (\mathbf{u}_1^1 - \mathbf{u}_1^0) + i\mathbf{k} (p_1^\theta + \mathbf{B}_0 \cdot \mathbf{B}_1^\theta) \Delta t - i(\mathbf{B}_0 \cdot \mathbf{k}) \mathbf{B}_1^\theta \Delta t = 0, \quad (2.7)$$

$$\mathbf{B}_1^1 - \mathbf{B}_1^0 + i\mathbf{B}_0 (\mathbf{k} \cdot \mathbf{u}_1^\theta) \Delta t - i(\mathbf{B}_0 \cdot \mathbf{k}) \mathbf{u}_1^\theta \Delta t = 0, \quad (2.8)$$

where $\mathbf{u}^\theta = \theta \mathbf{u}^1 + (1-\theta) \mathbf{u}^0$, and $p^\theta = \theta p^1 + (1-\theta) p^0$. Various values of θ and ϕ correspond to well known algorithms. With $\theta=1$ and $\phi=0$, one has the explicit leapfrog or Courant scheme; with $\theta=1/2$ and $\phi=1/2$, the implicit Crank-Nicholson scheme; and with $\theta=1$ and $\phi=1$, the implicit, backward Euler scheme. For simplicity, a system of units is chosen in which $c=4\pi$.

With the time variation, $e^{i\omega t}$, the finite difference approximation to the time derivative becomes,

$$\frac{\rho_1^1 - \rho_1^0}{\Delta t} = i\omega \rho_1 e^{i\omega t} \frac{\sin \frac{\omega \Delta t}{2}}{\frac{\omega \Delta t}{2}}. \quad (2.9)$$

The intermediate value, $\mathbf{u}^\theta$, can be written,

$$\begin{aligned} \mathbf{u}_1^\theta &= 1/2 (\mathbf{u}_1^1 + \mathbf{u}_1^0) + (\theta - 1/2) (\mathbf{u}_1^1 - \mathbf{u}_1^0), \\ &= \mathbf{u}_1 e^{i\omega t} \left[\cos \frac{\omega \Delta t}{2} + 2i(\theta - 1/2) \sin \frac{\omega \Delta t}{2} \right] \end{aligned} \quad (2.10)$$

The form of the characteristic equation [Friedrichs and Kranzer, 1958] is preserved if one defines a numerical frequency Ω , and wave number κ ,

$$\Omega = \frac{2}{\Delta t} \tan \left[\frac{\omega \Delta t}{2} \right],$$

$$\kappa^\theta = k \left[1 + i(\theta - 1/2)\Omega \Delta t \right],$$

$$\kappa^\phi = k \left[1 + i(\phi - 1/2)\Omega \Delta t \right],$$

With the numerical frequency and wave number, Eqs. (2.5-2.8) can be written

$$-\Omega \rho_1 + \rho_0 (\kappa^\theta \cdot u_1) = 0, \quad (2.11)$$

$$-\Omega p_1 + \gamma p_0 (\kappa^\theta \cdot u_1) = 0, \quad (2.12)$$

$$-\Omega \rho_0 u_1 + \kappa^\phi (p_1 + B_0 \cdot B_1) - (B_0 \cdot \kappa^\phi) B_1 = 0, \quad (2.13)$$

$$-\Omega B_1 + B_0 (\kappa^\theta \cdot u_1) - (B_0 \cdot \kappa^\theta) u_1 = 0, \quad (2.14)$$

The numerical phase velocity is defined by,

$$c^2 = \frac{\Omega^2}{\kappa^\theta \cdot \kappa^\phi} \quad (2.15)$$

The equations will have a solution of the form,

$$B_1 = B_1(t=0) e^{i(\omega t - kx)}$$

when the determinant of coefficients is zero,

$$\text{Det} = \pm p c^2 (\rho c^2 - B_n^2) \{ p c^4 - (p a^2 + B^2) c^2 + a^2 B_n^2 \} \quad (2.16)$$

where $a^2 = \gamma p_0 / \rho_0$ is the ordinary sound speed, B_n is the component of the magnetic field in the direction of the wave travel, $B_n = B \cdot k / |k|$, and $c = \omega/k$. The roots of the equation, $\text{Det}=0$, give the characteristic speeds. In the limit, $\Delta t=0$, the characteristic speeds are equal to the physical wave speeds.

The characteristic equation is eighth order, with four distinct roots. Each root corresponds to a possible independent

mode of the system of linear equations. The roots are the particle characteristics,

$$c = 0, \quad (2.17a)$$

Alfven or transverse waves,

$$c = \pm \frac{B_n}{\rho}, \quad (2.17b)$$

which cannot travel perpendicular to B, and the slow and fast waves, which are roots of the term in brackets in Eq. 2.16,

$$c^2 = \frac{1}{2} (a^2 + A^2 \pm \sqrt{(a^2 + A^2)^2 - 4a^2 A_n^2}), \quad (2.17c)$$

where $A^2 = B^2/\rho$, and $A_n^2 = B_n^2/\rho$. Parallel to the magnetic field, $A_n = A$, and the two solutions to Eq. (2.17c) are,

$$c = \pm a \quad (2.18a)$$

and

$$c = \pm A. \quad (2.18b)$$

Perpendicular to the magnetic field, the two roots are,

$$c=0; c=\pm(a^2 + A^2)^{1/2}. \quad (2.18c)$$

The last roots are magnetoacoustic waves.

Two and Two-and-One-Half-Dimensional Dispersion

The fully three dimensional formulation is less often used than reduced models. The models are called two-dimensional when variation is allowed in two space dimensions, and two components of B and u are advanced. The models are called two and one-half dimensional when variation is allowed in two space dimensions, and three components of B and u are advanced.

If one repeats the dispersion analysis for these two models, one finds that the two-and-one-half dimensional model yields the same characteristic equation as the three-dimensional model, but that the two-dimensional model yields the characteristic equation,

$$\text{Det}(2D) = \pm \rho c^2 \left\{ \rho c^4 - (\rho a^2 + B^2) c^2 + a^2 B_n^2 \right\}. \quad (2.16a)$$

In this equation, the transverse wave is absent. Since the transverse wave describes rotation of the magnetic field and flow in a plane perpendicular to the direction of propagation to the

wave, three components of $\mathbf{B}$ and $\mathbf{u}$ are required to describe it. Thus two-dimensional and two-and-one-half-dimensional flows have fundamentally different properties.

Numerical Stability

For the linearized equations, the group and phase velocities are equal

$$\frac{d\omega}{dk} = \frac{\omega}{k},$$

and thus the roots of the dispersion relation are the characteristic speeds at which information is propagated. The wave fronts therefore define the domain of influence of each point in the medium. The wave fronts also define the domain of dependence of the solution at each instant. For numerical stability, the domain of dependence must always exceed the domain of influence, a requirement that leads to the Courant-Friedrichs-Lewy condition on the time step. The numerical frequency is given by,

$$\Omega^2 = k^2(a^2 + A^2)(1 + i(\theta - 1/2)\Omega\Delta t)(1 + i(\phi - 1/2)\Omega\Delta t). \quad (2.18)$$

When $\theta = \phi = 1/2$ (the Crank-Nicholson scheme), $-\pi \leq \omega\Delta t \leq \pi$ for all values of the right hand side. As Δt increases, ω approaches the Nyquist frequency, $\omega \rightarrow \pm \pi/\Delta t$. Because ω is real for all values of k and Δt , the Crank-Nicholson scheme is unconditionally stable.

The Courant or leapfrog scheme corresponds to $\theta=1$, $\phi=0$. (One solves first the momentum equation u^1 . Then one uses u^1 to solve Faraday's law, the continuity equation, and the pressure equation for $\mathbf{B}^1$, ρ^1 and p^1). The dispersion relation for the Courant scheme is given by

$$\Omega^2 = \frac{k^2 c^2}{1 - \left[\frac{kc\Delta t}{2} \right]^2}. \quad (2.19)$$

The scheme is conditionally stable. When $kc\Delta t > 2$, Ω is pure

imaginary. Thus $\text{Re}(\omega) = \pm\pi$, and $\text{Im}(\omega)$ will include a positive root that corresponds to an exponentially growing mode. That is, when a signal propagates more than one wavelength in Δt , the scheme is unstable. The domain of influence exceeds the domain of dependence.

In general, when θ and ϕ are unequal to $1/2$, the frequency is complex. For example, when $\theta = \phi$, one can easily solve for the imaginary part of Ω ,

$$\text{Im}(\Omega) = \frac{k^2(a^2 + A^2)(\theta - 1/2) \frac{\Delta t}{2}}{1 + (\theta - 1/2)^2 k^2(a^2 + A^2) \left(\frac{\Delta t}{2}\right)^2} \quad (2.20)$$

When $\theta < 1/2$, $\text{Im } \Omega < 0$ (and thus $\text{Im } (\omega)$) and the modes will grow exponentially. When $\theta > 1/2$, $\text{Im } \Omega > 0$, and the modes will decay exponentially. When Δt is very large,

$$\text{Im}(\Omega) \frac{\Delta t}{2} = \frac{1}{(\theta - 1/2)}, \quad (2.21)$$

or for $\theta = 1$

$$\text{Im}(\tan \frac{\omega \Delta t}{2}) = 2. \quad (2.21)$$

Thus, implicit, backward Euler time differencing will damp modes; explicit time differencing will be unstable unless $\theta + \phi \geq 1$.

Numerical Dispersion and Implicit Differencing in Time

When the differencing in time is implicit with ϕ and θ both greater than one half, the solutions are stable with very large time steps. However, when the time step is large, the numerical solutions will give wave speeds that differ from the physical speeds. It is important to know what kinds of error will occur, and which parts of the solution are still correct if one wishes to use large time steps.

The discretization in time will cause the phase velocity for short wavelength modes to be smaller than the physical value.

Consider an implicit case with $\theta, \phi = 1/2$. The numerical wave speed is given by,

$$c' = \frac{\frac{2}{\Delta t} \tan^{-1} \left[\frac{kc\Delta t}{2} \right]}{k} . \quad (2.22)$$

Physically, the ratio of ω/k should be constant for all k . Numerically, ω cannot be larger than the Nyquist frequency, $\pi/\Delta t$. Thus, as k increases to its maximum value, $\pi/\Delta x$, ω/k approaches $\Delta x/\Delta t$ and $kc'\Delta t$ approaches two. Numerical dispersion will cause a train of short wavelength ripples to trail a disturbance.

However, for a given k the ordering of the wave speeds is preserved by the numerical solution. This can be shown by representing the wave speeds in a Friedrich's diagram, which depicts the wave fronts that result from a point disturbance such as would occur were a pebble dropped into a still pond of water. There are three wave fronts, two of which result from waves which propagate parallel to the magnetic field. When the sound speed is greater than the Alfvén speed, Figs. (2.1a,b) and (2.2), sound waves (outer curve) propagate slightly faster perpendicular to the field than along it. The slow wave (inner curve) and transverse wave (middle curve) propagate only along the magnetic field. When the time step is small, $kc\Delta t \leq 0.1$, the numerical wave speeds are equal to the analytical wave speeds, Fig. (2.1a), and they are well separated with $a=3\lambda$. However, when the time step is larger, $kc\Delta t \leq 1.0$, the numerical values for the sound speed are less than the analytical values, but the numerical values for the slow and transverse wave speeds are nearly equal to the analytical values, Fig. (2.1b).

Figure 2.1 The Friedrich's diagram for MHD is generated by plotting the zero contour Det , Eq. (2.10). The Alfvén speed is greater than the sound speed, $A=3a$. In (a), with $kc\Delta t \leq 0.1$, all wave speeds are resolved. In (b), with $kc\Delta t \leq 1$ and plotted to the same scale as (a), only the slow wave is the same as in (a).

When the time step is increased still further, $kc\Delta t \leq 10$, all waves are slowed, Fig. (2.2).

Figure 2.2 For the case shown in Fig. (2.1), but with $kc\Delta t \leq 10$, all waves are retarded, including the slow wave.

When the sound speed is less than the Alfvén speed, Fig. (2.3a,b), the sound waves (inner curve) propagate only along the magnetic field lines, while the fast waves propagate isotropically.

The transverse wave is similar to Fig. (2.1a). Once again, the solutions with $kc\Delta t \leq 0.1$, Fig.(2.3a), and $kc\Delta t \leq 1.0$, Fig.(2.3b), are compared. The large time step affects the faster speeds significantly, and the slower speeds essentially not at all.

Figure 2.3 The Friedrich's diagram is plotted with $a=2A$. In (a), with $kc\Delta t \leq 0.1$, all waves are resolved. In (b), with $kc\Delta t \leq 1$, the fast wave is retarded, but the transverse and slow waves have the same speeds as in (a).

One may conclude that implicit schemes used with large time steps will give accurate results for those modes that are resolved by the time step. Those modes that are not resolved are represented with significant error. In those cases where time scales are well separated, the time step can be chosen to resolve the time scale of interest, rather than the fastest time scale.

3. Energy Conservation and Time Centering

The nonlinear stability of an Eulerian, semi-discrete approximation to the Lundquist equations is considered. In the semi-discrete approximation, time is differenced but spatial variables are not. In these equations, errors due to approximating the time derivatives may be separated from those due to differencing space derivatives. These errors are shown to cause nonlinear instabilities in linearly stable, finite difference equations. Thus, the analysis shows that linear stability is a necessary, but not sufficient, condition for stability.

A scheme is nonlinearly stable if the solution is bounded in

the L^2 norm. If ϕ is the solution, then

$$\int \phi^2(t) dV = c \int \phi^2(t=0) dV \quad (31)$$

where c is a positive constant. Kinetic energy, for example, is quadratic in the velocity. If kinetic energy is found to increase without bound, the scheme is nonlinearly unstable. More generally, one may examine the stability of a numerical scheme using the energy method, which usually requires computing the norms of dependent variables other than the energy [Richtmyer and Morton, 1967]. In the case of MHD with the θ method, the energy integral can be computed and used to determine nonlinear stability.

A θ scheme for the nonlinear equations, which uses somewhat different time levels than Eqs. (2.5-2.8), but which can be analyzed similarly and shown to be linearly stable for $\theta, \phi \geq 1/2$, is written:

$$\rho^1 - \rho^0 + \nabla \cdot \mathbf{u}^0 \rho^1 \Delta t = 0 \quad (3.2)$$

for the mass continuity equation,

$$\rho^1 \mathbf{u}^1 - \rho^0 \mathbf{u}^0 + \nabla \cdot \mathbf{u}^0 \rho^1 \mathbf{u}^0 \Delta t = \left[(\mathbf{J}^0 \times \mathbf{B}^0) - \nabla p^0 \right] \Delta t \quad (3.3)$$

for the momentum equation,

$$\mathbf{B}^1 - \mathbf{B}^0 = -(\nabla \times \mathbf{E}^0) \Delta t \quad (3.4)$$

for Faraday's law, and

$$\rho^1 i^1 - \rho^0 i^0 + \nabla \cdot \mathbf{u}^0 \rho^1 i^0 = -p^0 (\nabla \cdot \mathbf{u}^0) \Delta t \quad (3.5)$$

for the internal energy.

These difference equations differ in two ways from ordinary methods. First, an internal energy equation is solved, rather than the usual total energy equation. This allows an analysis of the exchange of energy between components, as well as estimates of the error in each equation. Second, in Eqs. (3.2-3.5) the convection terms are implicit. By using implicit differencing, stable approximations to the convective derivative are obtained in a form suitable for centered differencing in space.

Explicit differencing of the convective derivative is more often used than implicit differencing to avoid having to invert a

matrix as required to evaluate implicit equations. With explicit methods, centered space differencing is linearly unstable. Linear stability is obtained by "upwinding" the convection term, which results in non-centered equations. Nonlinear stability is obtained by gradient limiting, by flux correction, or by the addition of dissipation.

To complete the semi-discrete equations, the current and electric field are given by Ampere's and Ohm's law,

$$\mathbf{J}^0 = \nabla \times \mathbf{B}^0, \quad (3.6)$$

$$\mathbf{E}^0 = -\mathbf{u}^0 \times \mathbf{B}^0. \quad (3.7)$$

From Eqs. (3.2-3.7), one can form a total energy equation and examine energy conservation.

Consider the change in kinetic energy from t to $t+\Delta t$,

$$\Delta E_K = \frac{1}{2} \left[\rho^1 (\mathbf{u}^1)^2 - \rho^0 (\mathbf{u}^0)^2 \right] \quad (3.8)$$

This term is now evaluated using the momentum equation, Eq.(3.3), to derive a kinetic energy equation like Eq.(1.15). So that a total energy equation like Eq.(1.18) is obtained when the kinetic energy is added to the magnetic and internal energy, an expression for the kinetic energy is constructed by projecting the momentum equation on to an intermediate value of the velocity,

$$\Delta E_K = \mathbf{u}^0 \left[\rho^1 \mathbf{u}^1 - \rho^0 \mathbf{u}^0 \right] + (\rho^1 - \rho^0) (\mathbf{u}^0)^2 - \epsilon_K \quad (3.9)$$

The error term ϵ_K is defined by equating the changes in kinetic energy given by Eqs. (3.8) and (3.9). The error in kinetic energy is

$$\epsilon_K = \left(\theta - \frac{1}{2} \right) \rho^0 (\mathbf{u}^1 - \mathbf{u}^0)^2 + (\rho^1 - \rho^0) (\mathbf{u}^1 \cdot \mathbf{u}^0) \cdot \mathbf{u}^1 \quad (3.10)$$

where $\mathbf{u}^1$ is defined by

$$\mathbf{u}^1 = \left(2\theta^2 - \frac{(\theta+1)}{2} \right) \mathbf{u}^1 + \left(-2\theta^2 + \frac{(\theta-1)}{2} \right) \mathbf{u}^0 \quad (3.11)$$

The first term in the error is positive definite, and therefore dissipative. It's contribution is stabilizing. The second term, which is absent in the Lagrangian form of the equations [Brackbill and Ruppel, 1986], appears to be of indefinite sign. It's

contribution may be stabilizing or destabilizing.

The term in brackets is evaluated by substituting from the momentum equation, Eq.(3.3). The momentum convection term, for example, yields

$$\mathbf{u}^0 \cdot \nabla \mathbf{u}^0 \rho^1 \mathbf{u}^0 = \frac{1}{2} \mathbf{u}^0 \cdot \mathbf{u}^0 \left[\nabla \cdot \mathbf{u}^0 \rho^1 \right] + \nabla \cdot \mathbf{u}^0 \rho^1 \frac{\mathbf{u}^0 \cdot \mathbf{u}^0}{2} \quad (3.12)$$

(It should be clear now why the momentum equation is projected on to $\mathbf{u}^0$.) Because the equations are not differenced in the spatial variable, there is no difficulty in using the product rule to derive Eq.(3.12). With effort, centered difference approximations can be derived that also satisfy the product rule, so that the identity is true for certain difference equations as well. Combining the first term in Eq.(3.12) and the second term in Eq.(3.9), and substituting from the mass continuity equation, Eq.(3.2), gives

$$\Delta E_K = \left[-\nabla \cdot \mathbf{u}^0 \rho^1 \frac{(\mathbf{u}^0)^2}{2} + \mathbf{E}^0 \cdot \mathbf{J}^0 - \mathbf{u}^0 \cdot \nabla p^0 \right] \Delta t - \epsilon_K \quad (3.11)$$

The first term in the brackets is due to the convection of kinetic energy and is in conservation form. The second term is due to work done on the fluid by the magnetic field and the third is due to work done by internal energy. These terms can be written in conservation form only after they are combined with corresponding terms in the magnetic and internal energy integrals.

At issue is the overall sign of the error. If it is positive, kinetic energy is dissipated by the error, and nonlinear stability in the sense of the energy method is achieved. In part, the sign of the error depends upon the value of θ . If one examines Eq.(3.13), one finds that for all values of θ , $\mathbf{u}'$ is different from zero and the sign and magnitude of the second term in Eq.(3.10) always must be taken into account. However, when θ equals one, $\mathbf{u}'$ is equal to $\mathbf{u}^1 - \mathbf{u}^0$, and the error in kinetic energy is given by,

$$\epsilon_K = \frac{1}{2} \rho^0 (\mathbf{u}^1 - \mathbf{u}^0)^2 \left[1 + \frac{2(\rho^1 - \rho^0)}{\rho^0} \right] \quad (3.14)$$

Thus, with backward Euler differencing the error in kinetic energy

is dissipative and stabilizing if the inequality

$$\frac{\rho^1 - \rho^0}{\rho^0} \geq -\frac{1}{2} \quad (3.15)$$

is satisfied. Here is a nonlinear stability condition similar to the linear stability conditions one usually encounters. Eq.(3.15) imposes a limit on the volumetric expansion that may occur in a rarefaction.

For other values of θ , however, the stability properties are less definite. For example, when θ is equal to zero, the error in kinetic energy is

$$\epsilon_K = -\frac{1}{2} [\rho^0 (u^1 - u^0)^2 + (\rho^1 - \rho^0) ((u^1)^2 - (u^0)^2)] \quad (3.16)$$

This error may be of either sign. If one satisfies Eq.(3.15), the coefficient of the second term is less than that of the first, and the error is evidently destabilizing. One might ask if dissipation can be added to the momentum equation to guarantee nonlinear stability. The answer is evidently no. Any dissipative term of the necessary form would violate conservation of momentum. One could add diffusion, such as that produced by enough viscosity to give net dissipation, but to assure that it would be sufficient for stability would require too much diffusion nearly everywhere.

Consider a flow with $B=0$. Since there is no contribution to the error from the internal energy equation, the kinetic energy error determines the sign of the change in the total energy. Thus, with θ equal to 1 and ϕ equal to zero, the total energy change is negative definite and the method is nonlinearly stable. However, with θ equal to zero and ϕ equal to one the total energy change may be positive and the scheme may be nonlinearly unstable, even though it is linearly stable!

To complete the energy integral, the change in magnetic field energy from t to $t+\Delta t$ is calculated by projecting the change in magnetic field intensity on to the average value of B over the interval,

$$\Delta E_b = \frac{1}{2} (\mathbf{B}^1 + \mathbf{B}^0) \cdot (\mathbf{B}^1 - \mathbf{B}^0). \quad (3.17)$$

Once again, the energy change is replaced by one obtained by projecting the difference in $\mathbf{B}$ on to an intermediate value appearing in the other equations,

$$\Delta E_b = \mathbf{B}^\phi \cdot (\mathbf{B}^1 - \mathbf{B}^0) \cdot \varepsilon_K \quad (3.18)$$

where the error in magnetic field energy is given by

$$\varepsilon_K = (\phi - \frac{1}{2}) (\mathbf{B}^1 - \mathbf{B}^0)^2 \quad (3.19)$$

Substituting from Eqs. (3.3), (3.5), and (3.6) above, and using several vector identities, one derives the change in magnetic field energy,

$$\Delta E_b = - \left[\nabla \cdot (\mathbf{E}^0 \times \mathbf{B}^\phi) + \mathbf{E}^0 \cdot \mathbf{J}^\phi \right] \Delta t - \varepsilon_K \quad (3.20)$$

The term in brackets includes the Poynting flux and work done by the field on the fluid just as in Eq. (1.16). The last term is the $O(\Delta t^2)$ numerical error introduced by lack of centering in the time differencing.

The change in total energy is the sum of changes in magnetic, internal, and kinetic energies,

$$\Delta E = - \left[\nabla \cdot \mathbf{u}^0 \rho^0 \frac{(\mathbf{u}^0)^2}{2} + \nabla \cdot (\mathbf{E}^0 \times \mathbf{J}^\phi) - \nabla \cdot \mathbf{u}^0 p^\phi \right] \Delta t - \varepsilon_K - \varepsilon_b \quad (3.21)$$

The term in brackets is in conservation form as in Eq. (1.18). When this term is integrated over the domain, all volume contributions cancel leaving, by Gauss's theorem, only a surface flux which depends upon the boundary conditions for the problem. The error terms, however, do give a volume contribution that determines the nonlinear stability of the solutions. If the error in kinetic and magnetic energies is positive, the solutions are stable. Exact energy conservation and positive internal energy result if these errors are added to the internal energy.

It is possible that the errors in magnetic and kinetic energies may have different signs. Consider an explicit scheme, such as the Courant scheme, $\theta=1$, $\phi=0$. The error in the kinetic

energy is dissipative, but the error in the magnetic energy is not. Therefore, the change in total energy over an isolated domain because of the error will be negative definite only when the flow is super-Alfvenic,

$$u^2 > \frac{B^2}{\rho}. \quad (322)$$

Otherwise, the apparent energy transfer from the flow to the magnetic field will cause the total energy to increase, and the solution will be nonlinearly unstable.

The leapfrog scheme is also explicit, but the offset in alternate steps yields $\theta, \phi = 1/2$. With the leapfrog scheme, the last term will contribute an error of indefinite sign. According to this analysis, the leapfrog scheme may or may not be nonlinearly stable. Experience suggests that convective transport will not be stable unless the method of differencing introduces diffusion.

To summarize, explicit schemes for magnetohydrodynamic flow are nonlinearly unstable, except for super-Alfvenic flow. Time-centered and forward-differenced implicit Lagrangian schemes are nonlinearly stable [Brackbill and Ruppel, 1986]. The convective derivative in the momentum equation contributes energy errors that are destabilizing. Only backward Euler differencing introduces enough dissipation to compensate for these errors. However, backward Euler differencing in time is stable with centered differencing in space, so that the overall error may be less than in explicit methods that are stabilized by "upwind" differencing of the convection term.

4. Spatial Differencing

To solve the Lundquist equations numerically, one must discretize in space as well as in time. If one wishes to difference the Eulerian form of the equations, Eqs. (1.2-1.7), an excellent reference for MHD is Roberts and Potter (1968). Here, a moving mesh method is described that allows one to use an adaptive mesh [Brackbill and Saltzman, 1982]. The formulation is given for flows in two space dimensions. The formulation for flows in three space

dimensions is given in Brackbill (1976).

There are at least two ways to write the difference equations for moving mesh calculations. For single step methods, Eqs.(1.2-1.7) can be generalized to moving coordinate frames [Brackbill,1976] For two step methods, including the arbitrary-Lagrangian-Eulerian (ALE) method developed by Hirt et al (1974), the equations of motion are solved in Lagrangian form followed by a separate calculation of convective transport.

In an ALE calculation, the Lagrangian equations are solved on a mesh of arbitrarily-shaped, quadrilateral zones. From time to time, or as often as every cycle, the data is transferred from the Lagrangian to a new grid that is generated by some external means. The transfer from one mesh to the other is calculated either by solving the convective transport equations, by a global rezoning [Dukowicz,1987], or by moving particles [Eastwood,1987].

The Lagrangian Difference Equations

The spatial derivatives are evaluated by a coordinate transformation. A comprehensive reference for such methods is Thompson et al (1985). The equations are derived by mapping each quadrilateral zone in physical space on to a unit square in natural space. In two dimensions, the transformation is given by inverting the bilinear interpolation formula,

$$x = \xi' \left[x_{1+1,j} (1-\eta') + x_{1+1,j+1} \eta' \right] + (1-\xi') \left[x_{1,j} (1-\eta') + x_{1,j+1} \eta' \right] \quad (4.1)$$

where $\xi' = \xi - 1$, $\eta' = \eta - j$. The derivative $\partial x / \partial \xi$, for example, at the center of a cell $\xi', \eta' = 1/2$, is given by the difference equation,

$$\frac{\partial x}{\partial \xi} = \frac{1}{2} \left[x_{1+1,j} + x_{1+1,j+1} - x_{1,j+1} - x_{1,j} \right]. \quad (4.2)$$

During Lagrangian motion of the zone, it is postulated that the natural coordinates are unchanging even as the physical coordinates are changing. Thus, the flow velocity at each point, which is derived by differentiating Eq.(4.2), is given by

$$u = \xi' \left[u_{1+1,j} (1-\eta') + u_{1+1,j+1} \eta' \right] + (1-\xi') \left[u_{1,j} (1-\eta') + u_{1,j+1} \eta' \right] \quad (4.3)$$

To evaluate derivatives of the velocity, one may differentiate

using the chain rule to obtain, for example,

$$\frac{\partial u}{\partial x} = \frac{\frac{\partial u}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial u}{\partial \eta} \frac{\partial x}{\partial \xi}}{J}, \quad (4.4)$$

where J is the Jacobian of the transformation

$$J = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta}. \quad (4.5)$$

If the derivative of the velocity is approximated by it's average value over a cell (or it's value at the center of a cell in Cartesian coordinates), it may be expressed as a linear equation,

$$\frac{\partial u}{\partial x_\alpha} = \sum_v d_{cv}^\alpha u_v^\beta, \quad (4.6)$$

where v denotes a vertex, c denotes a cell, and α and β denote the component directions. The coefficients, d_{cv} , in Eq. (4.4) describe the geometry of the zone.

In this ALE formulation, density, internal energy, and magnetic field intensity are stored at the centers of cells. Their evolution is calculated from the equations,

$$\rho_c^1 - \rho_c^0 + \rho_c^1 \sum_v d_{vc} \cdot u_v^\theta \Delta t = 0, \quad (4.7)$$

$$B_c^1 - B_c^0 + \left[B_c^1 \sum_v d_{vc} \cdot u_v^\theta - B_c^\theta \cdot \sum_v d_{vc} u_v^\theta \right] \Delta t = 0, \quad (4.8)$$

and

$$\rho_c^0 (i_c^1 - i_c^0) + p_c^\theta \sum_v d_{vc} \cdot u_v^\theta \Delta t = 0. \quad (4.9)$$

The momentum is calculated from the expression,

$$M_v (u_v^1 - u_v^0) - \sum_c \left\{ d_{vc} \left[p_c^\theta + \frac{1}{2} B_c^1 \cdot B_c^0 \right] - B_c^\theta \cdot d_{vc} B_c^\theta \right\} v_c \Delta t = 0. \quad (4.10)$$

The summation over c accumulates contributions from those cells adjacent to vertex v . The vertex mass, M_v , defined by,

$$M_v = \frac{1}{4} \sum_c \rho_c V_c \quad (4.11)$$

is a lumped mass matrix.

Using the geometric coefficients, one can easily demonstrate conservation of momentum and angular momentum [Brackbill, 1987]. The proofs follow from the summation rules,

$$\sum_c d_{vc} = \sum_v d_{vc} = 0, \quad (4.11)$$

where v is summed over the vertices of c , and c is summed over the cells adjacent to vertex v .

A total energy integral is formed by repeating the derivation in Section 3. If one neglects surface contributions and defines the components of the energy by the following quadrature rules,

$$\Delta E_B = \frac{1}{2} \sum_c (B_c^1)^2 V_c^1 - (B_c^0)^2 V_c^0, \quad (4.12)$$

$$\Delta E_I = \sum_c M_c (i_c^1 - i_c^0), \quad (4.13)$$

$$\Delta E_K = 1/2 \sum_v M_v ((u_v^1)^2 - (u_v^0)^2), \quad (4.14)$$

the total energy change is given by

$$\Delta E_B + \Delta E_I + \Delta E_K = -(\theta - \frac{1}{2}) \sum_v M_v (u_v^1 - u_v^0)^2 - (\phi - \frac{1}{2}) \sum_c V_c (B_c^1 - B_c^0)^2 \quad (4.15)$$

The Lagrangian equations are dissipative and, therefore, nonlinearly stable when $\theta, \phi \geq 1/2$.

Total energy conservation can be imposed by adding the dissipated magnetic and kinetic energy in Eq.(4.15) back into the internal energy.

Reasoning

From time to time, the Lagrangian grid is replaced by one that is supplied by some external generator. The new grid will generally have the same number of cells as the old grid, but it will be displaced. To transfer the data from the old grid to the new one,

and to conserve mass, momentum, magnetic flux and total energy, one typically solves convective transport equations in flux-divergent form. These equations are written,

$$M_c^1 - M_c^L = \Delta t \int_{s(v_c)^L} \mathbf{n} \cdot \mathbf{U} \rho_c^L dS, \quad (4.16)$$

$$M_v^1 \mathbf{u}_v^1 - M_v^L \mathbf{u}_v^L = \Delta t \int_{s(v_v)^L} \mathbf{n} \cdot \mathbf{U} \rho^L \mathbf{u}^L dS, \quad (4.17)$$

$$\mathbf{B}_c^1 V_c^1 - \mathbf{B}_c^L V_c^L = \Delta t \int_{s(v_c)^L} \mathbf{n} \cdot \mathbf{U} \mathbf{B}^L dS, \quad (4.18)$$

and

$$M_c^1 e_c^1 - M_c^L e_c^L = \Delta t \int_{s(v_c)^L} \mathbf{n} \cdot \mathbf{U} \rho^L e^L dS, \quad (4.19)$$

where the cell energy is given by,

$$e_c = i_c + \frac{1}{2} \frac{B_c^2}{\rho_c} + \frac{1}{8} \rho_c V_c \sum_v (u_v)^2. \quad (4.20)$$

The superscript L denotes the data from the Lagrangian calculation.

$\mathbf{U} \Delta t$ is the displacement of each vertex in moving from the old grid to the new one.

Note that the control volume for the momentum, which is constructed by joining cell centers to the centers of cell edges, is not the same as for any of the other variables. Because the control volumes are not the same, positivity of the internal energy cannot be guaranteed in calculations where the total energy is conserved. In such calculations, the internal energy is reconstructed by subtracting the magnetic and kinetic energies from the total energy. Since the control volumes for the two variables are not the same, the resulting internal energy may not be positive, especially in high speed flow problems.

The flux across the control volumes can be calculated in various ways. One may use van Leer's gradient limiter [Hawley *et al.* 1984], Lax-Wendroff or interpolated donor cell [Lax and

Wendroff,1960] flux corrected transport [Zalesak,1987], or Smolarkiewicz's method [Smolarkiewicz,1984]. Implicit schemes with centered spatial differencing are presently under development.

Numerical Examples of Convection

To move the grid points where they are needed, or to add points, one must transfer the data from one set of grid points to another. Essentially, the data must be transferred as accurately as is practical. Inaccuracies in conserving mass, momentum, energy and magnetic flux, and the diffusion of gradients must be minimized. A comparison of several approximations to convection applied to modeling the earth's magnetosphere is given in Brackbill (1985).

Another illustration is given by the rotation of a plasma eddy in a magnetic field. An initially uniform magnetic field is swept along and distorted by the plasma. The field lines bend as they are excluded from the center of the eddy and concentrated on the periphery [Moffatt, 1978]. As the gradients in the magnetic field intensity are steepened by the distortion of the field, currents are created that oppose further distortion. These currents are diminished by diffusion, which permits relative motion between the plasma and the field and thus reduces the stress. If there is numerical diffusion, it's presence can be detected by a decrease in the current and the maximum magnetic field.

The data for the eddy problem are:

Initial density: $\rho_0 = 1 \text{ g/cm}^3$

Sound speed: $a = 1 \text{ cm/s}$

Angular velocity: $\Omega = 0.5$ ($u_{\max} = 0.5 a$)

Field intensity: $B_{y0} = 0.01G$

Number of grid points: $N_x = N_y = 25$ (50)

Mesh spacing: $\Delta x = \Delta y = 0.08 \text{ cm}$ (0.04 cm)

The initial velocity field is shown in Fig.(4.1).

Figure 4.1 The initial velocity for the eddy is plotted.

The motion of the eddy is subject to a Kelvin-Helmholtz instability, which forms secondary eddies as it grows. These are evident as "wings" in the density contours, Fig.(4.2), after two rotation periods.

Figure 4.2 At two rotations of the eddy, contours of constant density are plotted. Secondary eddies are visible as "wings".

With donor cell differencing, Fig.(4.3a), and with van Leer differencing, Fig.(4.3b), [Hawley et al, 1984] the magnetic field exhibits similar distortion. However, the gradients in the field are greater with van Leer, and the maximum value of the magnetic field is half again as large as with donor cell.

Figure 4.3 Contours of constant vector potential represent the magnetic field with donor cell, (a), and van Leer, (b), differencing.

The current with donor cell differencing, Fig.(4.4a), and with van Leer differencing, Fig.(4.4b), exhibit a similar spatial distribution. However, the maximum current with donor cell is only two thirds the value with van Leer. Donor cell differencing is more diffusive than van Leer differencing.

Figure 4.4 Contours of constant current density are plotted for donor cell differencing, (a), and van Leer differencing, (b).

The quantitative results are compared for a coarse and a fine

mesh calculation by scaling them to the best result, a calculation with van Leer on a 48x48 mesh. The results are:

<u>Mesh Size</u>	<u>Differencing</u>	<u>I_{max}</u>	<u>B_{max}</u>
24x24	donor	0.22	0.45
48x48	donor	0.64	0.75
24x24	van Leer	0.32	0.57
48x48	van Leer	1.00	1.00

The two methods appear to scale similarly with the number of mesh points, but the donor cell method requires one and one-half times as many cells in each coordinate direction to give the same accuracy. There are more accurate algorithms for fluid flows, such as the PPM method, [Colella and Woodward, 1984] which have not yet been applied to MHD.

5. Adaptive Zoning

Field-Reversed Theta-Pinch

The method of adaptive zoning is illustrated by the application of the moving mesh method to modeling the field-reversed theta-pinch [Milroy and Brackbill, 1982]. In a field-reversed theta-pinch, a region of closed magnetic field lines is created when magnetic field lines in one direction tear and reconnect to magnetic field lines in the opposite direction. The tearing and reconnection are due to resistive diffusion that is caused by resistance to electrical current. With resistivity, η , the ideal Ohm's law, Eq. (1.4), is replaced by

$$\mathbf{E} + \frac{\mathbf{u} \times \mathbf{B}}{c} = \eta \mathbf{J} . \quad (5.1)$$

Resistive diffusion unfreezes the magnetic field from the plasma, and relative motion is possible.

In the model, the resistivity is, itself, proportional to the current, and resistive diffusion occurs most rapidly where the current is highest. In the field-reversed theta-pinch, the external currents that compress and heat the plasma are larger at the ends than in the center of the pinch. Thus, more reconnection occurs at the ends than elsewhere.

After it is formed, the closed field line region contracts axially until equilibrium is reached between magnetic and pressure forces. The final shape of the plasma is much like a football.

The stages in the evolution of the field-reversed theta-pinch are illustrated in Fig.(5.1). In each figure, the axis of symmetry is below and the inner surface of the coaxial coil above. In (a), which depicts the initial plasma and magnetic field, the axial variation in the coil current produces a corresponding axial variation in the magnetic field intensity. The plasma, which is represented by tracer particles, is separated from the coil by a vacuum region. The interface between the plasma and the vacuum moves during the formation of the closed field region.

As current increases in the coils, the plasma responds by contracting toward the axis. Reconnection begins on the right first, as shown in (b) at $t=2.4\mu s$. Some of the plasma is ejected from the open right end by the increasing magnetic field. By $t=3.6\mu s$, (c), the field has also reconnected on the left, and a closed field region has been created at the center. By $12\mu s$, (d), the plasma is nearly in equilibrium.

Adaptive Grid

The free-boundary at the vacuum-plasma interface and the gradient-dependent, reconnection process make resolving the field-reversed theta-pinch dynamics challenging. The moving interface prevents simply using a finely-zone, Eulerian mesh. The singular, internal boundary prevents simply using body-fitted coordinates. More control of internal mesh spacing than is afforded by the boundary conditions is needed. The successful calculation shown in Fig.(5.1) required the adaptively zoned grid shown in Fig.(5.2).

The adaptive grid is generated by minimizing a composite functional,

$$I = I_s + I_v \quad (5.2)$$

where I_s is the smoothing functional that generates body-fitted coordinates,

$$I_s = \iint dx dy \left\{ \nabla \xi^2 + \nabla \eta^2 \right\} , \quad (5.3)$$

and I_v is a weighted volume functional that adapts the grid to the data [Brackbill and Saltzman, 1982],

$$I_v = \iint dx dy wJ . \quad (5.4)$$

The weight, $w(x,y)$, is proportional to the current density in the plasma,

$$w = \frac{J^2}{B^2} . \quad (5.5)$$

In the vacuum region, where J is zero, the grid is simply boundary-fitted. In the plasma, where J is not zero, the weighting causes the mesh to cluster where the current is highest giving extra resolution where it is needed.

In the calculation of the dynamics of the field-reversed, theta-pinch, the mesh is regenerated each time step and the data is convected from cell to cell by solving Eqs.(4.16-4.19) with interpolated donor cell differencing.

Figure 5.1 The magnetic field lines and the plasma, which is represented by tracer particles, in a field-reversed theta-pinch are shown at $t=0$, (a), $t=2.4\mu s$, (b), $t=3.6\mu s$, (c), and $t=12\mu s$, (d).

(a)

(b)

(c)

(d)

Figure 5.2 The adaptive grid for the field-reversed theta-pinch calculation (Fig.(6.1)) is shown at $t=0$, (a), $t=2.4\mu s$, (b), $t=3.6\mu s$, (c), and $t=12\mu s$, (d). The zones are smallest where the plasma current is largest.

6. The Solenoidal Condition

The solenoidal condition on the magnetic field, Eq.(1.6), is an initial condition for the Lundquist equations. If one forms the divergence of Faraday's law, Eq.(1.5), one finds an initially solenoidal field remains solenoidal.

Unfortunately, this is not true for numerical calculations of MHD flow in two and three dimensions. This is one important aspect of numerical magnetohydrodynamics that is not the same in two and three dimensions as it is in one. Either through errors in boundary conditions, or errors in the finite difference approximation to Faraday's law, the field may develop a non-solenoidal component in time. An error in solenoidality will cause the magnetic field to exert a non-physical force along field lines. To correct this error, solenoidality is often imposed by solving a potential equation for a correction to the magnetic field [Brackbill,1980],

$$\mathbf{B}' = \mathbf{B} - \nabla \psi, \quad \nabla \cdot \nabla \psi = \nabla \cdot \mathbf{B}. \quad (6.1)$$

To reduce the expense of solving the potential equation, one can either solve it only at intervals of five or ten cycles, or solve it only approximately each cycle. For example, if the equations are solved by Jacobi iteration, each iteration may be written,

$$\psi^{(l+1)} = \psi^{(l)} + d \left[\nabla \cdot \nabla \psi^{(l)} - \nabla \cdot \mathbf{B} \right], \quad (6.2)$$

where d is the relaxation parameter with dimensions $(\text{length})^2$. If Eq.(6.2) were iterated only once each cycle, and $\psi^{(0)}$ were assumed to be zero, the potential would be given by,

$$\psi^{(1)} = -d \nabla \cdot \mathbf{B}. \quad (6.3)$$

A method developed by Marder (1987) to control charge imbalance in plasma simulation codes, seems to offer a way to make the evolutionary equations self-correcting. The correction is as follows: One adds a term to Faraday's law which causes the decay of $\text{div}(\mathbf{B})$,

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} + \frac{d}{\Delta t} \nabla \nabla \cdot \mathbf{B}. \quad (6.4)$$

In fact, the additional term is equivalent to order Δt^2 to the potential correction term when iterated only once, as can be shown by substituting Eqs. (6.1) and (6.3) into (6.4). In the form given by Eq.(6.4), however, the effect of the potential correction is clearer, and the equations are simpler to difference, to apply boundary conditions to, and to solve. The additional term alters neither the stability nor the solubility of the Lundquist equations. It makes no difference to the exact equations, because $\text{div}(\mathbf{B})$ is zero anyway, and it causes the error in the difference equations to diffuse away. Now the evolution equation for $\mathbf{B}$ contains a self-correcting term,

$$\frac{\partial(\nabla \cdot \mathbf{B})}{\partial t} = \nabla \cdot d \nabla (\nabla \cdot \mathbf{B}) \quad (6.5)$$

that causes errors in the solenoidality to diffuse away. d is the diffusivity. One can choose d arbitrarily, but it should be as large as possible without being so large that it imposes a limit on the time step when differenced explicitly.

The correction term appears to be nonlinearly stable as well for it increases dissipation,

$$\begin{aligned} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} &= -\mathbf{B} \cdot \nabla \times \mathbf{E} + d \mathbf{B} \cdot \nabla (\nabla \cdot \mathbf{B}) \\ &= -\nabla \cdot \mathbf{E} \times \mathbf{B} + \nabla \cdot d \mathbf{B} \nabla \cdot \mathbf{B} - \mathbf{E} \cdot \mathbf{J} - d(\nabla \cdot \mathbf{B})^2 \end{aligned} \quad (6.6)$$

The first terms are in conservation form. The third is the work done by the magnetic field on the fluid. The last is the dissipation in magnetic field energy due to the diffusion of the divergence of the field.

To illustrate the effect of the correction in Eq. (6.1), a model problem with an initially large error in the solenoidal condition is solved. At the start of the calculation, the magnetic field reverses direction at the mid-plane, as shown in Fig. (6.1).

Figure 6.1 The initial magnetic field is represented by vectors.

The initial conditions for the problem are:

Number of mesh points: $N_x = N_y = 50$

Mesh spacing: $\Delta x = \Delta y = 0.04$ cm

Initial density: $\rho_0 = 1$ g cm⁻³

Initial pressure: $p_0 = (\gamma-1)$ ergs cm⁻³, $\gamma = 5/3$

Initial magnetic field: $B_{y0} = \pm 1$ G

The left boundary is a flux surface, with reflection symmetry assumed. All boundaries allow no slip. Except on the left boundary, $\nabla \cdot \mathbf{B}$ is assumed to be zero exterior to the boundary.

The contours of constant vector potential, $\mathbf{A}$, obtained by solving the equation

$$\nabla \times \mathbf{A} = \mathbf{B}$$

for $\mathbf{A}$, represent the magnetic field lines. The field lines depict a tangential field at the midplane in Fig. (6.2) that is not evident in Fig. (6.1).

Figure 6.2 Contours of constant vector potential represent the initial magnetic field lines.

The difference is that the field lines correspond to a field that is solenoidal, in which there is a current sheet at the midplane. Because the reversal occurs discontinuously, and the tangential field everywhere is zero, there is no current sheet in the non-solenoidal field. The divergence of $\mathbf{B}$ is comparable to $B/\Delta x$, as shown in Fig. (6.3).

Figure 6.3 Contours of the divergence of the initial magnetic field are shown.

If the field were solenoidal, the force due to the current would cause motion in the plasma and magnetic field to reduce the stress. Eventually, current flows only at the boundary. One can compare the solution with and without the correction in Eq.(6.1) to see which is more physical.

With d equal to zero in Eq.(6.1), the divergence in $\mathbf{B}$ after many signal transit times is more than one-half its initial value, Fig.(6.4a). With d equal to $\Delta x^2/\Delta t$ in Eq.(6.1), there is a correction to reduce the divergence of $\mathbf{B}$. At the same time as above, the maximum value of the divergence is one twenty-fifth its initial value, Fig.(6.4b)

Figure 6.4 The divergence of B without correction, (a), and with correction, (b), is shown. The maximum value in (a) is one-half that in Fig. (6.3), and more than ten times that in (b).

Field-aligned streamers are evident in the velocity, Fig. (6.5a), with maximum values approaching the Alfvén speed. The velocity field, Fig. (6.5b), is very different from that shown in Fig. (6.5a). The maximum velocity is one-fifth as large, and the motion is concentrated in two eddies near the right boundary.

Figure 6.5 The plasma velocity without correction, (a), and with correction, (b), is shown.

The field lines without correction are distorted, Fig. (6.6a), but with correction have approached a force-free configuration.

Figure 6.6 The magnetic field lines without correction, (a), and with correction, (b), are plotted.

Other differences are not shown. Without correction, the magnetic field lines are inconsistent with the magnetic vector field, there are large currents that are aligned along the velocity streamers, and have no apparent cause. These currents are ten times as large as the currents in the corrected case, which also are limited to the vicinity of the right boundary. The variation in density in the uncorrected case is fifteen per cent, and in the corrected case less than four percent (not shown).

With the correction to impose solenoidality on the solution, the nonphysical or unresolved model problem is made more physical, and the results are made more plausible. In other cases, the correction appears to restore symmetry to the problem. For example, the magnetic field lines in the eddy problem without correction are less symmetric than those shown with correction. The field in the eddy problem is weak, but the fluid flow is unstable so that small errors in symmetry will grow in time. The symmetry breaking term in the equations is the component of the Maxwell stress, $\mathbf{E} \cdot \nabla \mathbf{B}$, in the momentum equation due to errors in maintaining solenoidality.

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