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Model of Emittance Growth in a Self-Pinched Beam*

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Abstract

A semi-phenomenological formula is proposed for the change of emittance of a self-pinched beam which is not matched to its equilibrium radius. Near equilibrium this formula, coupled with an envelope equation, yields the damped sausage oscillations observed in simulation and experiments. For a beam which is injected cold (no transverse velocity spread), the formula coincides with the analytically calculated initial growth of emittance. The basic theory is developed here and used to compute the linear damping rate for several current profiles. The resultant non-linear increase in equilibrium quantities is also calculated in lowest order of the degree of mismatch.

Introduction

The modeling of emittance change and growth is an essential aspect of the analysis of non-equilibrium states of a self-pinched beam. A semi-phenomenological formula for emittance growth proposed in this paper is very useful for computing the damping of pulsations at injection, beam head erosion, unstable growth of the sausage mode, and the dynamics of converging heavy ion beam envelopes. The development in z of a beam envelope can be followed in general by an equation for the r.m.s. radius R which depends on the emittance E and an average of the betatron frequency $k_B(r,z)$. Since the emittance represents a smoothed phase area, it is often assumed to be constant in z . For many applications this is an insufficient model since it

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leads to pulsations which are not damped. Particle simulations and experiments in which self generated forces dominate exhibit damping to an equilibrium state with an associated increase in emittance. The physical origin of damping is the non-harmonic character of the pinch force, which causes a spread of orbital frequencies and consequent phase mixing of a disturbance. In order to model damping effects in 1-D envelope calculations, variation of emittance must be incorporated. Unfortunately, no general calculation of this effect from first principles is known, although we do know certain properties which an emittance equation must exhibit. In addition, the rate of emittance growth can be evaluated in the special limit of a cold beam. Our approach then is to construct a phenomenological expression for dE/dz which satisfies the known general properties, reduces to the calculated cold limit and makes predictions in agreement with simulation. The proposed formula contains two adjustable constants, one which is adjusted to match results from simulations, and the other is derivable from the cold limit case.

Background

An envelope equation, together with some general properties of the beam emittance, can be derived from appropriate moments of the particle equations of motion. We consider a set of similar particles moving together in z (a thin disk). For simplicity, the energy and longitudinal component of particle velocity are taken to be constant. The i^{th} particle in the disk is then characterized by its transverse coordinates $r_i(z)$, and the dimensionless transverse velocity is defined by

$$v_i = r_i' \quad (1)$$

where the "prime" notation will henceforth represent the derivative d/dz . The

equation of motion of particles under a magnetic pinch force is

$$\ddot{x}_i = -k_\beta^2 (|x_i|, z) x_i, \quad (2)$$

where the betatron frequency is defined by

$$k_\beta^2 = \frac{q}{\beta \gamma m c^2} B_0, \quad (3)$$

Here q and $\beta \gamma m c^2$ are respectively the charge and the momentum of the beam particles. For a self pinch, the magnetic field is given by Ampere's Law, which may be written in the form

$$B_0 = \frac{2 I_r}{rc}, \quad (4)$$

where I_r is the current inside a disk of radius r :

$$I_r = \int_0^r dr' 2\pi r' J(r', z) \quad (5)$$

A more general treatment which includes other effects in the equations of motion is given in Ref. (1).

A description of the beam development in terms of a few macroscopic averages is clearly much simpler than the computation of all the particle orbits. However, for such a description to be useful, one must be able to write down a closed set of equations for the averaged quantities. We proceed by first introducing the rms radius and velocity of the N -particle disk.

$$R^2 = \overline{r^2} = \frac{1}{N} \sum_{i=1}^N |x_i|^2, \quad (6)$$

$$V^2 = \overline{v^2} = \frac{1}{N} \sum_{i=1}^N |v_{xi}|^2 \quad . \quad (7)$$

By forming the virial moment and the energy moment of the equations of motion, we obtain two equations relating R and V to averages which involve the betatron frequency:

$$\frac{R^{2''}}{2} - V^2 = - \overline{k_\beta^2 r^2} \quad , \quad (8)$$

$$V^{2'} = - \overline{k_\beta^2 r^2} \quad . \quad (9)$$

It is convenient at this point to introduce the emittance variable, which is proportional to the random component of the velocity:

$$E^2 = R^2 (V^2 - R^{1/2}) \quad . \quad (10)$$

Eqs. (8) and (9) can be recast into two equations with E replacing V as a dependent variable. The envelope equation, which is obtained directly from the virial moment [Eq. (8)] is

$$R'' = \frac{E^2}{R^3} - \frac{\overline{k_\beta^2 r^2}}{R} \quad . \quad (11)$$

It describes the evolution of the disk under two opposing effects: the emittance term which represents a pressure causing the beam to expand, and the magnetic pinch term representing the mutual attraction of the particles. With suitable manipulations of Eqs. (8) and (9), a formal equation for emittance

change can be derived:

$$E^{2'} = R^{2'} \overline{k_\beta^2 r^2} - R^2 \overline{k_\beta^2 r^{2'}} \quad (12)$$

Equations (11) and (12) are closed if the two averages over betatron frequencies can be specified. The quantity $\overline{k_\beta^2 r^2}$ involves only a spatial average of the magnetic field and is in general easy to evaluate. For a purely self-pinch beam, (no plasma currents) it has the constant value

$$\begin{aligned} \overline{k_\beta^2 r^2} &= \int_0^\infty dr \left(\frac{2\pi r J}{I} \right) \left(\frac{q}{\beta \gamma m c^2} \right) \left(\frac{2 I_r}{rc} \right) r \\ &= \left(\frac{q I}{\beta \gamma m c^3} \right) \int_0^\infty dr \left(\frac{\partial}{\partial r} \frac{I_r}{I} \right) \left(\frac{2 I_r}{I} \right) \\ &= \left(\frac{q I}{\beta \gamma m c^3} \right) \left[\frac{I_r}{I} (r = \infty) \right]^2 = \frac{q I}{\beta \gamma m c^3} = \frac{I}{I_A} \quad , \end{aligned} \quad (13)$$

where $I_A = \frac{\beta \gamma m c^3}{q}$ is the Alfvén current and I is the net beam current. If there is some degree of current neutralization with the net current given by $I_n = (1-f) I$, and if the net current profile is the same as that of the beam, then

$$\overline{k_\beta^2 r^2} = (1-f) \frac{I}{I_A} \quad (14)$$

On the other hand the quantity $\overline{k_\beta^2 r^{2'}}$ requires detailed knowledge of the velocity distribution of particles and its evaluation can be performed

only under special circumstances. If k_β^2 is flat (independent of r), then we have

$$\overline{k_\beta^2 r'^2} = k_\beta^2 R'^2, \quad ,$$

and

$$\overline{k_\beta^2 r^2} = k_\beta^2 R^2 \quad .$$

So Eq. (12) gives $E^{2'} = 0$. If the beam motion is self-similar, i.e. a particle velocity has the form

$$v_i = \frac{R'}{R} v_i + (\text{isotropically distributed part}) \quad ,$$

then

$$\overline{k_\beta^2 r'^2} = \overline{k_\beta^2 2rr'} = \overline{(k_\beta^2 2r^2) \frac{R'}{R}} = \frac{R'^2}{R^2} \overline{k_\beta^2 r^2} \quad ,$$

and again $E^{2'} = 0$. Thus emittance is conserved if the magnetic force is harmonic or if the beam motion is self similar. However, in general neither assumption can be made and emittance is not conserved.

Emittance Growth Formulae

(a) Near Equilibrium limit

A beam which is slightly mismatched will pulsate. Emittance variation manifests itself in the damping of the pulsations and a small increase in equilibrium quantities. A previous study² of damping near equilibrium in simulations and experiments led to the construction of phenomenological formula

$$E^{2'} = -\alpha \overline{k_\beta^2 r^2} R^2 \frac{R''}{R} \quad (15)$$

where the constant α is adjusted to give the best agreement with simulation. For a Bennett profile truncated at twice the nominal radius, it was found that $\alpha \approx 0.5$.

In addition to agreement with simulation near equilibrium, the following arguments are proposed in favor of Eq. (15):

- (a) Correct dimensions
- (b) Correct scaling with current
- (c) Proportional to disruption rate ($\propto R^n$).
- (d) Prediction of irreversible damping of oscillations in linear regime (near equilibrium) as observed.
- (e) Simple and useful
- (f) An obvious physical mechanism exists (phase mixing)

The phenomenological approach has been adopted because no rigorous derivation of emittance change from first principles in this limit is known to date.

(b) Near ballistic limit

We next consider the cold beam limit, where the particle trajectories converge nearly ballistically (either the field is weak or the applicable range is short). This case is important for heavy ion beam propagation in a gas-filled fusion reactor³. The assumption of a weak field permits the use of a perturbative approach for the calculation of particle trajectories and in this limit the emittance growth expression of Eq. (12) can be evaluated explicitly. The details of this calculation will be presented in the next section, but the result is summarized in the following equation:

$$E^{2'} = -B \frac{k_{\beta}^2 r^2 R^2 R^n}{k_{\beta}^2 r^2 R/E} \quad (15)$$

where B is a dimensionless constant which depends on the beam profile. We

find $B = 0.3$ for a parabolic profile, $B = .78$ for a Gaussian profile, and $B = .85$ for a Bennett profile truncated at twice the nominal Bennett radius.

(c) General Case

We note that the denominators of Eqs. (15) and (16) are essentially reciprocal. However at equilibrium they are equal since the envelope equation gives

$$\overline{k_B^2 r^2} \frac{R}{E} - \frac{E}{R} = - \frac{R^2 R''}{E} = 0 \quad (17)$$

Hence in this limit Eqs. (15) and (16) would have the same form except for the constants A and B . We propose to use the following formula in general:

$$E^{2'} = - \frac{\overline{k_B^2 r^2} R^2 R''}{\frac{1}{A} \left(\frac{E}{R} \right) + \frac{1}{B} \left(\frac{\overline{k_B^2 r^2} R}{E} \right)} \quad (18)$$

with the constant A determined by the relation

$$\frac{1}{A} + \frac{1}{B} = \frac{1}{\alpha} \quad (19)$$

As $E/R \rightarrow 0$, the term proportional to B^{-1} dominates and Eq. (18) approaches the known cold limit [Eq. (16)]. Near equilibrium, the phenomenological result [Eq. (15)] is recovered.

At the present stage of development, we view Eq. (18) as a phenomenological formula with one free parameter α which is adjusted to fit simulations. Further study may show that A and B are actually equal, in which case Eq. (18) is completely determined. The example of the Bennett profile truncated at twice the Bennett radius lends some support to this conjecture,

since (crudely determined) $\alpha \approx 0.5$, and $B = 0.85$, yields $A \approx 1.2$. If A is indeed equal to B , then the denominator of Eq. (18) is a minimum at equilibrium. The resulting formula is esthetically more pleasing and non-linear calculations are simplified, but we have no firm justification for this at present. A weak justification will be presented in the final section.

Calculation of Emittance Growth in the Cold-Beam, Weak-Field Limit

The emittance growth expression [Eq. (12)] can be evaluated in the special limit of a cold beam ($E_0 = 0$), and weak field. By weak field, we mean that the domain of z considered is such that $k_\beta z \ll 1$. Under these assumptions, particles move on nearly straight line trajectories, with

$$r_i = r_{i0} \left(1 - \frac{z}{L} \right) + O(\epsilon) \quad (20)$$

where $k_\beta^2 z^2 \propto \epsilon \ll 1$. Initially, the beam may be converging ($0 < L < \infty$), straight ($L = \infty$), or diverging ($L < 0$). Similarly, the beam envelope has the form

$$R = R_0 \left(1 - \frac{z}{L} \right) + O(\epsilon) \quad (21)$$

It is convenient to introduce the variable

$$u_i = r_i / R \quad (22)$$

which gives the position of a particle relative to the mean radius. To leading order u_i does not change as the disk propagates:

$$u_i = u_{i0} + O(\epsilon) \quad (23)$$

Since the particle orbits are simple, the self-pinch field is also simple. Assuming that the net current has the same radial profile as the beam current, we have from Eqs. (3) - (5)

$$R^2 k_\beta^2 = \frac{2 I_r(u)}{I_A u^2} + O(\epsilon^2) \quad (24)$$

The leading term on the rhs is by definition of order ϵ and is again independent of z to 0 of Eq. (24).

The calculation of emittance growth can be performed most conveniently if we rewrite Eq. (12) in the form

$$\mathcal{E}^{2'} = 2 \overline{(R^2 k_\beta^2)} u (R'r - Rr') \quad . \quad (25)$$

While Eq. (25) is completely general, we calculate only the lowest non-vanishing order terms in ϵ . Noting that $(R^2 k_\beta^2)$ and u are both z -independent, we take the derivative in z of Eq. (25) to obtain

$$\mathcal{E}^{2''} = 2 \overline{(R^2 k_\beta^2)} u (R''r - Rr'') \quad . \quad (26)$$

Substituting from the equations of motion [Eqs. (1) and (2)] and the envelope equation [Eq. (11)] with $E = 0$, we obtain

$$\begin{aligned} \mathcal{E}^{2''} &= 2 \overline{(R^2 k_\beta^2)} u \left(-\frac{\overline{k_\beta^2 r^2}}{R} r + R \overline{k_\beta^2 r} \right) \\ &= -2 \left(\overline{k_\beta^2 r^2} \right)^2 + 2 R^2 \left(\overline{k_\beta^4 r^2} \right) \quad . \end{aligned}$$

Rewriting this equation in a more suggestive form, we have

$$\mathcal{E}^{2''} = 2 \left(\overline{k_\beta^2 r^2} \right)^2 \Lambda \quad , \quad (27)$$

where

$$\Lambda \equiv \frac{\overline{k_\beta^4 r^2} R^2}{\left(\overline{k_\beta^2 r^2} \right)^2} - 1 \quad . \quad (28)$$

In the next section, we will evaluate the parameter Λ for several current profiles. Here, we note several properties:

- (a) Λ depends only on the shape, but not the magnitude of the current.
- (b) Λ is nonnegative, since by Schwartz inequality

$$\left(\overline{k_{\beta}^2 r^2} \right)^2 \leq \overline{k_{\beta}^4 r^2} \overline{r^2} \quad . \quad (29)$$

- (c) Λ vanishes for a flat profile.

Λ is z -independent, so we can integrate Eq. (27) once to obtain

$$E^{2'} = 2 \left(\overline{k_{\beta}^2 r^2} \right)^2 \Lambda z \quad , \quad (30)$$

and integrate again to get

$$E^2 = \overline{k_{\beta}^2 r^2}^2 \Lambda z^2 \quad . \quad (31)$$

The vanishing constants of integration in Eqs. (30) and (31) result from the initial conditions assumed. Since Λ is positive, E^2 increases with z as expected.

Eqs. (30) and (31) combined with Eq. (11) in the $E \rightarrow 0$ limit lead immediately to the emittance growth formula of Eq. (16), with the identification

$$B = 2 \sqrt{\Lambda} \quad . \quad (32)$$

Calculation of Λ

The parameter Λ is readily calculated for several profiles $J(r)$. We recall that Λ vanishes for flat J , since this yields constant k_B . The magnitude of Λ is related in general to how strongly $J(r)$ deviates from flat profile.

(a) Parabolic profile: $(R^2 = a^2/3)$

$$J(r) = \frac{2I}{\pi a^2} \left(1 - \frac{r^2}{a^2}\right) \quad (0 \leq r \leq a) \quad , \quad (33)$$

$$k_B^2 r = \frac{2I}{I_A r} = \left(\frac{4I}{I_A r}\right) \left(\frac{r^2}{a^2} - \frac{r^4}{2a^4}\right) \quad , \quad (34)$$

$$\Lambda + 1 = \frac{\overline{k_B^4 r^2 R^2}}{\overline{k_B^2 r^2}^2} = \frac{1}{(I/I_A)^2} \int_0^a dr \left(\frac{2\pi r J}{I}\right) (k_B^2 r)^2 R^2 \quad . \quad (35)$$

Straightforward evaluation of the integral gives

$$\Lambda + 1 = \frac{46}{45} \quad , \quad (36)$$

$$B = 2\sqrt{\Lambda} = 0.298142 \quad . \quad (37)$$

(b) Gaussian profile:

$$J = \frac{I}{\pi R^2} e^{-r^2/R^2} \quad ; \quad (38)$$

$$k_B^2 r = \frac{2 I_A r}{I_A r} = \left(\frac{2 I}{I_A} \right) \left(\frac{1 - e^{-r^2/R^2}}{r} \right) \quad (39)$$

Setting $x = r^2/R^2$, we have

$$\begin{aligned} \Lambda + 1 &= \int_0^\infty dx e^{-x} \left(\frac{4}{x} \right) (1 - e^{-x})^2 \\ &= 4 \ln \left(\frac{4}{3} \right) = 1.150728 \end{aligned} \quad (40)$$

$$B = 0.776475 \quad (41)$$

(c) Truncated Bennett:

$$J = \left(\frac{1 + n^2}{n^2} \right) \frac{1}{\pi a^2} \left(1 + \frac{r^2}{a^2} \right)^{-2} H(na - r) \quad (42)$$

where H is a step function at $r = na$:

$$k_B^2 r = \left(\frac{1 + n^2}{n^2} \right) \left(\frac{2 I}{I_A} \right) \left(\frac{r^2/a^2}{1 + r^2/a^2} \right) \quad (43)$$

and

$$R^2 = \int_0^{na} dr \left(\frac{2 \pi r J}{I} \right) r^2 = a^2 \left[\left(\frac{1 + n^2}{n^2} \right) \ln (1 + n^2) - 1 \right] \quad (44)$$

We find

$$\Lambda + 1 = \frac{2}{3} \left(1 + \frac{3}{n^2} \right) \left[\left(\frac{1 + n^2}{n^2} \right) \ln (1 + n^2) - 1 \right] \quad (45)$$

The numerical values of Λ and B for several values of n are given in Table

1. For the case $n = 2$, we find $B = 0.85$. A simulation of damping close to equilibrium for this profile has given the approximate value $\alpha = 0.5$. Hence, from Eq. (19), we get $A = 1.215$

Table 1 (Truncated Bennett Profile Results)

n	R/a	Λ	B
0.1	.07059	4.47×10^{-6}	4.23×10^{-3}
1.0	.6215	.0301	.3471
1.5	.8382	.0928	.6092
2.0	1.0059	.1804	.8495

Damping of Sausage Oscillations

The phenomenological expression for emittance change may be used to calculate the damping of sausage mode oscillations resulting from a mismatch at injection. The basic equations are

$$R'' = \frac{E^2}{R^3} - \frac{\overline{k_\beta^2 r^2}}{R} \quad , \quad (11)$$

$$E^{2'} = - \frac{\overline{k_\beta^2 r^2} R^2 R''}{\frac{1}{B} \frac{\overline{k_\beta^2 r^2} R}{E} + \frac{1}{A} \frac{E}{R}} \quad (18)$$

The emittance at injection has the known value E_0 ; the corresponding equilibrium radius R_0 is determined from Eq. (11) setting $R'' = 0$:

$$0 = \frac{E_0^2}{R_0^3} - \frac{\overline{k_\beta^2 r^2}}{R_0} \quad . \quad (46)$$

The beam is injected with a small mismatch

$$R(0) = R_0 + \delta R(0) \quad , \quad (47a)$$

$$R'(0) = \delta R'(0) \quad , \quad (47b)$$

$$\delta E^2(0) = 0 \quad . \quad (47c)$$

The linearized equations of motion are

$$\delta R'' = \frac{\delta E^2}{R_0^3} - \frac{3 E_0^2}{R_0^4} \delta R + \frac{\overline{k_\beta^2 r^2}}{R_0^2} \delta R \quad , \quad (48)$$

$$\delta E^2' = - \frac{\alpha R_0}{E_0} \frac{\overline{k_B^2 r^2}}{R_0^2} \delta R'' \quad (49)$$

Using the equilibrium condition [Eq. (46)] to eliminate stray factors of E_0 , we write these equations in the form

$$\left(\frac{\delta R}{R_0}\right)'' + 2 \frac{\overline{k_B^2 r^2}}{R_0^2} \left(\frac{\delta R}{R_0}\right) - \frac{\overline{k_B^2 r^2}}{R_0^2} \left(\frac{\delta E^2}{E_0^2}\right) = 0 \quad (50)$$

$$\sqrt{2} \left(\frac{\delta R}{R_0}\right)'' - \frac{R_0}{\sqrt{2} \overline{k_B^2 r^2}} + \left(\frac{\delta E^2}{E_0^2}\right)' = 0 \quad (51)$$

We seek solutions of the form

$$\frac{\delta R}{R_0}, \frac{\delta E^2}{E_0^2} \propto e^{-i\Omega z} \quad (52)$$

Defining

$$\omega_0^2 = 2 \frac{\overline{k_B^2 r^2}}{R_0^2} \quad (53)$$

we get

$$\left(\omega_0^2 - \Omega^2\right) \left(\frac{\delta R}{R_0}\right) - \frac{\omega_0^2}{2} \left(\frac{\delta E^2}{E_0^2}\right) = 0 \quad (54)$$

$$- \Omega^2 \frac{\sqrt{2} \alpha}{\Omega_0} \left(\frac{\delta R}{R_0}\right) - i \Omega \left(\frac{\delta E^2}{E_0^2}\right) = 0 \quad (55)$$

The frequencies Ω are determined from the condition that there be a

non-trivial solution i.e.

$$\begin{vmatrix} \Omega_0^2 - \Omega^2 & -\frac{\Omega_0^2}{2} \\ -\frac{\sqrt{2}\alpha\Omega^2}{\Omega_0} & -i\Omega \end{vmatrix} = \left(\Omega^2 - \Omega_0^2\right) i\Omega - \frac{\alpha}{\sqrt{2}} \Omega^2 \Omega_0 = 0. \quad (56)$$

The solution $\Omega = 0$ is not of interest since it corresponds to a shift in equilibrium quantities (i.e. $\delta E^2(0) \neq 0$). The oscillatory modes satisfy

$$\left(\frac{\Omega}{\Omega_0}\right)^2 + \frac{i\alpha}{\sqrt{2}} \frac{\Omega}{\Omega_0} - 1 = 0, \quad (57)$$

yielding

$$\frac{\Omega}{\Omega_0} = -\frac{i\alpha}{\sqrt{8}} \pm \sqrt{1 - \frac{\alpha^2}{8}}. \quad (58)$$

The damping in a single period is therefore a factor

$$\exp\left(-\frac{2\pi\alpha\sqrt{8}}{\sqrt{1 - \alpha^2/8}}\right)$$

For $\alpha = 0.5$, the damping factor is 0.323520.

Net Emittance Gain of a Slightly Mismatched Beam

An initially mismatched beam eventually settles into an equilibrium state. A net change of emittance with an associated change of beam radius occurs in this process. If $R'(0) \neq 0$, the phenomenological model predicts a first order change. From Eq. (49), we obtain the total emittance gain as

$$\begin{aligned} \Delta E^2_{\text{eq}} &= -\alpha \sqrt{k_B^2 r^2} R_0^2 \int_0^\infty dz \delta R'' \\ &= +\alpha \sqrt{k_B^2 r^2} R_0^2 \delta R'(0) \end{aligned} \quad (59)$$

The model thus predicts a net emittance decrease if the beam is initially converging. This is a somewhat peculiar result, but it violates no known physical principles.

If $\delta R'(0) = 0$, but the initial radius is not matched, the net change in E^2 is of second order in $\delta R(0)$. This can be seen readily by introducing the quantity

$$Q = E^2 + \frac{k_B^2 r^2}{\left(\frac{1}{A} \frac{E}{R} + \frac{1}{B} \frac{R k_B^2 r^2}{E} \right)} R^2 R' \quad (60)$$

Since E^2 equals Q at $z = 0$ and ∞ , we have

$$\Delta E^2 = \Delta Q = \int_0^\infty dz Q' \quad (61)$$

But

$$Q' = \left(\frac{\frac{k_B^2 r^2}{R^2}}{\frac{1}{A} \frac{E}{R} + \frac{1}{B} \frac{k_B^2 r^2}{E} R} \right) R' \quad (62)$$

which is of second order in the perturbed quantities. To evaluate Q' , we need to consider only first order variations in R and E . The case

$$A = B = 2\alpha$$

is particularly simple since the variation of the denominator of Eq. (62) is of second order. Assuming this relation, we have

$$\Delta Q \approx \frac{\alpha}{\sqrt{k_B^2 r^2}} \int_0^\infty dz \, 2 R_0 (\delta R')^2 \quad (63)$$

The results of the previous section yield, for $\delta R'(0) = 0$

$$\delta R = \delta R(0) \exp \left(-\sqrt{\frac{\alpha^2}{8}} \Omega_0 z \right) \left[\cos \sqrt{1 - \frac{\alpha^2}{8}} \Omega_0 z + \frac{\sqrt{\alpha^2/8}}{\sqrt{1 - \alpha^2/8}} \sin \sqrt{1 - \frac{\alpha^2}{8}} \Omega_0 z \right] \quad (64)$$

Inserting the derivative of this expression into eq. (63) and performing the integration yields finally

$$\Delta E^2 = \Delta Q = 2 \frac{k_B^2 r^2}{R^2} [\delta R(0)]^2 \quad (65)$$

The predicted emittance gain is independent of α .

It is interesting that the same result may also be derived using the energy integral of the beam. The quantity

$$W = \frac{V^2}{2} - \frac{I}{I_A} \int_0^b \frac{dr}{r} \left(\frac{r}{I} \right)^2 \quad (66)$$

is readily shown to be a constant of the motion, regardless of changes in beam profile. Radius b is a cutoff at very large r , and does not affect the physics. If a profile shape is specified, then

$$W = \frac{V^2}{2} - \frac{I}{I_A} \left[\log \left(\frac{b}{R} \right) - \frac{C}{2} \right] \quad , \quad (67)$$

where C is a number which depends on the beam shape only.⁴ The value of C ranges from 0.0 to +.19. If the beam is injected at a neck, it can be plausibly assumed that the profile shape is nearly the same at injection and in the final equilibrium state. However, we need to make no assumptions about variations of the beam profile during the intermediate states. Applying Eq. (67) at $z = 0$ and $z = \infty$, we have

$$\frac{V_i^2}{2} - \frac{I}{I_A} \log \left(\frac{b}{R_i} \right) = \frac{V_f^2}{2} - \frac{I}{I_A} \log \left(\frac{b}{R_f} \right) \quad , \quad (68)$$

where initial and final values are denoted by the subscripts i and f respectively. Since $R'(0) = 0$ we have by definition

$$V_i^2 = \frac{E_0^2}{R_i^2} = \frac{I}{I_A} \frac{R_0^2}{R_i^2} \quad , \quad (69)$$

while the pinch condition yields

$$V_f^2 = \frac{I}{I_A} \quad . \quad (70)$$

Hence Eq. (68) becomes

$$\frac{R_0^2}{2 R_i^2} - \log \left(\frac{b}{R_i} \right) = \frac{1}{2} - \log \left(\frac{b}{R_f} \right) \quad (71)$$

Solving for R_f , we get

$$\frac{R_f^2}{R_0^2} = \frac{R_i^2}{R_0^2} \exp \left(\frac{R_0^2}{R_i^2} - 1 \right) \quad (72)$$

$$\begin{aligned} & \approx \frac{R_i^2}{R_0^2} \left[1 + \left(\frac{R_0^2}{R_i^2} - 1 \right) + \frac{1}{2} \left(\frac{R_0^2}{R_i^2} - 1 \right)^2 \right] \\ & = 1 + \frac{1}{2} \frac{R_i^2}{R_0^2} \left(\frac{R_0^2}{R_i^2} - 1 \right)^2 \quad (73) \end{aligned}$$

The net gain of emittance is therefore

$$\begin{aligned} \Delta E^2 &= (E_f^2 - E_0^2) = \frac{1}{I_A} (R_f^2 - R_0^2) \\ &= \frac{1}{I_A} \frac{1}{2} R_i^2 \left(\frac{R_0^2}{R_i^2} - 1 \right)^2 \approx 2 \frac{1}{I_A} [\delta R(0)]^2, \quad (74) \end{aligned}$$

which agrees with Eq. (65).

References

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