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**THE VELOCITY-SPACE PARTICLE LOSS
IN FIELD-REVERSED THETA PINCHES**

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THE VELOCITY-SPACE PARTICLE LOSS
IN FIELD-REVERSED THETA PINCHES

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A field-reversed theta pinch (F RTP) is a compact device for magnetic fusion. It has attracted much attention in recent years since encouraging experimental results have been obtained. However, the definite causes for the observed particle loss rate and plasma rotation are not well known. In this work, we study the velocity-space particle loss (VSPL), i.e., particle loss due to the existence of a loss region in velocity space, in F RTP's in order to have a better understanding about the characteristics of this device.

Based on two constants of motion, H and P_θ , where H is the total energy of a particle and P_θ is the canonical angular momentum, particle confinement criteria are derived, which impose constraints on H and P_θ . It is found that only a portion of (H, P_θ) space, or equivalently, of velocity space, is for confinement. A loss region in velocity space is found.

Effects of this loss region in velocity space are investigated. The velocity-space particle loss due to collisions is calculated from the Fokker-Planck equation. The scaling law for particle confinement is obtained. Without assuming any instability, the particle confinement time based on VSPL theory alone can account for the experimentally observed particle confinement time. This shows the importance of the velocity-space particle loss.

For the velocity distribution in collisional equilibrium, a simple analytical expression, V_0 , is verified to be a good representation of the actual velocity distribution in the presence of the loss region in velocity space. In steady-state, simple analytical approximate solutions for the whole particle distribution function, $f(\underline{r}, \underline{v})$, are obtained, from which the characteristics of the F RTP plasmas can be known.

It is shown that the VSPL model can account for the experimental observation in density profiles, particle confinement time, occurrence and spin-up of plasma rotation. Therefore, the VSPL has the possibility to be the dominant particle loss mechanism and the sole cause responsible for the observed plasma rotation in F RTP's.

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CHAPTER 1

INTRODUCTION

1.1 Objectives

Field-reversed fusion devices, e.g., field-reversed mirror (FRM),¹ field-reversed theta pinch (F RTP),² have received strong attention and have been studied intensively in recent years. The significant advantages of higher β and longer particle-confinement time of the plasma in field-reversed devices, which give higher fusion power density and improve the fusion energy gain, make them potential candidates for future fusion reactors. Among the field-reversed devices, the F RTP appears especially attractive because of encouraging experimental results,^{2,3} compact size, and simple geometry.

The F RTP is created by the technique indicated in Fig. 1.1.³ An initial bias field is frozen into a cold preionized plasma, whereupon the direction of the current in the theta pinch coil is quickly reversed by the main field discharge bank. The resulting oppositely directed fields tear and reconnect at the ends, generating a closed field line structure. The plasma then contracts axially until an equilibrium is reached. The geometric configuration produced is shown in Fig. 1.2. A typical magnetic field configuration and coil structure of F RTP's are shown in Fig. 1.3.

In recent F RTP experiments,^{2,3} the field-reversed magnetic configuration is maintained for a significant period. It, therefore,

Figure 1.1. Formation of the FRTP.³

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Figure 1.2 The geometry of F RTP configurations.

Figure 1.3 The magnetic field configuration and coil structure of a F RTP in the (r,z) plane.

becomes essential to study particle transport in a field-reversed magnetic configuration, which is very different from and more complicated than the case in a non-field-reversed configuration. It is also observed that the whole plasma is disrupted by a rotational instability which depends on the particle loss mechanism. This makes the collisional particle transport play a very important role in F RTP studies.

Collisional particle transport in a variety of fusion devices has been studied for a long time. Various simpler forms of the Fokker-Planck equation are reduced and solved from the original complicated equation in a 6-dimensional (x, v) phase space. An appropriate calculation of collisional particle transport in F RTP's, which should take the unusual field-reversed magnetic configuration and large gyro-excursion of particles (or small s -value, $s \equiv \text{plasma size/orbit size}$) into account, is essential to the further development of F RTP's. Therefore, the objective of this work is to study the particle loss mechanism in F RTP's and to gain an understanding of the characteristics (e.g., density profile, velocity distribution, scaling law for particle confinement, etc.) of F RTP's in a collisional regime.

1.2 Importance of the Work

Although field reversal has been achieved in most recent F RTP experiments,^{2,3} there is still no conclusive calculation on particle transport in this field-reversed magnetic configuration. Although improved particle confinement time observed in the same experiment,

more than 10 times longer in particle confinement time compared with that in an open-field system,² has given encouraging promise to F RTP's, the observed particle confinement time has not been well explained. There is a major difference in particle confinement time between the experiment value and prediction of classical diffusion.

Since particle transport is fundamentally important in studying particle and energy confinements, generation and the following effects of the electric field, the bases of other calculations (e.g., particle confinement time, energy gain, reactor design, etc.) are not strong enough unless the particle transport is well understood. Therefore, advances in further development and future utilization of F RTP's rely on the understanding of particle transport in the field-reversed configuration, which has not been encountered in conventional devices and has different characteristics from the open-field mirror systems.

As shown by Wang and Miley,⁴ the particle orbits are more complicated due to the field-reversed magnetic configuration than the drifted circular or helical orbits in a non-field-reversed configuration. In addition to the complexity of particle orbits, the small size of the system makes the orbital size relatively large, i.e., s -value small, suggesting that the finite-Larmor-radius effects may become important. These two characteristics of F RTP's imply that the guiding-center approximation is not applicable, especially for particle transport problem. Moreover, the effect of a combination of small size of the system and a field-reversed magnetic configuration is to imply large spatial variation of a magnetic field and, thus, of a particle density,

i.e., large ∇B and/or ∇n . This makes the MHD theory not applicable because of small scale lengths of the system. In fact, no existing simplified theory has solved the collisional particle transport problem by taking account of large gyroexcursion, large ∇B and large ∇n all at once. We, therefore, adopt a kinetic-theoretical approach in order to include these effects.

The most important distinction between field-reversed and non-field-reversed magnetic configurations is the difference in the criterion for particle confinement. The criterion for particle confinement in a field-reversed magnetic configuration is a relation between the total energy H and canonical angular momentum P_θ of the particle (see Chapter 3), instead of a relation between perpendicular velocity $v_\perp$ and parallel velocity $v_{\parallel}$ (with respect to $\underline{B}$).⁵ This is the most important characteristic of field-reversed configurations to observe. In F RTP's, the criteria for particle confinement break the symmetry of velocity distribution of the particle distribution function about magnetic field lines; hence, the distribution function $f(\underline{x}, \underline{v}, t)$ may no longer be reduced to $f(\underline{x}, v, \mu, t)$ where $\mu = \cos\theta$ and θ is the pitch angle. Much of the theoretical and numerical work that has been carried out for the study of $f(\underline{x}, v, \mu, t)$ does not apply to field-reversed configurations.

Two important effects immediately follow the criteria. First, only those particles in a certain region in (H, P_θ) space are confined; therefore, particle loss in (H, P_θ) space, similar to loss cone in $(v_\perp, v_{\parallel})$ space in a simple mirror machine, is expected to play an

important role in particle transport. Secondly, the velocity distribution of particle distribution function is not Maxwellian. To date, work has been carried out for collisional transport of test particles which are tenuous with respect to background particles, in (H, P_θ) space in the case of (i) Maxwellian background particles, and (ii) no transport of background particles,⁶ and for collisional particle transport of non-Maxwellian background particles in $(v_\perp, v_{||})$ space.^{7,8} But no work which considers both effects simultaneously has been reported. In this work we carry out the calculation for collisional transport of background particles in (H, P_θ) space, which incorporates the important characteristics of FRTP's, such that more reliable results can be obtained.

1.3 Accomplishments

The most fundamental constraints concerning particle transport in a plasma and overall plasma behavior in a fusion device are the particle confinement criteria, i.e., the conditions a particle has to satisfy in order to be kept in an interested spatial region. By considering single particle motion, the particle confinement criteria for axisymmetric field-reversed magnetic configurations are derived (see Chapter 3). From the studies on particle confinement criteria, a loss region in velocity space is found. Effects of this loss region in velocity space on particle confinement and plasma characteristics of FRTP's are investigated. Collisional particle losses are calculated; scaling laws for particle confinement are obtained (see Chapter 4). Approximate steady-state solutions to the Fokker-Planck kinetic equation,

subject to the loss boundary in velocity space, are obtained. Simple analytic expressions are found to be good representations of actual particle distribution functions in the collisional equilibrium (see Chapter 5), and these results should be very useful for reactor studies. In addition to the important effects on particle losses and plasma velocity distribution, the spin-up of plasma rotation observed in experiments can be a direct result of particle losses, merely based on the particle confinement criteria. In short, in the present work we have developed an improved understanding of the general characteristics of F RTP's in the collisional regime.

CHAPTER 2

LITERATURE REVIEW

2.1 Experiments

Most encouraging experiments of F RTP's were done in recent years as an effort to approach nuclear fusion by compact toroids. However, the early experiments can be traced back at least to the 1960's. In 1960, Green's experiments indicated the possibility of achieving stable confined plasmas with theta pinches.⁹ Equilibrium configuration of field-reversed magnetic fields was suggested.^{9,10} Bodin et al. performed a series of F RTP experiments;¹¹⁻¹⁴ rapid axial contraction and instabilities were found. High β (between 0.2 and 0.8) plasma was achieved.¹² The rotational instability of the whole plasma, which appeared after the axial contraction, was reported.¹³ It was also found that the plasma annulus broke up into a number of discrete rings along its axial length during the period following the radial implosion and before the end of the axial contraction, possibly due to the resistive instability.¹⁴ Eberhagen and Glaser observed similar plasma behavior, i.e., the plasma column breaking up into several discrete rings, but the observed formation time of the plasma rings was much longer than the tearing instability predicted.¹⁵ Eberhagen and Grossmann further showed experimentally the stabilizing effect of plasma rotation on the resistive tearing modes in their F RTP experiments, and also observed a much longer stable behavior of the plasma, at least an order of magnitude, than the tearing instability would predict.¹⁶ Es'kov et al.

did a series of experiments in compact toroids including F RTP's.¹⁷⁻¹⁹ Various control techniques in plasma heating, plasma confinement, formation of closed magnetic fields, etc. were studied. As to F RTP's, some important encouraging results were obtained. The plasma parameters, $T_e = 200-300$ eV, $T_i = 100$ eV, $n = (2-5) \times 10^{14} \text{ cm}^{-3}$, and stable confinement time $\sim 50 \mu\text{s}$ (the longest up to then), were reported.¹⁷ And, no hydromagnetic instability was observed.¹⁸ Linford et al.² and Armstrong et al.³ made quite detailed measurements in their field-reversed experiments, FRX-A and FRX-B. Plasma parameters, $T_e = 100 - 150$ eV, $T_i = 150 - 800$ eV, $n_m = (1-4) \times 10^{15} \text{ cm}^{-3}$, and stable time $\sim 50 \mu\text{s}$ were obtained.³ However, different from the observations of Es'kov et al., the destructive $m = 2$ rotational instability was observed in FRX-A and B experiments. More recently, Armstrong et al. achieved better plasma conditions, compared with FRX-A and B, in their larger experimental device FRX-C.²⁰ Longer stable time (75-80 μs) and larger closed-field region were achieved; however, $m = 2$ rotational instability still appeared. Aso et al. studied the end effects on the rotational instability in F RTP's by applying a very long solenoidal guide field at each end of the main theta pinch coil.^{21,22} They observed that the onset of the $m=2$ rotational instability was delayed significantly, and suggested that the rotational instability was driven by end-shortening effects.

In summary, many experiments in F RTP's have been carried out. Field-reversed configurations (FRC's) have been formed successfully, yet the control techniques in the formation of FRC's need to be explored

more. The improvement in plasma confinement has been achieved, yet the particle and energy loss mechanisms are not fully understood. Therefore, more experimental work is needed in order to establish reliable scaling laws for particle and energy confinements in FRC's.

2.2 Theoretical Studies

Particle transport in fully ionized plasmas in the presence of a magnetic field or a combination of magnetic and electric fields has been studied at least for decades. Both kinetic and hydrodynamic approaches have been used in studying the particle transport induced by collisions. There also has been a lot of work on particle transport induced by waves in plasmas; however, we concentrate our attention on collisional particle transport only.

Simon²³ has studied the particle diffusion across a magnetic field due to like particle collisions, based on the hydrodynamic approach, and has shown that the diffusion rate due to ion-ion collisions is comparable to or larger than that due to ion-electron collisions. Longmire and Rosenbluth²⁴ have studied the cross-field diffusion by treating collision-induced steps of guiding center as a random walk process. Rosenbluth et al.²⁵ have derived a Fokker-Planck kinetic equation for Coulomb interaction in a fully ionized plasma. This Fokker-Planck kinetic equation has been the basis for numerous theoretical calculations in the transport phenomena in plasmas. Since the Fokker-Planck equation is, in general, a fourth-order, three-dimensional, time-dependent, non-linear, partial differential equation and is very difficult to solve,

effort in simplifying this equation, according to the considered problem, is always tried. For example, Rosenbluth et al.²⁵ have first simplified the Fokker-Planck equation for distribution function $f(x, v, \mu, t)$ by assuming an axial symmetry in velocity space. Corman et al.²⁶ have derived, by assuming (1) Maxwellian background electrons and ion, (2) $v_{th,e} \gg v \gg v_{th,i}$, (3) axial symmetry in velocity space, (4) plane geometry in space, an energy-dependent Fokker-Planck equation for distribution function $f(x, E, \mu, t)$. And Tsuji et al.²⁷ have made a further simplification to derive a Fokker-Planck-type equation for distribution function $f(E, t)$.

Braginskii has derived a system of transport equations from the kinetic equation,²⁸ and has studied the transport phenomena in plasmas, based on these transport equations.^{28,29} Kaneko has obtained the transport coefficients of plasmas in which both the magnetic and electric fields are spatially uniform, and the electric field is perpendicular to the magnetic field.³⁰ Boeschoten has studied the plasma rotation caused by the cross-field ion current which can be produced by collisions (the so-called short-circuit effect).³¹ Pastukhov has obtained an approximate solution of the Fokker-Planck equation for the collisional loss of electrons from an adiabatic trap in a plasma with a positive potential, where a loss region in (v, μ) space (or equivalently, a hyperbolic loss region in velocity space) has to be imposed.³² Hamasaki and Davidson have done numerical simulations of theta-pinch implosion, based on the fluid equations, including classical and anomalous transport processes.³³ They have found that both classical and anomalous transport processes can play an important role during the implosion in their simulations.

They have also remarked that anomalous transport processes tend to dominate in the sheath region, although the physical mechanisms responsible for the anomalous transport have not been identified. Tuszewski and Lichtenberg have examined the collisional like-particle diffusion again within the scope of fluid equations.³⁴ They have shown that when the ratio of the ion gyroradius to the plasma transverse dimension is of the order of the fourth root of the electron-to-ion mass ratio, the ion-ion diffusion is significant. In some cases, the ion-ion collisional loss dominates the electron-ion collisional loss and decreases the plasma life time significantly. Auerbach has derived an expression for classical diffusion coefficient for field-reversed mirrors from MHD equations, where simplified Ohm's law has been employed.³⁵ The particle confinement time and reactor implications based on this classical diffusion coefficient has also been discussed.³⁶ Auerbach and Boozer have studied the classical diffusion in the presence of an X point (field-reversed configurations are examples of this).³⁷ The flow patterns of plasma fluid for a scalar resistivity and for a tensor resistivity have been studied; effects of an electric field in the direction of symmetry have been discussed. Hamasaki and Krall have performed a one-dimensional, MHD-like, numerical modelling of plasma diffusion in field-reversed configurations, in which some 2-D features are included.³⁸ The simulation has included both classical transport and anomalous transport, and have shown that the anomalous scaling of the plasma particle confinement time is R/ρ_i , where R = plasma major radius, ρ_i = ion Larmor radius. They have also reported that if

anomalous diffusion and subsequent plasma leakage is causing the rotation, the plasma life times observed in FRX experiments agree with the theory. Nguyen and Kammash have calculated the classical transport properties of a plasma in a field-reversed configuration.³⁹ The approach used has been similar to Auerbach and Condit's,^{35,36} but instead of using the simplified Ohm's law, improvement has been made by using the generalized Ohm's law. As in Refs. 35 and 36, all plasma properties have been assumed to be a function of magnetic flux only, and averaging over magnetic flux surfaces has been performed. Particle confinement times have been shown in good agreement with the FRX-B experiment if certain choice has been made for the plasma effective charge. Tuszewski and Linford have investigated particle transport in field-reversed configurations using a one-dimensional (radial), non-decaying magnetic field structure.⁴⁰ Various diffusion coefficients from models of ion-ion collisions, ion-electron collisions, and lower-hybrid-drift instability, have been compared with experimental data. Comparisons based on their model have suggested that the particle confinement times obtained with anomalous lower-hybrid-drift transport are in good agreement with the available numerical and experimental data.

In summary, the problem of understanding the collisional particle transport in plasmas has attracted considerable attention for many years. On theoretical studies, effort in both numerical simulations and theoretical formulations have been devoted to this problem. Although many studies in this area have been carried out for magnetic mirrors and

tokamaks for decades, the collisional transport in field-reversed configurations has been carried out mostly in recent years, especially after the success of simultaneously achieving field reversal, high β , and longer particle confinement time ($\sim 50 - 100 \mu s$) in the FRX experiments.^{2,3} Several possible mechanisms have been proposed to explain the experimental observations; however, the cause responsible for the observed plasma behavior have not been conclusively identified yet. More work is needed to understand and improve the plasma performance in field-reversed theta pinches in order to scale this attractive compact toroid up to fusion reactor.

CHAPTER 3

PARTICLE CONFINEMENT CRITERIA

Particle-confinement criteria for a fusion system are fundamentally important because they form the basis for many essential calculations, e.g., evaluation of particle-confinement time, particle distribution functions, possible loss-cone-like instability, etc. In previous studies, Wang and Miley⁴ obtained the confinement criteria for a limiting case of a spherical-Hill's vortex magnetic configuration; Lovelace, et al.⁴¹ obtained similar results when considering ion-ring equilibria where they considered ions with canonical angular momentum $P_\theta < 0$ (defining the z-component of the external magnetic field $B_{ex,z} > 0$). In this chapter, particle confinement criteria are derived for an axisymmetric system, and the results can be applied to calculations for Field Reversed Mirrors and Field-Reversed Theta Pinches. In Section 3.1, the general conditions for particle confinement are derived. In Section 3.2, the derivation is carried out for vanishing electric field. Criteria for closed-field and absolute confinements are obtained explicitly, including both the lower and upper bounds of P_θ/q where q is the charge of the particle, for a class of vortex-type field reversed magnetic configurations. The constraint on P_θ/q represents a necessary condition for particle confinement which reduces the confinement region in (H, P_θ) space; however, in previous studies,^{4,41,42} the lower bound on P_θ/q has not been imposed. In certain limits, the commonly used criterion for Hamiltonian, $H < -\omega_0 P_\theta$, where

$\omega_0 \equiv qB_0/mc$, is deduced from a more general form as a special case.

In this case, a new criterion, $-B_0 R_w^2/2c < P_\theta/q < 0$, where R_w = wall radius and B_0 = vacuum field, is found necessary. This also reduces the confinement region in (H, P_θ) space. In Section 3.3, the derivation including a nonvanishing electric field is carried out, and confinement criteria are obtained for several interesting cases.

3.1 General Basis

The Hamiltonian for a particle with mass m and charge q in an axisymmetric magnetic field $\underline{B} = \nabla \times \underline{A}$, $\underline{A} = A_\theta(r, z)\hat{\theta}$, and electric field $\underline{E} = -\nabla\phi$, $\phi = \phi(r, z)$, is given by

$$H = \frac{P_r^2}{2m} + \frac{P_z^2}{2m} + \frac{(P_\theta - \frac{q}{c}\Psi)^2}{2m r^2} + q\phi, \quad (3.1)$$

where c = speed of light, P_z , P_r , P_θ are the canonical momenta

$$P_z = m\dot{z}, \quad P_r = m\dot{r}, \quad P_\theta = m r^2 \dot{\theta} + \frac{q}{c} \Psi, \quad (3.2)$$

and

$$\Psi = r A_\theta, \quad (3.3)$$

where $\dot{X}$ stands for $\partial X/\partial t$. As was noted earlier,⁴³ since P_θ is a constant of motion, Eq. (3.1) can be viewed as

$$H = \left[\begin{array}{c} \text{effective kinetic} \\ \text{energy in } (r,z) \\ \text{motion, } T \end{array} \right] + \left[\begin{array}{c} \text{effective potential} \\ \text{energy in } (r,z) \\ \text{motion, } U \end{array} \right]$$

where

$$T(r,z) = \frac{P_r^2}{2m} + \frac{P_z^2}{2m}, \quad (3.5)$$

$$U(r,z) = \frac{(P_\theta - \frac{q}{c} \Psi)^2}{2m r^2} + q \phi. \quad (3.6)$$

The consideration of particle confinement is, therefore, equivalent to considering the (r,z) motion of a particle with total energy H moving in a potential $U(r,z)$.

The equipotential contours in (r,z) plane (or equipotential surfaces in (r, θ, z) configuration space) are determined by

$$U(r,z) = \frac{(P_\theta - \frac{q}{c} \Psi)^2}{2m r^2} + q \phi = \text{constant}. \quad (3.7)$$

A different value of the constant gives a different contour. Therefore, for a particle with total energy H and canonical angular momentum P_θ to be confined in certain spatial region R , the sufficient condition is that the equipotential contour $U(r,z) = H$ must locate within region R . In other words, if U_{crit} is the maximum permissible potential barrier, i.e., every point of the contour $U(r,z) = U_{\text{crit}}$ is located either inside R or on the boundary of R , then the sufficient condition is $H < U_{\text{crit}}$. If particle orbits are ergodic, then $H < U_{\text{crit}}$ is also the necessary condition for confinement. As shown numerically,⁴⁴ ergodic behavior of particle

orbits depends on the magnetic configuration, and ergodicity is more dominant for higher particle energy. Since the third constant of motion is not known, ergodicity is assumed for particles with energy $H > U_{\text{crit}}$. In this case, $H < U_{\text{crit}}$ is the necessary and sufficient condition for particle confinement.

Assume the system under consideration is bounded by $0 < r < R_w$ and $L_1 < z < L_2$. The conditions for absolute confinement (i.e., for particles to be confined in the system) are the following:

(i) There must be points (r_i, z_i) , $i = 1, 2, 3, 4$, on the equipotential contour, $U(r, z) = \text{constant}$, such that

$$\left. \begin{aligned} \frac{dz}{dr} \Big|_{(r_1, z_1)} = \frac{dz}{dr} \Big|_{(r_2, z_2)} = \frac{dr}{dz} \Big|_{(r_3, z_3)} = \frac{dr}{dz} \Big|_{(r_4, z_4)} = 0 \\ \text{where} \\ 0 \leq r_i \leq R_w, \quad L_1 \leq z_i \leq L_2, \quad i = 1, 2, 3, 4, \end{aligned} \right\} \quad (3.8)$$

see Fig. 1.

(ii) The total energy, H , must satisfy

$$H < U_{\text{crit}}, \quad (3.9)$$

where the critical potential is defined as

$$U_{\text{crit}} \equiv \max \left\{ U(r_i, z_i) \mid (r_i, z_i) \text{ satisfies Eq. (3.8)} \right\} \quad (3.10)$$

By Eq. (3.7), on the equipotential contour we have

$$\left(\frac{dz}{dr} \right)^{-1} = \frac{dr}{dz} = - \frac{\left(\frac{\partial U}{\partial z} \right)}{\left(\frac{\partial U}{\partial r} \right)} = - \frac{(P_\theta - \frac{g}{c} \psi) r \psi_z - m c r^3 \phi}{(P_\theta - \frac{g}{c} \psi) (\frac{c}{g} P_\theta - \psi + r \psi_r) - m c r^3 \phi_r}, \quad (3.11)$$

Figure 3.1. Points (r_i, z_i) , $i = 1, 2, 3, 4$, on the equipotential contour, $U(r, z) = \text{constant}$, represent the necessary condition, Eq. (3.8).

where subindices z and r stand for the corresponding partial derivatives.

In summary, the conditions (i) and (ii) are sufficient conditions for particle confinement. Since Eqs. (3.8) - (3.11) are very general, to get more explicit form for particle confinement criteria, the magnetic and electric fields need to be specified.

3.2 Derivation without Electric Field

In case there is no electric field, we can take $\phi = 0$. Furthermore, we consider the class of Hill's vortex-type field-reversed magnetic configurations given by

$$\Psi_{HV} = \begin{cases} -\frac{3B_0}{4R_s^2} r^2 (R_s^2 - r^2 - \zeta^2) & , r^2 + \zeta^2 \leq R_s^2, \\ \frac{B_0}{2} r^2 \left[1 - \frac{R_s^3}{(r^2 + \zeta^2)^{3/2}} \right] & , r^2 + \zeta^2 \geq R_s^2, \end{cases} \quad (3.12)$$

where $\zeta \equiv z/k$, $k \equiv z_s/R_s \equiv$ elongation factor, R_s and z_s are the maximum r - and z - coordinates of the separatrix, $0 \leq r < R_w$, $-L < z < L$, $R_s < R_w$, $z_s < L$. The magnetic configuration described by Eq. (3.12) is shown in Fig. 3.2. Since the conditions for particle confinement are determined by the effective potential, U , it is very instructive to study the spatial dependence of the effective potential, here, given by

$$U(r, z) = \frac{(P_\theta - \frac{q}{c} \Psi)^2}{2m r^2} \quad (3.13)$$

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Figure 3.2. The magnetic configuration described by Eq. (3.12).

Figures 3.3 - 3.5 are three-dimensional plots of $U(r,z)$ for various values of P_θ , and show the three types of effective potential.

Figure 3.3 shows a potential well with a hump in the central region, Fig. 3.4 a potential well without a central hump, and Fig. 3.5 a monotonically decreasing potential in the closed-field region as r or $|z|$ increases. From these plots, it is obvious that an equipotential contour (i.e., the intersection of a horizontal plane and the effective potential surface) can be closed only if $U(r,z)$ is one of the first two types. Equipotential contours of the third type of effective potential are open. Therefore, for confinement, a particle must have a canonical angular momentum, P_θ , such that the effective potential is a well-type.

Figure 3.6 shows typical equipotential contours for closed-field confinement (i.e., confined particles with orbits in the closed-field region) and absolute confinement (i.e., confined particles with orbits extend to the open-field region). Typical orbits of a closed-field confined particle and an absolutely confined particle are shown in Figs. 3.7 and 3.8 respectively, which justify the discussion about equipotential contours.

We now proceed to apply the conditions (i) and (ii) to the present case. For absolute confinement, the condition, $dr/dz = 0$, implies $r\zeta = 0$, which can always be satisfied by certain (r,ζ) in the open-field region where $r^2 + \zeta^2 > R_s^2$, $0 \leq r \leq R_w$, $-\zeta_{\max} \leq \zeta \leq \zeta_{\max} = L/k$. No constraint is obtained. However, the condition, $dz/dr = 0$, implies

Figure 3.3. Effective potential, $U(r, z)$, given by Eq. (3.13) is shown for $0.2 R_S \leq r \leq 1.2 R_S$, $-1.5 R_S \leq z \leq 1.5 R_S$. Parameters in Gaussian units are $m = 3.34 \times 10^{-24}$, $Q = 4.8 \times 10^{-10}$, $P_\theta = -9.3 \times 10^{-17}$, ψ given by Eq. (3.12), $B_0 = 8000$, $R_S = z_S = 6.0$.

Figure 3.4. Effective potential, $U(r, z)$. All parameters are the same as in Fig. 3.3 except $P_\theta = -8.5 \times 10^{-16}$.

Figure 3.5. Effective potential, $U(r,z)$. All parameters are the same as in Fig. 3.3 except $P_\theta = 1.37 \times 10^{-15}$.

Figure 3.6. Typical equipotential contours for a closed-field confined particle and an absolutely confined particle.

Figure 3.7. A typical orbit of a closed-field confined particle.

Figure 3.8. A typical orbit of an absolutely confined particle.

$$\frac{P_\theta}{g} + \frac{B_0}{2C} \left\{ 1 + \frac{R_s^3}{(r^2 + \zeta^2)^{3/2}} \left(\frac{3r^2}{r^2 + \zeta^2} - 1 \right) \right\} = 0 \quad (3.14)$$

which must be satisfied by certain (r, ζ) in the open-field region. This requires that

$$-\frac{B_0 R_w^2}{2C} \left(1 + \frac{2R_s^3}{R_w^3} \right) < \frac{P_\theta}{g} < 0 \quad (3.15)$$

as a necessary condition to be satisfied by absolutely confined particles.

In order to apply condition (ii), i.e., to calculate U_{crit} which is defined by Eq. (3.10), we need to know, depending on R_w and L , whether the radial confinement or axial confinement is more stringent. If the radial bound is most stringent, i.e., the maximum permissible equipotential surface touches the radial wall of the system first before it touches the axial wall, then

$$\begin{aligned} U_{\text{crit}} &= \min \{ U(r=R_w, z) \} \\ &= U(r, z) \Big|_{\text{at } \left(\frac{dr}{dz} = 0 \text{ and } r=R_w \right)} \\ &= U(R_w, 0) \\ &= \frac{1}{2mR_w^2} \left\{ P_\theta - \frac{gB_0}{2C} R_w^2 \left(1 - \frac{R_s^3}{R_w^3} \right) \right\}^2 \quad (3.16) \end{aligned}$$

On the other hand, if the axial bound is most stringent, then

$$\begin{aligned} U_{\text{crit}} &= \min \{ U(r, z = L) \} \\ &= U(r, z) \Big|_{\text{at } \left(\frac{dz}{dr} = 0, z = L \right)}. \end{aligned} \quad (3.17)$$

Using Eq. (3.14), Eq. (3.17) becomes

$$\begin{aligned} U_{\text{crit}} &= U(r = r_c, z = L) \\ &= \frac{1}{2mr^2} \left\{ P_\theta - \frac{qB_0 r_c^2}{2c} \left[1 - \frac{R_s^3}{(r_c^2 + \zeta_{\text{max}}^2)^{3/2}} \right] \right\}^2, \end{aligned} \quad (3.18)$$

where $\zeta_{\text{max}} \equiv LR_s/z_s$, r_c is the solution of

$$\frac{P_\theta}{q} + \frac{B_0}{2c} r_c^2 \left\{ 1 + \frac{R_s^3}{(r_c^2 + \zeta_{\text{max}}^2)^{3/2}} \left[\frac{3r_c^2}{r_c^2 + \zeta_{\text{max}}^2} - 1 \right] \right\} = 0, \quad (3.19)$$

and

$$0 < r_c < R_w. \quad (3.20)$$

As mentioned before, Eq. (3.15) is a necessary condition to be satisfied by P_θ/q of the absolutely confined particles. The actual lower and upper bounds of P_θ/q , which depend on whether the radial or axial confinement is most stringent, are derived as follows. If the radial confinement is most stringent, i.e., $U_{\text{crit}} = U(R_w, 0)$, then the condition, $dz/dr = 0$, must be satisfied by certain (r, ζ) in the system. This requires that

$$-\frac{B_0 R_w^2}{2c} \left(1 + \frac{2R_s^3}{R_w^3}\right) < \frac{P_\theta}{g} < 0$$

which is the same as Eq. (3.15). If the axial confinement is most stringent, i.e., $U_{\text{crit}} = U(r_c, \zeta_{\text{max}})$, then the corresponding requirement is

$$-\frac{B_0 \zeta_{\text{max}}^2}{8c} (\sqrt{137} - 11) \left\{ 1 + \frac{\sqrt{137} - 13}{(\sqrt{137} - 7)^{5/2}} \cdot \frac{16R_s^3}{\zeta_{\text{max}}^3} \right\} < \frac{P_\theta}{g} < 0, \quad (3.21)$$

or approximately,

$$-0.88 \frac{B_0 \zeta_{\text{max}}^2}{c} \left(1 - 0.43 \frac{R_s^3}{\zeta_{\text{max}}^3}\right) < \frac{P_\theta}{g} < 0. \quad (3.22)$$

For particles confined in the closed-field region, $dr/dz = 0$ implies $r\zeta = 0$. Again, this can always be satisfied by certain (r, z) in the closed-field region. The condition, $dz/dr = 0$, implies

$$\frac{P_\theta}{g} = -\frac{3B_0 r^2}{4cR_s^2} (3r^2 + \zeta^2 - R_s^2), \quad (3.23)$$

which must be satisfied by certain (r, z) in the closed-field region where $r^2 + \zeta^2 < R_s^2$, $0 \leq r \leq R_s$, $0 \leq \zeta \leq R_s$. This requires

$$-\frac{3B_0 R_s^2}{2c} \leq \frac{P_\theta}{g} \leq \frac{B_0 R_s^2}{16c}. \quad (3.24)$$

And the maximum permissible potential is given by

$$U_{\text{crit}} = \min \left\{ U(r, \zeta) \mid (r, \zeta) \text{ on separatrix} \right\} \\ = \frac{P_\theta^2}{2mR_s^2} . \quad (3.25)$$

As a limiting case, if $\zeta_{\text{max}} \gg R_s$ and $\zeta_{\text{max}} \gg r_c$, then Eq. (3.18) is simplified to

$$U_{\text{crit}} = - \frac{g B_0}{mc} P_\theta . \quad (3.26)$$

Note that $r_c < R_w$ and usually $R_s \sim R_w$; therefore, $\zeta_{\text{max}} \gg R_s$ also implies $\zeta_{\text{max}} \gg r_c$. Equation (3.26) is the previous confinement criterion^{4,41} and is used very commonly. In this limiting case, the condition, Eq. (3.14), reduces to

$$\frac{P_\theta}{g} + \frac{B_0}{2c} r^2 = 0$$

which requires

$$- \frac{B_0 R_w^2}{2c} < \frac{P_\theta}{g} < 0 . \quad (3.27)$$

Note that the lower bound of P_θ/g in Eq. (3.27) represents an additional constraint on the confinement criteria, which is not included in previous results.^{4,41} In fact, Eqs. (3.26) and (3.27) are applicable to the limiting case not only of the Hill's vortex magnetic configuration, but also of general magnetic configurations. This can be shown as follows. For an

arbitrary axisymmetric magnetic configuration with $E(r,L) = 0$, the condition $dz/dr = 0$ at (r,L) implies

$$(P_\theta - \frac{g}{c} \Psi) (P_\theta - \frac{g}{c} \Psi + \frac{g}{c} r \Psi_r) = 0. \quad (3.28)$$

In the case that the axial confinement is most stringent,

$$U_{\text{crit}} = U(r_c, L) = \frac{[P_\theta - \frac{g}{c} \Psi(r_c, L)]^2}{2 m r_c^2} \quad (3.29)$$

Since $U_{\text{crit}} \neq 0$, from Eq. (3.28), we have

$$P_\theta - \frac{g}{c} \Psi(r_c, L) + \frac{g}{c} r_c \Psi_r(r_c, L) = 0. \quad (3.30)$$

If the magnetic field approaches a uniform field at the ends (or at the mirror throats), i.e.,

$$\tilde{B}(r, \pm L) \approx B_0 \hat{z}, \quad B_0 = \text{constant}, \quad B_0 > 0, \quad (3.31)$$

then Eq. (3.30) gives the solution for r_c as

$$r_c = \left(-\frac{2c P_\theta}{g B_0} \right)^{1/2} \quad (3.32)$$

Since $0 < r_c < R_w$ for confined particles, Eq. (3.32) requires

$$-\frac{B_0 R_w^2}{2c} < \frac{P_\theta}{g} < 0. \quad (3.27)$$

Using Eqs. (3.31) and (3.32), Eq. (3.29) becomes

$$U_{\text{crit}} = -\frac{g B_0}{m c} P_\theta. \quad (3.26)$$

The above criteria derived from Eqs. (3.8) - (3.11) are the conditions for a particle having total energy H and canonical angular momentum P_θ to be confined. Since

$$H \equiv \frac{P_r^2}{2m} + \frac{P_z^2}{2m} + \frac{(P_\theta - \frac{e}{c}\Psi)^2}{2mr^2} + e\phi \geq \frac{(P_\theta - \frac{e}{c}\Psi)^2}{2mr^2} + e\phi, \quad (3.33)$$

only part of (H, P_θ) space is energetically accessible by the particles. Therefore, one more constraint that needs to be imposed when we deal with variables H and P_θ is

$$H \geq \frac{(P_\theta - \frac{e}{c}\Psi)^2}{2mr^2} + e\phi. \quad (3.34)$$

3.3 Derivation including Electric Field

For practical cases, axial confinement is typically most stringent. This implies that, for absolute confinement, in the presence of an electric field,

$$U_{\text{crit}} = U(r_c, L) = \frac{[P_\theta - \frac{e}{c}\Psi(r_c, L)]^2}{2mr_c^2} + e\phi(r_c, L), \quad (3.35)$$

where r_c is solved from

$$\begin{aligned} & [P_\theta - \frac{e}{c}\Psi(r_c, L)] [P_\theta - \frac{e}{c}\Psi(r_c, L) + \frac{e}{c}r_c\Psi_r(r_c, L)] \\ & + m e r_c^3 E_r(r_c, L) = 0, \end{aligned} \quad (3.36)$$

and E_r is the radial electric field. Since the magnetic and electric fields at $z = L$ (the actual axial wall or magnetic mirror throat) must be known in order to calculate U_{crit} , the following representative cases are considered.

Case 1: $B(r, L) = B_0 \hat{z}$, $B_0 > 0$, and $E(r, L) = E_0 \hat{r}$, where B_0 and E_0 are constants. This represents a uniform magnetic field and constant radial electric field at $z = L$. In this case, $\psi(r, L) = B_0 r^2/2$, $\phi(r, L) = -E_0 r$, and Eq. (3.36) becomes

$$g^2 B_0^2 r_c^4 - 4 m c^2 g E_0 r_c^3 - 4 c^2 P_0^2 = 0.$$

The solutions to this equation are

$$r_c = -\frac{C_1}{4} + \frac{C_2}{2} \pm \frac{C_3}{2} \quad \text{or} \quad -\frac{C_1}{4} - \frac{C_2}{2} \pm \frac{C_4}{2}, \quad (3.37)$$

where

$$C_1 = -\frac{4 m c^2 E_0}{g B_0^2}, \quad (3.37.1)$$

$$C_2 = \left(\frac{C_1^2}{4} + C_5 \right)^{1/2}, \quad (3.37.2)$$

$$C_3 = \left[\frac{3 C_1^2}{4} + 2 (C_5^2 - 4 C_6)^{1/2} \right]^{1/2}, \quad (3.37.3)$$

$$C_4 = \left[\frac{3 C_1^2}{4} - 2 (C_5^2 - 4 C_6)^{1/2} \right]^{1/2}, \quad (3.37.4)$$

$$\begin{aligned}
C_5 &= \frac{4CP_\theta}{8B_\theta} \left\{ \left[-\frac{m^2 C^2 E_\theta^2}{28B_\theta^3 P_\theta} + \left(\frac{1}{27} + \frac{m^4 C^6 E_\theta^4}{48^2 B_\theta^6 P_\theta^2} \right)^{1/2} \right]^{1/3} \right. \\
&\quad \left. + \left[-\frac{m^2 C^2 E_\theta^2}{8B_\theta^3 P_\theta} - \left(\frac{1}{27} + \frac{m^4 C^6 E_\theta^4}{48^2 B_\theta^6 P_\theta^2} \right)^{1/2} \right]^{1/3} \right\}, \quad (3.37.5) \\
C_6 &= -\frac{4C^2 P_\theta^2}{8^2 B_\theta^2}.
\end{aligned}$$

Asymptotic expressions for Eq. (3.37) are obtained as follows.

$$\gamma_c \approx \begin{cases} \gamma_{c1+} \equiv \frac{u}{2} (v + \sqrt{\frac{v}{2}}) \\ \gamma_{c2+} \equiv \frac{u}{2} (1 - v + \sqrt{1 + \frac{3}{2}v}) \end{cases} \left. \begin{array}{l} \text{if } \alpha \gg \frac{256}{27} \\ \text{and } E_\theta g > 0, \end{array} \right\} \quad (3.38)$$

$$\gamma_c \approx \begin{cases} \gamma_{c1-} \equiv \frac{u}{2} (v - \sqrt{\frac{v}{2}}) \\ \gamma_{c2-} \equiv \frac{u}{2} (1 - v - \sqrt{1 + \frac{3}{2}v}) \end{cases} \left. \begin{array}{l} \text{if } \alpha \gg \frac{256}{27} \\ \text{and } E_\theta g < 0, \end{array} \right\} \quad (3.39)$$

$$\gamma_{c3} \equiv \frac{u}{4} + \sqrt{-\frac{P_\theta}{8a}} \quad \text{if } \alpha < 1, \quad (3.40)$$

where $u \equiv mE_\theta/qa^2$, $a \equiv B_\theta/2c$, $\alpha \equiv m^4 E_\theta^4 / q^2 a^6 P_\theta^2$, $v \equiv \alpha^{-1/3}$.

These results make use of the properties that (i) r_c must be real and non-negative, and (ii) the asymptotic expressions must reproduce the previous result, Eq. (3.32), as $E_0 \rightarrow 0$. To see the physical meaning of the parameter α , we rewrite

$$\alpha = \frac{1}{\pi^2} \left(\frac{\Delta r}{r_0} \right)^4, \quad (3.41)$$

where

$$\Delta r \equiv \left(\frac{q E_0}{m} \right) \left(\frac{2\pi}{\omega} \right) \left(\frac{2\pi}{\omega} \right)$$

$\approx$ displacement of the particle in a gyroperiod due to

$$\vec{E} = E_0 \hat{r},$$

and r_0 is defined by

$$P_\theta = \frac{q}{c} \psi(r_0, L) = \frac{q B_0 r_0^2}{2c},$$

or

$$r_0 \equiv \left(\frac{2c}{B_0} \left| -\frac{P_\theta}{q} \right| \right)^{1/2}.$$

Since $P_\theta \equiv m r^2 \dot{\theta} + q\psi/c$ is a constant of motion, $r = r_0$ is the radial position along the particle orbit where $\dot{\theta} = 0$ and $z = L$ (for particles that can reach this point). As Eq. (3.41) shows, the parameter α measures the relative importance on particle confinement of the electric field to that of the magnetic field and canonical angular momentum, or, the importance on the particle motion of the electric force relative to the particle gyromotion. Therefore, smaller α implies that the electric field is less influential in determining particle confinement.

Substituting Eq. (3.38) into Eq. (3.35), we obtain, for $\alpha \gg (256/27)$ and $E_0 q > 0$,

$$\begin{aligned} U_{\text{crit}} &= \min \{ U(r_{c1+}, L), U(r_{c2+}, L) \} \\ &= U(r_{c2+}, L) \\ &= - \frac{2mc^2 E_0^2}{B_0} \end{aligned} \quad (3.42)$$

This negative value of U_{crit} can be qualitatively understood from the following fact. The condition $\alpha \gg (256/27)$ implies that the confinement is mainly determined by the electric field, while the condition $E_0 q > 0$ indicates that particles move radially outward under the electric force.

Similarly, Eq. (3.39) gives, for $\alpha \gg (256/27)$ and $E_0 q > 0$,

$$\begin{aligned} U_{\text{crit}} &= \min \{ U(r_{c1-}, L), U(r_{c2-}, L) \} \\ &= U(r_{c2-}, L) \\ &\approx \frac{2mc^2 E_0^2}{B_0^2} \left\{ \left(\frac{64}{49} + \frac{7}{4} \right) v - 2S_1 v^{3/2} \right\} \\ &\approx \left(\frac{64}{49} + \frac{7}{4} \right) \frac{2mc^2 E_0^2 v}{B_0^2}, \end{aligned} \quad (3.43)$$

where

$$S_1 = \begin{cases} 1 & \text{if } \theta P_\theta > 0, \\ -1 & \text{if } \theta P_\theta < 0. \end{cases}$$

Equation (3.40) gives, for $\alpha < 1$,

$$U_{\text{crit}} = \left(-\frac{g B_0 P_\theta}{2mc} \right) \left(1 + \frac{g}{d} + \frac{d}{32} \right) - \frac{mc^2 E_0^2}{B_0^2} \left(1 + \frac{4 S_2}{\alpha^{1/4}} \right), \quad (3.44)$$

where

$$d = 16 + 8 S_2 \alpha^{1/4} + \alpha^{1/2},$$

$$S_2 = \begin{cases} 1 & \text{if } g E_0 > 0, \\ -1 & \text{if } g E_0 < 0. \end{cases}$$

As before, the bounds of P_θ are obtained from the constraint, $0 < r_c < R_w$. The results corresponding to Eqs. (3.42) - (3.44) are as follows. For $\alpha \gg (256/27)$ and $E_0 q > 0$, no additional constraint on bounds of P_θ is necessary. For $\alpha \gg (256/27)$ and $E_0 q < 0$, we obtain

$$-1.22 (-mg E_0 R_w^3)^{1/2} < P_\theta < 1.22 (-mg E_0 R_w^3)^{1/2}. \quad (3.45)$$

For $\alpha < 1$,

$$-\frac{B_0}{2c} \left(R_w - \frac{m c E_0}{g B_0^2} \right)^2 < \frac{P_\theta}{g} < -\frac{m^2 c^3 E_0^2}{2 g^2 B_0^3}. \quad (3.46)$$

We note that Eqs. (3.44) and (3.46) reduce to the correct limits, Eqs. (3.26) and (3.27), as $E_0 \rightarrow 0$.

Case 2: $B(r, L) = B_0 \hat{z}$, $B_0 > 0$, and $E(r, L) = (E_0/R_w) r \hat{r}$, where B_0 and E_0 are constants. This case represents the situation at $z = 1$ where the magnetic field is uniform and the electrical field is radial

and linear in r (the net electric charge density, $\rho(r,L)$, is constant).

In this case, $\psi(r,L) = B_0 r^2/2$, $\phi(r,L) = -(E_0/R_w)r^2/2$, and E_0 is the radial electric field at $r = R_w$. Following the same approach as in case 1, we obtain

$$\gamma_c = \left(-\frac{2cP_0}{\mathcal{E}B_0} \right)^{1/2} \left(1 - \frac{4mc^2E_0}{\mathcal{E}B_0^2R_w} \right)^{-1/4}, \quad (3.47)$$

$$\begin{aligned} \mathcal{U}_{\text{crit}} = & -\frac{\mathcal{E}B_0P_0}{mc} \left(1 - \frac{4mc^2E_0}{\mathcal{E}B_0^2R_w} \right)^{-1/2} \\ & \cdot \left\{ \frac{1}{4} \left(1 - \frac{4mc^2E_0}{\mathcal{E}B_0^2R_w} \right) \cdot \left[1 + \left(1 - \frac{4mc^2E_0}{\mathcal{E}B_0^2R_w} \right)^{-1/2} \right]^2 - \frac{mc^2E_0}{\mathcal{E}B_0^2R_w} \right\}, \end{aligned} \quad (3.48)$$

$$-\frac{B_0R_w^2}{2c} \left(1 - \frac{4mc^2E_0}{\mathcal{E}B_0^2R_w} \right)^{1/2} < \frac{P_0}{\mathcal{E}} < 0, \quad (3.49)$$

and

$$\frac{E_0}{\mathcal{E}} < \frac{B_0^2R_w}{4mc^2}. \quad (3.50)$$

Also, Eq. (3.50) provides another necessary condition for confinement. If Eq. (3.50) is not satisfied, we conclude that the radial electric field at $z = L$ is so strong, compared with the magnetic confining force, that particles moving radially outward due to the electric force (with $qE_0 > 0$) are not confined.

Equation (3.48) and (3.49) illustrate the following points. Firstly, as the electric field strength changes, the confinement region in (H, P_θ) space changes; and Eqs. (3.48) and (3.49) reduce to Eqs. (3.26) and (3.27), respectively, as expected, when $E_0 \rightarrow 0$. Secondly, the presence of the electric field improves the confinement of species with $qE_0 < 0$ (these particles experience radially inward electric force) while worsening the confinement of species with $qE_0 > 0$. Thirdly, since the electric field appears in the confinement criteria only through the factor $mc^2 E_0 / qB_0^2 R_w$, it has a more important effect, due to the mass dependence, on the confinement of ions than it has on the confinement of electrons.

As in Section 3.2, the above conditions are for a particle having total energy H and canonical angular momentum P_θ to be confined.

Since

$$H \equiv \frac{P_r^2}{2m} + \frac{P_z^2}{2m} + \frac{(P_\theta - \frac{e}{c}\Psi)^2}{2m r^2} + e\phi \geq \frac{(P_\theta - \frac{e}{c}\Psi)^2}{2m r^2} + e\phi,$$

the energetic constraint, Eq. (3.34), has to be imposed, too.

3.4 Summary and Discussion

(I) The general form of confinement criteria given by Eqs. (3.8) - (3.11) can be applied to the calculations for field-reversed mirrors and field-reversed theta pinches. For a general class of vortex-type field-reversed magnetic configurations, Eq. (3.12), the confinement criteria in case $\underline{E}(r,L) = \underline{E}(R_w,z) = 0$ are summarized as follows. For a particle to be confined within the closed-field region, with $\underline{E} = 0$ on separatrix, the total energy H and canonical angular momentum P_θ need to satisfy Eq. (3.24), Eq. (3.34) and $H < U_{\text{crit}}$, where U_{crit} is given by Eq. (3.25). For absolute confinement, the following criteria need to be satisfied: If the radial confinement is most stringent, i.e., $U(R_w,0) < U(r_c,L)$, the criteria are Eq. (3.15), Eq. (3.34), and $H < U(R_w,0)$. If the axial confinement is most stringent, i.e., $U(R_w,0) > U(r_c,L)$, the criteria are Eq. (3.21), Eq. (3.34) and $H < U(r_c,L)$, where $U(R_w,0)$ and $U(r_c,L)$ are given by Eqs. (3.16) and (3.18), respectively.

(II) As a special case of the general expressions, the criteria for axial confinement where $\underline{B}(r,\pm L) \approx B_0 \hat{z}$, and $\underline{E}(r,\pm L) = 0$ are given by Eq. (3.27), Eq. (3.34), and $H < U_{\text{crit}} = -qB_0 P_\theta / mc$.

(III) Typical results for the confinement region in (H, P_θ) space are shown in Figs. 3.9 to 3.13. Figure 3.9 is for closed-field confinement. Figures 3.10 to 3.13 compare the exact expressions with the approximate limiting case for the absolute confinement. Figures 3.10 and 3.11 show the cases where the axial confinement is most stringent, while Figs. 3.12 and 3.13 are for the cases where the radial confinement is most stringent.

Figure 3.9. A typical confinement region in (H, P_θ) space for the closed-field confinement. System parameters, in Gaussian units, are $B_0 = 8500$, $q = 4.8 \times 10^{-10}$, $m = 3.34 \times 10^{-24}$, $R_w = 10$, $R_s = 5$, $L = 50$, $z_s = 40$, $r = 4$, $z = 0$.

Figure 3.10. Typical confinement regions in (H, P_θ) space, at a position (r, z) in the closed field region, for the exact axial confinement (dashed lines) and the approximate limiting case (solid lines). All parameters are the same as in Fig. 3.9.

Figure 3.11. Typical confinement region in (H, P_θ) space, at a position (r, z) in the open-field region, for the exact axial confinement (dashed lines) and the approximate limiting case (solid lines). All parameters are the same as in Figure 3.9, except $r = 7$.

Figure 3.12. Typical confinement regions in (H, P_θ) space, at a position (r, z) in the closed-field region, for the exact radial confinement (dashed lines) and the approximate limiting case (solid lines). All parameters are the same as in Fig. 3.10.

Figure 3.13. Typical confined regions in (H, P_θ) space, at a position in open-field region, for the exact radial confinement (dashed lines) and the approximate limiting case (solid lines). All parameters are the same as in Fig. 3.11.

(IV) In practical use of these confinement criteria, as shown by Figs. 3.9 to 3.13, Eq. (3.24) and the lower bound in Eq. (3.15) do not affect the confinement region in (H, P_θ) space because they will be satisfied automatically once the other criteria are satisfied. But the lower bound of P_θ/q in Eqs. (3.21) and (3.27) does represent a new, necessary criterion which reduces the confined region in (H, P_θ) space. This occurs in case of axial confinement and represents a new constraint which has not been imposed in previous studies.^{4,41,42}

(V) As both the criteria and Figs. 3.9 - 3.13 have shown, the confinement region in (H, P_θ) space is spatially dependent. Therefore, the particle distribution function, f , should have explicit spatial dependence other than through P_θ -dependence, in other words, $f = f(r, z, H, P_\theta)$. To make a direct comparison with the previous studies, define $f(H, P_\theta) \equiv \int_V dr f(r, z, H, P_\theta)$, where V is the system volume. Then, the confinement region for the whole system in (H, P_θ) space, i.e., the region where $f(H, P_\theta) \neq 0$ (assuming fast losses), can be compared, as shown in Fig. 3.14, for the limiting case mentioned in (II) between present and previous studies.⁴¹

(VI) As shown in Figs. 4.11 and 4.13, the new constraint mainly affects particles with large values of P_θ . Recalling the definition of P_θ , since P_θ is both position- and direction-dependent, for certain cases where particles are directed in θ -direction, e.g., for particle injections, ion rings, etc., the new criterion may affect a significant number of particles.

(VII) For $\tilde{E}(r, L) \neq 0$, confinement criteria for several interesting cases were obtained: (i) For $\tilde{B}(r, L) = B_0 \hat{z}$, $\tilde{E}(r, L) = E_0 \hat{r}$, and strong

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Figure 3.14. A schematic comparison between present and previous¹² studies, for the limiting case where $B(r, \pm L) \approx B_0 \hat{z}$.

shows the confinement region from Fig. 1 of Ref. 41.

shows the confinement region from present results.

$\underline{E}(r,L)$, i.e., $\alpha \gg (256/27)$: The confinement criteria are Eq. (3.34) and $H < U_{crit}$ where U_{crit} is given by Eq. (3.42) for $E_0 q > 0$; Eq. (3.34), Eq. (3.45) and $H < U_{crit}$ where U_{crit} is given by Eq. (3.43) for $E_0 q < 0$. (ii) for $\underline{B}(r,L) = B_0 \hat{z}$, $\underline{E}(r,L) = E_0 \hat{r}$ and weak $\underline{E}(r,L)$, i.e., $\alpha < 1$, the confinement criteria are Eq. (3.46), Eq. (3.34), and $H < U_{crit}$, where U_{crit} is given by Eq. (3.44). (iii) For $\underline{B}(r,L) = B_0 \hat{z}$, $\underline{E}(r,L) = (E_0/R_w) r \hat{r}$, the confinement criteria are Eq. (3.49), Eq. (3.50), Eq. (3.54), and $H < U_{crit}$, where U_{crit} is given by Eq. (3.48).

(VIII) Since only part of (H, P_θ) space is energetically allowed, the energetic constraint, Eq. (3.34) is a necessary condition to be satisfied. This constraint has not been stated explicitly in previous study.⁴

CHAPTER 4

VELOCITY-SPACE PARTICLE LOSS

In Chapter 3, we obtain the particle confinement criteria for axisymmetric field-reversed configurations (FRC's) from the two constants of motion, H and P_θ . It is found that only part of (H, P_θ) space is available for particle confinement, i.e., a loss region in (H, P_θ) space exists. As with the loss cone in velocity space for a mirror, a loss region in (H, P_θ) space may have a significant influence on plasma behavior in FRC's. In this chapter, we study the effect of this loss region on the collisional particle loss in FRTC's.

4.1 Loss Region in Velocity Space

Usually, the mathematical expressions describing particle collisions are expressed in terms of the velocity changes of the colliding particles, e.g., the collisional term in the Boltzmann kinetic equation or in the Fokker-Planck kinetic equation. It is also customary to study the particle distribution function in $(\underline{r}, \underline{v})$ phase space, especially in the collisional regime. Therefore, for the present study the particle confinement criteria are reformulated in the $(\underline{r}, \underline{v})$ phase space.

First, we transform the criteria for absolute confinement in (H, P_θ) space to $(\underline{r}, \underline{v})$ space. For simplicity, approximate particle confinement criteria with no electric field, Eqs. (3-26), (3-27), and (3-34), are used. Namely,

$$\frac{(P_\theta - \frac{e}{c} \Psi)^2}{2m\gamma^2} \leq H < -\frac{eB_0}{mc} P_\theta, \quad (4.1)$$

and

$$-\frac{B_0 R_w^2}{2c} < \frac{P_\theta}{g} < 0. \quad (4.2)$$

Using the definitions of H and P_θ , Eqs. (3.1) and (3.2), the above criteria, Eqs. (4.1) and (4.2), are transformed to (r, v) space and given by

$$v_r^2 + (v_\theta - v_{\theta 0})^2 + v_z^2 \leq v_c^2, \quad (4.3)$$

and

$$-\frac{g B_0 r}{mc} \left(\frac{\Psi}{B_0 r^2} + \frac{R_w^2}{2r^2} \right) < v_\theta < -\frac{g\Psi}{mcr} \quad \text{for } g > 0, \quad (4.4.1)$$

$$-\frac{g\Psi}{mcr} < v_\theta < -\frac{g B_0 r}{mc} \left(\frac{\Psi}{B_0 r^2} + \frac{R_w^2}{2r^2} \right) \quad \text{for } g < 0, \quad (4.4.2)$$

where

$$v_r = \dot{r} = \frac{\partial r}{\partial t}, \quad v_\theta = r\dot{\theta} = r \frac{\partial \theta}{\partial t}, \quad v_z = \dot{z} = \frac{\partial z}{\partial t}, \quad (4.5)$$

$$v_{\theta 0} = -\frac{g B_0 r}{mc}, \quad (4.6)$$

$$v_c = \frac{|g| B_0 r}{mc} \left(1 - \frac{2\Psi}{B_0 r^2} \right)^{1/2}. \quad (4.7)$$

Equations (4.3) and (4.4) are the corresponding particle confinement criteria in (r, v) space. Equation (4.3) defines the interior of a sphere in velocity space. The sphere is centered at $(0, v_{\theta 0}, 0)$ with radius v_c in the orthogonal (v_r, v_θ, v_z) space. Equation (4.4) defines an infinite slab between two planes, $v_\theta = -q\Psi/mcr$ and $v_\theta = -q(\Psi + B_0 R_w^2/2)/mcr$ in

(v_r, v_θ, v_z) space. The confinement region in velocity space is given by the intersection of the sphere and the infinite slab.

From Eq. (4.3) we have

$$-\frac{qB_0 r}{mc} \left(1 + \frac{|q|}{q} \sqrt{1 - \frac{2\Psi}{B_0 r^2}} \right) < v_\theta$$

$$< -\frac{qB_0 r}{mc} \left(1 - \frac{|q|}{q} \sqrt{1 - \frac{2\Psi}{B_0 r^2}} \right). \quad (4.8)$$

We compare Eq. (4.4) with Eq. (4.8). It can be easily shown that for $q > 0$ we have

$$-\frac{q\Psi}{mcr} > -\frac{qB_0 r}{mc} \left(1 - \frac{|q|}{q} \sqrt{1 - \frac{2\Psi}{B_0 r^2}} \right), \quad (4.9.1)$$

and

$$-\frac{qB_0 r}{mc} \left(\frac{\Psi}{B_0 r^2} + \frac{R_w^2}{2r^2} \right) \begin{matrix} \geq \\ \leq \end{matrix} -\frac{qB_0 r}{mc} \left(1 + \frac{|q|}{q} \sqrt{1 - \frac{2\Psi}{B_0 r^2}} \right), \quad (4.9.2)$$

while for $q < 0$ we have

$$-\frac{q\Psi}{mcr} < -\frac{qB_0 r}{mc} \left(1 + \frac{|q|}{q} \sqrt{1 - \frac{2\Psi}{B_0 r^2}} \right), \quad (4.10.1)$$

and

$$-\frac{qB_0 r}{mc} \left(\frac{\Psi}{B_0 r^2} + \frac{R_w^2}{2r^2} \right) \begin{matrix} \geq \\ \leq \end{matrix} -\frac{qB_0 r}{mc} \left(1 - \frac{|q|}{q} \sqrt{1 - \frac{2\Psi}{B_0 r^2}} \right), \quad (4.10.2)$$

where the direction of inequality or equality in Eqs. (4.9.2) and (4.10.2) depends on the position. Therefore, depending on the position r , the absolute-confinement region in (v_r, v_θ, v_z) space can be either the entire sphere described by Eq. (4.3) or the sphere with a portion being cut off by $v_\theta = -qB_0 r(\Psi/B_0 r^2 + R_w^2/2r^2)/mc$ plane.

Particles with velocity vector outside this confinement region are not confined. Figures 4.1 and 4.2 show the two cases of absolute-confinement region in velocity space for $q > 0$ and $q < 0$, respectively.

For the closed-field confinement, the criteria are Eqs. (3.24) and (3.34), i.e.,

$$\frac{(P_\theta - \frac{g}{c}\Psi)^2}{2m r^2} \leq H < \frac{P_\theta^2}{2m R_s^2} \quad (4.11)$$

Similarly, we transform Eq. (4.11) to $(\underline{r}, \underline{v})$ space and obtain, with no electric field,

$$v_r^2 + \frac{(v_\theta - v_{\theta 1})^2}{a^2} + v_z^2 < v_{c1}^2, \quad (4.12)$$

where

$$a = \left(1 - \frac{r^2}{R_s^2}\right)^{1/2}, \quad (4.13)$$

$$v_{\theta 1} = \frac{g r \Psi}{m c a^2 R_s^2}, \quad (4.14)$$

$$v_{c1} = \frac{|g \Psi|}{m c a R_s}. \quad (4.15)$$

Equation (4.12) defines the interior of an ellipsoid in (v_r, v_θ, v_z) space. The ellipsoid has the center at $(0, v_{\theta 1}, 0)$ with radii v_{c1} , $a v_{c1}$, and v_{c1} in the v_r -, v_θ -, and v_z - directions, respectively, shown as Fig. 4.3. It may be worth noting that, for a Hill's vortex magnetic configuration described

$$O' = (0, -\frac{qB_0 r}{mc}, 0)$$

$$A = (0, -\frac{qB_0 r}{mc} (\frac{\psi}{B_0 r^2} + \frac{R_w^2}{2r^2}), 0)$$

$$v_c = \frac{|q|B_0 r}{mc} (1 - \frac{2\psi}{B_0 r^2})^{1/2}$$

Figure 4.1. Two possibilities of the typical absolute-confinement region in velocity space for positively charged particles; (a) a cut-off sphere (b) entire sphere.

$$O' = (0, -\frac{qB_0 r}{mc}, 0)$$

$$A = (0, -\frac{qB_0 r}{mc} (\frac{\psi}{B_0 r^2} + \frac{R_w^2}{2r^2}), 0)$$

$$v_c = \frac{|q|B_0 r}{mc} (1 - \frac{2\psi}{B_0 r^2})^{1/2}$$

Figure 4.2. Two possibilities of the typical absolute-confinement region in velocity space for negatively charged particles: (a) a cut-off sphere (b) entire sphere.

Figure 4.3. A typical ellipsoid for the closed-field confinement region in velocity space (shown for $q > 0$), where $v_{\theta 1}$ and v_{cl} are defined by Eqs. (4.11) and (4.12), respectively.

by Eq. (3.12), the ratio of the critical speed for the closed-field confinement, v_{c1} , to the critical speed for the absolute confinement, v_c , is given by

$$\frac{v_{c1}}{v_c} = \frac{3r}{4R_s} \left(\frac{R_s^2 - r^2 - \zeta^2}{R_s^2 - r^2} \right)^{1/2} \left(\frac{3}{2} + \frac{R_s^2}{R_s^2 - r^2 - \zeta^2} \right)^{-1/2} \quad (4.16)$$

which is smaller than $3\sqrt{2}/4\sqrt{5}$, since in the closed-field region $r^2 + \zeta^2 < R_s^2$. We also notice the spatial dependence of $v_{\theta 0}$, v_c , $v_{\theta 1}$ and v_{c1} through r and Ψ . That is, the loss region in velocity space is position (of the particle considered) dependent.

4.2 Corresponding Boundary-Value Problem

We have obtained the confinement region in velocity space for both the absolute confinement and closed-field confinement. At present, we are interested in the absolutely confined particles. As shown in Sec. 4.1, the confinement region in velocity space for these particles are either a sphere or a cut-off sphere, depending on the positions of particles. For mathematical simplicity, the cut-off portion is neglected, i.e., the entire confinement sphere is used for particles at all positions. Thus, from Eq. (4.3) the loss boundary in velocity space is given by a spherical surface, defined by

$$v_r^2 + (v_\theta - v_{\theta 0})^2 + v_z^2 = v_c^2, \quad (4.18)$$

where $v_c(r, z)$ is given by Eq. (4.7). This loss boundary affects the velocity distribution of confined particles. Mathematically, it represents a boundary condition in velocity space for the particle distribution

function. With a fast loss of unconfined particles, i.e., all the unconfined particles lost out of the system immediately, the constraints on the particle distribution functions, $f(\underline{r}, \underline{v}, t)$, are $f=0$ on the loss boundary S_v and f finite for all $\underline{v}$. As given by Eq. (4.7), the radius of confinement sphere in velocity space (we call it critical speed), v_c , is spatially dependent, through the dependences on r and Ψ . Figure 4.4 shows the spatial dependence of $v_c(r, z)$ for a Hill's vortex magnetic configuration given by Eq. (3.12). As one can see, the critical speed peaks in the vicinity of the field null ($B=0$) and falls off away from the field null. Especially, the critical speed decreases to zero at the z -axis ($r=0$). We know that the smaller critical speed, thus, the smaller confinement region in velocity space means the larger loss probability. Therefore, we expect that in collisional regime the particle density is smaller near the z -axis (r small) and in the open-field region, compared with the density in the field-null region.

The Fokker-Planck equation describing the time rate of change of f due to collisions is, in general, given by

$$\left(\frac{\partial f_a}{\partial t}\right)_c = \sum_b \Gamma_{ab} \left\{ -\frac{\partial}{\partial v^i} \left(f_a \frac{\partial h_{ab}}{\partial v^i} \right) + \frac{1}{2} \frac{\partial^2}{\partial v^i \partial v^j} \left(f_a \frac{\partial^2 g_b}{\partial v^i \partial v^j} \right) \right\} \quad (4.19)$$

where the summation on the repeated indices i and j are implied,

$$h_{ab} = \frac{m_a + m_b}{m_b} \int d\underline{v}' f_b(\underline{v}') |\underline{v} - \underline{v}'|^{-1}, \quad (4.20)$$

$$g_b = \int d\underline{v}' f_b(\underline{v}') |\underline{v} - \underline{v}'|, \quad (4.21)$$

Figure 4.4: The spatial dependence of the critical speed $v_c(r, z)$ for a Hill's vortex magnetic configuration $\Psi_{\text{HV}}(r, z)$ are given by Eqs. (4.7) and (3.12), respectively. Parameters for this figure, in Gaussian units, are $B_0 = 8000$, $R_s = z_s = b$, $q = 4.8 \times 10^{-10}$, and $m = 3.34 \times 10^{-24}$.

$$\Gamma_{ab} = \frac{4\pi g_a^2 g_b^2}{m_a^2} \ln \Lambda_{ab}, \quad (4.22)$$

$$\ln \Lambda_{ab} = \text{Coulomb Logarithm} = \ln \frac{m_{ab} v_{ab}^2 \lambda_D}{2 g_a g_b}, \quad (4.23)$$

$$\lambda_D = \text{Debye Length} = \left(\frac{T}{4\pi n_e e^2} \right)^{1/2}, \quad (4.24)$$

$$m_{ab} = \frac{m_a m_b}{m_a + m_b}, \quad v_{ab} = |v_a - v_b|, \quad (4.25)$$

n_e and $(-e)$ are the electron number density and electron charge, respectively, and the summation on b in Eq. (4.19) runs over all species. The functions, g and h , are called the Rosenbluth potentials.

Here, our main concern is the velocity-space particle loss due to collisions, i.e., the collision induced particle loss due to the existence of the loss region in velocity space. In particular, we are interested in the confinement of ions. Since the energy and velocity changes of ions due to ion-electron collisions are much less effective than ion-ion collisions because of the small ratio of m_e/m_i , the ion diffusion in velocity space mainly comes from ion-ion collisions. In other words, the electron contribution to the ion diffusion in velocity space is negligible. Furthermore, without losing the generality we consider a two-component plasma, i.e., only one ion species in the plasma. Equation (4.19) is, then, simplified; the summation over all species b is dropped off Eq. (4.19).

However, the equation is still a nonlinear (in f), fourth-order differential, integral equation. Further simplifications will be made as follows.

For convenience, we first shift the origin of the cartesian coordinates (v_r, v_θ, v_z) from $(0,0,0)$ to $(0, v_{\theta 0}, 0)$, i.e.,

$$\left. \begin{aligned} u_1 &= v_r, \\ u_2 &= v_\theta - v_{\theta 0}, \\ u_3 &= v_z, \end{aligned} \right\} \quad (4.26)$$

in a new cartesian velocity coordinates (u_1, u_2, u_3) . The loss boundary in velocity space, Eq. (4.18), becomes

$$u \equiv |u| = v_c(r) \quad (4.27)$$

Noticing that Eq. (4.27) is spatially dependent but isotropic, we further change the velocity coordinates from the shifted cartesian (u_1, u_2, u_3) to spherical (v, θ_v, ϕ_v) by

$$\left. \begin{aligned} v &= (u_1^2 + u_2^2 + u_3^2)^{1/2}, \\ \theta_v &= \tan^{-1} \left(\frac{u_2}{u_1} \right), \\ \phi_v &= \cos^{-1} \left(\frac{u_3}{v} \right). \end{aligned} \right\} \quad (4.28)$$

The loss boundary then becomes

$$v \equiv |\underline{v}| = v_c(\underline{r}), \quad (4.29)$$

and, the constraints on the distribution function are

$$\left. \begin{aligned} f(\underline{r}, \underline{v}, t) &= 0 \quad \text{for } v \geq v_c(\underline{r}), \\ f &\text{ finite for all } \underline{v}. \end{aligned} \right\} \quad (4.30)$$

In a collisional equilibrium, the governing equation, Eq. (4.19) together with the isotropic boundary condition, Eq. (4.30), imply that f must be isotropic, i.e., $f = f(\underline{r}, v, t)$.

After the above manipulations, Eq. (4.19) is simplified to

$$\begin{aligned} \left(\frac{\partial f}{\partial t} \right)_c &= \nabla \cdot \left\{ \left[-\frac{2}{v} \frac{\partial h}{\partial v} - \frac{\partial^2 h}{\partial v^2} + \frac{2}{v} \frac{\partial^3 g}{\partial v^3} \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \frac{\partial^4 g}{\partial v^4} \right] f + \left[-\frac{\partial h}{\partial v} \right. \right. \\ &\quad \left. \left. - \frac{1}{v^2} \frac{\partial g}{\partial v} + \frac{2}{v} \frac{\partial^2 g}{\partial v^2} + \frac{\partial^3 g}{\partial v^3} \right] \right. \\ &\quad \left. \frac{\partial f}{\partial v} + \left[\frac{1}{2} \frac{\partial^2 g}{\partial v^2} \right] \frac{\partial^2 f}{\partial v^2} \right\}, \end{aligned} \quad (4.31.1)$$

where

$$g = 4\pi \left\{ \int_0^v f(r, v', t) v \left(1 + \frac{v'^2}{3v^2}\right) v'^2 dv' + \int_v^\infty f(r, v', t) \left(1 + \frac{v^2}{3v'^2}\right) v'^3 dv' \right\}, \quad (4.32)$$

$$h = 8\pi \left\{ \int_0^v f(r, v', t) \frac{v'^2}{v} dv' + \int_v^\infty f(r, v', t) v' dv' \right\}, \quad (4.33)$$

$$\Gamma = \frac{4\pi g^4 \ln \Lambda}{m^2} \quad (4.34)$$

Equation (4.31.1) can also be written as

$$\left(\frac{\partial f}{\partial t}\right)_c = \Gamma \left\{ 4\pi f + \frac{1}{v^2} \frac{\partial g}{\partial v} \frac{\partial}{\partial v} + \frac{1}{2} \frac{\partial^2 g}{\partial v^2} \frac{\partial^2}{\partial v^2} \right\} f, \quad (4.31.2)$$

here, instead of using both g and h , only g is explicitly used.

Before making further simplifications, we introduce the following model distribution function, $f_0(r, v, t)$, given by

$$f_0(r, v, t) = n(r, z, t) C_1(r, z) \left\{ 1 - \frac{v^2}{v_c^2(r, z)} \right\} \cdot U_s(v_c - v) \exp \left\{ - \frac{v^2}{\beta(r, z) v_c^2(r, z)} \right\}, \quad (4.35)$$

where $n(r, z, t)$ is the number density, $U_s(x)$ is the heavy-side unit step function,

$$U_s(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ 0 & \text{for } x < 0, \end{cases} \quad (4.36)$$

$C_1(r, z)$ is the local normalization constant,

$$C_1(r, z) = \left\{ \int_0^{v_c(r, z)} 4\pi v^2 \left(1 - \frac{v^2}{v_c^2}\right) e^{-\frac{v^2}{\beta v_c^2}} dv \right\}^{-1}, \quad (4.37)$$

$v_c(r, z)$ is given by Eq. (4.7), and $\beta(r, z)$ is a parameter describing the thermal speed of the model distribution function, i.e., a parameter related to the local temperature $T(r, z)$. We notice that the model distribution function, f_0 , is chosen such that (i) the constraints on the distribution function, Eq. (4.30), are satisfied, (ii) the exponential behavior which is an important characteristic of the collisional equilibrium distribution in the case where no loss region exists in velocity space, i.e., the Maxwellian distribution, is retained, (iii) the normalization $n(r, z, t) = \int f_0 dv$ is satisfied, and (iv) the spatial variations of the number density and temperature are allowed.

We now linearize Eq. (4.31) by substituting the model distribution function for f in evaluating the Rosenbluth potentials, g and h . This linearizing process is shown later to be a valid procedure. After the linearization, we further change the velocity variable v to a dimensionless velocity v' by

$$\tilde{v}'(r, z, v) \equiv \frac{v}{v_c(r, z)} \quad (4.38)$$

This change of velocity variable is not only for convenience but also essential in simplifying the problem, as it will become evident later.

After all the above manipulations and simplifications, we finally obtain the Fokker-Planck equation and the boundary condition imposed by the loss region in velocity space as follows.

$$\left(\frac{\partial F(r, z, v', t)}{\partial t} \right)_c = \frac{\Gamma n(r, z, t)}{v_c^3(r, z)} D\{F\}, \quad (4.39)$$

with conditions

$$F(r, z, v', t) = 0 \quad \text{for } v' \geq 1, \quad (4.40)$$

$$F \text{ finite for } 0 \leq v' \leq 1, \quad (4.41)$$

where

$$F(r, z, v', t) \equiv f(r, z, v = v' v_c, t), \quad (4.42)$$

$D(v', \beta)$ = a second-order differential operator

$$\begin{aligned} &\equiv \left[-\frac{2}{v'} \frac{dH}{dv'} - \frac{d^2 H}{dv'^2} + \frac{2}{v'} \frac{d^3 G}{dv'^3} + \frac{1}{2} \frac{d^4 G}{dv'^4} \right] \\ &+ \left[-\frac{dH}{dv'} - \frac{1}{v'^2} \frac{dG}{dv'} + \frac{2}{v'} \frac{d^2 G}{dv'^2} + \frac{d^3 G}{dv'^3} \right] \frac{\partial}{\partial v'}, \\ &+ \left[\frac{1}{2} \frac{d^2 G}{dv'^2} \right] \frac{\partial^2}{\partial v'^2} \end{aligned} \quad (4.43.1)$$

$$= 4\pi V_o(v') + \frac{1}{v'^2} \frac{dG}{dv'} \frac{\partial}{\partial v'} + \frac{1}{2} \frac{d^2 G}{dv'^2} \frac{\partial^2}{\partial v'^2} \quad (4.43.2)$$

$$V_0(v', \beta) = C_2(1-v'^2) U_5(1-v') \exp\left(-\frac{v'^2}{\beta}\right), \quad (4.44)$$

$$C_2(\beta) = \left\{ \int_0^1 4\pi u^2 (1-u^2) e^{-\frac{u^2}{\beta}} du \right\}^{-1}, \quad (4.45)$$

$$G(v', \beta) = 4\pi C_2 \left\{ v' \int_0^{v'} (1-u^2) e^{-\frac{u^2}{\beta}} \left(1 + \frac{u^2}{3v'^2}\right) u^3 du \right. \\ \left. + \int_{v'}^1 (1-u^2) e^{-\frac{u^2}{\beta}} \left(1 + \frac{v'^2}{3u^2}\right) u^3 du \right\}, \quad (4.46)$$

$$H(v', \beta) = 8\pi C_2 \left\{ \frac{1}{v'} \int_0^{v'} (1-u^2) e^{-\frac{u^2}{\beta}} u^2 du \right. \\ \left. + \int_{v'}^1 (1-u^2) e^{-\frac{u^2}{\beta}} u du \right\}, \quad (4.47)$$

Γ and $v_c(r, z)$ are given by Eqs. (4.34) and (4.7), respectively. We also obtain the following useful relations:

$$C_1(r, z, \beta) = \frac{C_2(\beta)}{v_c^3(r, z)}, \quad (4.48)$$

and

$$F_0(r, z, v', \beta, t) \equiv f_0(r, z, v = v' v_c, \beta, t) = \frac{n(r, z, t)}{v_c^3(r, z)} V_0(v', \beta) \quad (4.49)$$

As Eq. (4.39) shows, the Fokker-Planck collisional operator is separated into two factors: an algebraic factor $\Gamma n/v_c^3$ and a differential operator D . The operator D , given by Eq. (4.43), is a linear second-order differential operator (instead of the original fourth-order differential integral operator), and depends only on the dimensionless variable v' and the parameter β .

Equations (4.39) - (4.41) govern the evolution of the distribution function. Since Eq. (4.39) is a diffusion-type equation, i.e., with first-order time derivative and second-order "spatial" (in velocity space) derivative, this set of equations describe a diffusion problem with a boundary condition. It is, therefore, very useful to study the following boundary-value problem:

$$D \{ V(v') \} = \lambda V(v') \quad \text{for } 0 \leq v' \leq 1, \quad (4.50)$$

$$V(1) = 0, \quad (4.51)$$

$$V(v') \text{ finite for all } 0 \leq v' \leq 1. \quad (4.52)$$

i.e., to obtain the eigenfunctions, V 's, and corresponding eigenvalues, λ 's, of the operator D subject to the same boundary condition.

For convenience, unless it is specified, we omit the prime notation of v' in the following through this chapter. Carrying out the differentiations in Eq. (4.43.2), we obtain

$$D(v, \beta) = A(v, \beta) \frac{\partial^2}{\partial v^2} + B(v, \beta) \frac{\partial}{\partial v} + C(v, \beta), \quad (4.53)$$

where

$$A(v, \beta) = \frac{4\pi}{3v^3} \left\{ \int_0^v V_0(u, \beta) u^4 du + v^3 \int_v^1 V_0(u, \beta) u du \right\}, \quad (4.54)$$

$$B(v, \beta) = \frac{4\pi}{3v^3} \left\{ \int_0^v \left(3v - \frac{u^2}{v} \right) u^2 V_0(u, \beta) du + 2v^2 \int_v^1 V_0(u, \beta) u du \right\}, \quad (4.55)$$

$$C(v, \beta) = 4\pi V_0(v, \beta), \quad (4.56)$$

and $V_0(v, \beta)$ is given by Eq. (4.44). We may also carry out the integrations in Eqs. (4.54) - (4.55) in terms of the error function, $\text{erf}(x)$, defined as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad (4.57)$$

and the exponential function. Then we have,

$$\begin{aligned} A(v, \beta) = \frac{4\pi}{3} C_2(\beta) \left\{ \frac{3\sqrt{\pi}}{8} \beta^{5/2} \left(1 - \frac{5}{2} \beta \right) \frac{1}{v^3} \text{erf} \left(\frac{v}{\sqrt{\beta}} \right) \right. \\ \left. + \left[\frac{3}{4} \beta^2 \left(\frac{5\beta}{2} - 1 \right) \frac{1}{v^2} + \frac{15}{4} \beta^{5/2} - \frac{\beta^2}{2} \right. \right. \\ \left. \left. + \beta + \frac{\beta^2}{2} e^{-\frac{1}{\beta}} \right] e^{-\frac{v^2}{\beta}} \right\}, \quad (4.58) \end{aligned}$$

$$\begin{aligned}
B(v, \beta) = \frac{4\pi}{3} C_2(\beta) \left\{ \frac{\sqrt{\pi}}{8} \left[(6-9\beta) \beta^{3/2} \frac{1}{v^2} - \left(3-\frac{15}{2}\beta\right) \right. \right. \\
\left. \cdot \beta^{5/2} \frac{1}{v^4} \right] \operatorname{erf}\left(\frac{v}{\sqrt{\beta}}\right) + \left[\frac{3}{4} \beta^2 \left(1-\frac{5}{2}\beta\right) \frac{1}{v^3} \right. \\
\left. \left. + \left(\frac{\beta}{2} - \frac{\beta^2}{4} - \frac{15}{4} \beta^{5/2} + \beta^2 e^{-\frac{1}{\beta}}\right) \frac{1}{v} \right] e^{-\frac{v^2}{\beta}} \right\}, \quad (4.59)
\end{aligned}$$

$$C(v, \beta) = 4\pi C_2(\beta) (1-v^2) e^{-\frac{v^2}{\beta}}, \quad (4.60)$$

and $C_2(\beta)$ is given by Eq. (4.45).

The exact solutions for the boundary-value problem described by Eqs. (4.50) - (4.52) is very difficult to obtain analytically, since a special function, $\operatorname{erf}(v/\sqrt{\beta})$, appears in the coefficients of the operator D . It is, therefore, solved numerically as follows.

Using the central-difference scheme of the finite difference method,⁴⁵ Eq. (4.50) becomes

$$\begin{aligned}
(2C_n + hB_n) V_{n+1} + (2h^2 A_n - 4C_n) V_n \\
+ (2C_n - hB_n) V_{n-1} = 2h^2 \lambda V_n \quad \text{for } 1 \leq n \leq m-1, \quad (4.61)
\end{aligned}$$

with $V_m = 0$ and $V_0 = 2V_1 - V_2$, where m = number of intervals, $h = 1/m$ = width of each interval.

$$v_n = nh \quad \text{for } 0 \leq n \leq m, \quad (4.62)$$

$$A_n = A(v_n), B_n = B(v_n), C_n = C(v_n), V_n = V(v_n). \quad (4.63)$$

Here, for convenience the dependence on the parameter β is not explicitly written out for A , B , C and V in Eq. (4.63), but we keep in mind their dependences on β . The end condition, $V_m = 0$, is given by the boundary condition $V(1) = 0$, and the other end condition, $V_0 = 2V_1 - V_2$, is obtained by linear extrapolation of the solutions. The extrapolation to $v = v_0$ is necessary for the numerical solutions, as we can see by setting $n = 1$ in Eq. (4.61). Since we know that the eigenfunctions (the distribution functions) are well behaved, this linear extrapolation is expected to be a good approximation, as proved later by the solutions obtained. Eq. (4.61) can be rewritten in a matrix form which is more suitable for computer work. It, then, reads as follows.

$$M_{(m-1) \times (m-1)} V_{(m-1) \times 1} = \lambda V_{(m-1) \times 1}, \quad (4.64)$$

where the matrices $M_{(m-1) \times (m-1)}$ and $V_{(m-1) \times 1}$ are given by Eqs. (4.65) and (4.66), respectively:

$$V_{(m-1) \times 1} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{m-1} \end{bmatrix}. \quad (4.66)$$

$$M_{(m-1) \times (m-1)} = \frac{1}{2h^2}$$

$$\begin{bmatrix} 2h^2 A_1 - 2hB_1 & 2hB_1 & 0 & \dots & 0 \\ 2C_2 - hB_2 & 2h^2 A_2 - 4C_2 & 2C_2 + hB_2 & \dots & 0 \\ 2C_3 - hB_3 & 2h^2 A_3 - 4C_3 & 2C_3 + hB_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2C_{m-2} - hB_{m-2} & 2h^2 A_{m-2} - 4C_{m-2} & 2C_{m-2} + hB_{m-2} & \dots & 0 \\ 2C_{m-1} - hB_{m-1} & 2h^2 A_{m-1} - 4C_{m-1} & 2C_{m-1} + hB_{m-1} & \dots & 0 \end{bmatrix}$$

(4.65)

TABLE 4.1 Comparisons of the largest fifteen eigenvalues of Eq. (4.61) for $m = 50$ and $m = 100$, where m is the number of intervals, $\beta = 1$ in both cases.

mode number, k	k-th eigenvalue, λ_k	
	$m = 50$	$m = 100$
1	-0.5124	-0.5140
2	-11.73	-11.76
3	-29.44	-29.56
4	-54.24	-54.55
5	-85.93	-86.70
6	-124.4	-125.8
7	-169.3	-172.1
8	-220.5	-225.3
9	-277.9	-285.4
10	-340.9	-352.3
11	-409.3	-425.9
12	-482.6	-506.0
13	-560.6	-593.0
14	-642.6	-686.0
15	-728.4	-785.6

The matrix eigenvalue problem, Eq. (4.64), is solved numerically. As one may expect, for a smaller h (larger m), more eigenvectors are obtained, and higher accuracy for the solutions. As $h \rightarrow 0$ ($m \rightarrow \infty$), infinite eigenvectors with corresponding eigenvalues for the matrix equation, Eq. (4.61), are expected. To assure that satisfactory results are obtained, Eq. (4.64) is solved for two cases: $m = 50$ and $m = 100$. Results are compared. Table 4.1 compares the largest fifteen eigenvalues obtained in these two cases where $\beta = 1$ has been chosen. For the first five modes ($1 \leq k \leq 5$), the difference between the two cases is $< 0.5\%$; for the first ten modes ($1 \leq k \leq 10$), it is $< 3\%$. Since we are mainly interested in the lower modes (small k 's), these comparisons show that the eigenvalues obtained for $m = 50$ or $m = 100$ are different from those obtained for $m \rightarrow \infty$ for less than a few percents. Therefore, we know that, in numerically solving Eq. (4.61), $m = 100$ indeed gives satisfactorily accurate results. In the following, all results are obtained for $m = 100$.

Figure 4.5 shows the first four eigenfunctions of Eq. (4.50), V_k , $k = 1, 2, 3, 4$ (i.e., eigenfunctions with the largest four eigenvalues), for the case $\beta = 0.1$. The corresponding eigenvalues are $\lambda_1 = -9.150 \times 10^{-4}$, $\lambda_2 = -10.57$, $\lambda_3 = -20.00$ and $\lambda_4 = -33.96$. Fig. 4.6 shows the fifth to eighth eigenfunctions with corresponding eigenvalues $\lambda_5 = -52.30$, $\lambda_6 = -74.85$, $\lambda_7 = -101.6$ and $\lambda_8 = -132.4$, also for the $\beta = 0.1$ case. Similarly, Figs. 4.7 and 4.8 show the first four and the fifth to eighth eigenfunctions, respectively, for $\beta = 1.0$. Fig. 4.9 plots the eigenvalues λ_k versus the mode number k for the lowest 35 modes for $\beta = 0.1$ and $\beta = 1.0$. From these figures, we notice the following general characteristics of the eigenfunctions and eigenvalues of Eq. (4.50).⁽¹⁾

Figure 4.5. The first four eigenfunctions of Eq. (4.50) for the case $\beta = 0.1$. The corresponding eigenvalues are $\lambda_1 = -9.150 \times 10^{-4}$, $\lambda_2 = -1.057 \times 10^1$, $\lambda_3 = -2.000 \times 10^1$, and $\lambda_4 = -3.396 \times 10^1$.

Figure 4.6. The fifth to eighth eigenfunctions of Eq. (4.50) for the case $\beta = 0.1$. The corresponding eigenvalues are $\lambda_5 = -5.230 \times 10^1$, $\lambda_6 = -7.485 \times 10^1$, $\lambda_7 = -1.016 \times 10^2$, and $\lambda_8 = -1.324 \times 10^2$.

Figure 4.7. The first four eigenfunctions of Eq. (4.50) for the case $\beta = 1.0$. The corresponding eigenvalues are $\lambda_1 = -5.140 \times 10^{-1}$, $\lambda_2 = -1.176 \times 10^1$, $\lambda_3 = -2.956 \times 10^1$, and $\lambda_4 = -5.455 \times 10^1$.

Figure 4.8. The fifth to eighth eigenfunctions of Eq. (4.50) for the case $\beta = 1.0$. The corresponding eigenvalues are $\lambda_5 = -8.670 \times 10^1$, $\lambda_6 = -1.258 \times 10^2$, $\lambda_7 = -1.721 \times 10^2$, and $\lambda_8 = -2.253 \times 10^2$.

Figure 4.9. Eigenvalues versus mode numbers for the lowest 35 modes for $\beta = 0.1$ and $\beta = 1.0$.

The eigenfunctions behave like damped oscillators: the amplitude decreasing with increasing v and the frequency increasing with the mode number.(2)

For the k -th mode, there are k zeroes for $V_k(v) = 0$.(3) For the same mode, the larger β , the smaller eigenvalue. Namely, for any mode k , if

$\beta_1 > \beta_2$, then the eigenvalues for the two cases have the relation

$\lambda_k(\text{for } \beta_1) < \lambda_k(\text{for } \beta_2)$. Although only the results for $\beta = 0.1$

and $\beta = 1.0$ are shown here, the above characteristics of the solutions of the eigen-problem described by Eqs. (4.50) - (4.52) are quite general.

Results for other β -values behave the same way, which are not shown here.

In addition to the numerical solutions, power-series solutions of the eigenvalue problem are also sought. We expand Eq. (4.44) in a power series:

$$V_0(v) = \sum_{j=0}^{\infty} \frac{C_2 (-1)^j}{j! \beta^j} \left[v^{2j} - v^{2(j+1)} \right] U_5(1-v) \quad (4.67)$$

Substituting this into Eqs. (4.53) - (4.56) and carrying out the integrations, we find that Eq. (4.50) can be written as:

$$R(v) \frac{\partial^2 V}{\partial v^2} + \frac{1}{v} P(v) \frac{\partial V}{\partial v} + \frac{1}{v^2} Q(v) V = 0, \quad (4.68)$$

where

$$R(v) \equiv \sum_{j=0}^{\infty} R_j v^j = 1 - \frac{5}{14} v^2 + \sum_{j=1}^{\infty} \frac{10}{3} \frac{(-1)^{j+1}}{j! \beta^j} \cdot \left[\left(\frac{1}{2j+5} - \frac{1}{2j+2} \right) v^{2j} + \left(\frac{1}{2j+4} - \frac{1}{2j+7} \right) v^{2j+2} \right], \quad (4.69)$$

$$\begin{aligned}
P(v) &\equiv \sum_{j=0}^{\infty} P_j v^j \\
&= \frac{2}{3} - \frac{v^2}{7} + \sum_{j=1}^{\infty} \frac{10}{3} \cdot \frac{(-1)^{j+1}}{j! \beta^j} \left[\left(\frac{3}{2j+3} \right. \right. \\
&\quad \left. \left. - \frac{1}{2j+5} - \frac{2}{2j+2} \right) v^{2j} + \left(\frac{1}{2j+7} \right. \right. \\
&\quad \left. \left. - \frac{3}{2j+5} + \frac{2}{2j+4} \right) v^{2j+2} \right], \tag{4.70}
\end{aligned}$$

and

$$\begin{aligned}
Q(v) &\equiv \sum_{j=0}^{\infty} Q_j v^j \\
&= \left(\frac{5\lambda}{2\pi C_2} - 10 \right) + 10 v^2 + \sum_{j=1}^{\infty} \frac{10 (-1)^{j+1}}{j! \beta^j} \\
&\quad \cdot (v^{2j} - v^{2j+2}) \tag{4.71}
\end{aligned}$$

Before proceeding further, attention is called to the fact that $v = 0$ is a regular singular point of differential equation, Eq. (4.50) or Eq. (4.68). This can be easily seen from Eq. (4.68), as the equation is purposely put into standard form. Of course, it can also be shown by taking $v \rightarrow 0$ in Eq. (4.50) together with Eqs. (4.53) - (4.56).

Since $v = 0$ is a regular singular point, we have to expand

$$V(v) = \sum_{k=0}^{\infty} a_k v^{k+s}, \tag{4.62}$$

where s is a constant to be determined. Following procedures of the Frobenius method,⁴⁶ we obtain two possible values for s :

$$s = s_1 = \frac{1}{6} \left\{ 1 + \left[1 + 90 \left(-\frac{\lambda}{\pi C_2} + 4 \right) \right]^{1/2} \right\}, \quad (4.73)$$

or

$$s = s_2 = \frac{1}{6} \left\{ 1 - \left[1 + 90 \left(-\frac{\lambda}{\pi C_2} + 4 \right) \right]^{1/2} \right\}. \quad (4.74)$$

and the coefficients a_k 's are given by

$$a_k = -\frac{1}{f(s+k)} \sum_{n=1}^k g_n(s+k) a_{k-n} \text{ for } k \geq 1, \quad (4.75)$$

where

$$f(x) = x^2 + (P_0 - 1)x + Q_0, \quad (4.76)$$

$$g_n(x) = R_n(x-n)^2 + (P_n - R_n)(x-n) + Q_n, \quad (4.77)$$

P_j, Q_j, R_j are defined by Eqs. (4.69) - (4.71), respectively, and a_0 is an arbitrary constant. However, for our problem we find that $s = s_2$ is not an eligible solution. The desired solution is given by $s = s_1$ only. This is explained in the following.

Since the eigenfunctions are the normal velocity distributions of particles under collisions, the eigenfunctions must be regular, i.e., no singularity. In addition to this, as Eq. (4.39) shows, a positive eigenvalue for the operator D means that the collisions result in a net increase of total number of particles, while a zero eigenvalue implies the constancy of total number of particles. In our case, a loss region in velocity space exists and no particles are generated as a result of collisions; hence, collisions always result in particle losses. Therefore, both positive and zero eigenvalues are impossible. The eigenvalues must be negative, i.e., $\lambda < 0$. We, therefore, know that for our problem the eigenfunctions must be regular and the eigenvalues must be negative. From Eqs. (4.73) and (4.74), we have $s_1 > 0$ and $s_2 < 0$ since $\lambda < 0$. The negativeness of s , i.e., $s = s_2$, gives eigenfunctions with a singularity at $v = 0$, as can be seen from Eq. (4.72). Therefore, the eligible solutions are given by $s = s_1$ only, i.e.,

$$V(v) = \sum_{k=0}^{\infty} a_k v^{-k+s_1}, \quad (4.78)$$

where

$$a_k = -\frac{1}{f(s_1+k)} \sum_{n=1}^k g_n(s_1+k) a_{k-n}, \quad k \geq 1, \quad (4.79)$$

a_0 is an arbitrary constant, $f(x)$, $g_n(x)$ and s_1 are given by Eqs. (4.76), (4.77) and (4.73), respectively. Here, the a_k 's depend on λ through s_1 . The eigenvalues, λ 's, are determined by the boundary condition $V(1) = 0$,

i.e., by

$$\sum_{k=0}^{\infty} a_k(\lambda) = 0. \quad (4.80)$$

In many situations, the power-series form, Eqs. (4.78), may be desired because of its easy manipulations in integrations and differentiations. Eqs. (4.78) and (4.79) are especially convenient when specific eigenvalues are known and the corresponding eigenfunctions are desired.

4.3 Collisional Particle Loss Rate

We have studied the loss boundary in velocity space and the characteristics of the collisional operator subject to the loss boundary. We now calculate the particle loss rate, $\partial n / \partial t$, due to collisions.

Substituting Eq. (4.50) into Eq. (4.39) and integrating Eq. (4.39) over the whole velocity space, we obtain the collisional particle loss rate for the k -th eigen velocity distribution, i.e., for $F(r, z, v, t) = n(r, z, t) V_k(v) / v_c^3(r, z)$, where F is defined by Eq. (4.40), $V_k(v)$ is the normalized k -th eigenfunction and n is the number density,

$$\left(\frac{\partial n(r, z, t)}{\partial t} \right)_c = \lambda_k \Gamma \frac{n^2(r, z, t)}{v_c^3(r, z)} \quad (4.81)$$

Here, λ_k is the eigen value of the k -th mode, Γ and v_c are given by Eqs. (4.34) and (4.7), respectively. From this equation, we see that, for the more negative λ_k , the particle loss rate is larger. That is, the more negative eigenvalue, the faster particle loss, thus, the faster decay of the

corresponding mode. Therefore, in a collisional equilibrium, i.e., for $t \gg \tau_c$ where τ_c is the characteristic collisional time scale for the species under consideration, the dominant velocity distribution is the fundamental mode, $V_1(v)$, which has the least negative eigenvalue λ_1 . In other words, in collisional equilibrium, the particle distribution function, $F(r, z, v, t)$ should have $F \approx n(r, z, t) V_1(v) / v_c^3(r, z)$. Here, as a reminder, we recall that v is the dimensionless velocity variable normalized by the local critical speed $v_c(r, z)$.

As shown in the previous section, in solving the loss-boundary problem and obtaining the eigenfunctions and eigenvalues, a linearization of the Fokker-Planck equation has been made. The model velocity distribution has been used to evaluate the Rosenbluth potentials and has simplified the problem appreciably. Now, we want to justify our previous linearizing process, i.e., the use of the model velocity distribution $V_0(v)$. Since we are concerned about the particle loss in collisional equilibrium, as shown above, the dominant velocity distribution we obtain from the linearized problem is the fundamental mode, $V_1(v)$. Therefore, to justify the linearization we have to show that $V_0(v)$ is not significantly different from $V_1(v)$. Figure 4.10 compares $V_0(v)$ with $V_1(v)$ for the case of $\beta = 0.1$, where both are normalized to unity at $v = 0$. Figure 4.11 compares $4\pi v^2 V_0(v)$ with $4\pi v^2 V_1(v)$ also for $\beta = 0.1$, where the normalizations,

$$\int_0^1 4\pi v^2 V_0(v) dv = \int_0^1 4\pi v^2 V_1(v) dv = 1, \text{ have been made. In this } \beta\text{-small case,}$$

$V_0(v)$ and $V_1(v)$ are almost exactly identical; no difference can be seen in Figs. 4.10 and 4.11. For other β -values, Figs. 4.12 - 4.17 show the

Figure 4.10. A comparison between the model velocity distribution, $V_0(v)$, and the fundamental mode of eigenfunctions, $V_1(v)$, for $\beta = 0.1$, where v is the dimensionless speed (normalized by v_c), and the normalizations of $V_0(0) = V_1(0) = 1$ have been made.

Figure 4.11. A comparison between $4\pi v^2 V_0(v)$ and $4\pi v^2 V_1(v)$ for $\beta = 0.1$, where v is the dimensionless speed (normalized by v_c), and the normalizations of $\int_0^1 4\pi v^2 V_0(v) dv = \int_0^1 4\pi v^2 V_1(v) dv = 1$ have been made.

Figure 4.12. A comparison between $V_0(v)$ and $V_1(v)$, similar to Fig. 4.10 except for $\beta = 0.5$.

Figure 4.13. A comparison between $4\pi v^2 V_0(v)$ and $4\pi v^2 V_1(v)$, similar to Fig. 4.11 except for $\beta = 0.5$.

Figure 4.14. A comparison between $V_0(v)$ and $V_1(v)$, similar to Fig. 4.10 except for $\beta = 1.0$.

Figure 4.15. A comparison between $4\pi v^2 V_0(v)$ and $4\pi v^2 V_1(v)$, similar to Fig. 4.11 except for $\beta = 1.0$.

Figure 4.16. A comparison of $V_0(v)$ and $V_1(v)$, similar to Fig. 4.10 except for $\beta = 5.0$.

Figure 4.17. A comparison between $4\pi v^2 V_0(v)$ and $4\pi v^2 V_1(v)$, similar to Fig. 4.11 except for $\beta = 5.0$.

similar comparisons for $\beta = 0.5, 1.0$ and 5.0 . From these figures, the following conclusion can be drawn: (1) For smaller β -values, the difference between V_0 and V_1 , or, between $4\pi v^2 V_0$ and $4\pi v^2 V_1$ is smaller. (2) In a wide range of β -values, the comparison between V_0 and V_1 is favorable. (3) In collisional equilibrium, the simple analytic model velocity distribution, $V_0(v)$, given by Eq. (4.44) is a good representation of the actual velocity distribution which is expected to be between V_0 and V_1 . (4) The use of the model distribution function, f_0 , given by Eq. (4.35) in linearizing the Fokker-Planck equation is valid.

Since the equilibrium distribution, $V_{eq}(v)$, is expected to be between $V_0(v)$ and $V_1(v)$, we calculate the particle loss rate at both $V_{eq}(v) = V_0(v)$ and $V_{eq}(v) = V_1(v)$. For $V_{eq}(v) = V_1(v)$, the particle loss rate is given by Eq. (4.78) with λ_k equal to λ_1 . Table 4.2 gives the eigenvalue of the fundamental mode, λ_1 , for some typical β -values.

TABLE 4.2 The eigenvalue of the fundamental mode, λ_1 , for different β -values.

λ_1	β
-9.15×10^{-4}	0.1
-3.16×10^{-1}	0.5
-5.14×10^{-1}	1.0
-6.67×10^{-1}	2.5
-7.22×10^{-1}	5.0

In calculating the particle loss rate, let us go back to the usual velocity variable for a moment, i.e., in Eqs. (4.82) - (4.87) the variable v is the usual dimensional velocity variable (not normalized by v_c). We first rewrite the Fokker-Planck equation, Eq. (4.31), in the following form:

$$\left(\frac{\partial f}{\partial t} \right)_c = \nabla_{\underline{v}} \cdot (\gamma \nabla_{\underline{v}} f + \alpha \underline{v} f), \quad (4.82)$$

where

$$\gamma = \frac{\Gamma}{2} \frac{\partial^2 g}{\partial v^2}, \quad (4.83)$$

$$\alpha = \frac{\Gamma}{v^3} \left(\frac{v^2}{2} \frac{\partial^3 g}{\partial v^3} + v \frac{\partial^2 g}{\partial v^2} - \frac{\partial g}{\partial v} - v^2 \frac{\partial h}{\partial v} \right). \quad (4.84)$$

Then, integrating Eq. (4.79) over the velocity space, and making use of the divergence theorem, we obtain

$$\left(\frac{\partial n(r, z, t)}{\partial t} \right)_c = \left\{ 4\pi v^2 \left(\gamma \frac{\partial f}{\partial v} + \alpha v f \right) \right\}_{v=v_c}. \quad (4.85)$$

Evaluating the right-hand side of Eq. (4.85) for $f = f_0$, where the model distribution function $f_0(r, z, v, t)$ is given by Eq. (4.35), we obtain

$$\left(\frac{\partial n(r, z, t)}{\partial t} \right)_c = -\frac{32}{3} \pi^2 \Gamma C_2^2 C_3 e^{-\frac{1}{\beta}} \frac{n^2(r, z, t)}{v_c^3(r, z)}, \quad (4.86)$$

where

$$C_3(\beta) = \int_0^1 (1-u^2) u^4 e^{-\frac{u^2}{\beta}} du, \quad (4.87)$$

Γ , C_2 and v_c are given by Eqs. (4.34), (4.45) and (4.7), respectively.

Equation (4.86) gives the particle loss rate for $f = f_0$.

As expected, Eqs. (4.81) and (4.86) have the same spatial dependence n^2/v_c^3 . The dependence of the partial loss rate on n^2 comes from the fact that the collisional frequency is proportional to n^2 . The dependence on v_c^{-3} comes from the fact that the confinement probability of colliding particles is proportional to the volume ($\propto v_c^3$) of the confinement sphere in velocity space. Let us come back to the previous dimensionless velocity variable (with the prime notation omitted). We can combine these two cases in the following form:

$$\left(\frac{\partial \eta(r, z, t)}{\partial t} \right)_c = - C_4 \frac{\eta^2(r, z, t)}{v_c^3(r, z)}, \quad (4.88)$$

where

$$C_4 = \begin{cases} -\lambda_1 \Gamma & \text{if } V_{ef}(v) = V_i(v), \end{cases} \quad (4.89.1)$$

$$\begin{cases} \frac{32}{3} \pi^2 \Gamma C_2^2 C_3 e^{-\frac{1}{\beta}} & \text{if } V_{ef}(v) = V_o(v). \end{cases} \quad (4.89.2)$$

Note that C_4 is always positive since λ_k is negative for all k .

As mentioned before, the parameter β describes the thermal spread of the model distribution function f_0 ; hence, β is related to the temperature. As usual, the temperature T is defined by

$$T \equiv \frac{m}{3} \langle \underline{v}_{rm} \cdot \underline{v}_{rm} \rangle, \quad (4.90)$$

where $\underline{v}_{rm}$ is the random velocity, i.e., the velocity relative to the local average velocity (i.e., the fluid velocity). Using this definition, we obtain the temperature of the model distribution function, T_0 , given as follows.

$$T_0 = \frac{4\pi}{3} C_2 C_3 m v_c^2. \quad (4.91)$$

Thus, T_0 is related to β through C_2 and C_3 . It may be worth mentioning that, in addition to v_c , β can be a function of position, too. For example, for $\beta = 1.0$, we have $C_2 = 0.8921$, $C_3 = 0.03354$, thus,

$$T_0 = 0.125 m v_c^2 \quad \text{for } \beta = 1.0. \quad (4.92)$$

For deuterium, $m = 2m_p$ (m_p = mass of a proton) and $q = +e$, in a Hill's vortex magnetic configuration, Ψ_{HV} , with $B_0 = 5$ kG and $R_S = z_S = 4$ cm, the maximum critical speed (at the field null) is 8.5×10^7 cm/s which gives $T_0 = 1.83$ keV by Eq. (4.92). If for the same case but $\beta = 0.1$ ($C_2 = 6.681$, $C_3 = 0.001577$), then $T_0 = 0.644$ keV. We notice that, although the temperatures in these two examples are not very high (in the order of 1 keV), they do not imply the particle velocity (or energy) small.

small. Low temperatures only imply the average random velocity small. If the average velocity is large, the particle velocity (or energy) can be high for small temperatures.

As Eq. (4.88) shows, the particle loss rate depends on β . For $V_{eq}(v) = V_0(v)$ and, as usual, taking $\ln \Lambda$ as a constant, the β -dependence of Eq. (4.88) becomes

$$\left(\frac{\partial n(r, z, t)}{\partial t} \right)_c \propto C_2^2 C_3 e^{-\frac{1}{\beta}} \quad (4.93)$$

Figure 4.18 shows the β -dependence of the collisional particle loss rate. For larger β -values, the loss rates are larger. This is because there are more high-energy particles which have higher loss probability (in considering VSPL) when suffering collisions. Although the Coulomb cross section decreases for higher energy, the net result from the increase of β -value is the increase of VSPL rate, as shown by Fig. 4.18.

The local particle confinement time, τ_p , is defined by $\tau_p = -n/(\partial n/\partial t)$. From Eq. (4.88), this is given by

$$\tau_p = \frac{v_c^3}{C_4 n} \quad (4.94)$$

For example, for $V_{eq} = V_0$, $m = 3.34 \times 10^{-24}$, $q = 4.8 \times 10^{-10}$, $N = 2 \times 10^{15}$, $B_0 = 5 \times 10^3$ and $R_s = 4$ (all in Gaussian units), the local particle confinement time at the field null is $350 \mu s$ if $\beta = 1.0$ and $1.1 s$ if $\beta = 0.1$, where $\ln \Lambda = 13.7$ has been taken.

4.4 Applications and Comparisons

We have obtained the local particle loss rate, $(\partial n/\partial t)_c$, due to the velocity-space particle loss (VSPL). In order to evaluate the overall

Figure 4.18. The β -dependence of the collisional particle loss rate given by Eq. (4.93).

confinement property of a system, we like to calculate the average particle confinement time (sometimes simply called the particle confinement time).

As usual, the average particle confinement time, $\langle \tau_p \rangle$, is defined as

$$\langle \tau_p \rangle \equiv - \frac{N}{\left(\frac{\partial N}{\partial t} \right)} = - \frac{\int dr \int dz \, 2\pi r n}{\int dr \int dz \, 2\pi r \left(\frac{\partial n}{\partial t} \right)} \quad (4.95)$$

where the integrations are carried over the interested volume in the configuration space. To actually calculate $\langle \tau_p \rangle$, the density, n , and magnetic flux function, Ψ , must be known.

Consider the commonly used, one-dimensional, field-reversed, rigid-rotor profiles:³

$$n(r) = n_m \operatorname{sech}^2 \left[K \left(\frac{r^2}{R^2} - 1 \right) \right], \quad (4.96)$$

and

$$\tilde{B}(r) = B_0 \tanh \left[K \left(\frac{r^2}{R^2} - 1 \right) \right], \quad (4.97)$$

where n_m is the maximum density occurring at the plasma major radius $r = R (=R_s/\sqrt{2})$, B_0 is a constant, and K is the reversal factor satisfying

$$K = \tanh^{-1} \left[- \frac{B(r=0)}{B_0} \right]. \quad (4.98)$$

The corresponding magnetic flux function is given by

$$\Psi(r) = \frac{B_0 R^2}{2K} \left\{ \ln \left[\cosh \left(K \left(\frac{r^2}{R^2} - 1 \right) \right) \right] - \ln (\cosh K) \right\} \quad (4.99)$$

Figure 4.19 shows the one-dimensional (radial) profiles of $n(r)$, $B(r)$ and $\Psi(r)$, i.e., Eqs. (4.96); (4.97) and (4.99), for the case where $R = 3.6$ cm and $K = 1.5$. Now we can apply the VSPL model to this equilibrium. Using Eqs. (4.88), (4.96) and (4.99), the average particle confinement time is calculated to be (averaged over the interval $[0, r]$)

$$\langle \tau_p \rangle \Big|_0^r = \frac{R^2 |\omega_{c0}|^3}{2 K n_m} \left\{ \tanh K + \tanh \left[K \left(\frac{r^2}{R^2} - 1 \right) \right] \right\} \cdot \left\{ \int_0^r \frac{C_4 u \operatorname{sech}^4 \left[K \left(\frac{r^2}{R^2} - 1 \right) \right] du}{\left\{ u^2 - \frac{R^2}{K} [\ln \cosh (K (\frac{r^2}{R^2} - 1)) - \ln \cosh K] \right\}^{3/2}} \right\}^{-1} \quad (4.100)$$

where $\omega_{c0} = qB_0/mc$ and C_4 is given by Eq. (4.89). Figure 4.20 shows the r -dependence of $\langle \tau_p \rangle \Big|_0^r$ for a typical case where C_4 (or β) = constant, $R = 3.6$ cm and $K = 1.5$. As Fig. 4.20 shows, the $\langle \tau_p \rangle \Big|_0^r$ is almost constant as long as $r > R_s$. This is because the particle density in the open-field region is so low (compared with the density in the closed-field region) that it has only negligible contribution to the overall particle confinement of the system. The particle confinement is mainly determined by the closed-field region. For this reason, we can take the particle confinement time as

$$\begin{aligned} \langle \tau_p \rangle &= \langle \tau_p \rangle \Big|_0^{R_s} \\ &= \frac{R^2 |\omega_{c0}|^3 \tanh K}{K n_m} \left\{ \int_0^{R_s} C_4 y(r) dr \right\}^{-1} \quad (4.101) \end{aligned}$$

Figure 4.19. The radial profiles of the density $n(r)$, magnetic field $B(r)$ and magnetic flux function $\psi(r)$ of a field-reversed equilibrium, i.e., Eqs. (4.96), (4.97), and (4.99). Here $R = 3.6$ cm and $K = 1.5$ are chosen.

Figure 4.20. The dependence of the average particle confinement time on the interval of averaging, i.e., Eq. (4.100). Here, $C_r = C_4 R_0^2 |W_{c0}|^3 / 2Kn_m$ and parameters, $R_0 = 3.6$ cm, $K = 1.5$ are chosen.

where

$$y(r) \equiv \frac{r \operatorname{sech}^4 \left[K \left(\frac{r^2}{R^2} - 1 \right) \right]}{\left\{ r^2 - \frac{R^2}{K} \left[\ln \cosh \left(K \left(\frac{r^2}{R^2} - 1 \right) \right) - \ln \cosh K \right] \right\}^{3/2}} \quad (4.102)$$

We like to compare the predicted $\langle \tau_p \rangle$'s based on the present VSPL model with those observed in experiments. Recalling the definition of C_4 , Eq. (4.89), we know that $C_4 = C_4(\beta)$ and β can be spatially dependent. Since in experiments^{3,20} no detailed temperature profile is available, we define an average value of C_4 as follows.

$$\bar{C}_4 \equiv \frac{\int_0^{R_s} C_4 y(r) dr}{\int_0^{R_s} y(r) dr} \quad (4.103)$$

For convenience, we further define $\bar{\beta}$ such that

$$\bar{C}_4 = C_4(\bar{\beta}), \quad (4.104)$$

i.e., $\bar{\beta}$ is the spatially averaged β -value for the VSPL rate. (We recall that β is a quantity related to the thermal spread of the particle distribution relative to the loss boundary in velocity space.) For example, if β is not spatially dependent, i.e., $\beta = \text{constant}$, then $\beta = \bar{\beta}$ and $\bar{C}_4 \equiv C_4(\bar{\beta}) = C_4(\beta) = \text{constant}$. Using the definition of $\bar{\beta}$, Eq. (4.101) can be written as

$$\langle \tau_p \rangle_{VSPL} = \frac{R^2 |\omega_{co}|^3}{n_m C_4(\bar{\beta})} \cdot \frac{\tanh K}{K \int_0^{R_s} y(r) dr} \quad (4.105)$$

It is noted that the value of the integral, $\int_0^{R_s} y dr$, depends on the reversal factor K only and is independent of R_s (or R). Figs. 4.21 and 4.22 show the scaling of $\langle \tau_p \rangle_{VSPL}$ with the reversal factor K . It is shown that $\langle \tau_p \rangle_{VSPL}$ increases with increasing K , and is very sensitive to K . This suggests that a very effective way to improve particle confinement is to increase the reversal factor K .

We calculate the predicted $\langle \tau_p \rangle$'s for the FRX-A, FRX-B and FRX-C experiments, based on Eq. (4.105), and compare them with the experimental $\langle \tau_p \rangle$'s. Table 4.3 shows the comparisons of $\langle \tau_p \rangle$ between the VSPL model and various FRX experiments. (Other experiments are not compared in Table 4.3, since no sufficient data are available.) In Table 4.3, the value of the reversal factor K is inferred from a and R by Eq. (A13) of Ref. 3, i.e.,

$$\frac{\tanh K}{2K} = \frac{a}{R^2} (R^2 - a^2)^{1/2}, \quad (4.106)$$

where a is the plasma minor radius, and $r = R$ is the field null. In these experiments, temperature is the measured quantity relating to β . As mentioned before, the VSPL occurs mainly in the region where v_c is small, in the present one-dimensional rigid-rotor case, the small- r region.

Therefore, invoking Eq. (4.91), an approximate value of $\bar{\beta}$ is determined from

Figure 4.21. $\langle \tau_p \rangle_{\text{VSPL}}$ scales with reversal factor K , Eq. (4.105). Here $\langle \tau_p \rangle_{\text{VSPL}}$ is normalized by its value at $K = 1$.

Figure 4.22. $\langle \tau_p \rangle_{VSPL}$ scales with reversal factor K , Eq. (4.105). Here $\langle \tau_p \rangle_{VSPL}$ is normalized by its value at $K = 1$.

$$T_i = \frac{4\pi}{3} C_2(\bar{\beta}) C_3(\bar{\beta}) m [v_c(r \approx 1 \text{ cm})]^2, \quad (4.107)$$

where T_i is the reported ion temperature in the FRX experiments. The values of $\bar{\beta}$ are also given in Table 4.3. Then, the predicted $\langle \tau_p \rangle_{\text{VSPL}}$'s are calculated from Eq. (4.105) and compared with the corresponding experimental values, $\langle \tau_p \rangle_{\text{exp}}$'s. Here $\ln \Lambda$ is, as usual, assumed to be a constant, and the expression,

$$\ln \Lambda = 23 - \ln \left[\frac{\sqrt{2 n_m} Z^3}{T_i^{3/2}} \right], \quad (4.108)$$

has been used, where Z is the charge state, n_m and T_i are in cm^{-3} and eV; respectively.

While the comparisons of $\langle \tau_p \rangle$ in Table 4.3 agree well except for one of the three FRX-A experiments, the following points are worth noticing: (1) parametric studies show that the predicted $\langle \tau_p \rangle$ is sensitive to the reversal factor K ; the larger K , the longer $\langle \tau_p \rangle$. The reasons for this are as follows. For larger K , more reversed magnetic flux is trapped. This implies that the confinement region in velocity space is larger, i.e., larger v_c ; hence, the loss probability due to collisions is smaller. In addition to this, i.e., more trapped reversed magnetic flux, the density profile is more peaked around the field null for a larger K . Since the VSPL mainly occurs in the v_c -small region, plus the collision frequency is proportional to n^2 , the more peaked n implies the smaller collision frequency in the VSPL-dominant region, thus, smaller particle loss rate.

TABLE 4.3 Comparisons of $\langle \tau_p \rangle$ between the VSPL model and various experiments. In these experiments, $m = 3.34 \times 10^{-24}$, and $q = 4.8 \times 10^{-10}$ (in Gaussian units).

Experiment	n_m (10^{15} cm^{-3})	T_i (eV)	R (cm)	a (cm)	K*	B_0 (kG)	$\langle \tau_p \rangle_{\text{exp}}$ (μs)	$\bar{\beta}$	$\langle \tau_p \rangle_{\text{VSPL}}$ (μs)
FRX-A ³	2.6	170	3.3	1.2	1.3	5.5	35	0.15-0.16	31-21
FRX-A ³	2.0	170	3.6	1.2	1.4	5.5	58	0.15-0.16	67-46
FRX-A ³	2.3	170	3.6	0.9	2.0	5.5	48	0.15-0.16	460-310
FRX-B ³	3.5	200	3.8	1.9	0.7	6.5	39 ± 15	0.13-0.14	17-10
FRX-C ²⁰	4.0	250	8.5	+	0.7^{++}	7.3	$100-120^{**}$	0.13-0.14	144-86

* Calculated from Eq. (A13) of Ref. 3: $(\tanh K)/2K = (R^2 - a^2)^{1/2} a/R^2$, where a is the experimental value of the minor radius.

** Assume $\langle \tau_p \rangle_{\text{exp}} \approx \tau_s$ where τ_s is the plasma stable time in the experiment.

+ Not available.

++ Since the value of minor radius, a, is not available, K is chosen to be equal to the value in FRX-B.

Adding these two effects together, the $\langle \tau_p \rangle$ is sensitive to K and $\langle \tau_p \rangle$ is larger for a larger K . (2) the much longer predicted particle confinement time for the third set of FRX-A experimental data in Table 4.3, $\langle \tau_p \rangle_{\text{VSPL}} = 310 - 460 \mu\text{s} \gg \langle \tau_p \rangle_{\text{exp}} = 48 \mu\text{s}$, is possibly due to the overestimate of K . As noted from Table 4.3, all experimentally measured parameters are almost constant for all three FRX-A experiments, except the minor radius a . Since the value of reversal factor is inferred from a and R by Eq. (4.106), not a direct measurement of $B(r=0)/B_0$, the value of $K = 2.0$ for the third set of FRX-A data is possibly overestimated, especially, when compared with $K = 1.3$ and $K = 1.4$ of the other two sets of FRX-A data. For comparison, if we use $K = 1.4$ in the third set of FRX-A data, instead of $K = 2.0$ inferred from a and R , then $\langle \tau_p \rangle_{\text{VSPL}} = 61 - 42 \mu\text{s}$ which is very close to the reported value, $\langle \tau_p \rangle_{\text{exp}} = 48 \mu\text{s}$. (3) since no detailed temperature profile is reported in the experiments, there is an uncertainty in the evaluation of $\bar{\beta}$. Nevertheless, for the small values of $\bar{\beta}$ in Table 4.3 ($\bar{\beta} = 0.13 - 0.16$), the short $\langle \tau_p \rangle_{\text{VSPL}}$'s predicted show the importance of velocity-space particle loss. (4) in using the one-dimensional rigid-rotor profiles, the axial variations of n and B are neglected.

From the confinement region in velocity space, as shown in Figs. 4.1, since this is the most fundamental constraint on the particle distribution, it is immediately known that the ion species will rotate in the negative θ -direction. To show the effect of VSPL on plasma rotation, consider an ion distribution originally at rest in the laboratory frame with a Maxwellian velocity distribution, i.e.,

$$f_i(r, z, \underline{v}, t=0) = n(r, z) C(r, z) \cdot \exp \left\{ -\frac{m}{2 T_i(r, z)} (\underline{v}_r^2 + \underline{v}_\theta^2 + \underline{v}_z^2) \right\}, \quad (4.109)$$

where C is the normalization factor. Here, we have come back to the usual velocity variable $\underline{v}$. Since unconfined particles which have velocities larger than u are lost out of the system in a time period about L/u , where L is the system length, the remaining particles have an angular speed (or rotational frequency) at $t \sim L/u$ given by

$$\omega_{\text{rot}}(r, z, t \sim \frac{L}{u}) = \frac{\int_Q d\underline{v} \left(\frac{\underline{v}_\theta}{r} \right) f_i}{\int_Q d\underline{v} f_i}, \quad (4.110)$$

where Q is the volume in velocity space where either $v < u$ or $\underline{v}$ locates inside the confinement sphere. Using the symmetry about $\underline{v}_\theta$ -axis, Eq. (4.110) is explicitly given by the following:

$$\omega_{\text{rot}}(r, z, t \sim \frac{L}{u}) = 0 \quad (4.111.1)$$

if $v > v_c + |\underline{v}_{\theta 0}|$,

$$\omega_{\text{rot}}(r, z, t \sim \frac{L}{u}) = \frac{\int_{Q_1} d\underline{v} f_1}{\int_{Q_1} d\underline{v} f_2 + \int_{Q_2} d\underline{v} f_2} \quad (4.111.2)$$

if $v_c - |\underline{v}_{\theta 0}| \leq v \leq v_c + |\underline{v}_{\theta 0}|$, and

$$\omega_{\text{rot}}(r, z, t \sim \frac{L}{u}) = \frac{\int_{Q_3} d\underline{v} f_1}{\int_{Q_3} d\underline{v} f_2} \quad (4.111.3)$$

if $v < v_c - |\underline{v}_{\theta 0}|$, where

$$\int_{Q_1} d\underline{v} = \int_{S_1}^{S_2} ds \int_0^{w_1} dw + \int_{S_2}^{S_3} ds \int_{w_2}^{w_1} dw, \quad (4.112)$$

$$\int_{Q_2} d\tilde{v} = \int_{S_2}^{S_4} ds \int_0^{w_2} dw, \quad (4.113)$$

$$\int_{Q_3} d\tilde{v} = \int_{S_1}^{S_5} ds \int_0^{w_1} dw, \quad (4.114)$$

$$f_1 = \pi(r, z) C(r, z) \frac{1}{r} s w e^{-\frac{m}{2T} (s^2 + w^2)}, \quad (4.115)$$

$$f_2 = \pi(r, z) C(r, z) w e^{-\frac{m}{2T} (s^2 + w^2)}, \quad (4.116)$$

$$S_1 = v_{\theta 0} - v_c, \quad (4.117)$$

$$S_2 = -u, \quad (4.118)$$

$$S_3 = \frac{1}{2v_{\theta 0}} (u^2 + v_{\theta 0}^2 - v_c^2), \quad (4.119)$$

$$S_4 = v, \quad (4.120)$$

$$S_5 = v_{\theta 0} + v_c, \quad (4.121)$$

$$w_1 = [v_c^2 - (s - v_{\theta 0})^2]^{1/2}, \quad (4.122)$$

and

$$W_2 = (u^2 - s^2)^{1/2} \quad (4.123)$$

Usually, a parameter α is defined as

$$\alpha = -\omega_{rot} / \Omega_i^* \quad (4.124)$$

where

$$\Omega_i^* = -\frac{c}{g_i r B_z n_i} \frac{\partial P_i}{\partial r} \quad (4.125)$$

is the ion diamagnetic drift frequency. For the FRX experiments characterized by one-dimensional, radial, rigid-rotor $n_i(r)$ and $B_z(r)$, Eqs. (4.96) and (4.97), Eq. (4.125) becomes

$$\Omega_i^* = 4. \times 10^5 \frac{K T_i (eV)}{Z [R(cm)]^2 B_0 (kG)} \quad (4.126)$$

where Z is the charge state of ions. In obtaining Eq. (4.115), as in Ref. 3, T_i is assumed to be a constant.

Equations (4.111) and (4.126) are calculated for a set of FRX-A parameters; typical results are shown in Fig. 4.23. It is noted that, for a rigidly rotating plasma, a typical theoretical threshold for rotational instability is $\alpha \approx 1.4$.⁴⁷ In experiments, at present, there are no satisfactory data for the plasma rotation available.³ From the above calculations, we notice the following points: (1) The VSPL has an effect in spinning up the plasma, since ω_{rot} starts with zero at $t = 0$ and increases with time. (2) The α -values at $r = R_s$ and $r = 1$ cm are close to the theoretically predicted threshold. This shows that the VSPL can

Figure 4.23. Typical temporal behavior of α_{VSPL} from Eqs. (4.111) and (4.126). This figure is shown for FRX-A parameters: $T_i = 170 \text{ eV}$, $R = 3.3 \text{ cm}$, $L = 100 \text{ cm}$, $B_0 = 5500 \text{ G}$, $m = 3.34 \times 10^{-24} \text{ g}$, $n_m = 2.6 \times 10^{15} \text{ cm}^{-3}$ and $K = 1.3$. Here, $t \sim L/u$ has been taken to be $t = L/2u$.

have significant contribution to the rotational instability. Furthermore, the higher α -values in the $|\Psi|$ -small region (i.e., near z-axis or separatrix) suggest the possibility that instabilities may occur in this region first. (3) In the above, ω_{rot} is calculated for the initially rest Maxwellian distribution, based on the loss of initially unconfined particles. Therefore, further collisional VSPL should result in further increase of ω_{rot} . (4) The confinement sphere in velocity space implies that a plasma will evolve, due to collisions, toward a rigidly rotating state with angular frequency ω_{co} . However, as pointed out in (2), stability theories predict that instabilities are very probable to occur before reaching that state.

At present, there are several theoretical models for particle confinement in FRTP's. Based on classical particle diffusion and using the simple Ohm's law, together with some other assumptions, Auerbach and Condit obtained,³⁶ for a Hill's vortex equilibrium

$$\begin{aligned} \langle \tau_p \rangle_{\text{cl}} &= 4.41 \frac{k^2}{1+4k^2} \frac{R^2 \sigma_0}{c^2} \\ &= 4.26 \times 10^{-7} \frac{k^2}{1+4k^2} \frac{[T_e(\text{eV})]^{3/2} [R(\text{cm})]^2}{Z_{\text{eff}} \ln \Lambda} \end{aligned} \quad (4.127)$$

where k is the elongation factor defined before, σ_0 is the Spitzer conductivity at the field null ($r = R$; $z = 0$), and Z_{eff} is the plasma effective charge. By a similar approach but using the generalized form of Ohm's law, Nguyen and Kammash obtained,³⁹ for a Hill's vortex magnetic configuration

$$\langle \tau_p \rangle_{\alpha} = \frac{6.95 R^2 \sigma_0}{(4 + \alpha^2) c^2} \left\{ (2.33 - 1.5 \mu) \cdot \left[1 + \frac{3}{2} \frac{P_{e0}}{P_0} (\mu - 1) \right]^{-1} \right\}, \quad (4.128)$$

where α is the ellipticity parameter (a parameter related to k) of the Hill's vortex configuration, μ is a parameter characterizing the density and temperature profile (both α and μ are defined in Ref. 39), and quantities with subindex 0 refer to the values at the field null. Considering a one-dimensional (radial) anomalous transport model, in which some of two-dimensional features are included due to the lower-hybrid-drift instability with wave energy bound, Tuszewski and Linford obtained⁴⁰

$$\langle \tau_p \rangle_{LHD} = \frac{4}{\sqrt{\pi}} \cdot \frac{(m_i/m_e)^{1/2}}{(1 + T_e/T_i) \omega_{ci0}} \cdot \frac{(1 - \beta_s)^{3/2} \langle \beta \rangle}{\beta_s} \cdot S w^3, \quad (4.129)$$

where $\beta = P/(B_0^2/8\pi)$, $\langle \beta \rangle$ is the β -value averaged over the closed-field region ($0 \leq r \leq R_s$), $S = R/\rho_{i0}$, w is a parameter determined self consistently from the model, and the subindex s refers to values at $r = R_s$.

Now we want to compare the VSPL model with the above three models. First, we consider the first two models. Basically, the physical mechanism responsible for the first two models is the electron-ion collisions. One important scaling characteristic of this process is that $\langle \tau_p \rangle_{c1}$ is almost independent of T_i , since the relative velocity for electron-ion collisions is mainly determined by the electron velocity. However, this predicted general tendency does not agree with the FRX experiments, although

the two classical diffusion models have agreements with some of the FRX-experimental data if certain choice is made for the plasma effective charge, Z_{eff} . In experiments,⁴⁰ for $T_e \approx \text{constant}$, particle confinement times increase with decreasing ion temperatures, which does not agree with the prediction by classical particle diffusion. For the VSPL model, particle confinement times increase with decreasing temperature, as shown by Fig. 4.18, since there are less high energy particle for lower temperatures, thus, lower VSPL rates. It is noted that the classical particle diffusion in the first two models is not included in the VSPL model, and represents an addition mechanism for particle loss.

In summary we find that (1) the two classical-particle-diffusion models^{36,39} do not predict the observed temperature dependence of particle confinement times in the FRX experiments,⁴⁰ (2) the VSPL model predicts that particle confinement times increase with decreasing ion temperatures, which is consistent with the FRX experiments, (3) the particle loss due to classical diffusion is an additional loss mechanism (in addition to the VSPL), and does not imply any contradiction to the VSPL model, and (4) classical diffusion does not seem to be a dominant particle loss mechanism for the whole parameter range of FRX experiments, since it fails to predict the T_i -dependence of $\langle \tau_p \rangle$.

Next, we compare the VSPL model with the anomalous transport model⁴⁰ based on the lower-hybrid-drift instability (let us call it LHD model). As mentioned in Ref. 40, $\langle \tau_p \rangle_{\text{LHD}}$ depends on three parameters, S , x_s ($\equiv R_s/R_w$) and w . Since all these three parameters, together with other parameters, are iteratively determined self-consistently within the LHD model, no explicitly scaling law for $\langle \tau_p \rangle_{\text{LHD}}$ is obtained. Only typical results

for certain parameters are reported. In addition, in LHD model only the region outside the field null ($R \leq r$) is considered, and the communication between the two regions, $r < R$ and $r > R$, is ignored. Therefore, we only make qualitative comparisons between the two models. From Figs. 4, 6, and 8 of Ref. 40, we know the following. First, $\langle \tau_p \rangle_{\text{LHD}}$ increases with increasing S . By the definition of S , increasing S means increasing R , or decreasing T_i , or increasing B_0 . In any case, $\langle \tau_p \rangle_{\text{VSPL}}$ increases, too. Second, $\langle \tau_p \rangle_{\text{LHD}}$ increases with increasing x_s . Since x_s is linked to the ratio of trapped magnetic flux ($\int_0^R 2\pi r B_z dr$) to the external magnetic flux ($\pi R^2 B_0$), increasing x_s means increasing the trapped magnetic flux, thus, increasing the reversal factor K . This, in turn, increases $\langle \tau_p \rangle_{\text{VSPL}}$. Third, $\langle \tau_p \rangle_{\text{LHD}}$ increases with increasing w . By the definition of w , increasing w implies decreasing $|(\partial n / \partial r) / n|$ on the separatrix. Since in LHD model $n = n(\Psi)$, the increasing w in LHD model corresponds to decreasing $|(\partial n / \partial r) / n|$, or increasing n , at $r = R_s$ and $r = 0$ in VSPL model; therefore, $\langle \tau_p \rangle_{\text{VSPL}}$ decreases. This difference in scaling with w is the only qualitative difference between $\langle \tau_p \rangle_{\text{LHD}}$ and $\langle \tau_p \rangle_{\text{VSPL}}$. The reason for this difference is the following. In LHD model particle loss flux is proportional to $|(\partial n / \partial r)^3 / n^2|$, while particle loss in VSPL model is proportional to n^2 . It may be worth noticing that none of the above classical-diffusion and LHD models predicts the occurrence of plasma rotation, while the VSPL model does. For convenience, the above comparisons among various models are summarized in Table 4.4.

4.5 Summary and Discussion

In Chapter 3, we have found that there is a loss region in (H, P) space

Table 4.4. Comparisons among various theoretical models for particle confinement in F RTP's.

	FRX Experiments	Auerbach & Condit ³⁶	Nguyen & Kammash ³⁹	Tuszewski & Linford ⁴⁰	VSPL
Variable(s) in configuration space	(r,z)	Ψ	Ψ	Υ	r or (r,z)
Physical mechanism	To be identified	Classical diffusion with simple Ohm's law	Classical diffusion with generalized Ohm's law	Lower-hybrid- drift insta- bility	Velocity-space- particle loss
Explain avail- able $\langle \tau_p \rangle_{\text{exp}}$	---	? ⁺	? ⁺	Yes	Yes
Occurrence of Rotation	Yes	No	No	No	Yes
$\langle \tau_p \rangle$ behavior with:					
Increasing R	Increase	Increase	Increase	Increase	Increase
Increasing T_i	Decrease	Unchange	Unchange	Decrease	Decrease
Increasing B_0	Increase	Increase	Increase	Increase	Increase
Increasing x_s	Not Available	Unchange*	Unchange*	Increase	Increase
Increasing w	Not Available	Increase	Increase	Increase	Decrease

+ Can have agreements with some of the available $\langle \tau_p \rangle_{\text{exp}}$ if certain choice is made for the plasma effective charge. However, the T_i -dependence of $\langle \tau_p \rangle$ does not agree with $\langle \tau_p \rangle_{\text{exp}}$.

* In these models, $\langle \tau_p \rangle$ is independent of x_s .

for particle confinement in field-reversed configurations. In this chapter, we study the collisional particle loss due to the existence of the loss region in (H, P) space and their effect on the particle distribution function.

In Section 4.1, we change the particle confinement criteria from (H, P_0) space to velocity space, and obtain the corresponding criteria in velocity space. For absolutely confined particles, the criteria are given by Eqs. (4.3) and (4.4); for closed-field confined particles, Eq. (4.12). The loss boundary in velocity space is found.

In Section 4.2, we study the effect of the loss region in velocity space on the particle distribution function f . The loss boundary in velocity space imposes a boundary condition, Eq. (4.30), on the particle distribution function, and the Fokker-Planck equation, Eq. (4.19), describes the time rate of change of the particle distribution function due to collisions. These constitute a boundary-value problem for f . Using a model distribution function f_0 , given by Eq. (4.35), to linearize the Fokker-Planck equation, and making coordinate and variable transformations, we arrive at a linear, separable form for the interested boundary value problem. The corresponding eigenvalue problem for the velocity distribution, Eqs. (4.50) - (4.52), is solved numerically. The eigenfunctions and corresponding eigenvalues are obtained. The power series form of the eigenfunctions is also obtained. It is found that the model distribution function is a good approximation of the fundamental mode. Therefore, the simple, analytic model distribution function, f_0 , is a good representation of the actual particle distribution function in collisional equilibrium, and provides a convenient tool for many other calculations, e.g., particle confinement time, stability analysis, etc.

In Section 4.3, the collisional particle loss rate due to the existence of the loss region in velocity space, or simply called the velocity-space particle loss (VSPL), is obtained for the particle distribution in collisional equilibrium, as given by Eq. (4.88). The local particle confinement time, Eq. (4.84), is also obtained.

In Section 4.4, the VSPL model is applied to commonly used, field-reversed, rigid-rotor, radial profiles. The scaling law for the average particle confinement time, $\langle \tau_p \rangle$, is obtained, as given by Eq. (4.100) or Eq. (4.105). It is found that a very effective way to improve particle confinement is to increase the reversal factor, i.e., to increase trapped magnetic flux in the closed-field region. Comparisons between the VSPL model and FRX-A, FRX-B and FRX-C experiments are made. Favorable comparisons are shown, except for one experiment where the unfavorable comparison is possibly due to the inaccuracy in estimating the reversal factor K . Nevertheless, the importance of the velocity-space particle loss is shown. Comparison with other theoretical models, including two classical-particle-diffusion models^{36,39} and on anomalous transport model⁴⁰ based on the lower-hybrid-drift instability with wave energy bound, are also made. The only qualitative difference between the VSPL and LHD models is their scalings with $(\partial n / \partial r)$ and n at $r = R_s$. At present, there are no sufficient experimental data to clarify this. Therefore, both the VSPL and LHD models seem to be eligible candidates for particle confinement in FTRP's. But, the VSPL model is the only one predicts that the plasma rotation can be a direct result of the VSPL mechanism, moreover, the rotational frequency directly from VSPL can be in the order of the threshold rotational frequency for the $m = 2$ rotational instability.

The loss boundary in velocity space, which imposes a boundary condition in velocity space for the particle distribution function, is position-dependent. Therefore, the particle distribution function, f , is not separable in $\underline{r}$ and $\underline{v}$, i.e., f can not be written in a form of $W(\underline{r},t)V(\underline{v})$. After the transformations used in Section 4.2, the particle distribution function becomes separable and can be written in a form of $f = W(\underline{r},t)V(\underline{v}')$, where $\underline{v}'$ is the new velocity variable in the new velocity coordinates. This makes the unseparable Fokker-Planck equation, Eq. (4.19), in $(\underline{r},\underline{v})$ space become separable in $(\underline{r},\underline{v}')$ space, and the problem is simplified significantly.

In Sections 4.3 and 4.4, the particle loss rate and confinement time are obtained for the particle distribution function in collisional equilibrium, i.e., $f=f_{eq}$. If f is not in collisional equilibrium, i.e., f is not isotropic with respect to the center $(0,v_{\theta 0},0)$, or f is isotropic but has higher modes in its velocity distribution, then the particle loss rate is even larger than that for $f=f_{eq}$.

The velocity-space particle loss increases as the plasma temperature increases. Therefore, for reactor-grade plasmas, the velocity-space particle loss is expected to have an even more important effect on the particle confinement than it is for the previous experiments.

From the confinement region in velocity space, it is known that, if a plasma originally is at rest in the laboratory frame with $\overline{v}_i \equiv \int \underline{v} f_i d\underline{v} = 0$ and $\overline{v}_e \equiv \int \underline{v} f_e d\underline{v} = 0$, the collisional particle loss will make the ion (electron) fluid start rotating in the negative (positive) θ -direction, and the rotation speed will be accelerated toward a negative (positive) value $-qB_0 r/mc$. Although the friction between the electron and

ion fluids does exist when the two fluids rotate in the opposite θ -directions, the friction force is not expected to stop the ion rotation. The reasons for this are the following: (1) The momentum (or energy) change of ions due to collisions with electrons is ineffective. (2) As the collisional losses of electrons and ions keep occurring, the density product, $n_i n_e$, decreases, and the relative velocity between the two fluids, v_{rel} , increases. Since the friction force is roughly proportional to $n_i n_e / v_{rel}$, it decreases as a result of the losses.

Therefore, just based on the particle confinement criteria which are the most fundamental constraints on particle confinement, the following important conclusion can be drawn: The confinement region in velocity space shows that the ion species tends to rotate with a negative v , i.e., in the negative θ -direction, due to the collisional VSPL. Calculations based on the VSPL show the importance of VSPL on ion rotation, and results are consistent with the experimental observation: the occurrence and spin-up of ion rotation in the negative θ -direction.^{2,3,13,16,20-22}

In this chapter, the assumption of a fast loss of unconfined particles is made, i.e., unconfined particles are assumed to be lost instantaneously. As the loss boundary in velocity space, Fig. 4.1, shows, there are both fast and slow unconfined particles. In reality, the fast unconfined particles (with large v) and the slow unconfined particles (with small v) have different, nonzero lost times, where $v = |v|$, v is the particle velocity in the laboratory frame (i.e., 0-coordinates in Fig. 4.1), and the lost time is defined as the time period needed for an unconfined particle to hit the wall. The effects of this can be two-fold. Firstly, the nonzero lost times imply that there are some unconfined particles in the system, i.e., the

unconfined particles still have some contribution to the plasma (or to the particle distribution function). Secondly, the different lost times imply that, in collisional equilibrium, there are more slow unconfined particles in the system than fast unconfined particles, because the slow unconfined particles both travel more slowly and have higher probabilities to suffer collisions. These two effects can be included, effectively, by the following modifications of the confinement region in velocity space, at least as an approximation. (1) The center of the confinement sphere in velocity space, $(0, v_{\theta 0}, 0)$, is modified to be $(0, v'_{\theta 0}, 0)$ with $|v'_{\theta 0}| < |v_{\theta 0}|$. (2) The radius of confinement sphere is modified to a value of v_C' , where $v_C' \geq v_C$. Since the assumption of a fast loss is good for the fast unconfined particles, and there are less fast (or high energy) particles than slow (or low energy) particles, the relatively important effect of the non-instantaneous loss of unconfined particles is the presence of the slow unconfined particles in the system. Therefore, if without the assumption of a fast loss, the results in this chapter are still valid, with the substitution of $v_{\theta 0}'$ for $v_{\theta 0}$.

CHAPTER 5

PARTICLE DISTRIBUTION FUNCTION IN THE PRESENCE
OF THE VELOCITY-SPACE PARTICLE LOSS

In Chapter 4, the velocity distribution of the confined particles in collisional equilibrium, in the presence of the velocity-space particle loss, has been obtained. The VSPL rate for any given density profile has been calculated. The next question arises may be the following: What density profiles are possible to be maintained? or what density profiles are reasonable to be assumed? In this chapter, we answer this question by solving the kinetic equation for the steady-state particle distribution function.

5.1 Formulation of the Problem

The steady-state kinetic equation governing the particle distribution function, $f(\underline{r}, \underline{v})$, with no electric field is

$$\underline{v} \cdot \frac{\partial f(\underline{r}, \underline{v})}{\partial \underline{r}} + \frac{q}{mc} \underline{v} \times \underline{B} \cdot \frac{\partial f}{\partial \underline{v}} = \left(\frac{\partial f}{\partial t} \right)_c + S(\underline{r}, \underline{v}), \quad (5.1)$$

where $S(\underline{r}, \underline{v})$ is the source rate, and $(\partial f / \partial t)_c$ is the Fokker-Planck collisional term given by Eq. (4.19). As before, assuming a fast loss of the unconfined particles, the boundary condition, $f(\underline{r}, \underline{v}) = 0$ if $\underline{v}$ is outside the confinement sphere in velocity space, has to be imposed. Of course, certain boundary conditions for $f(\underline{r}, \underline{v})$ in configuration space have to be imposed, too, which have not been specified yet.

Let us choose the cylindrical spatial coordinates, (r, θ, z) , and the Cartesian velocity coordinates, (v_r, v_θ, v_z) . Because of the boundary condition in velocity space, it is more convenient to translate the velocity coordinates by

$$\underline{u} = \underline{v} - v_{\theta 0}(r) \hat{\theta}, \quad (5.2)$$

where $v_{\theta 0}(r)$ is the rotational speed, and $\hat{\theta}$ is the unit vector in the θ -direction. Invoking the axisymmetry of the system, after the translation of Eq. (5.2), Eq. (5.1) becomes (the derivation is given in Appendix A)

$$\begin{aligned} & u_r \frac{\partial f(r, z, \underline{u})}{\partial r} + u_z \frac{\partial f}{\partial z} + \frac{q}{mc} \underline{u} \times \underline{B} \cdot \frac{\partial f}{\partial \underline{u}} \\ & - u_r \frac{\partial v_{\theta 0}}{\partial r} \frac{\partial f}{\partial u_\theta} + \frac{v_{\theta 0}}{r} (v_{\theta 0} + u_\theta) \frac{\partial f}{\partial u_r} + \frac{q}{mc} v_{\theta 0} \\ & \cdot (B_z \frac{\partial f}{\partial u_r} - B_r \frac{\partial f}{\partial u_z}) = \left(\frac{\partial f}{\partial t} \right)_c + S(r, z, \underline{u}), \end{aligned} \quad (5.3)$$

where

$$f(r, z, \underline{u}) = f(r, z, \underline{v} = \underline{u} + v_{\theta 0} \hat{\theta}), \quad (5.4)$$

$$S(r, z, \underline{u}) = S(r, z, \underline{v} = \underline{u} + v_{\theta 0} \hat{\theta}), \quad (5.5)$$

and $(\partial f / \partial t)_c$ is still given by Eq. (4.19), just replacing v by u . The boundary condition in velocity space becomes

$$f(r, z, \underline{u}) = 0 \quad \text{for} \quad u \equiv |\underline{u}| = v_c(r, z). \quad (5.6)$$

The problem, now, becomes to find the possible steady-state solutions for $f(r, z, \underline{u})$ which satisfies Eqs. (5.3) and (5.6), with a proper source rate, $S(r, z, \underline{u})$. Since the problem, Eqs. (5.3) and (5.6), is so complicated, it does not seem possible to obtain exact, analytic solutions. In the following, we try to obtain approximate solutions to this problem.

5.2 Approximate Steady-State Particle Distribution Function

In chapter 4, we have shown that, based on the Fokker-Planck equation, the particle distribution function in collisional equilibrium is

$$f(r, \underline{u}, t) = f(r, z, u, t) \approx f_0(r, z, u, t), \quad (5.7)$$

where $f_0(r, z, u, t)$ is given by Eq. (4.35). If the velocity diffusion due to collisions is the dominant process both for the particle loss and for determining the velocity distribution of the confined particles, then it is reasonable to approximate the particle distribution function in the present case, $f(r, z, u)$, as

$$f(r, z, \underline{u}) \approx f(r, z, u) \approx f_0(r, z, u), \quad (5.8)$$

where

$$f_o(r, z, u) = n(r, z) C_1(r, z) \left\{ 1 - \frac{u^2}{v_c^2(r, z)} \right\} \cdot U_s(v_c - u) \cdot \exp \left\{ - \frac{u^2}{\beta(r, z) v_c^2(r, z)} \right\}, \quad (5.9)$$

and all other notations are defined as before. As shown in Chapter 4, $f(r, z, \underline{u}, t) = f(r, z, u, t)$ is an exact expression for the problem described by the Fokker-Planck equation, Eq. (4.19), and the isotropic boundary condition, Eq. (4.30). However, in the present case where the streaming terms, i.e., the left-hand side of Eq. (5.3), are included, $f(r, z, \underline{u}) = f(r, z, u)$ is only an approximate expression. This is because, although the boundary condition, Eq. (5.6), and the collisional term, $(\partial f / \partial t)_c$, are speed-dependent only, the stream terms are direction-dependent. Nevertheless, if the velocity diffusion due to collisions is the dominant process, the approximation, Eq. (5.8) is good.

Now, we proceed with the following procedures.

(1) For $f = f(r, z, u)$, the third term in the left-hand side of Eq. (5.3) vanishes, since $\underline{u} \times \underline{B} \cdot \partial f / \partial \underline{u} = 0$ for an isotropic $f(r, z, u)$.

(2) For $f = f(r, z, u)$ and $v_{\theta 0} \propto r$ (e.g., in the present case, $v_{\theta 0} = -\omega_{c0} r$), the fourth term and the sixth term in the left-hand side of Eq. (5.3) cancel each other, since

$$\frac{\partial f}{\partial u_\theta} = \frac{u_\theta}{u} \frac{\partial f}{\partial u}, \quad \frac{\partial f}{\partial u_r} = \frac{u_r}{u} \frac{\partial f}{\partial u}, \quad \frac{\partial v_{\theta 0}}{\partial r} = \frac{v_{\theta 0}}{r}. \quad (5.10)$$

(3) Use $f = f_0$ to evaluate the Rosenbluth potentials for the collisional term.

(4) Make the transformation,

$$\underline{v}' = \underline{v}'(r, z, \underline{u}) \equiv \frac{\underline{u}}{v_c(r, z)}, \quad (5.11)$$

such that the Fokker-Planck collisional term becomes separable in the variables $\underline{r}$ and $\underline{v}'$.

(5) In the dimensionless velocity, $\underline{v}'$, Eq. (5.8) is rewritten as

$$f(r, z, \underline{v}') \approx f(r, z, v') \approx f_0(r, z, v'), \quad (5.12)$$

where

$$f(r, z, v') \equiv f(r, z, u = v' v_c), \quad (5.13)$$

$$f_0(r, z, v') = \frac{n(r, z)}{v_c^3(r, z)} V_0(v'), \quad (5.14)$$

and $V_0(v')$ is given by Eq. (4.44). Since $f = f_0(r, z, v')$ is in a separable form in (r, z) and v' , it is natural to assume that the source rate, $S(r, z, v')$, is also in a separable form, i.e.,

$$S(r, z, \underline{v}') \approx S(r, z, v') \approx \frac{n_s(r, z)}{v_c^3(r, z)} V_s(v'), \quad (5.15)$$

where

$$S(r, z, \underline{v}') \equiv S(r, z, \underline{u} = \underline{v}' v_c), \quad (5.16)$$

n_s and $V_s(v')$ are the number density and the velocity distribution (normalized by $\int_0^1 4 v'^2 V_s(v') dv' = 1$) of the source rate, respectively.

For convenience, define

$$W(r, z) \equiv \frac{n(r, z)}{v_c^3(r, z)}, \quad (5.17)$$

and

$$W_s(r, z) \equiv \frac{n_s(r, z)}{v_c^3(r, z)}, \quad (5.18)$$

such that we can write

$$f(r, z, v') \approx W(r, z) V_o(v'), \quad (5.19)$$

and

$$S(r, z, v') \approx W_s(r, z) V_s(v'). \quad (5.20)$$

(6) Substitute Eqs. (5.19) and (5.20) into the kinetic equation.

After carrying out the above manipulations, the kinetic equation, Eq. (5.3), can be written as

$$\begin{aligned}
& \frac{v_c^4}{W^2} \left(v_r' \frac{\partial W}{\partial r} + v_z' \frac{\partial W}{\partial z} \right) + \frac{g}{mc} \frac{v_{\theta 0} v_c^2}{W} \frac{1}{v' V_0} \frac{\partial V_0}{\partial v'} \\
& \cdot \left\{ v_r' \left(B_z + \frac{mc v_{\theta 0}}{g r} \right) - v_z' B_r \right\} \\
& = \frac{\Gamma}{V_0} D\{V_0\} + \frac{v_c^3 W_s}{W^2} \frac{V_s}{V_0}, \tag{5.21}
\end{aligned}$$

where D is the second-order differential operator defined by Eq. (4.43).

Based on this equation, the approximate steady-state particle distribution is obtained as follows.

For FRTP's, the axial length is usually much longer than the radius. First, consider an infinitely long system ($L \rightarrow \infty$). For an infinitely long system, we have $\partial n / \partial z = 0$ and $B_r = 0$. In this case, Eq. (5.21) simplifies to

$$\begin{aligned}
& v_r' \left\{ \frac{v_c^4}{W^2} \frac{dW}{dr} - \frac{g^2 B_0}{m^2 c^2} \frac{v_c^2 r}{W} (B_z - B_0) \frac{1}{v' V_0} \frac{\partial V_0}{\partial v'} \right\} \\
& = \frac{\Gamma}{V_0} D\{V_0\} + \frac{v_c^3 W_s}{W^2} \frac{V_s}{V_0}, \tag{5.22}
\end{aligned}$$

where $v_{\theta 0} = -\omega_{c0} r$ has been used. It may be worth mentioning that the third term in the left-hand side of Eq. (5.22) comes from the rotation, specifically, from the fifth term in the left-hand side of Eq. (5.3). In case $v_{\theta 0} = 0$, then this term vanishes, i.e., $(B_z - B_0)$ is replaced by B_z .

Examine Eq. (5.22). The bracket in the left-hand side and the whole right-hand side are functions of r and v' only. Therefore, the only

possibility to satisfy Eq. (5.22) is to have

$$\frac{v_c^4}{W^2} \frac{dW}{dr} - \frac{g^2 B_0}{m^2 c^2} \frac{v_c^2 r}{W} (B_z - B_0) \frac{1}{v' V_0} \frac{\partial V_0}{\partial v'} = 0 \quad (5.23)$$

and

$$\frac{\Gamma}{V_0} D\{V_0\} + \frac{v_c^3 W_s}{W^2} \frac{V_s}{V_0} = 0 \quad (5.24)$$

Physically, Eqs. (5.23) and (5.24) mean that the collisional loss is compensated by the source, while the drift due to the magnetic field and rotation is balanced by the density-gradient drift.

Since, for $V_0(v')$ given by Eq. (4.44),

$$\frac{1}{v' V_0} \frac{\partial V_0}{\partial v'} = (-2) \left(\frac{1}{\beta} + \frac{1}{1-v'^2} \right) \quad (5.25)$$

As Figs. 4.10 - 4.17 have shown, for the particles having a velocity distribution $V_0(v')$, most particles are in the velocity range $0 \leq v \leq 0.8$. In this range, we have $1 \leq 1/(1-v^2) \leq 2.78$. It is, therefore, a good approximation to take

$$\frac{1}{v' V_0} \frac{\partial V_0}{\partial v'} \approx (-2) \left(\frac{1}{\beta} + 2 \right), \quad (5.26)$$

especially for $\beta \ll 1$, which is true for many interested cases. Using Eq. (5.26), Eq. (5.23) becomes

$$\frac{dw}{dr} = (-2) \left(\frac{1}{\beta} + 2 \right) \frac{g^2 B_0}{m^2 c^2} \frac{r(B_z - B_0)}{v_c^2} W \quad (5.27)$$

Furthermore, examining Eq. (5.24), it is found that, in order to satisfy this equation, we must have

$$\frac{v_c^3 W_s}{W^2} = -\nabla \frac{D\{V_0\}}{V_s} = C_6, \quad (5.28)$$

where C_6 is a constant.

To fully determine the density profile, of course, a boundary condition is needed. Say, if the density at the field null, $n(R)$, is known, then $W(R) = n(R)/v_c^3(R)$. Integrating Eq. (5.27) and using this condition, we obtain

$$W(r) = \frac{n(R)}{v_c^3(R)} \exp \left\{ - \int_R^r \left(\frac{2}{\beta} + 4 \right) \frac{r' [B_z(r') - B_0]}{B_0 r'^2 - 2\psi(r')} dr' \right\} \quad (5.29)$$

By Eq. (5.17), this gives

$$n(r) = n(R) \frac{[B_0 r^2 - 2\psi(r)]^{3/2}}{[B_0 R^2 - 2\psi(R)]^{3/2}} \cdot \exp \left\{ - \int_R^r \left(\frac{2}{\beta} + 4 \right) \frac{r' [B_z(r') - B_0]}{B_0 r'^2 - 2\psi(r')} dr' \right\} \quad (5.30)$$

Substituting Eq. (5.29) into Eq. (5.19) and changing the variable v' back to u , or, directly substituting Eq. (5.30) into Eq. (5.9), we finally obtain the steady-state particle distribution function, explicitly given by

$$\begin{aligned}
 f(r, u) \approx & \frac{C_2 n(R)}{v_c^3(R)} \exp \left\{ - \int_R^r \left(\frac{2}{\beta} + 4 \right) \frac{r' [B_z(r') - B_0]}{B_0 r'^2 - 2 \psi(r')} dr' \right\} \\
 & \cdot U_s(v_c - u) \left\{ 1 - \frac{u^2}{v_c^2(r)} \right\} \\
 & \cdot \exp \left\{ - \frac{u^2}{\beta(r) v_c^2(r)} \right\},
 \end{aligned} \tag{5.31}$$

where v_c , $U_s(x)$ and C_2 are defined by Eqs. (4.7), (4.36) and (4.45), respectively.

As to the distribution function for the source rate, which is required to maintain the steady state, we go back to Eq. (5.28). Recalling that $V_s(v')$ is normalized, Eq. (5.28) implies $C_6 = -\alpha \Gamma$,

$$V_s(v') = \frac{1}{\alpha} D \{ V_0 \}, \tag{5.32}$$

and

$$W_s(r) = -\alpha \Gamma \frac{W^2(r)}{v_c^3(r)}, \tag{5.33}$$

where

$$\alpha \equiv \int_0^1 4\pi v'^2 D\{V_0(v')\} dv' < 0 \quad (5.34)$$

We may use the results in Section 4.2, $V_0(v') \approx V_1(v')$ and $D\{V_1\} = \lambda_1 V_1$, to obtain $\alpha = \lambda_1$,

$$V_s(v') \approx V_0(v'), \quad (5.35)$$

and

$$W_s(r) \approx -\lambda_1 \Gamma \frac{W^2(r)}{v_c^3(r)} \quad (5.36)$$

Similarly, substituting Eqs. (5.35) and (5.36) into Eq. (5.20) and changing v' back to u , we obtain the distribution function for the source rate, given by

$$\begin{aligned} S(r, u) \approx & - \frac{\lambda_1 \Gamma C_2 \pi^2(R)}{v_c^6(R)} \cdot \frac{1}{v_c^3(r)} \\ & \cdot \exp \left\{ - \int_R^r \left(\frac{4}{\beta} + 8 \right) \frac{r' [B_z(r') - B_0]}{B_0 r'^2 - 2\psi(r')} dr' \right\} \\ & \cdot U_s(v_c - u) \cdot \left\{ 1 - \frac{u^2}{v_c^2(r)} \right\} \exp \left\{ - \frac{u^2}{\beta v_c^2(r)} \right\} \end{aligned} \quad (5.37)$$

Now we consider a finite-length FRTP. Since the axial length is still much longer than the radius, we have $B_z \gg B_r$. Therefore, we use the approximation,

$$v_z B_r \ll v_r B_z. \quad (5.38)$$

In addition, since we know that, in the above limiting case (i.e., an infinitely long system), the kinetic equation is separable, we carry out the same separation-of-variables attempt for this case. With the approximation,

$$\frac{v_z}{v_r} \frac{\partial \eta}{\partial z} \approx \pm \frac{\partial \eta}{\partial z}, \quad (5.39)$$

following the similar procedures as for the above case, Eq. (5.21) is again separated into a set of simpler equations. Three of the four separated equations are the same as before, i.e., Eqs. (5.26), (5.32) and (5.33); the only one changed is Eq. (5.27) which is now given by

$$\frac{\partial W}{\partial r} + a_1 \frac{\partial W}{\partial z} = \frac{q^2 B_0}{m^2 c^2} (-2) \left(\frac{1}{\beta} + 2 \right) \frac{r(B_z - B_0) W}{v_c^2}, \quad (5.40)$$

where $a_1 = \pm 1$, and the two cases for different values of a_1 have equal probabilities.

Equation (5.40) is a first-order, linear, partial differential equation and can be solved by the standard method.⁴⁸ The general solution is given by

$$W(r, z) = P(a_1 r - z) Q(r, z), \quad (5.41)$$

where $P(a_1 r - z)$ is an arbitrary function of $a_1 r - z$,

$$Q(r, z) \equiv g(r, C_7 = a_1 r - z), \quad (5.42)$$

$g(r, C_7)$ is an indefinite integral defined by

$$g(r, C_7) = \exp \left\{ \int h(r, C_7) dr \right\}, \quad (5.43)$$

$$h(r, C_7) \equiv \gamma(r, z = a_1 r - C_7), \quad (5.44)$$

and

$$\gamma(r, z) = -\left(\frac{2}{\beta} + 4\right) \frac{r [B_z(r, z) - B_0]}{B_0 r^2 - 2\psi(r, z)}. \quad (5.45)$$

To determine the proper form of the function $P(a_1 r - z)$ in the general solution, Eq. (5.43), for the present case, we need to employ the solution for the previous limiting case. Since Eq. (5.43) should reproduce Eq. (5.38) by taking $a_1 = 0$, we have

$$P(a_1 r - z) = \text{constant} \quad (5.46)$$

If, say, the density at the field null, $n(R,0)$, is known, then

$$W(r,z) = \frac{n(R,0)}{v_c^3(R,0)} Q(r,z). \quad (5.47)$$

By Eq. (5.17),

$$n(r,z) = n(R,0) \frac{v_c^3(r,z)}{v_c^3(R,0)} Q(r,z), \quad (5.48)$$

where $Q(r,z)$ is defined by Eq. (5.44), and the integration limits of Eq. (5.45) is, now, from R to r .

Since the other three separated equations are the same as the previous case, from Eqs. (5.38) and (5.39), the solutions for $f(r,z,u)$ and $S(r,z,u)$ are finally given by

$$f(r,z,u) \approx \frac{C_2 n(R,0)}{v_c^3(R,0)} Q(r,z) U_5(v_c(r,z) - u) \cdot \left\{ 1 - \frac{u^2}{v_c^2(r,z)} \right\} \exp \left\{ -\frac{u^2}{\beta v_c^2(r,z)} \right\}, \quad (5.49)$$

and

$$\begin{aligned}
S(r, z, u) \approx & (-\lambda_1) \Gamma C_2 \frac{\pi^2(R, 0)}{v_c^6(R, 0)} \frac{Q(r, z)}{v_c^3(r, z)} \\
& \cdot U_s(v_c(r, z) - u) \cdot \left\{ 1 - \frac{u^2}{v_c^2(r, z)} \right\} \\
& \cdot \exp \left\{ - \frac{u^2}{\beta v_c^2(r, z)} \right\} .
\end{aligned} \quad (5.50)$$

5.3 Applications and Comparisons

For a Hill's vortex configuration, Ψ_{HV} , given by Eq. (3.12), the magnetic field is given by $B = B_r r + B_z z$, where

$$B_r = \begin{cases} -\frac{3B_0}{2kR_s^2} r\zeta & \text{if } r^2 + \zeta^2 \leq R_s^2, \\ -\frac{3B_0 R_s^3}{2k} \frac{r\zeta}{(r^2 + \zeta^2)^{5/2}} & \text{if } r^2 + \zeta^2 \geq R_s^2, \end{cases} \quad (5.51)$$

$$B_z = \begin{cases} -\frac{3B_0}{2R_s^2} (R_s^2 - 2r^2 - \zeta^2), & \text{if } r^2 + \zeta^2 \leq R_s^2, \\ \frac{B_0}{2} \left\{ 2 + \frac{R_s^3 (r^2 - 2\zeta^2)}{(r^2 + \zeta^2)^{5/2}} \right\}, & \text{if } r^2 + \zeta^2 \geq R_s^2, \end{cases} \quad (5.52)$$

$\zeta = z/k$, and k is the elongation factor. We now apply the results obtained in Section 5.2 to this configuration.

For the infinitely long system, if $\beta = \text{constant}$, Eq. (5.30) gives

$$n(r) = C_8 \left(\frac{r}{R_s} \right)^6 \left(\frac{5R_s^2}{3r^2} - 1 \right)^{-\left(\frac{1}{\beta} + \frac{1}{2}\right)} \quad (5.53.1)$$

for $r \leq R_s$, and

$$n(r) = n(R_s) \left(\frac{r}{R_s} \right)^{-\left(\frac{1}{\beta} + 3.5\right)} \quad (5.53.2)$$

for $r \geq R_s$, where C_8 is a constant determined by any known value of n in the region $r \leq R_s$. In FRX experiments, the observed rotational speed is in the order of the ion diamagnetic drift speed, i.e., $\omega_{\text{rot}} \approx 0(\omega_{D_i})$

$\ll \omega_{C_{O_i}}$, where $\omega_{\text{rot}} = -v_{\theta 0, \text{exp}}/r \approx \text{constant}$ and $\omega_{D_i} = (c/qn_i r B)(-\partial p_i / \partial r)$.

As mentioned before, if $|v_{\theta 0}| \ll |\omega_{C_{O_i}}| r$, the fifth term in the left-hand side of Eq. (5.3) can be neglected, and the factor $(B_z - B_0)$ in the steady-state solutions is replaced by B_z . To compare with the experimental density profile, we also apply Eq. (5.30) to this case. The density calculated is then given by

$$n(r) = n(R_o) 2^{7 + \frac{2}{\beta}} 7^{-4.3 - \frac{1.4}{\beta}} \cdot \left(\frac{r}{R_s} \right)^{5.4 + \frac{1.2}{\beta}} \cdot \left(5 - \frac{3r^2}{R_s^2} \right)^{4.3 + \frac{1.4}{\beta}} \quad (5.54.1)$$

for $r \leq R_s$, and

$$n(r) = n(R_s) \left(\frac{R_s}{r}\right)^{\frac{1}{\beta} + \frac{7}{2}} \cdot \exp \left\{ -\frac{2}{3} \left(\frac{1}{\beta} + 2\right) \left(\frac{r^3}{R_s^3} - 1\right) \right\} \quad (5.54.2)$$

for $r \geq R_s$.

Figure 5.1 shows a typical radial density profile from FRX experiments.³ Figures 5.2 and 5.3 show the calculated radial density profiles described by Eqs. (5.55) and (5.56), respectively. For comparison, the experimental results from Fig. 5.1 is also shown in Fig. 5.2, where the peak density of the experimental data is chosen to locate at the field null ($r = R_s/\sqrt{2}$). (From Table 4.3, $\beta \approx 0.16$ for FRX experiments.) The comparison agrees qualitatively. The density profiles are peaked in the field-null region, and fall off as moving away from the field null. Quantitatively, the comparison is not favorable in the small- r region because of the fast-loss assumption. The reason for this is the following: As $r \rightarrow 0$, since $v_c \rightarrow 0$, even slow particles are not confined. By the fast-loss assumption, they are lost instantaneously and do not contribute to the particle density. However, in reality, these unconfined slow particles need some time to get lost and still have contribution to the particle density. Therefore, the calculated particle density tend to be underestimated in the region where v_c is small. This accounts for the unfavorable comparison in r -small region. From Figs. 5.2 and 5.3, it is noticed that in the presence of rotation the density peak moves away from the field null and toward the separatrix. Since $(\partial n / \partial t)_{vspl} \propto (n^2 / v_c^3)$,

Figure 5.1. A radial density profile (measured at 10^{-6} s after implosion) from FRX experiments.³

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Figure 5.2. Calculated radial density profiles for Hill's vortex configurations, Eq. (5.53). The experimental profile from Fig. 5.1 is also shown for comparison.

Fig. 5.3. Calculated radial density profiles for Hill's vortex configurations, Eq. (5.54). The experimental profile from Fig. 5.1 is also shown for comparison.

the profile with the presence of rotation has higher particle loss rates because the density peak moves to a v_c -smaller region. Therefore, rotation not only increases the possibility for instabilities but also increases the particle loss rate. We also note that the steady-state density profiles are not sensitive to the values of β , but, as shown before, the particle loss rates are indeed sensitive to the values of β . Figure 5.4 (taken from Ref. 49) shows the change of density profile due to anomalous transport. Comparing Figs. 5.3 and 5.4, it is shown that both the anomalous transport and VSPL have the same effect on particle density, i.e., to reduce particle density in the r -small region and near the separatrix.

For the finite-length system with $\omega_{\text{rot}} \ll \omega_{\text{co}}$, Eq. (5.49) gives, for $r \leq R_s$,

$$n(r, z) \approx C_s \left\{ (d_1)^{\gamma_1} (d_2)^{\gamma_2} (d_3)^{\gamma_3} \right\}^{\gamma_4}, \quad (5.55)$$

where

$$d_1(r, z) = \left| \frac{2A_1 r + A_2 - A_3}{2A_1 r + A_2 + A_3} \right|, \quad (5.56)$$

$$d_2(r, z) = \frac{r^2}{A_1 r^2 + A_2 r + A_4}, \quad (5.57)$$

$$d_3(r, z) = \left| \frac{1}{R_s^2} (A_1 r^2 + A_2 r + A_4) \right|, \quad (5.58)$$

A. DENSITY PROFILE

B. DIAMAGNETIC FREQUENCY PROFILE

Figure 5.4. Profile modification by
anomalous transport.⁴⁹

$$\gamma_1(r, z) = \frac{A_3}{3(a_1 r - z)} \left[\frac{2R_5}{5k^2 R_5^2 - 3(a_1 r - z)^2} - \frac{1}{k^2 + 1} \right], \quad (5.59)$$

$$\gamma_2(r, z) = \frac{3[k^2 R_5^2 - (a_1 r - z)^2]}{2[5k^2 R_5^2 - 3(a_1 r - z)^2]}, \quad (5.60)$$

$$\gamma_3 = \frac{2k^2 + 1}{2k^2 + 2}, \quad (5.61)$$

$$\gamma_4 = \frac{\beta}{2} + 4, \quad (5.62)$$

$$A_1 = -3 \left(1 + \frac{1}{k^2} \right), \quad (5.63)$$

$$A_2(r, z) = \frac{6}{k^2} (a_1 r - z), \quad (5.64)$$

$$A_3(r, z) = 2\sqrt{3} \left[5R_5^2 + \frac{5}{k^2} R_5^2 - \frac{3}{k^2} (a_1 r - z)^2 \right]^{1/2}, \quad (5.65)$$

$$A_4(r, z) = 5R_5^2 - \frac{3}{k^2} (a_1 r - z)^2, \quad (5.66)$$

and C_5 is a constant. Say, if the density at the field null, $n(R,0)$, is known, then

$$C_5 = n(R,0) \left\{ d_1^{\gamma_1} d_2^{\gamma_2} d_3^{\gamma_3} \right\}^{\gamma_4} \bigg|_{(R,0)} \quad (5.67)$$

Figures 5.5 and 5.6 show the density profiles described by Eq. (5.57) for $\beta = 0.16$ and $\beta = 1.0$, respectively. The experimental axial density profile from FRX experiments is shown in Fig. 5.7. Comparing the calculated profiles, Fig. 5.5, with the experimental results, Figs. 5.1 and 5.7, it is noted that both the radial and axial density profiles from the present model agree favorably with the experimental results, except in the v_c -small region where the calculated density appear to be smaller than the experimental data for the reason explained before, i.e., because of the fast-loss assumption.

5.4 Discussion

In the presence of the VSPL, approximate solutions for the steady-state particle distribution function are obtained. Under certain approximations and transformations, the separation-of-variables techniques are applied to the very complicated problem, Eq. (5.3) with a boundary condition Eq. (5.6). Then, the separated equations are solved easily and analytically by invoking some previous results from Chapter 4. For an infinitely long system, an approximate steady-state particle distribution is given by Eq. (5.31), with a source distribution, Eq. (5.37). For a finite-length system, the corresponding expressions are Eqs. (5.49) and (5.50), respectively.

Figure 5.5. The density profile, $n(r,z)$, in the closed-field region of a Hill's vortex magnetic configuration from the present model, i.e., $n(r,z)$ given by Eq. (5.53). Here $\beta = 0.16$ have been used.

Figure 5.6. Same as Fig. 5.5 except $\beta = 1.0$.

Figure 5.7. Axial density profile from FRX experiments.³

The source distribution obtained from solving the kinetic equation, which is required to maintaining the steady-state particle distribution, has the same velocity distribution as the particles, i.e., $S \propto V_0$. Since the source particles also subject to the same loss boundary in velocity space, this required velocity distribution of the source is easy to maintain.

Both the radial and axial profiles of the calculated particle density agree, qualitatively, with the experimental observation. Quantitatively, the theoretical calculations are underestimated in the region where v_c is small, due to the fast-loss assumption.

The characteristics of F RTP's in steady states can be obtained through the particle distribution function. The analytical expressions for the distribution function is especially convenient in many other studies, e.g., parametric reactor studies, stability studies where an equilibrium particle distribution is needed, etc.

In many studies, either $T = T(\psi)$ or $T = \text{constant}$ has been assumed,^{35-37,39-40,49} where T is the temperature, ψ is the magnetic flux defined by Eq. (3.3). That is, the temperature has been assumed to be either a constant or a function of the magnetic flux only. However, as shown before, the confinement region in velocity space is dependent of v_c , where v_c is an explicit function of both r and ψ . Since the confinement criteria are the most fundamental constraints on the confined particles, the particle distribution is dependent of the confinement region in velocity space, thus, $v_c(r, \psi)$. Therefore, the temperature should have explicit dependences on r and ψ . The additional dependence on r should be taken into account, i.e., it is more reliable to assume $T = T(r, \psi)$ than $T = T(\psi)$ or $T = \text{constant}$. Not only for the temperature, the same argument is also

valid for the particle density, n .

Usually, as may be suggested by the experimental data, the particle density profile is arbitrarily assumed to be certain functions of ψ .^{39,40} In this chapter, the density profile is obtained as a function of ψ from theoretical calculations. This gives a good indication on the relation between n and ψ .

CHAPTER 6

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS FOR FUTURE WORK

6.1 Summary and Conclusions from the Present Study

In this work, the velocity-space particle loss (VSPL), i.e., the particle loss due to the existence of a loss region in velocity space, in field-reversed theta pinches (F RTP's) is studied. Important findings from the present work are summarized as follows.

The particle confinement criteria for F RTP's are obtained and given by inequalities for the particle total energy, H , and canonical angular momentum, P_θ . Both the lower and upper bounds of H and P_θ are found necessary to impose, i.e., $H_{\min} < H < H_{\max}$ and $P_{\theta,\min} < P_\theta < P_{\theta,\max}$. It is found that only a portion of (H, P_θ) space, or equivalently, of velocity space, is available for particle confinement. A loss region in velocity space is found.

As in the case of the loss cone in $(v_\parallel, v_\perp)$ space for simple mirrors, the loss region in velocity space for F RTP's has very important effects on the plasma behavior. First, particles are lost due to collisions by being scattered into the loss region in velocity space. Secondly, the velocity distribution of the confined particles in collisional equilibrium is no longer a pure Maxwellian, since particles with certain velocities are not confined.

For particle confinement, the velocity-space particle loss due to collisions is calculated from the Fokker-Planck equation. The scaling law for particle confinement is obtained. Without assuming any instability, the

particle confinement time based on the velocity-space particle loss alone can account for the experimentally observed particle confinement time. This provides one more possibility for the particle loss mechanism in FRTP's. This also shows the importance of the velocity-space particle loss.

The velocity-space particle loss increases as the plasma temperature increases. Therefore, for reactor-grade plasmas, the velocity-space particle loss is expected to have an even more important effect on the particle confinement than it is for the previous experiments.

For the velocity distribution in collisional equilibrium, a simple analytical expression, V_0 , is verified to be a good representation of the actual velocity distribution in the presence of the loss region in velocity space. In steady-state, simple analytical approximate solutions for the whole particle distribution function, $f(\underline{r}, \underline{v})$, are obtained, from which the characteristics of the FRTP plasmas can be known. The analytical expressions for the distribution function are especially convenient in many other studies, e.g., parametric reactor studies, stability studies where an equilibrium particle distribution is needed, etc.

It is shown that both the temperature and the density should have explicit dependences on r and Ψ , while they are usually assumed to be functions of Ψ only. In this study, the density is obtained as an explicit function of Ψ from theoretical calculations. This gives a good indication in the relation between the density and the magnetic flux function. In comparison with the experimental observation, both the radial and axial density profiles agree qualitatively; however, the calculated density is underestimated in the v_c -small region because of the fast-loss assumption.

It is shown that, if a plasma originally is at rest in the laboratory frame, the collisional velocity-space particle loss will make the ion (electron) fluid start rotating in the negative (positive) θ -direction, and the rotation speed will be accelerated toward a negative (positive) value $-qB_0r/mc$. Calculations based on the velocity-space particle loss are consistent with the experimental observation: the occurrence and spin-up of ion rotation in the negative θ -direction. Therefore, the velocity-space particle loss alone can explain the occurrence and spin-up of plasma rotation in experiments.

As the rotational speed increases, the location of the peak density moves radially outward, thus, larger density gradient at the separatrix. The combination of the higher rotational speed (or rotational energy) and the larger density gradient at the separatrix represents a stronger driving force for instabilities. This may have contribution to the experimentally observed rotational instability which occurs after the rotational speed exceeds certain value.

In conclusion, the velocity-space particle loss is shown to be an important particle loss mechanism for F RTP plasmas. Based on this VSPL model alone, the experimental results, including radial density profile, axial density profile, particle confinement time, occurrence of rotation, and spin-up of rotation, can be explained. Therefore, the VSPL model has the possibility to be the dominant mechanism for particle loss and the sole cause for plasma rotation. This model also provides an alternative explanation for the observed plasma behavior in F RTP's, in addition to some other models, e.g., the diffusion model based on lower-hybrid drift wave,⁴⁰ and the torsional wave model induced by end-shortening.⁴⁹

6.2 Recommendations for Future Work

A few of further improvements and applications of the present work are listed as follows.

(1) Modify the fast-loss assumption. In the present study, the fast-loss assumption has been made, which may not be good for slow particles. To improve the present model, one may want to modify this assumption.

(2) Incorporate the spatial diffusion with the velocity-space particle loss. As shown in Chapter 5, when the streaming terms in the kinetic equation, i.e., the terms associated with $\partial f / \partial \underline{r}$ and $\partial f / \partial \underline{v}$, are included, the particle distribution function, f , is not exactly isotropic. To improve the present results, one may want to get more accurate solution for f , under the combination of the spatial diffusion due to various driving forces (e.g., $\partial T / \partial \underline{r}$, $\partial n / \partial \underline{r}$, $\underline{E}$) and the velocity-space particle loss due to the loss region in velocity space.

(3) Study source effects on a background plasma. In practice, there are restrictions on an external source distribution, arising from particle injections. A realistic source that can be achieved (as a function of position and velocity) may not be ideal for particle confinement. It is, therefore, important to study source effects on a background plasma such that conclusions for more realistic situations can be obtained.

(4) Find a self-consistent magnetic field. In the present model, a known static magnetic configuration has been assumed; the consistency between the assumed B field and the B field produced by plasma current profile, which is obtained from the solution of particle distribution function, has not been pursued. Since the steady-state density profile,

current profile, velocity distribution, etc., depend on magnetic configuration, it may be desirable to obtain a self-consistent B field so that a better model can be available for further development of F RTP type reactor studies.

DERIVATION OF EQUATION (5.3)

Let us start with the kinetic equation,

$$\begin{aligned} \frac{\partial f(\underline{r}, \underline{v}, t)}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{r}} + \frac{q}{m} (\underline{E} + \frac{\underline{v}}{c} \times \underline{B}) \cdot \frac{\partial f}{\partial \underline{v}} \\ = \left(\frac{\partial f(\underline{r}, \underline{v}, t)}{\partial t} \right)_c + S(\underline{r}, \underline{v}, t) \end{aligned} \quad (\text{A.1})$$

We want to change Eq. (A.1) from the laboratory frame to a frame moving with a velocity $\underline{V}(\underline{r}, t)$. The velocity in the new coordinates, $\underline{u}$, is given by

$$\underline{u} = \underline{u}(\underline{r}, \underline{v}, t) \equiv \underline{v} - \underline{V}(\underline{r}, t), \quad (\text{A.2})$$

or

$$\underline{v} = \underline{u} + \underline{V}(\underline{r}, t) \quad (\text{A.2})$$

For convenience in notation, we designate $g(\underline{r}, \underline{u}, t)$ as the particle distribution function in the moving frame, i.e.,

$$g(\underline{r}, \underline{u}, t) \equiv f(\underline{r}, \underline{u} + \underline{V}(\underline{r}, t), t) = f(\underline{r}, \underline{v}, t) \quad (\text{A.3})$$

Namely, $g(\underline{r}, \underline{u}, t)$ is obtained by directly replacing $\underline{v}$ by $\underline{u} + \underline{V}(\underline{r}, t)$ in $f(\underline{r}, \underline{v}, t)$. Similarly, we designate the source rate as $h(\underline{r}, \underline{u}, t)$, i.e.,

$$h(\underline{r}, \underline{u}, t) \equiv S(\underline{r}, \underline{u} + \underline{V}(\underline{r}, t), t) = S(\underline{r}, \underline{v}, t). \quad (\text{A.4})$$

Consider the first term in Eq. (A.1),

$$\left. \frac{\partial f(\underline{r}, \underline{v}, t)}{\partial t} \right|_{(\underline{r}, \underline{v})} = \left. \frac{\partial f(\underline{r}, \underline{v}, t)}{\partial t} \right|_{(\underline{r}, \underline{v})} \quad (\text{A.5})$$

which is the rate of change of f with respect to t while $\underline{r}$ and $\underline{v}$ are held constant. The subindices are the variables which are held constant. Now, we want to express this by $g(\underline{r}, \underline{u}, t)$. Since $\underline{u} = \underline{u}(\underline{r}, \underline{v}, t)$, and using Eqs. (A.2) and (A.3), we have Eq. (A.5) rewritten as

$$\begin{aligned} & \left. \frac{\partial f(\underline{r}, \underline{v}, t)}{\partial t} \right|_{(\underline{r}, \underline{v})} \\ &= \left. \frac{\partial f(\underline{r}, \underline{u} + \underline{V}, t)}{\partial t} \right|_{(\underline{r}, \underline{u} + \underline{V})} \\ &= \left. \frac{\partial g(\underline{r}, \underline{u}(\underline{r}, \underline{v}, t), t)}{\partial t} \right|_{(\underline{r}, \underline{v})} \end{aligned}$$

$$\begin{aligned}
&= \left. \frac{\partial g(r, u, t)}{\partial t} \right|_{(r, u)} + \left. \frac{\partial g(r, u, t)}{\partial u} \right|_{(r, t)} \cdot \left. \frac{\partial u(r, v, t)}{\partial t} \right|_{(r, v)} \\
&= \left. \frac{\partial g(r, u, t)}{\partial t} \right|_{(r, u)} + \left. \frac{\partial g(r, u, t)}{\partial u} \right|_{(r, t)} \cdot \left(- \left. \frac{\partial V(r, t)}{\partial t} \right|_{(r)} \right) \\
&= \frac{\partial g(r, u, t)}{\partial t} - \frac{\partial V(r, t)}{\partial t} \cdot \frac{\partial g(r, u, t)}{\partial u} \quad (A.6)
\end{aligned}$$

Similarly, we obtain

$$\frac{\partial f(r, v, t)}{\partial t} = \frac{\partial g(r, u, t)}{\partial r} - \frac{\partial V(r, t)}{\partial r} \cdot \frac{\partial g(r, u, t)}{\partial u} \quad (A.7)$$

and

$$\frac{\partial f(r, v, t)}{\partial v} = \frac{\partial g(r, u, t)}{\partial u} \quad (A.8)$$

Furthermore, since the collision process depends only on the relative velocity between the two colliding particles, the collisional term, $(\partial f / \partial t)_c$, is unchanged, i.e.,

$$\left(\frac{\partial f(\underline{r}, \underline{v}, t)}{\partial t} \right)_c = \left(\frac{\partial g(\underline{r}, \underline{u}, t)}{\partial t} \right)_c \quad (\text{A.9})$$

Substituting Eq. (A.2) and Eqs. (A.4) - (A.9) into Eq. (A.1), we obtain the kinetic equation in the moving frame:

$$\begin{aligned} & \frac{\partial g(\underline{r}, \underline{u}, t)}{\partial t} + \underline{V}(\underline{r}, t) \cdot \frac{\partial g}{\partial \underline{r}} + \underline{u} \cdot \frac{\partial g}{\partial \underline{r}} \\ & - \left(\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \frac{\partial \underline{V}}{\partial \underline{r}} \right) \cdot \frac{\partial g}{\partial \underline{u}} - \underline{u} \cdot \frac{\partial \underline{V}}{\partial \underline{r}} \cdot \frac{\partial g}{\partial \underline{u}} \\ & + \frac{q}{m} \left(\underline{E} + \frac{1}{c} \underline{u} \times \underline{B} + \frac{1}{c} \underline{V} \times \underline{B} \right) \cdot \frac{\partial g}{\partial \underline{u}} \\ & = \left(\frac{\partial g(\underline{r}, \underline{u}, t)}{\partial t} \right)_c + h(\underline{r}, \underline{u}, t), \end{aligned} \quad (\text{A.10})$$

where the notation,

$$\underline{P} \cdot \frac{\partial \underline{R}}{\partial \underline{Q}} \cdot \frac{\partial \underline{Y}}{\partial \underline{X}} \equiv \sum_{i=1}^3 \sum_{j=1}^3 P_i \frac{\partial R_j}{\partial Q_i} \frac{\partial Y}{\partial X_j}, \quad (\text{A.11})$$

has been used.

For a system under the following conditions: (1) steady-state, i.e., $\partial/\partial t = 0$, (2) axisymmetric, i.e., $\partial/\partial \theta = 0$, (3) no macroscopic electric field, i.e., $\underline{E} = 0$, (4) the moving frame with a velocity $\underline{V}_0 = v_{\theta 0}(r) \hat{\theta}$,

where, $v_{\theta 0}$ is a function of r only, then, Eq. (A.10) reduces to, after some straight forward calculations,

$$\begin{aligned}
 & u_r \frac{\partial g(r, z, u)}{\partial r} + u_z \frac{\partial g}{\partial z} + \frac{\hbar}{mc} \underline{u} \times \underline{B} \cdot \frac{\partial g}{\partial \underline{u}} \\
 & - u_r \frac{\partial v_{\theta 0}}{\partial r} \frac{\partial g}{\partial u_{\theta}} + \frac{v_{\theta 0}^2}{r} \left(1 + \frac{u_{\theta}}{v_{\theta 0}} \right) \frac{\partial g}{\partial u_r} \\
 & + \frac{\hbar}{mc} v_{\theta 0} \left(B_z \frac{\partial g}{\partial u_r} - B_r \frac{\partial g}{\partial u_z} \right) = \left(\frac{\partial g}{\partial t} \right)_c + h(r, z, u) \quad \text{(A.12)}
 \end{aligned}$$

Finally, changing g and h back to the original notations, f and S , respectively, we obtain Eq. (5.3).

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