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**VERTICAL COHERENT INSTABILITIES
IN BUNCHED-PARTICLE BEAMS**

Ronald D. Ruth

July 1981

ACCELERATOR DEPARTMENT

**BROOKHAVEN NATIONAL LABORATORY
UPTON, NEW YORK 11973**

VERTICAL COHERENT INSTABILITIES IN BUNCHED-PARTICLE BEAMS*

Ronald D. Ruth

DISCLAIMER

July 1981

***This is the text of a dissertation submitted for partial fulfillment of the degree of Doctor of Philosophy
at the State University of New York at Stony Brook.**

ACCELERATOR DEPARTMENT

**BROOKHAVEN NATIONAL LABORATORY
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Abstract of the Dissertation
Vertical Coherent Instabilities in Bunched-Particle Beams

By

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1981

The purpose of this paper is to study the vertical coherent instabilities which occur in bunched particle beams. The problem is complicated by the fact that the velocity of a single particle in a bunch is not constant, but rather consists of an equilibrium velocity and an oscillation about that. This "synchrotron" oscillation occurs at a frequency which is in general much less than the other characteristic frequencies of the system: the revolution frequency and the transverse betatron frequencies. The approach used here to study coherent instabilities illuminates the effect of the synchrotron frequency in setting the time scale for an instability, without making restrictive assumptions on the relative size of the synchrotron frequency and the coherent frequency shift (or growth rate).

To accomplish this we use the single particle equations of motion and the Vlasov equation to obtain an integral equation which governs the coherent motion of a single bunch of particles. The solutions to the integral equation are then examined in two different regions.

For the case of a bunch long compared to the characteristic wavelength of the instability we find two regions of instability. The first is a region of slow blowup at low currents (growth rate less than the synchrotron frequency). This instability is basically the head-tail effect which in this case is sensitive to the local cancellation of the impedance at positive and negative frequencies. The second is a region of fast blowup (growth rate much greater than the synchrotron frequency). This is a coasting-beam-like instability sensitive to the peak current in a bunch. Connecting the two regions is a striking transition, a "phase transition", in which the growth rate increases by two orders of magnitude.

For the case of a bunch shorter than the characteristic wavelength of the instability the behavior is quite different. For low currents the head-tail effect is again important; however, in this case moments of the impedance times a bunch factor couple to drive the instability. For larger currents there is a new instability (the parallel to the fast blowup for a long bunch) which arises even for zero chromaticity.

Table of Contents

List of Illustrations.....	vi
List of Symbols.....	vii
Acknowledgments.....	ix
I. Introduction.....	1
II. Equations of Motion.....	6
Single Particle Equations of Motion.....	6
The Linearized Vlasov Equation.....	15
The Coherent Force.....	16
III. The Integral Equation.....	21
Transforming the Vlasov Equation.....	21
The Gaussian Distribution and Kernel.....	28
The "Dispersion Integral" $B(p, p', \lambda)$	29
IV. Long Bunch - Short Wavelength.....	32
High Frequency Fast Blowup.....	33
The Dispersion Relation.....	36
Slow Blowup and the Head-Tail Effect.....	42
The General Solution for Broad Band Impedance at High Frequency.....	49
Discussion - The "Phase Transition".....	56
V. Short Bunch - Long Wavelength.....	62
The Head-Tail Effect.....	64
The Coupling of Modes: A More General Solution.....	70
Discussion.....	75
VI. Conclusion & Summary.....	77
References.....	83
Appendix 1.....	84

List of Illustrations

- 1) The Coordinate System.
- 2) Upper Bound on the Growth Rate for ISABELLE [$\sigma(\omega_c + \omega_\xi) = 35$]
- 3) The Stability Diagram for Fast Blowup. (Map of $b(35, \lambda)$)
- 4) Growth rate vs Current for Fast Blowup. (ISA parameters)
- 5) Imaginary Part of $G_{HT}(\omega)$ for $\sigma\omega_c = 30$ and $\sigma\omega_\xi = +5$
- 6) Growth Rate vs Current, ISA Fast Blowup and the Head-Tail Effect.
- 7) Growth Rate vs Current, Equation (161), Locally Pure Imaginary $G(\omega_c)$, Chromaticity, $\xi = 0$
- 8) The Phase Transition. Growth rate vs Current. (ISA impedance).
- 9) The Phase Transition. Growth rate vs Current ISA-like Impedance with $\text{Arg } G(\omega_c) = 45^\circ$.
- 10) Upper Bound on Growth Rate Evaluated at $\sigma(\omega_c + \omega_\xi) = 4$

Table 1. ISABELLE Parameters for the Injected Bunch.

List of Symbols
(in order of appearance)

s	distance along reference orbit
ρ	radius of curvature
$\vec{r}_0(s)$	the reference orbit
$\vec{a}, \vec{b}, \vec{c}$	unit vectors
x, y	horizontal and vertical excursion
p_s, p_x, p_y	canonical momenta
A_s, A_x, A_y	vector potential
ϕ	scalar potential
H	Hamiltonian
v_0	synchronous velocity
R	Average Radius
C	circumference
ω_0	v_0/R , revolution frequency
c	speed of light
γ	$[1 - v^2/c^2]$
K_1, K_2	focusing functions
Δp	$(p_s - p_0)$
δ	$s - v_0 t$
γ_T	transition energy
v_x, v_y	horizontal and vertical
	betatron tune
η	$(1/\gamma_T^2 - 1/\gamma^2)$, frequency slip factor
ω_s	synchrotron frequency
ξ	$\frac{\Delta p}{p} \frac{\Delta v}{v}$, the chromaticity
τ	$(s - v_0 t)/v_0$, longitudinal position (time units)
p_τ	$\Delta p v_0$, canonical momenta

H_c	coherent Hamiltonian
F	vertical coherent force
J_y	action variable (vertical)
θ_y	angle variable (vertical)
J_s	action variable (longitudinal)
θ_s	angle variable (longitudinal)
a	$1 - \xi/\eta$
$\tilde{\tau}$	amplitude variable ($\tau = \tilde{\tau} \cos \theta_s$)
$\dot{\tilde{\tau}}$	$-\omega_s \tilde{\tau} \sin \theta_s = -\eta \Delta p/p$
ψ	Distribution function - phase space
ω	the coherent frequency
$g(t)$	wake field function
$G(\omega)$	fourier transform of $g(t)$
$Z_{\perp}(\omega)$	transverse impedance
Γ	width of $G(\omega)$
ω_c	cutoff frequency
$D(\tau)$	dipole density
$D(k)$	Fourier transform of $D(\tau)$
λ	$(\omega + \omega_y)/\omega_s$
N	number of particles in a bunch
σ	τ_{rms}
σ_o	$\tau \omega_o$ rms
β	$\text{Im}(\lambda)$
I	Current
$B(pp', \lambda), C(pp', \lambda)$	"Dispersion integral's"
ω_{ξ}	$\xi/\eta \omega_y$
Δ	$\alpha(\omega_c + \omega_{\xi})$
$G_{\pm}$	$G(\omega_c \pm \omega_{\xi})$
C_{μ}	short bunch matrix elements

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1. INTRODUCTION

A particle in an accelerator experiences focusing forces which localize it both transverse to the direction of motion and along the direction of motion. In the absence of other complications the motion is essentially a 3 dimensional harmonic oscillation which can be made stable by a proper choice of frequencies. In addition to the externally applied fields, there are also electromagnetic fields produced by the beam of particles and the beam environment. One might expect that the effect of these fields is simply to modify the externally applied fields slightly, but there is an additional complication.

Consider a collection of particles oscillating coherently in the transverse external focusing field. This oscillation induces fields in the surrounding environment which in turn can act back on the oscillating beam. Qualitatively, if these fields are shifted in phase, the coherent oscillation can "force itself" and thus lead to a coherent instability. This type of instability is the subject of this paper.

Coherent instabilities have been observed in essentially all accelerators and consequently have been studied by many people.¹⁻⁷ In fact questions of the coherent stability of a beam of particles have taken a place along side questions of single

particle stability and now constitute a major consideration for the design of any accelerator.

With the upcoming new accelerators, ISABELLE at Brookhaven National Laboratory, and LEP at CERN, we are led to reformulate the theory of transverse coherent instabilities. In addition, the recent discovery of a "high frequency fast blowup" region for longitudinal instabilities by Wang and Pellegrini⁴, and the discovery of a new vertical instability at PETRA^{6,7} point to the need for a more precise formulation of the transverse problem.

In the case of longitudinally uniform beams (coasting beams) the analysis is straightforward and well understood. For bunched beams, however, the problem is much more mathematically complicated, and due to this complication many approximations have been used to understand the mechanism of bunched beam instability.

We can divide the problem into two categories: coupled bunch modes and single bunch phenomena. Coupled bunch modes of oscillation have been well understood within the approximation that the bunches are rigid.² The additional complication of non-rigid oscillations can also be included but will not be considered here.

The problem of the non-rigid transverse oscillation of a single bunch has been formulated by C. Pellegrini³ within the approximation that the coherent force is in some sense small compared to the longitudinal focusing force. However, the

problem is still quite complex since it involves the solution of an integral equation. Qualitatively, the circulation longitudinally of the particles within the bunch can lead to the following circumstance: a particle in the tail of the bunch, which is being forced by the field left by the head of the bunch, can return to the head of the bunch, thus in turn forcing particles which lie at the tail. This internal feedback can lead to an instability, the head-tail effect.

The bunched beam problem has also been formulated by F. Sacherer⁵ with an emphasis on a quantum-mechanical-like perturbation theory. A formula is obtained for the coherent frequency in terms of the (presumed known) eigenfunctions of the problem. However, this approach is primarily useful as a phenomenological tool and does not "diagonalize" the integral equation.

The mechanism of instability for a coasting beam is quite different. In this case the beam is uniform longitudinally, so it is not surprising that the instability is in the form of a plane wave. Since the fields due to resistivity in the vacuum chamber (or other objects in the environment) shift the phase of the field which acts back on the beam, the wave can be unstable.

The reason that the bunched beam problem is more difficult than the coasting beam problem is that in a bunched beam the solution must be localized longitudinally at least as much as the

normal bunch. This means that the solution is a superposition of plane waves, a wave packet, rather than just a plane wave, and therefore the eigenfunction of the problem is not transparent. An additional complication, which in turn is essential for the head-tail effect, is the circulation of the particles from the head to the tail of the bunch.

With all this complication we can still ask the question: can we imagine an instability for a bunched beam which is different from the head-tail effect? The answer to this question is yes.

To do this imagine a bunch of particles which is long in some sense. In addition assume that the growth time is short compared to the period of longitudinal circulation (the synchrotron period). In this case the mechanism for the head-tail effect is absent, and the approximation used to study the effect is not valid. (The coherent force associated with such an instability must be large compared to the longitudinal focusing force.) On the other hand we can imagine a coasting-beam-like instability in this case with the solution being a wave packet rather than a plane wave. However, in order to discover such an instability, we must formulate the problem so that the coasting beam limit and the head-tail limit are transparent, and avoid any restrictive assumptions on the relative size of the synchrotron frequency and the frequency shift due to the coherent force.

With this in mind we present a new formulation of the transverse coherent instability for bunched beams. Since we are primarily interested in single bunch effects, we will emphasize wake fields which are short compared to the distance between bunches. The main points are:

1) to investigate instabilities which might have rise times which are less than a synchrotron period (a coasting beam like instability for a bunch long compared to the characteristic wavelength of the instability).

2) to study the transition between the head-tail region (slow blowup) and the new region of the coasting-beam-like instability (fast blowup).

3) to investigate the corresponding instability in the case when the bunch length is less than the characteristic wavelength of the instability.

To begin, in the next section we introduce equations of motion, first those for a single particle and then the equation for coherent motion, the Vlasov equation.

II. THE EQUATIONS OF MOTION

The Single Particle Equations of Motion

In this section we introduce the equation of motion for a particle in a circular accelerator. Since this subject has been treated in great detail elsewhere,⁸ we will only touch on a few main points.

The particles circulating in an accelerator are subjected to transverse forces which keep them close to an ideal orbit. In addition during acceleration they are "bunched" longitudinally so that as we move with the particle, the motion is roughly that of a three dimensional harmonic oscillator. Since the particle is bent by a vertical magnetic field and since the longitudinal oscillations change the energy, there is coupling between the 3 degrees of freedom.

To treat the problem analytically we assume that the particle moves in the vicinity of a planar reference orbit and is characterized by the Frenet-Serret curvilinear coordinates,

Fig. 1. The Coordinate System.

s = distance along the reference orbit.

$\rho(s)$ = local radius of curvature of the reference orbit.

$\vec{r}_0(s)$ = the reference orbit.

$\vec{b} = d\vec{r}_0/ds$ = the unit tangent vector.

$\vec{a}$ = unit vector in the orbit plane perpendicular to $\vec{b}$ and directed outward.

$\vec{c} = \vec{a} \times \vec{b}$ = the binormal or vertical unit vector.

We also have the following relations:

$$\frac{d\vec{a}}{ds} = \frac{\vec{b}}{\rho} \quad \frac{d\vec{b}}{ds} = \frac{-\vec{a}}{\rho} \quad \frac{d\vec{c}}{ds} = 0 \quad (1)$$

An arbitrary point near the reference orbit is characterized by the coordinates (x, s, y) :

$$\vec{r} = \vec{r}_0(s) + x \vec{a} + y \vec{c} \quad (2)$$

To obtain the equations of motion we will use the Hamiltonian formulation. It is easy to show that in the new coordinate system the Hamiltonian is given by⁸

$$H = e\phi + c \left[m^2 c^2 + \frac{(p_s - e/cA_s)^2}{(1+x/\rho)^2} + (p_x - e/cA_x)^2 + (p_y - e/cA_y)^2 \right]^{1/2} \quad (3)$$

where

$$p_s = \vec{p} \cdot \vec{b} (1 + x/\rho), \quad p_x = \vec{p} \cdot \vec{a}, \quad p_y = \vec{p} \cdot \vec{c}$$

are the canonical momentum, and the vector potential A_s ,

A_x, A_y is given by the analogous relation for $\vec{A}$.^{*} Once we specify the fields via the vector and scalar potentials, the equations of motion are given by Hamiltons equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = - \frac{\partial H}{\partial q_i} \quad (4)$$

The basic external fields in an accelerator are the bending field, a dipole magnetic field; the focusing field, a quadrupole magnetic field; and the accelerating electric field, which we will represent by a wave. If we choose a gauge in which $\phi = 0$, the vector potential is given by

$$\begin{aligned} -e/c A_s &= p_0 (x/\rho + \frac{K_1 x^2}{2} + \frac{K_2 y^2}{2} + \dots) + \underbrace{\frac{eV}{2\pi v_0} [\cos(\omega_0 t - s/R) - 1]}_{\text{Accelerating or bunching}} (1+x/\rho) \\ -e/c A_x &= -\frac{x}{2R} \frac{eV}{2\pi v_0} \sin(\omega_0 t - s/R) \\ -e/c A_y &= \frac{-y}{2R} \frac{eV}{2\pi v_0} \sin(\omega_0 t - s/R) \end{aligned} \quad (5)$$

where $V/2\pi R$ is the peak electric field, v_0 is the velocity of the particle which is synchronous longitudinally, $R = C/2\pi$ where, C is the circumference, $\omega_0 \equiv v_0/R$ is the revolution frequency,

* Note that $A_s = \vec{A} \cdot \vec{b} (1 + x/\rho)$.

$p_0 \equiv \gamma m v_0$ is the momentum of the synchronous particle, ρ is the bending radius, and K_1, K_2 are related to the gradients of the magnetic field. For the purposes of this paper we take p_0 to be a constant so that the effect of the accelerating field is just to induce a longitudinal oscillation.

The usual approach⁸ at this point is to let s be the independent variable rather than the time, t . This is useful in solving the single particle motion because ρ_1, K_1 , and K_2 are periodic functions of s , and this allows the use of Floquet theory in the solution of the resulting differential equations. However, for our purposes it is more convenient to derive equations with time as the independent variable. First let us go to a moving coordinate system by the canonical transformation:

$$\begin{pmatrix} p_s \\ s \end{pmatrix} \rightarrow \begin{pmatrix} \Delta p \\ \delta \end{pmatrix} \text{ such that } \begin{aligned} p_s &= p_0 + \Delta p \\ \delta &= s - v_0 t \end{aligned} \quad (6)$$

the generating function⁹ is

$$F_2(s, \Delta p, t) = (p_0 + \Delta p)s - \Delta p v_0 t \quad (7)$$

so that

$$H' = H + \partial F_2 / \partial t = H - \Delta p v_0 \quad (8)$$

Since p_0 is large compared to e/cA_s , p_x and p_y , we can expand the square root to arrive at

$$H = E_0 + \frac{\Delta p^2}{2\gamma m} \frac{1}{\gamma^2} - \frac{eV\delta^2}{4\pi R^2} - v_0 \Delta p x / \rho + \frac{\gamma m v_0^2 K_1 x^2}{2} + \frac{p_x^2}{2\gamma m} + \dots \quad (9)$$

where $\gamma^2 \equiv 1/(1 - v_0^2/c^2)$. We have kept terms up to 2nd order and suppressed the y variables to simplify the expression.

Recall that we are primarily interested in studying coherent motion. We expect that the coherent forces due to the charge distribution will be much less than the external transverse forces. Since this is true we expect instabilities to occur on a time scale long compared to the period of oscillation in the transverse direction. Thus, we expect that only the gross characteristics of the transverse oscillation will be relevant. Consistent with these considerations we replace the true "Betatron" motion, which one obtains from the rigorous solution of the equations of motion, by harmonic oscillations. In our Hamiltonian this corresponds to the replacement $v_0^2 K_1 \rightarrow \omega_x^2$ and $\rho = \text{constant} = R$, and with this simplification the problem is just two coupled harmonic oscillators. We could diagonalize this motion to obtain the solution; however, it is more convenient to perform a canonical transformation first to illustrate the effects of the coupling.

$$\begin{pmatrix} x \\ p_x \\ \Delta p \\ \delta \end{pmatrix} \rightarrow \begin{pmatrix} z \\ p_z \\ \Delta p' \\ \delta' \end{pmatrix} \quad \text{so that} \quad \begin{aligned} p_z &= p_x \\ z &= x - \frac{v_0 \Delta p}{\gamma m \omega_x^2} \\ \Delta p &= \Delta p' \\ \delta' &= \left(\delta - \frac{v_0}{\gamma m \omega_x^2} p_z \right) \end{aligned} \quad (10)$$

The generating function is

$$F_2(x, p_z, \delta, \Delta p') = \left(x - \frac{v_o \Delta p'}{\gamma m \omega_x^2} \right) p_z + \delta \Delta p' \quad (11)$$

which yields

$$H = - \eta \frac{\Delta p^2}{2\gamma m} - \frac{\gamma m \omega_s^2}{2\eta} \left(\delta' + \frac{v_o p_z}{\gamma m \omega_x^2} \right)^2 + \frac{\gamma m \omega_x^2 z^2}{2} + \frac{p_z^2}{2\gamma m} \quad (12)$$

where

$$\eta \equiv \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}, \quad \omega_s^2 = \frac{\eta eV}{2\pi R^2 \gamma m}, \quad \gamma_T = \gamma_x, \quad \nu_x = \omega_x / \omega_o \quad (13)$$

We have introduced some standard notation here: η , the frequency slip factor; γ_T , the transition energy; ω_s , the synchrotron frequency; ν_x , the horizontal betatron tune. All of these quantities have more precise definitions, and their values must be calculated in a rigorous way for any accelerator; the definitions above simply give the values within our simplified model. The coupling thus leads to two main effects:

(1) Transition, the negative mass effect. This comes about due to the change in sign of the factor η depending upon whether γ is greater than or less than the "transition energy", γ_T . If we set $z, p_z = 0$, we see that we have chosen the sign of V so that

the oscillations are stable for $\eta > 0$.

(2) Dispersion. For $\omega_s \rightarrow 0$ the x motion has been decomposed into two parts, one part proportional to Δp , and the other an harmonic betatron oscillation at frequency ω_x . Since ω_s is typically much less than ω_y , this transformation effectively diagonalizes the motion. If we now recall the vertical (y, p_y) motion which we wish to study in connection with coherent instabilities, and suppress the horizontal betatron oscillations (z, p_z) , we have

$$H = - \left(\eta \frac{\Delta p^2}{2\gamma m} + \frac{\gamma m \omega_s^2}{2\eta} \delta^2 \right) + \frac{p_y^2}{2\gamma m} + \frac{\gamma m \omega_y^2}{2} y^2 \quad (14)$$

Before proceeding to study coherent effects we must include another complication. There are third and higher order terms in the Hamiltonian which we have so far neglected. Some of these arise naturally due to the variation of the particle energy with Δp , while others are due to additional higher order terms in A_s . The additional terms (in particular the sextupole term $A_s \sim x^3 - 3xy^2$) are inserted to control the natural terms. For our purposes the details of these additional terms are not important, so we include their effect by the addition of the term

$$- \gamma m \omega_y^2 (\eta - \xi) \frac{\Delta p}{p_0} y^2 \quad (15)$$

where ξ , the chromaticity, is defined by

$$\nu = \frac{\omega_y}{\omega_0}, \quad \frac{\Delta \nu}{\nu} = \xi \frac{\Delta p}{p} \quad (16)$$

This just gives the change in the "tune", ν , with momentum. If we introduce the new variables

$$\tau \equiv \frac{s - v_o t}{v_o} = s/v_o, \quad p_\tau \equiv \Delta p v_o \quad (17)$$

then we find the single particle hamiltonian,

$$H_o = - \left(\eta \frac{p_\tau^2}{2\gamma m v_o^2} + \frac{\gamma m \omega_s^2 v_o^2 \tau^2}{2\eta} \right) + \frac{p_y^2}{2\gamma m} + \frac{\gamma m \omega_y^2}{2} \left[1 - 2(\eta - \xi) \frac{p_\tau}{v_o p_o} \right] y^2 \quad (18)$$

In order to study coherent effects we must include the force on a particle due to the collective motion (in our case vertical) of the entire beam. This we accomplish by including an extra term in the hamiltonian,

$$H_c = - y F(\tau, t) \quad (19)$$

where F is the vertical coherent force. However, before discussing the particular form of the coherent force it is useful to rewrite H_o . First go to action angle variables in the y motion given by

$$y = \sqrt{2J_y / \gamma m \omega_y} \cos \theta_y, \quad p_y = - \sqrt{2J_y \omega_y \gamma m} \sin \theta_y \quad (20)$$

Then we have

$$H_o = - \eta \frac{p_\tau^2}{2p_o v_o} - \frac{p_o v_o \omega_s^2 \tau^2}{2\eta} + \omega_y \left(1 - (\eta - \xi) \frac{p_\tau}{v_o p_o} \right) J_y \quad (21)$$

In the above expression we have averaged over θ_y , so there is another fluctuating term which has been dropped. This term contributes to changes in frequency which are of higher order and

so we neglect it. Let us now examine the equations of motion.

We have

$$\begin{aligned}\dot{\theta}_y &= \omega_y [1 - (\eta - \xi) p_\tau / v_o p_o] \\ \dot{J}_y &= 0 \\ \dot{\tau} &= - \eta p_\tau / p_o v_o - \omega_y (\eta - \xi) J_y / v_o p_o \\ \dot{p}_\tau &= + p_o v_o \omega_s \tau / \eta\end{aligned}\quad (22)$$

which yields

$$\ddot{\tau} + \omega_s^2 \tau = 0$$

and

$$\dot{\theta}_y = \omega_y (1 + a \dot{\tau}) + \eta \frac{a^2 \omega_y^2}{v_o p_o} J_y \quad (23)$$

where

$$a \equiv 1 - \xi / \eta \quad (24)$$

So we have a simple harmonic oscillation longitudinally and transversely a nonlinear oscillator with a periodically modulated frequency. We have completely integrated the single particle motion; however, it is convenient to go to a coordinate system that reflects this. Consider the following canonical transformation

$$\begin{aligned}p_\tau &= p'_\tau - a \omega_y J'_y \\ J_y &= J'_y \\ \tau' &= \tau \\ \theta'_y &= \theta_y - a \omega_y \tau\end{aligned}\quad (25)$$

Then the new hamiltonian is

$$H = - \frac{\eta}{2 p_o v_o} p_\tau'^2 - \frac{p_o v_o \omega_s^2 \tau'^2}{2 \eta} + \omega_y J_y' + \frac{\eta a^2 \omega_y^2 J_y'^2}{2 v_o p_o} \quad (26)$$

The last term is of higher order and so we will neglect it. These coordinates reflect the longitudinal motion transparently, and identify the term which should be neglected. The variation in frequency of the y motion is now "built into" the coordinate, θ_y^1 . Since the longitudinal motion is just harmonic, it is convenient to again use action angle variables:

$$\tau = \sqrt{2nJ_s/p_o v_o \omega_s} \cos\theta_s, \quad p_\tau = -\sqrt{2J_s \omega_s p_o v_o / n} \sin\theta_s \quad (27)$$

To simplify the notation we introduce $\tilde{\tau}(J_s) = \sqrt{2nJ_s/p_o v_o \omega_s}$. Thus, we arrive at the hamiltonian we will use to study coherent effects (dropping the primes)

$$\begin{aligned} H &= H_o + H_c \\ H_o &= \omega_y J_y - \omega_s J_s \\ H_c &= -\sqrt{2J_y/\gamma m \omega_y} \cos(\theta_y + a \omega_y \tilde{\tau} \cos\theta_s) F(\tilde{\tau} \cos\theta_s, t) \end{aligned} \quad (28)$$

The Vlasov Equation

In order to study the coherent motion of a collection of particles (ignoring collisions) we will employ the Vlasov Equation. This is most easily introduced via Liouville's theorem. Let ψ be the distribution function in phase space. Thus ψ depends upon all the coordinates, their canonical momenta, and the time t . Then we have

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + [\psi, H] = 0 \quad (29)$$

where $[,]$ is the Poisson bracket. Since the Hamiltonian depends upon ψ through the solution of Maxwell's equations with the relevant boundary conditions, this equation must be supplemented by Maxwell's equations to find the consistent solution. However, since we are primarily interested in stability, we will take the standard approach of assuming that we have already solved the E.M. problem. In addition we linearize Eq. (29) by the substitution

$$\psi = \psi_0 + \psi_1 e^{-i\omega t} \quad (30)$$

where ψ_0 is chosen so that $[\psi_0, H_0] = 0$. With this substitution we have

$$-i\omega \psi_1 + [\psi_1, H_0] + [\psi_0, H_c] = 0 \quad (31)$$

where we have dropped the nonlinear term $[H_c, \psi_1]$ and suppressed the factor $e^{-i\omega t}$. If we can find a solution to Eq. (31) with $\text{Im}(\omega) > 0$, the system is unstable since a small perturbation will grow in time.

The Coherent Force

The electromagnetic fields produced by the interaction of the beam with its environment can be quite complicated in general. Since in this paper we are primarily interested in the coherent dynamics of a beam of particles, we will not treat the

electrodynamics in detail. Instead we consider a smooth force given by

$$\frac{F(\theta, t)}{\gamma m} \equiv \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dt' \int_0^{2\pi} d\theta' g(\theta - \theta', t - t') D_1^L(\theta', t') \quad (32)$$

where $\theta \equiv s/R$ and D_1^L is the vertical dipole density in the lab frame. In terms of the perturbed distribution function,

$$D_1^L(\theta, t) = \int y \psi_1^L(\theta, p_\theta, y, p_y, t) dp_\theta dy dp_y \quad (33)$$

If we let

$$g(\theta, t) \equiv \sum_k \int_{-\infty}^{\infty} d\omega G_k(\omega) e^{i(k\theta - \omega t)} \quad (34)$$

and

$$D_1^L(\theta, t) \equiv \sum_k \int_{-\infty}^{\infty} d\omega D_k^L(\omega) e^{i(k\theta - \omega t)} \quad (35)$$

then we have

$$\frac{F(\theta, t)}{\gamma m} = \sum_k \int_{-\infty}^{\infty} G_k(\omega) D_k^L(\omega) e^{i(k\theta - \omega t)} d\omega \quad (36)$$

Now we need to express the force in terms of the moving coordinate system and the distribution function in that system. Recall that $\tau = (\theta - \omega_0 t)/\omega_0$; with this change of variables plus the change of integration variable $\omega = \omega' + k\omega_0$ we find

$$\frac{F}{\gamma m} = \sum_k \int_{-\infty}^{\infty} G_k(k\omega_0 + \omega') D_k^L(k\omega_0 + \omega') e^{i(k\omega_0 \tau - \omega' t)} d\omega' \quad (37)$$

Inverting equation (35) we have

$$D_k^L(k\omega_0 + \omega') = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dt \int y \psi_1^L(\theta, p_\theta, y, p_y) e^{-i[k\theta - (k\omega_0 + \omega')t]} d\theta dp_\theta dy dp_y \quad (38)$$

Under a canonical transformation the volume element is invariant so the new distribution function is simply obtained by the

coordinate replacement.

$$\text{i.e.} \quad \psi_1(\tau, p_\tau, y, p_y) = \psi_1^L[\omega_0(\tau+t), R p_0 + p_\tau / \omega_0, y, p_y] \quad (39)$$

so

$$\begin{aligned} D_k^L(k\omega_0 + \omega') &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dt \, y \psi_1(\tau, p_\tau, y, p_y, t) e^{-i(k\omega_0 \tau - \omega' t)} d\tau dp_\tau dy dp_y \\ &\equiv D(k, \omega') \end{aligned} \quad (40)$$

If we now let $\psi \sim e^{-i\omega t}$, then $D(k, \omega') = D(k) \delta(\omega' - \omega)$ where

$$D(k) = \frac{1}{2\pi} \int y \psi_1 e^{ik\omega_0 \tau} d\tau dp_\tau dy dp_y \quad (41)$$

So we find for the force in this case

$$\frac{F(\tau) e^{-i\omega t}}{\gamma m} = \sum_k G_k(k\omega_0 + \omega) D(k) e^{i(k\omega_0 \tau - \omega t)} \quad (42)$$

There are basically two types of forces which can be studied; those which decay in a time long compared to a revolution, the long wake, and those which decay in a time less than one revolution, the short wake. We are primarily interested in the latter. If we also restrict ourselves to a single bunch of particles with an extent less than the circumference, then the periodicity is not important, and we can make the replacement

$$\sum_{k=-\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty} dk \quad (43)$$

We recall the sum only when it plays a significant role in the discussion.

To obtain the standard impedance consider the effect of a structure localized azimuthally in the accelerator (for example,

a cavity resonator). Then in this case

$$g(\vartheta, t) = \delta(\vartheta) g(t) \quad (44)$$

and so

$$G_k(k\omega_0 + \omega) = G(k\omega_0 + \omega) \quad (45)$$

i.e., the only dependence on the mode number k is through the frequency in the moving frame. With these changes the force becomes

$$\frac{F(\tau)e^{-i\omega\tau}}{\gamma m} = \int_{-\infty}^{\infty} G(k\omega_0 + \omega) D(k) e^{i(k\omega_0\tau - \omega t)} dk. \quad (46)$$

If we now apply the standard definition of the transverse impedance¹⁰, $Z_{\perp}(\omega)$, we find that

$$G(\omega) = \frac{ie^2\omega_0^2 Z_{\perp}(\omega)}{2\pi m \gamma c}. \quad (47)$$

This is the form of the force which we will use in the Vlasov equation. It is useful at this point to note some properties of $G(\omega)$. We have

$$G(\omega) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} g(t) e^{i\omega t} dt. \quad (48)$$

However, from causality we have

$$g(t \leq 0) = 0 \quad (49)$$

so

$$G(\omega) = \frac{1}{(2\pi)^2} \int_0^{\infty} g(t) e^{i\omega t} dt. \quad (50)$$

So G is analytic on the upper half plane and

$$G^*(\omega) = G(-\omega^*) . \quad (51)$$

As an example consider

$$g(t) = Ae^{-\Gamma t} \sin \omega_R t \quad (52)$$

Then we find

$$G(\omega) = \frac{-G_o \omega_c^2}{2i\Gamma\omega + (\omega^2 - \omega_c^2)} \quad (53)$$

where $G_o = G(\omega=0) = \frac{1}{(2\pi)^2} \int_0^\infty g(t) dt$

and

$$\omega_c^2 = \omega_R^2 + \Gamma^2 \quad (54)$$

We continue the discussion of the transverse impedance in Appendix 1 where we discuss a model impedance for the accelerator ISABELLE. In the next section we study the Vlasov Equation with the single particle equations of motion in order to formulate an equation which describes the coherent oscillations of a particle bunch.

III. THE INTEGRAL EQUATION

Transforming the Vlasov Equation

We are now prepared to write the Vlasov equation for our problem and transform it so that it is more amenable to solution. Using Eq. (31) together with our Hamiltonian in Eq. (28) we can write the linearized Vlasov Eq.

$$\begin{aligned}
 -i\omega\psi_1 + \omega_y \frac{\partial\psi_1}{\partial\theta_y} - \omega_s \frac{\partial\psi_1}{\partial\theta_s} + \\
 - \frac{\partial\psi_0}{\partial J_y} \sqrt{2J_y/\gamma m\omega_y} \sin(\theta_y + a\omega_y \tilde{\tau} \cos\theta_s) F(\tilde{\tau} \cos\theta_s) = 0
 \end{aligned}
 \tag{55}$$

We have neglected the coherent force in the longitudinal direction because $\omega_y J_y \ll \omega_s J_s$ for any given particle. Our task now is to solve this equation; to find the eigenvalue ω , and eigenfunction ψ_1 . If any solution has $\text{Im}(\omega) > 0$, then the system is unstable. The growth rate in this circumstance is just $\text{Im}(\omega)$.

Remember that $[\psi_0, H_0] = 0$. This means that ψ_0 is a general function of the integrals of the motion, in particular $\psi_0 = \psi_0(J_s, J_y)$. In the coasting beam case $\psi_0 = \psi_0(p_\tau, J_y)$ since there is no longitudinal focusing. This difference leads to the coupling of different revolution modes, $\exp(ik\omega_0\tau)$, and so increases the difficulty of the problem. On the other hand, this complication is

essential because the solution must lead to a wave packet not a plane wave.

We expect an integral equation due to the form of the force so we seek an equation for $D(k)$ of the form

$$D(k) = \int_{-\infty}^{\infty} dk' A(k, k') D(k'). \quad (56)$$

Since equation (55) is a linear integro differential equation, such an integral equation should exist. In the coasting beam limit this must reduce to a dispersion relation of the standard form.

To begin this process we search for a solution in the neighborhood of $\omega \sim -\omega_y$. This means we let

$$\psi_1 = \text{Re} e^{-i\theta_y} \quad (57)$$

so that

$$R = \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta_y} \psi_1 d\theta_y. \quad (58)$$

Then from Eq. (55) we obtain the following:

$$-i(\omega + \omega_y)R - \omega_s \frac{\partial R}{\partial \theta_s} + \frac{\partial \psi_0}{\partial J_y} \sqrt{2J_y / \gamma m \omega_y} \frac{e^{-ia\omega_y \tilde{\tau} \cos \theta_s}}{2i} F(\tilde{\tau} \cos \theta_s) = 0. \quad (59)$$

Since $\tilde{\tau} = \sqrt{2\eta J_s / p_0 v_0 \omega_s}$, R is a function of the variables J_y , J_s , and θ_s . The force is written in terms of $y\psi_1$, the dipole density. If we examine the definition of $D(k)$ in Equation (41), we find it convenient to define

$$\begin{aligned}
f &\equiv \int_0^{2\pi} \int_0^{\infty} \sqrt{2J_y/\gamma m \omega_y} \cos(\theta_y + a\omega_y \tilde{r} \cos \theta_s) \psi_1 dJ_y d\theta_y. \\
&= \pi e^{ia\omega_y \tilde{r} \cos \theta_s} \int_0^{\infty} \sqrt{2J_y/\gamma m \omega_y} R dJ_y
\end{aligned} \quad (60)$$

so that

$$D(k) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} f e^{-ik\omega_0 \tilde{r} \cos \theta_s} d\theta_s dJ_s. \quad (61)$$

This suggests we multiply Eq. (59) by $\sqrt{2J_y/\gamma m \omega_y}$ and integrate over J_y . We find

$$-i[\omega + \omega_y + a\omega_y \omega_s \tilde{r} \sin \theta_s] f - \omega_s \frac{\partial}{\partial \theta_s} f + \frac{i\rho_0(J_s)F}{2\omega_y \gamma m} = 0 \quad (62)$$

where we have made use of

$$\int_0^{\infty} 2\pi J_y \frac{\partial \psi_0}{\partial J_y} = - \int_0^{2\pi} \int_0^{\infty} dJ_y d\theta_y \psi_0 \equiv \rho_0(J_s). \quad (63)$$

We will normalize ρ_0 such that

$$\int_0^{\infty} \int_0^{2\pi} \rho_0 d\theta_s dJ_s = N = \# \text{ of particles}. \quad (64)$$

If we drop the s index since the y variables are integrated out and let $\lambda \equiv (\omega + \omega_y)/\omega_s$, we have

$$\{-i(\lambda + a\omega_y \tilde{r} \sin \theta) - \frac{\partial}{\partial \theta}\} f = \frac{-i\rho_0 F(\tilde{r} \cos \theta)}{2\omega_y \omega_s \gamma m} \quad (65)$$

At this stage it is possible to introduce synchrotron harmonics ($f \sim e^{-im\theta}$); however, we will delay their introduction so that we can consider shifts in frequency which might be large compared to the synchrotron frequency.

It is easy to see that the periodic solution to Eq. 65 is given by

$$f = \frac{ie^{-i\lambda\theta + i\omega_y \tilde{\tau} \cos\theta} \rho_0}{(e^{2\pi i\lambda} - 1)} \int_{\theta}^{\theta+2\pi} \frac{F(\tilde{\tau} \cos\theta')}{2\omega_y \omega_s \gamma m} e^{i\lambda\theta' - i\omega_y \tilde{\tau} \cos\theta'} d\theta'. \quad (66)$$

If we change variables and note that

$$\begin{aligned} \tilde{\tau} \cos(\theta+t) &= (\tilde{\tau} \cos\theta) \cos t + (-\omega_s \tilde{\tau} \sin\theta) \sin t / \omega_s \\ &= \tau \cos t + \dot{\tau} / \omega_s \sin t, \end{aligned} \quad (67)$$

then the expression for f becomes

$$f = \frac{i\rho_0}{e^{2\pi i\lambda} - 1} \int_0^{2\pi} dt e^{i\lambda t} F(\tau \cos t + \dot{\tau} / \omega_s \sin t) \times \quad (68)$$

$$\times e^{-i\omega_y \tau (\cos t - 1)} e^{-i\omega_y \dot{\tau} / \omega_s \sin t}$$

where we have recalled that $\tau = \tilde{\tau} \cos\theta$ and $\dot{\tau} = -\omega_s \tilde{\tau} \sin\theta$ and written f in terms of τ , $\dot{\tau}$ to simplify notation. We are really interested in $D(k)$ which is given by Eq. (61). So if we write ρ_0 in terms of τ and $\dot{\tau}$, substitute the force from Eq. (46), and operate with

$\frac{1}{2\pi} \int d\tau d\dot{\tau} e^{-ik\omega_y \tau}$, we obtain

$$D(k) = \frac{-i}{(1 - e^{2\pi i\lambda})} \int_0^{2\pi} dt e^{i\lambda t} \int_{-\infty}^{\infty} dk' \frac{G(\omega + k'\omega_0)}{\omega_y \omega_s 4\pi} D(k') \times \quad (69)$$

$$\times \iint d\tau d\dot{\tau} \rho_0 e^{i(k'\omega_0 - a\omega_y)\tau \cos t} e^{i(k'\omega_0 - a\omega_y)\dot{\tau} / \omega_s \sin t} e^{-i(k\omega_0 - a\omega_y)\tau}.$$

The transformation to τ , $\dot{\tau}$ is not canonical so we must take care to absorb the Jacobian into ρ_0 .

So we have an integral equation for $D(k)$. Before proceeding with more simplification it is useful to see how the above result reduces to the coasting beam result. We know that to solve the problem correctly, ω and therefore λ must have a positive imaginary part corresponding to instability. We discover stability or damping by analytic continuation. We must consider the limit $\omega_s \rightarrow 0$. In this limit the imaginary part of $\lambda \rightarrow +\infty$. This means that the entire contribution to the integral over t is in the neighborhood of $t = 0$. So we can expand the integral about $t = 0$; the integral is simple, and we find

$$D(k) = \int_{-\infty}^{\infty} dk' \frac{G(\omega + k'\omega_o)}{4\pi\omega_y} \int \frac{dt d\tau \rho_o e^{i(k'-k)\omega_o \tau}}{\omega + \omega_y + (k'\omega_o - a\omega_y)\tau} D(k') \quad (70)$$

as ($\omega_s \rightarrow 0$).

So in the limit as $\omega_s \rightarrow 0$ we obtain the familiar pole in the integration over τ which yields a region of stability. To complete the coasting beam limit we let $\rho_o(J_s) \rightarrow \rho_o(p_\tau)$. In this case the integral over τ is just $2\pi\delta(k-k')$ and we find

$$1 = \frac{G(\omega + k\omega_o)}{2\omega_y} \int_{-\infty}^{\infty} \frac{\rho_o(\tau) d\tau}{\omega + \omega_y + (k\omega_o - a\omega_y)\tau} \quad (71)$$

which is just the familiar dispersion relation for a coasting beam. Since the system is periodic in this limit, k must be an integer.

Equation (70) is useful for understanding the revolution mode coupling since the coupling is due to the Fourier transform of ρ_0 . If ρ_0 has an rms width of σ in τ space, then the modes that are coupled are $\Delta k \sim 1/\omega_0 \sigma$. But $\omega_0 \sigma$ is just the rms spread in azimuth. So if we want $\Delta k \lesssim 1$, i.e. no coupling, then $\omega_0 \sigma \gtrsim 1$. If we take 5σ to be the full bunch length, then we find that for decoupling the beam must occupy almost all of the circumference ($5\omega_0 \sigma \gtrsim 5 \sim 2\pi$). Physically this makes sense; in order for modes to completely decouple we must have a true coasting beam.

Returning to Eq. (69) we make the following change of variables

$$k'\omega_0 - a\omega_y = p'\omega_0 \quad (72)$$

but $\omega_y = v\omega_0$ so $p' = k' - av$ (73)

so let $\bar{D}(p) = D(p + av)$. (74)

With these changes Eq. 69 becomes

$$\begin{aligned} \bar{D}(p) = & \frac{-i}{1-e^{2\pi i\lambda}} \int_0^{2\pi} dt e^{i\lambda t} \int_{-\infty}^{\infty} dp' \frac{G(\omega + p'\omega_0 + a\omega_y)}{4\pi\omega_y s} \bar{D}(p') \times \\ & \times \int d\tau d\tau' \rho_0 e^{ip'\omega_0 \tau \cos t + ip'\omega_0 \tau' \sin t / \omega_s - ip\omega_0 \tau} \end{aligned} \quad (75)$$

The last integral in Eq. (75) can be done easily by recognizing that it is of the form

$$\iint dx dy f(\sqrt{x^2+y^2}) e^{i(k_1 x + k_2 y)} = 2\pi \int r dr f(r) J_0(|k|r) \quad (76)$$

where $|k| = \sqrt{k_1^2 + k_2^2}$ and $r = \sqrt{x^2 + y^2}$.

Using Eq. 76 we find an equation of the form

$$\bar{D}(p) = \int_{-\infty}^{\infty} dp' A(p, p') \bar{D}(p') \quad (77)$$

with

$$A(p, p') = \frac{G(\omega + p' \omega_0 + a \omega_y)}{4\pi \omega_y s} x \quad (78)$$

$$\frac{-i}{(1 - e^{2\pi i \lambda})} \int_0^{2\pi} e^{i\lambda t} \int_0^{\infty} \tilde{t} d\tilde{t} \rho_0(\tilde{t}) J_0[K_{pp'}(t) \omega_0 \tilde{t}]$$

where

$$K_{pp'}(t) = [(p - p')^2 - 2pp'(\cos t - 1)]^{1/2} \quad (79)$$

and ρ_0 is renormalized so that

$$\int_0^{\infty} \tilde{t} d\tilde{t} \rho_0(\tilde{t}) = N. \quad (80)$$

The above result follows exactly from the Vlasov equation in Eq. (55). The problem has been reduced to the solution of an integral equation with eigenvalue equal to unity. The eigenvalue of Eq. (77) will have λ as a free parameter, so that after the integral equation is diagonalized we are left with an equation of the form

$$1 = E(\lambda) \quad (81)$$

where $E(\lambda)$ is the eigenvalue. Thus, after diagonalization, we obtain an equation of the same form as the normal dispersion relation in coasting beam theory [see Eq. (71)]. In order to gain more insight into the integral equation, let us now specialize to a gaussian distribution for ρ_0 .

The Gaussian Distribution and Kernel

By a gaussian bunch we mean a distribution function given by

$$\begin{aligned} \rho_0 &= N/\sigma^2 e^{-\tilde{\tau}^2/2\sigma^2} \\ &= N/\sigma^2 e^{-\tau^2/2\sigma^2} e^{-\tilde{t}^2/2\omega_s^2\sigma^2} \end{aligned} \quad (82)$$

in other words gaussian in τ and $\tilde{t}$. Physically this is gaussian in azimuth and gaussian in $\Delta p/p$. The normalization is

$$\int_0^\infty \tilde{\tau} d\tilde{\tau} \rho_0(\tilde{\tau}) = N. \quad (83)$$

The last integral in Eq. (78) can be done easily with this distribution:

$$\begin{aligned} I &= \int_0^\infty \tilde{\tau} d\tilde{\tau} \rho_0(\tilde{\tau}) J_0[K_{pp'}(\tilde{\tau})\tilde{\tau}] \\ &= N e^{-\frac{(p-p')^2\sigma_0^2}{2} + \sigma_{pp'}^2(\cos\theta-1)} \end{aligned} \quad (84)$$

where $\sigma_0 = \sigma\omega_0$ = rms spread in azimuth. So we find that the

kernel can be written

$$A(p, p') = N \frac{G(\omega + p'\omega_o + a\omega_y)}{4\pi\omega_y\omega_s} e^{-(p-p')\sigma_o^2/2} \times B(pp', \lambda) \quad (85)$$

with

$$B = \frac{-i}{1 - e^{2\pi i \lambda}} \int_0^{2\pi} e^{i\lambda t} e^{pp'\sigma_o^2(\cos t - 1)} dt \quad (86)$$

Before proceeding to the solution of the integral equation it is useful to examine the factor $B(pp', \lambda)$.

The "Dispersion Integral" $B(pp', \lambda)$

First notice that B can be written exactly in terms of a series with modified Bessel functions,

$$B = e^{-pp'\sigma_o^2} \sum_k \frac{I_k(pp'\sigma_o^2)}{\lambda - k} \quad (87)$$

From this form it is apparent that B is real when λ is real.

Recall that $\lambda \equiv (\omega + \omega_y)/\omega_s$, so this series is just the development in terms of synchrotron harmonics. The usual approach^{3,5} considers one or two terms in this sum. The justification for this is that if $\lambda \sim k$, then only the k^{th} term contributes. If we select only one term in the sum, we find

$$\begin{aligned} e^{-(p-p')^2\sigma_o^2/2} B &\rightarrow e^{-(p^2+p'^2)\sigma_o^2/2} \frac{I_k(pp'\sigma_o^2)}{\lambda - k} \\ &= e^{-(p^2+p'^2)\sigma_o^2/2} \frac{(\text{sign } pp')^k}{\lambda - k} I_k(|p||p'|\sigma_o^2). \end{aligned} \quad (88)$$

Apart from the sign, this is a function of $|p|$ and $|p'|$. So positive and negative p contribute equally in the integral equation.

To contrast with this case consider the situation when $|pp'|\sigma_0^2 \gg 1$ and $\text{Im}(\omega) > \omega_s$, i.e. $\text{Im}(\lambda) > 1$ (fast blow up). Then the Bessel function in the numerator is large up to $k \sim pp'\sigma_0^2$, and no one denominator is small. This means that we must include all the terms in the sum. In practice though we must consider the integral for B . In the above situation the factor B together with the bunch factor $e^{-(p-p')^2\sigma_0^2/2}$, decouples positive and negative p . This is easy to see if $\text{Im}(\lambda) \gg p\sigma_0$. In this regime

$$B \approx \frac{1}{\lambda} \quad (89)$$

and the factor $e^{-(p-p')^2\sigma_0^2/2}$ decouples positive and negative p . If $\text{Im}(\lambda) \lesssim p\sigma_0$, this is also true but slightly more difficult to see. First change B to the form

$$B = \frac{1}{\sin \lambda \pi} \int_0^\pi \cos \lambda t e^{-pp'\sigma_0^2(\cos t + 1)} dt \quad (90)$$

for $\text{Im}(\lambda) > 1$ and $pp' < 0$

$$B \approx i e^{i\lambda\pi} \int_0^\pi e^{-i\lambda t} e^{pp'\sigma_0^2(\cos t + 1)} dt \quad (91)$$

Look in the neighborhood of $p = -p' = p_0$. Including the bunch factor we find

$$e^{-p_0^2\sigma_0^2} i e^{i\lambda\pi} \int_0^\pi e^{-i\lambda t} e^{p_0^2\sigma_0^2 \cos t} dt \quad (92)$$

The first factor in the integral is increasing while the second is decreasing, so we can use the saddlepoint method to evaluate the integral. The result is

$$e^{p_o^2 \sigma_o^2 (\cos t_o - 1)} \frac{i\lambda(\pi - t_o)}{ie} \int_{-\infty}^{\infty} e^{-p_o^2 \sigma_o^2 \cos t_o t^2/2} dt \quad (93)$$

where $\sin t_o = \text{Im}(\lambda)/p_o^2 \sigma_o^2$, $0 \leq t_o < \pi/2$.

The region we are interested in is $t_o \ll 1$, so Eq. (93) becomes

$$\frac{i\alpha(\pi - t_o)}{ie} \times e^{-\beta(\pi - \beta/p_o^2 \sigma_o^2)} \int_{-\infty}^{\infty} e^{-p_o^2 \sigma_o^2 t^2/2} dt \quad (94)$$

where $\alpha + i\beta \equiv \lambda$ ($\beta > 0$). (95)

The magnitude of the above expression has the factor $e^{-\pi\beta}$, whereas for p and p' of like sign this factor is missing, so when $p_o \sigma_o > \text{Im}(\lambda) > 1$, we again have a decoupling of positive and negative p .

In addition, it is useful to note that B has the following upper bounds:

$$|B| \leq \frac{\pi I_o (pp' \sigma_o^2)}{1 - e^{-\pi\beta}} e^{-pp' \sigma_o^2} \quad (96)$$

and (97)

$$|B| \leq 1/\beta.$$

These will be useful in the next section for determining a threshold for fast blowup ($\text{Im}\lambda > 1$).

IV. LONG BUNCH - SHORT WAVELENGTH

In this section we consider the case of a long bunch. We know that in the limit of a long bunch with $\omega_s \rightarrow 0$ that we obtained the normal coasting beam results. For a finite bunch length we must solve the integral equation; however, in spite of this there is a possible simplification. Consider a bunch with an rms bunch length of σ (in seconds). Then the Fourier transform of the unperturbed distribution has a width $\sim 1/\sigma$. Qualitatively, the perturbed distribution, the eigenfunction of our problem, must have an extent equal to or less than the extent of the unperturbed bunch. This means that the Fourier transform of the perturbed distribution should also have a width $\gtrsim 1/\sigma$.

Normally, in coasting beam theory, the impedance drives one particular lab coherent frequency $= (n\omega_0 + \omega)$, where ω is the coherent frequency in the rotating frame, ω_0 is the revolution frequency and n is the revolution mode number. Since different modes are coupled in a bunched beam, this must lead to a band of frequencies in the lab of width $\gtrsim 1/\sigma$. This sounds difficult; since the impedance varies with frequency, different modes in the bunch will be excited with different strengths. However, if

impedance is that of a very broad band resonance, so that the width of $Z_{\perp}(\omega)$ is greater than $1/\sigma$, then each mode in the perturbation in the neighborhood of the resonance is excited by approximately the same impedance. So in this regime we expect a simplification to a coasting beam like result except with a wave packet excited rather than a plane wave.

High Frequency Fast Blow Up¹¹

We have seen that in the fast blow up regime the integral B decouples the positive and negative frequencies in the integral equation. Now we would like to consider a broad band impedance at high frequency. This should lead to some simplification since this impedance is approximately constant over a broad range.

Let Γ be the width of $G(\omega)$ and let G be peaked at the cutoff frequency of the vacuum chamber, ω_c . We will consider the case when the bunch length is much longer than the wakefield, i.e.

$$\Gamma \gg 1/\sigma \quad (98)$$

and

$$\omega_c \gg 1/\sigma. \quad (99)$$

Now we search for a solution to equation (77) in the neighborhood of the cutoff frequency. Then the kernel in Eq. (85) is the product of a slowly varying function,

$$G(\omega + p'\omega_0 + a\omega_y) \times B(pp', \lambda), \quad (100)$$

and a sharply spiked function,

$$e^{-(p-p')^2 \sigma_o^2 / 2} \quad (101)$$

We will therefore approximate the kernel by

$$A(p, p') \approx \frac{NG(\omega_c)}{4\pi\omega_y\omega_s} e^{-(p-p')^2 \sigma_o^2 / 2} \times B(p_o^2, \lambda) \quad (102)$$

$$\text{where } p_o = (\omega_c - a\omega_y - \omega) / \omega_o \quad (103)$$

or since $\omega \sim -\omega_y$, and $a = (1 - \xi/\eta)$

$$p_o = (\omega_c + \xi/\eta \omega_y) / \omega_o = (\omega_c + \omega_\xi) / \omega_o \quad (104)$$

We should restrict the range of integration to be over the width of G; however, since $\Gamma \gg 1/\sigma$, we can extend the range of

integration with little error. We are left with

$$\bar{D}(p) = C \int_{p_o - \Gamma/\omega_o}^{p_o + \Gamma/\omega_o} e^{-(p-p')^2 \sigma_o^2 / 2} \bar{D}(p') dp' \quad (105)$$

$$\text{where } C = \frac{NG(\omega_c)}{4\pi\omega_y\omega_s} B(p_o^2, \lambda) \quad (106)$$

If we extend the range of integration, we find

$$\bar{D}(p_o + p) = C \int_{-\infty}^{\infty} e^{-(p-p')^2 \sigma_o^2 / 2} \bar{D}(p_o + p') dp' \quad (107)$$

The above equation can be solved. Substitute $\bar{D}(p) \sim e^{-ip\omega_o q}$ and we find

$$1 = C \int_{-\infty}^{\infty} dk e^{-k^2 \sigma_o^2 / 2 - ik\omega_o q} \quad (108)$$

$$1 = C \frac{\sqrt{2\pi}}{\sigma_o} e^{-q^2 / 2\sigma_o^2}$$

Thus, for the eigenfunction corresponding to $e^{-ip\omega_0 q}$, we find the eigenvalue given in Eq. (108). Since we are searching for instability, select the largest eigenvalue ($q = 0$) then using Eq. (106) and (47) we obtain

$$1 = ie \frac{I_{\text{peak}} Z_1(\omega_c)}{m_0 \gamma c 4\pi v \omega_s} B(p_0^2, \lambda) \quad (109)$$

where
$$I_{\text{peak}} = \frac{eN}{\sigma \sqrt{2\pi}} . \quad (110)$$

Notice that the solution of the integral equation has brought in a factor of the local current (the peak current since we selected the largest eigenvalue). This makes sense physically because a broad band impedance corresponds to a very local interaction within the bunch.

The next question that arises is the shape of the instability. Remember we searched for a solution in the neighborhood of the cutoff frequency, ω_c . The range of frequencies excited is roughly the width of the impedance in this region. So the Fourier transform of the dipole density is

$$D(p + a v) = \bar{D}(p) \sim e^{-ip\omega_0 q}, \quad p_0 - \Gamma/\omega_0 < p < p_0 + \Gamma/\omega_0 \quad (111)$$

From equation (41) we introduce the dipole density in τ space

$$D(\tau) = \int_{-\infty}^{\infty} e^{+ip\omega_0 \tau} D(p) dp \quad (112)$$

$$D(\tau) = \int_{-\infty}^{\infty} e^{ip\omega_0 \tau} \bar{D}(p - a v) dp = e^{ia\omega_0 \tau} \int_{-\infty}^{\infty} e^{ip\omega_0 \tau} \bar{D}(p) dp.$$

In the case above

$$D_y(\tau) = e^{i\omega_y \tau} \int_{p_o - \Gamma/\omega_o}^{p_o + \Gamma/\omega_o} dp e^{ip\omega_o(\tau-q)} \quad (113)$$

For $\Gamma \rightarrow \infty$ D becomes a delta function; however, for finite Γ we have

$$D_y(\tau) \sim e^{i\omega_y \tau} e^{ip_o \omega_o(\tau-q)} \frac{\sin[\Gamma(\tau-q)]}{\omega_o(\tau-q)} \quad (114)$$

The eigenfunction above is a spiked function of width $\sim 1/\Gamma$ with a position that depends on the parameter q . The parameter q is simply the "index" which indicates which eigenfunction and eigenvalue we are talking about. With the approximations we have made, our index is a continuous variable. From Eq. (114), you see that the eigenfunctions are a set of near delta functions centered at the value q in τ space. The corresponding eigenvalue is calculated with Eq. (109) by including the factor $e^{-q^2/2\sigma^2}$ from Eq. (108). The general solution to the problem is given by a linear superposition of the eigenfunctions. However, when dealing with questions of stability, we will always select the eigenfunction and eigenvalue which leads to the largest growth rate. In the next section we calculate the coherent frequency λ from Eq. (109).

The Dispersion Relation

It is convenient for the following discussion to define

$$\text{Re } i\theta \equiv \frac{i e I_{\text{peak}} Z_c(\omega_c)}{m_o \gamma c 4 \pi \nu \omega_s} \quad (115)$$

$$\text{and} \quad \Delta = p_o \sigma_o \quad (116)$$

with these changes we may rewrite Eq. (109)

$$1 \approx \text{Re}^{i\theta} \frac{i}{(e^{2\pi i\lambda} - 1)} \int_0^{2\pi} e^{i\lambda t} e^{\Delta^2(\cos t - 1)} \quad (117)$$

At large growth rates ($\text{Im}\lambda \gg \Delta$), we can take the leading term to obtain

$$\lambda \approx \text{Re}^{i\theta} \quad (118)$$

or

$$\omega + \omega_y \approx \frac{ieI_{\text{peak}} Z_1(\omega_c)}{m_o \gamma c 4\pi v} \text{ for } \text{Im}(\omega/\omega_s) > \Delta \quad (119)$$

Since we have upper bounds on the integral B, it is easy to find upper bounds on the growth rate. These are given by:

$$\begin{aligned} \text{for } r < 1 \quad \beta &\leq 1/\pi \ln \left(\frac{1}{1-r} \right) \\ \text{where} \quad r &\equiv \text{Re}^{-\Delta^2} \pi I_o(\Delta^2) \approx R \frac{\sqrt{2\pi}}{2\Delta}, \quad \Delta \gg 1 \end{aligned} \quad (120)$$

$$\text{and} \quad \beta \leq R.$$

For proton machines Δ is typically large. In Fig. 2 we plot the 2 upperbounds evaluated for $\Delta = 35$, which corresponds to the value for ISABELLE (see Appendix 1). As you see, there is a sharp rise at the intersection of the 2 bounds. This occurs at $r \approx 1$ and we shall call this the threshold for fast blowup. This does not mean that fast blowup necessarily occurs at $r > 1$, but it is useful as a rule of thumb for determining stability. In terms of conventional quantities this "threshold" is given by

Fig. 2 Upper Bound on the Growth Rate for
ISABELLE ($\sigma(\omega_c + \omega_\xi) = 35$)

$$1 \equiv \frac{eI_{\text{peak}} |Z_{\perp}(\omega_c)|}{4\sqrt{2\pi} \nu \eta \sigma_{\epsilon} |\omega_c + \omega_{\xi}| m_o \gamma c} \quad \text{"Threshold"} \quad (121)$$

where
$$\eta \sigma_{\epsilon} = \eta \left(\frac{\Delta p}{p} \right)_{\text{rms}} = \omega_s \sigma. \quad (122)$$

In order to obtain more precise results we must compute B. The most useful approach at this point is to plot a "stability diagram." From Equation (117) we have

$$b(\Delta, \lambda) = re^{i\theta} \quad (123)$$

where

$$\frac{1}{b(\Delta, \lambda)} = 2\Delta B(\Delta^2, \lambda) / \sqrt{2\pi} \quad (124)$$

We now can plot a map of b in the complex plane identifying contours of constant growth rate. Given the current and impedance of a machine we can find the point in the map which solves Eq. (123). Using Eq.'s (115), (120) we have

$$b(\Delta, \lambda) = \frac{ieI_{\text{peak}} Z_{\perp}(\omega_c)}{4\nu\sqrt{2\pi} m_o \gamma c |\omega_s \sigma(\omega_c + \omega_{\xi})|} \quad (125)$$

In Fig. 3 we plot contours of constant $\text{Im}(\lambda)$ for b (35, λ). The unit circle in this diagram corresponds to what we have called the threshold for fast blowup. With the aid of this map we can plot the growth rate vs. current for any impedance. To illustrate this, consider an impedance which has equal real and imaginary

Fig. 3 The Stability Diagram for Fast Blowup.
(Map of $b(35, \lambda)$)

parts. If we draw a ray in Fig. 3 at 45° , then the points of intersection yield a graph of β vs. r , or β vs $1/I_{th}$; I_{th} being the threshold current (corresponding to $r = 1$).

The results given in Fig. 4 illustrate the above procedure. The curves drawn are valid for $\beta \geq 1$, but we have extended them to lower values to suggest the slow blow up regime. However, from the analysis of the factor $B(pp', \lambda)$ we know that for slow blowup ($\text{Im} \lambda < 1$) if $p \sim p_0$ contributes to the integral, then $p \sim -p_0$ also contributes. Since it is the $\text{Im}(G) [\text{real } Z_{\perp}(\omega)]$ which causes instability, and since $\text{Im}G(-p) = -\text{Im}G(p)$, we expect that this additional contribution to the integral equation will cause a cancellation and contribute to stability. This argument indicates that the slow blowup regime should be even more stable than the curve in Fig. 4 suggests.

Fig. 4 Growth Rate vs. Current for Fast Blowup. (ISA parameters)

Slow Blowup And The Head-Tail Effect

In order to treat the head-tail regime ($\lambda \sim k$) or the slow blowup regime in general ($\text{Im}\lambda < 1$), it is necessary to include the coupling between positive and negative frequencies in the integral equation. We have seen that in the head-tail regime, positive and negative frequencies contribute equally. In this section we give a general treatment (Gaussian bunch) of the slow blowup regime. We begin with the integral equation

$$\bar{D}(p') = \int_{-\infty}^{\infty} A(p, p') \bar{D}(p') dp' \quad (126)$$

where from Eq. 85 we have

$$A(pp') = \frac{NG(p'\omega_o - \omega_\xi)}{4\pi\omega_y \omega_s} e^{-(|p| - |p'|)^2 \sigma_o^2 / 2} C(pp', \lambda), \quad (127)$$

$$\text{where } C(pp', \lambda) \equiv e^{-|pp'| \sigma_o^2} \left(\frac{i}{e^{2\pi i \lambda} - 1} \right) \int_0^{2\pi} e^{i\lambda t} e^{pp' \sigma_o^2 \cos t} dt. \quad (128)$$

From this starting point we would like to formulate the problem so that we treat the coupling of opposite signs of frequency on equal footing with like signs. We proceed by first rewriting the integral equation as

$$\bar{D}(p) = \int_0^{\infty} A(p, p') \bar{D}(p') dp' + \int_0^{\infty} A(p, -p') \bar{D}(-p') dp' \quad (129)$$

We need $\bar{D}(-p)$; using the same equation we have

$$\bar{D}(-p) = \int_0^{\infty} A(-p, p') \bar{D}(p') dp' + \int_0^{\infty} A(-p, -p') \bar{D}(-p') dp' \quad (130)$$

and of course we need only p (and p') > 0 now. Let us define

$$\vec{D} = \begin{pmatrix} \bar{D}(p) \\ \bar{D}(-p) \end{pmatrix}. \quad (131)$$

Then we can write the above equation as

$$\vec{D}(p) = \int_0^\infty \frac{Ne^{-(p-p')^2 \sigma_0^2/2}}{4\pi\omega_y \omega_s} M(p,p') \vec{D}(p') dp' \quad (132)$$

where

$$M(p,p') = \begin{pmatrix} G(p'\omega_0 - \omega_\xi) C(pp', \lambda) & G^*(p'\omega_0 + \omega_\xi) C(-pp', \lambda) \\ G(p'\omega_0 - \omega_\xi) C(-pp', \lambda) & G^*(p'\omega_0 + \omega_\xi) C(pp', \lambda) \end{pmatrix} \quad (133)$$

and we have used

$$G^*(\omega) = G(-\omega) \text{ for } \omega \text{ real.} \quad (134)$$

Thus, when we fold the negative onto the positive we get a 2×2 matrix within our original integral equation.

For the fast blowup regime with a long bunch ($\sigma\omega_c \gg 1$) the off diagonal terms are small. If we neglect them, we obtain the kernel which we diagonalized approximately in the previous section. Notice that there is also a parallel problem with $G^*(p'\omega_0 + \omega_\xi)$ in the kernel rather than $G(p'\omega_0 - \omega_\xi)$. This other problem corresponds to a stable solution, and so we cannot consistently solve it within the fast blowup approximation.

On the other hand it is instructive now to make the head-tail approximation to contrast with fast blowup. From Eq. (128) we have

$$C(\pm pp', \lambda) \equiv C_{\pm} = \frac{ie^{-pp'\sigma_0^2}}{(e^{2\pi i\lambda} - 1)} \int_0^{2\pi} e^{i\lambda t} e^{\pm pp'\sigma_0^2 \cos t} dt \quad (135)$$

$$C_{\pm} = e^{-pp'\sigma_0^2} \sum_{\ell} \frac{(\pm 1)^{\ell} I_{\ell}(pp'\sigma_0^2)}{\lambda - \ell}.$$

The head-tail approximation consists of searching for a solution $\lambda \sim k + \delta_k$ where $\delta_k \ll 1$. In this case

$$C_{\pm} \approx e^{-pp'\sigma_0^2} \frac{(\pm 1)^k I_k(pp'\sigma_0^2)}{\delta_k} \quad (136)$$

Using Eqs. 129, 130 it is easy to show that

$$\delta_k D_{HT}(p) = \int_0^{\infty} \frac{Ne^{-(p-p')^2 \sigma_0^2 / 2}}{4\pi \omega_y \omega_s} G_{HT}(p' \omega_0) I_k(pp'\sigma_0^2) D_{HT}(p') dp' \quad (137)$$

$$\text{and} \quad D(p) - (-1)^k D(-p) = 0. \quad (138)$$

$$\text{where} \quad D_{HT} = [D(p) + (-1)^k D(-p)], \quad (139)$$

$$G_{HT} = G(p' \omega_0 - \omega_{\xi}) + G^*(p' \omega_0 + \omega_{\xi}). \quad (140)$$

So for k even the solutions are even functions of p , while for k odd they are odd functions of p . Notice the form of the head-tail impedance G_{HT} . If $\xi = 0$ ($\omega_{\xi} = 0$), then the kernel is real (and can easily be symmetrized) so we find a well known result³: no head-tail instability for $\xi = 0$.

This problem is now very similar to the fast blowup problem which we solved in the previous section. Recall that the important aspect of the approximate solution was that the local

value of the impedance drove the instability. Once again consider a high frequency broad band impedance. If G is a broad band impedance, then G_{HT} has a shape sensitive to the slope of G . In Fig. (5) we sketch an example using the ISA broad band impedance from Appendix 1. You see that $\text{Im } G_{HT}$ changes sign in the neighborhood of ω_c , the resonant frequency of G . The higher frequency region corresponds to instability and is also broad in the neighborhood of another frequency, ω_H . $\text{Im } G_{HT}(\omega_H) \approx 2\omega_\xi/\omega_c \text{Im } G(\omega_c)$. With this in mind we search for a solution in the neighborhood of ω_H .

Since the argument of the Bessel function is large in this case, we use the leading term of the asymptotic form and approximate the kernel by

$$\frac{N e^{-(p-p')^2 \sigma^2/2}}{4\pi\omega_y \omega_s} \frac{1}{\sqrt{2\pi}\sigma_o p_o} G_{HT}(\omega_H), \quad (p_o = \omega_H/\omega_o) \quad (141)$$

Within the slowly varying approximation we can diagonalize exactly as in the fast blowup case. If we select the largest eigenvalue, we find

$$\frac{\omega + \omega_y - k\omega_s}{\omega_s} = \frac{N G_{HT}(\omega_H)}{4\pi\omega_y \sigma \omega_H \omega_s \sqrt{2\pi} \omega_o \sigma} \sqrt{2\pi} \quad (142)$$

In terms of more conventional quantities we have

$$\frac{\omega + \omega_y - k\omega_s}{\omega_s} = \frac{i I_{\text{peak}} e^{Z_\perp^{HT}(\omega_H)}}{4\pi v \sqrt{2\pi} (\sigma \omega_s) \omega_H m_o \gamma c} \quad (143)$$

where $Z_\perp^{HT}(\omega) = Z_\perp(\omega - \omega_\xi) - Z_\perp^*(\omega + \omega_\xi)$ (144)

Fig. 5 Imaginary Part of $G_{HT}(\omega)$ for $\sigma\omega_c = 30$
and $\sigma\omega_\xi = +5$.

Thus we have found another solution of the head-tail effect valid for a gaussian bunch and a broad band impedance at high frequency. We see again that the solution depends upon the local value of the impedance much like a coasting beam; however, in this case the instability is a sensitive function of the chromaticity, ξ . This is caused by the cancelation of real part of $Z_{\perp}$ at positive and negative frequencies. There is, however, an important new effect here. The beam is unstable for ξ positive at frequencies slightly greater than ω_c while for ξ negative, the instability occurs at frequencies somewhat less than ω_c . This is caused by the long bunch effect.

The factor $\exp(-(p-p')^2\sigma_0^2/2)$ is so sharply peaked that the instability is sensitive to the local cancelation of $Z_{\perp}(\omega-\omega_{\xi})$ and $Z_{\perp}(\omega+\omega_{\xi})$. In Fig. (6) we plot the fast blowup results together with those for the head tail effect. Notice that above threshold the predicted growth rates differ by two orders of magnitude. Thus the head-tail approximation has completely obscured the long bunch/fast blowup results. In order to avoid this we must use a more accurate representation of the integral C.

Fig. 6 Growth Rate vs. Current, ISA Fast Blowup and the Head-tail Effect.

The General Solution For Broad Band Impedance At High Frequency

The next problem is to formulate solutions to the integral equation which are valid in both the fast and slow blow up regimes. It is, of course, easy to state the problem. Solve the integral equation in (126) with the kernel given in (127). However, in order to understand physically how the two regimes connect it is still useful to consider the high frequency broad band problem. This has lead to relatively simple solutions both for the fast blow up regime and for the head-tail regime, because the integral equation in this limit reduces to a dispersion relation with a factor of the peak current. For the general problem we can again make this simplification. This eliminates the complication of the integral equation while retaining the effects of the synchrotron frequency. This should help us understand how the head-tail results can go smoothly into the fast blow up regime. The key to this, of course, is the inclusion of all the synchrotron harmonics when we evaluate the integral B. To begin, let us restate the problem. We have:

$$\vec{D}(p) = \int_0^{\infty} \frac{Ne^{-(p-p')^2 \sigma_0^2 / 2}}{4\pi \omega_y s} \times M(p, p') \vec{D}(p') dp' \quad (145)$$

where

$$M(p, p') = \begin{pmatrix} G(p'\omega_0 - \omega_\xi) C(pp', \lambda) & G^*(p'\omega_0 + \omega_\xi) C(-pp', \lambda) \\ G(p'\omega_0 - \omega_\xi) C(-pp', \lambda) & G^*(p'\omega_0 + \omega_\xi) C(pp', \lambda) \end{pmatrix} \quad (146)$$

We can restrict the problem to a broad band impedance by assuming $G(p'\omega_0)$ has a broad maximum at $p'\omega_0 \approx \omega_c$. Once again we search for a solution near $p_c = \omega_c/\omega_0$. Since the first factor is sharply spiked, we can again approximate the kernel by factoring out the slowly varying part, which in this case is a 2×2 matrix. We can now diagonalize as we did in the fast blowup case.

Selecting the largest eigenvalue $(\sqrt{2\pi}/\sigma\omega_0)$ we obtain the following 2×2 problem.

$$\vec{D} = \frac{N}{4\pi\omega_y\omega_s} \frac{\sqrt{2\pi}}{\sigma\omega_0} M(p_c, p_c) \vec{D} \quad (147)$$

where M is given by equation (146). Again this is in the form of a dispersion relation except that we must diagonalize the 2×2 problem. So the frequency λ is obtained by

$$\text{Det}\left[1 - \frac{N}{4\pi\omega_y\omega_s} \frac{\sqrt{2\pi}}{\sigma\omega_0} M(p_c, p_c)\right] \quad (148)$$

or absorbing the constants into the definition of G we obtain

$$1 - C_+(G_- + G_+^*) - G_-G_+(C_-^2 - C_+^2) = 0. \quad (149)$$

Equation (149) is a dispersion relation for the frequency λ . Although it is slightly more complicated due to its quadratic nature, it yields solutions which are valid with no restrictions on the size of the real and imaginary parts of λ .

To understand the transition region between fast and slow blow up it is convenient to evaluate some approximate expressions for $C_{\pm}$ which include all the synchrotron harmonics, yet yield the correct results in the slow blow up limit.

We have

$$C_{\pm} = \frac{ie^{-\Delta^2}}{e^{2\pi i\lambda} - 1} \int_0^{2\pi} e^{i\lambda t} e^{\pm \Delta^2 \cos t} dt \quad (150)$$

or

$$C_{\pm} = \frac{e^{-\Delta^2}}{\sin \lambda \pi} \int_0^{\pi} \cos \lambda t e^{\mp \Delta^2 \cos t} dt \quad (151)$$

Since $\Delta \gg 1$,

$$C_{-} \approx \frac{e^{-\Delta^2} I_0(\Delta^2)}{\sin \lambda \pi} + O(\lambda/\Delta)^2 \quad (152)$$

$$\approx \frac{\sqrt{2\pi}}{2\Delta} \frac{1}{\sin \lambda \pi}, \quad (153)$$

on the other hand,

$$C_{+} = e^{-\Delta^2} \left\{ \cot \lambda \pi \int_0^{\pi} \cos \lambda t e^{\Delta^2 \cos t} dt + \int_0^{\pi} \sin \lambda t e^{\Delta^2 \cos t} dt \right\} \quad (154)$$

$$C_{+} \approx \frac{\sqrt{2\pi}}{2\Delta} \left[\cot \lambda \pi + \frac{2\lambda}{\sqrt{2\pi}\Delta} + O(\lambda/\Delta)^2 \right]. \quad (155)$$

The previous expressions are valid $0 < |\lambda| \ll \Delta$. This regime includes the slow blowup and overlaps the fast blow up regime. To understand the approximations in the context of synchrotron harmonics, express $C_{\pm}$ as

$$C_{\pm} = e^{-\Delta^2} \sum_k \frac{I_k(\Delta^2)(\pm 1)^k}{\lambda - k} \quad (156)$$

Since Δ is very large. $I_k \approx I_0 \approx \frac{1}{\sqrt{2\pi}} \frac{e^{\Delta^2}}{\Delta}$ for k small.

$$C_{\pm} \approx \frac{\pi}{\sqrt{2\pi\Delta}} \frac{1}{\pi} \sum_K \frac{(\pm 1)^k}{\lambda - k} \quad (157)$$

so
$$C_+ \approx \frac{\sqrt{2\pi}}{2\Delta} \operatorname{ctn} \lambda\pi \quad (158)$$

and
$$C_- \approx \frac{\sqrt{2\pi}}{2\Delta} \frac{1}{\sin \lambda\pi} \quad (159)$$

The factor $\Delta = \sigma\omega_c$ which plays a very important role can be thought of in two ways. First it is the spread in frequency in units of the synchrotron frequency, and second it is the number of wavelengths which lie within the bunch. These two are linked due to the circulation of the particle in longitudinal phase space. The effect of Δ being large is to excite many synchrotron harmonics equally (up to $k \sim \Delta^2$); however, in Eq. (157) we approximate this situation by exciting all the harmonics equally.

Comparing (153), (155), and (158), (159) you see that this yields the leading term in the expression for the integral $C_{\pm}$. So in this regime we do not excite just one synchrotron harmonic, we excite all equally and then correct the expression to order λ/Δ by the extra term in Eq. (155). It is this difference in the expressions for C_+ and C_- that

ultimately leads to the fast blowup regime which is only weakly dependent on ξ , the chromaticity.

To see the initiation of the fast blowup regime let $\omega_{\xi} = 0$ and let $G(\omega_c)$ be pure imaginary. Then we have

$$1 - \left(G \frac{\sqrt{2\pi}}{2\Delta}\right)^2 \left(1 - \frac{4\lambda \operatorname{ctn} \lambda \pi}{\sqrt{2\pi} \Delta}\right) = 0. \quad (160)$$

Search for a pure imaginary solution ($\lambda = i\beta$), then

$$\frac{1}{r} = 1 - \frac{4\beta}{\sqrt{2\pi} \Delta} \operatorname{cth} \pi \beta. \quad (161)$$

In Fig. (7) we plot β vs r (which is just β vs I/I_{th}). You see that the curve has the shape characteristic of the fast blow up regime; however, notice that in this case $\beta \rightarrow 0$ as $r \rightarrow 1$. So there appears to be a genuine threshold for the chromaticity "independent" instability at $r = 1$.

To see this examine the slow blow up regime with Eq. (149).

Let

$$\omega_{\xi} = 0$$

$$\begin{aligned} C_+ &= \frac{\sqrt{2\pi}}{2\Delta} \operatorname{ctn} \lambda \pi \\ C_-^2 - C_+^2 &= \left(\frac{\sqrt{2\pi}}{2\Delta}\right)^2 \left(1 - \frac{4\lambda \operatorname{ctn} \lambda \pi}{\sqrt{2\pi} \Delta}\right) = \left(\frac{\sqrt{2\pi}}{2\Delta}\right)^2 \end{aligned} \quad (162)$$

then we have

$$1 - \operatorname{ctn} \lambda \pi 2r \cos \theta - r^2 = 0 \quad (163)$$

where $\theta = \operatorname{Arg} (G)$

So below threshold ($r < 1$) we have

$$\tan \lambda \pi = \frac{2r \cos \theta}{1 - r^2} \quad (\omega_{\xi} = 0) \quad (164)$$

Fig. 7 Growth Rate vs. Current, Equation (161), Locally Pure Imaginary $G(\omega_c)$ Chromaticity, $\xi = 0$.

which implies that $\text{Im}\lambda$ is real for $r < 1$ (within the above approximation). If we now let ω_ξ be nonzero, we find an expression which is valid for slow blowup.

$$\tan \lambda \pi = \frac{G(\omega_c - \omega_\xi) + G^*(\omega_c + \omega_\xi)}{1 - G_- G_+^*} \quad (165)$$

where we have absorbed the factor $\sqrt{2\pi}/2\Delta$ into G . The above solution reduces to the head-tail result for $\lambda \sim \text{integer}$, but is valid so long as we are below the fast blowup threshold. In order to see the transition to fast blow up we must keep more terms in the dispersion relation. To do this we use a modification of the mapping approach used in the previous section. It is straightforward to evaluate C_+ and C_- numerically. Given $C_\pm$ we find the zeros of the expression.

$$1 - C_+(G_- + G_+^*) - G_- G_+(C_-^2 - C_+^2) = 0. \quad (166)$$

again numerically. It is then possible to select those solutions with the largest growth rate, since these are the most troublesome.

In the next section we discuss the solutions of the dispersion relation above and conclude the discussion of long bunches.

**In this expression

$$G_\pm = \frac{N \sqrt{2\pi}}{4\pi\omega_y \omega_s \sigma \omega_o} \quad G(\omega_c \pm \omega_\xi) = \frac{i e I_{\text{peak}} Z_\perp(\omega_c \pm \omega_\xi)}{4\pi v \omega_s \omega_o \gamma_c} .$$

Discussion - The "Phase Transition"

In Figs. (8) and (9) we plot the $\text{Im}(\omega/\omega_s)$ vs I/I_{th} for two cases: the case corresponding to ISABELLE in which $\text{Arg } G(\omega_c) \sim 20^\circ$ due to the dominating effect of the space charge and an ISA-like case for $\text{Arg } G(\omega_c) = 45^\circ$. In both cases $\sigma\omega_c = 30$ while $\sigma\omega_\xi = 5$, the values for ISABELLE.

The striking feature of these curves is the sharp transition that occurs just above the "threshold" current. We will refer to this as the "phase transition" for the following reason. In the slow blowup regime the physics is well described by head-tail modes. They are still coupled, but the coupling is weak and the physics is described basically by the isolated modes. On the other hand in the fast blowup regime all these modes are coupled in the sum. In fact, if we think in terms of the expression in Equation (158) then all of the modes add to yield $\cot n\lambda\pi$ which in the fast blowup regime goes to -1 , the leading term in Eq. (155).

The curves plotted in Figs. (8) and (9) are valid throughout the range of currents. However, in obtaining the simple dispersion relation we have approximated the kernel in order to understand the behavior for a broad band impedance. Do we expect the same behavior if the impedance is not broadband?

Fig. 8 The Phase Transition. Growth Rate vs. Current. (ISA Impedance).

Fig. 9 The Phase Transition. Growth Rate vs. Current, ISA-like Impedance with $\text{Arg } G(\omega_c) = 45^\circ$.

The key to the answer to this question lies in the factor $C_+(pp', \lambda) = B(pp', \lambda)$ which reduces the dispersion integral in the coasting beam limit. If the impedance is large around ω_c and $\sigma\omega_c \gg 1$, then the number of wavelengths within one bunch length is large and the factor B yields a coasting beam-like threshold or a phase transition to fast blowup. If the impedance in this case is not broad band then this will effect the numerical results, however, the qualitative nature of the transition should be comparable to that obtained in the previous section. We can conjecture that there will basically be an additional numerical factor in all the results if we solve the integral equation exactly in the case of a long bunch.

We also should ask another question. What if the bunch length is not long compared to the characteristic wavelength; will there still be a phase transition? For a partial answer to this question consider again the upper bounds given in equation (120). We have seen in the previous section that the qualitative nature of the instability was given by the upperbound curve for the growth rate. Now evaluate the upper bound for $\sigma(\omega_c + \omega_\xi) = \Delta = 4$. Fig (10) shows the intersection of the two bounds for this case. This curve clearly indicates that the transition is disappearing as $\Delta \rightarrow 1$. So we expect qualitatively different results for $\sigma\omega_c \lesssim 1$.

Fig. 10 Upper Bound on Growth Rate Evaluated
at $\sigma(\omega_c + \omega_\xi) = 4$

In the next section we reformulate the problem with a view towards the solution in the regime $\sigma\omega_c \lesssim 1$. The aim is to discover the new instability for a short bunch which corresponds to the phase transition for a long bunch.

V. SHORT BUNCH - LONG WAVE LENGTH*

In the previous sections we have studied coherent instabilities in the regime in which the bunch length is much longer than the wavelength of coherent oscillations. If ω_c is the frequency at which the impedance is large, and σ is the rms bunch-length in seconds, then this condition is given by:

$$\sigma\Gamma \gg 1, \sigma\omega_c \gg 1, \text{ long bunch/short wavelength} \quad (167)$$

If ω_c is given by the cut off frequency of the vacuum chamber, then this is applicable mainly to proton machines where Eq. (167) is well satisfied. In the case of a short bunch/long wavelength we have

$$\sigma\omega_c \ll 1, \sigma\Gamma \ll 1, \quad (168)$$

and we expect the solution of the integral equation to be qualitatively different. To see this let us examine the kernel in Eq. (85). In the long bunch case the is the product of a slowly varying function, $G(p\omega_0 - \omega_c) B(pp', \lambda)$, and a sharply spiked function, $\exp [-(p-p')^2 \sigma_0^2 / 2]$. This means that the solution is sensitive to the local value of the impedance. We have seen that this is true for the fast and slow blow up in the previous

*The results presented in this section are preliminary results of a collaboration with J.M. Wang.

sections.

In the short bunch case, the bunch factor, $\exp [-(p-p')^2 \sigma_o^2 / 2]$ is much broader than the impedance. So in this case we expect that the average value of the impedance will be important. To see this explicitly consider the limiting case $\sigma_o \rightarrow 0$. Then the integral equation becomes

$$\lambda \bar{D}(p) = N \int_{-\infty}^{\infty} \frac{G(p' \omega_o - \omega_{\xi})}{4\pi \omega_y \omega_s} \bar{D}(p') dp' \quad (169)$$

which yields

$$\lambda = \int_{-\infty}^{\infty} \frac{NG(p \omega_o - \omega_{\xi})}{4\pi \omega_y \omega_s} dp. \quad (170)$$

Thus, we see that in the point bunch case, it is the average value of the impedance that is most important rather than the local value.

Another important point which is evident from equation (170) is that in the point bunch case, for a physically reasonable short wake impedance, the frequency shift and the growth rate are both zero. This follows if we require that in the time domain the force has a finite rise time. This is easily seen by considering the inverse fourier transform:

$$g(t) \equiv \int_{-\infty}^{\infty} G(\omega) e^{-i\omega t} d\omega \quad (171)$$

so that if $g(t=0) = 0$, then

$$\int_{-\infty}^{\infty} G(\omega) d\omega = 0. \quad (172)$$

Of course if we include space charge this is not true, but for electrons this term is very small and so we will assume $G(\omega)$ to satisfy Eq. (172). Physically this is reasonable; the point bunch can leave a wake field, but due to the finite rise time this field cannot act on the point bunch.

In the next section we illustrate a new method for solving the integral equation in the long wave length regime by first considering the head tail effect.

The Head-Tail Effect - Short Bunch/Long Wavelength

By the head tail effect we mean solutions to the integral equation such that $\lambda \sim k + \delta_k$ where $\delta_k \ll 1$. We expect this to be the case for small currents. From equation (127) we have

$$\bar{D}(p) = \int_{-\infty}^{\infty} \frac{NG(p'\omega_o - \omega_{\xi})}{4\pi\omega_y\omega_s} e^{-(p^2+p'^2)\sigma_o^2/2} B'(pp', \lambda) \bar{D}(p') dp' \quad (173)$$

where

$$B'(pp', \lambda) = \frac{-i}{1-e^{2\pi i\lambda}} \int_0^{2\pi} e^{i\lambda t} e^{pp'\sigma_o^2 \cos t} dt \quad (174)$$

$$= \sum_{\ell} \frac{I_{\ell}(pp'\sigma_o^2)}{\lambda - \ell} \quad (175)$$

So for the head tail approximation we have

$$B'(pp', \lambda) \approx \frac{I_k(pp'\sigma_o^2)}{\delta_k} \quad (176)$$

and equation (173) becomes

$$\delta_k \bar{D}(p) = \int_{-\infty}^{\infty} \frac{NG(p' \omega_o - \omega_{\xi})}{4\pi \omega_y \omega_s} e^{-(p^2 + p'^2) \sigma_o^2 / 2} I_k(pp' \sigma_o^2) \bar{D}(p') dp' \quad (177)$$

If G is peaked at a value p_o , then by the short bunch/longwavelength regime we mean $p_o \sigma_o < 1$; so we expect an expansion of the Bessel function to be useful.

With this change equation (177) takes the form

$$\delta_k \bar{D}(p) = \sum_{\ell} \frac{(p_o / \sqrt{2})^{k+2\ell}}{[\ell! (k+\ell)!]^{1/2}} e^{-p^2 \sigma_o^2 / 2} \underbrace{\int_{-\infty}^{\infty} \frac{NG(p' \omega_o - \omega_{\xi})}{4\pi \omega_y \omega_s} e^{-p'^2 \sigma_o^2 / 2} \frac{(p' \sigma_o / \sqrt{2})^{k+2\ell}}{[\ell! (k+\ell)!]^{1/2}} \bar{D}(p') dp'}_{\equiv V_{\ell}} \quad (178)$$

where we have rewritten the equation in a suggestive form. If we define V as given above, it is easy to see that V satisfies

$$\delta_k V_{\ell} = \sum_{\ell'=0}^{\infty} M_{\ell \ell'} V_{\ell'} \quad (179)$$

where

$$M_{\ell \ell'} = \int_{-\infty}^{\infty} \frac{NG(p \omega_o - \omega_{\xi})}{4\pi \omega_y \omega_s} e^{-p^2 \sigma_o^2 / 2} \frac{(p \sigma_o / \sqrt{2})^{2(k+\ell+\ell')}}{[\ell! (k+\ell)!]^{1/2} [\ell'! (k+\ell')!]^{1/2}} \quad (180)$$

Thus, we have written the integral equation in terms of an infinite dimensional matrix equation. The eigenvalues are given by

$$\det (M - \delta_k) = 0 \quad (181)$$

So far nothing has been gained; however, in the case of a short bunch we will see that it is possible to truncate the matrix at a finite dimension with little error. So in this case we expect considerable simplification. To see this first rewrite equation (180) as

$$M_{\ell\ell'} = C_{2(k+\ell+\ell')/2}^{(k+\ell+\ell')} [\ell!(k+\ell)!]^{1/2} [\ell'!(k+\ell')!]^{1/2} \quad (182)$$

where

$$C_\mu = N \int_{-\infty}^{\infty} \frac{G(p\omega_o - \omega_\xi)}{4\pi\omega_y\omega_s} (p\sigma_o)^\mu e^{-p^2\sigma_o^2} dp \quad (183)$$

So, if C_μ does not increase strongly with increasing μ , then the matrix can be truncated. However, before considering the truncated problem in more detail it is instructive to examine the matrix M .

Notice that since $G^*(\omega) = G(-\omega)$ and since M contains only even moments; for $\omega_\xi = 0$, M is Hermitian and so has real eigenvalues. This well known result always obtains when the approximation $\lambda \sim k + \delta_k$ is used to simplify the integral equation.

It is also instructive to approach the problem naively by truncating the matrix to be 1×1 . In this case we find

$$(\omega + \omega_y - k\omega_s) = N \int_{-\infty}^{\infty} \frac{v(p' \omega_o - \omega_s)}{4\pi\omega_y} \frac{(p\sigma_o)^{2k}}{k! 2^k} e^{-p^2 \sigma_o^2 / 2} dp \quad (184)$$

for the k th head tail mode. We can also calculate the eigenfunctions. It is easy to show that if we find an eigenvector V , then the corresponding solution in the integral representation is

$$\bar{D}(p) = \sum_{\ell=0}^{\infty} (p\sigma_o)^{k+2\ell} \frac{e^{-p^2 \sigma_o^2 / 2}}{[2^{k+2\ell} \ell! (k+\ell)!]^{1/2}} V_{\ell} \quad (185)$$

For the 1×1 problem this yields

$$\bar{D}(p) = \frac{(p\sigma_o)^k}{(2^k k!)^{1/2}} e^{-p^2 \sigma_o^2 / 2} \quad (186)$$

So for this naive (1×1) case the eigenfunctions are moments of the gaussian distribution. These have been referred to as "Hermite modes" and as "normal modes of a gaussian bunch" in the context of longitudinal instabilities¹², a term which has also been taken over into the parallel usage for transverse instabilities. This is very misleading in that these modes do not diagonalize the head-tail mode problem. In fact using these modes to calculate frequency shifts and growth rates can be quite misleading.

To illustrate this consider the impedance given in Eq. (53). In order to restrict consideration to the long wavelength-short bunch region it is necessary to modify this impedance by cutting it off above ω_c (say at $2\omega_c$) so that the high frequency tail

doesn't effect the calculation. We already know that the high frequency part of the impedance is best treated by the short wavelength approach discussed previously.

With this in mind we let

$$G(\omega) = \begin{cases} \frac{-G(0)\omega_c^2}{2i\Gamma\omega + (\omega^2 - \omega_c^2)}, & |\omega| < 2\omega_c \\ 0, & |\omega| > 2\omega_c \end{cases} \quad (187)$$

With this impedance it is obvious that for large μ , $|C_u| \sim (\sigma\omega_c)^\mu$; therefore, for $\sigma\omega_c$ sufficiently small we can easily truncate the dimension of the matrix. Let us truncate above C_2 for illustration and diagonalize for the cases $k = 0$ and $k = 1$.

The solutions in this case are given by

$$\delta_0 = C_0/2 \pm \frac{1}{2} [C_0^2 + C_2^2]^{\frac{1}{2}} \quad (188)$$

$$(Truncated) \quad (189)$$

$$\delta_1 = C_2/2$$

On the otherhand for the (1×1) problem we have

$$\begin{aligned} \delta_0 &= C_0 \\ \delta_1 &= C_2/2 \quad (1 \times 1) \end{aligned} \quad (190)$$

So we see that if we truncate in a consistent fashion we introduce another solution to the $k = 0$ problem. Therefore, even if $\text{Im}(C_0) < 0$ ($\xi > 0$), mode zero can still have an unstable solution with a somewhat smaller frequency shift. This also shows that if the higher head-tail modes are excited, we must diagonalize a larger matrix in order to calculate the frequency

shifts for the lowest modes. Of course, the eigenfunctions to the truncated problem are not simple Hermite modes but are linear combinations of them with the coefficients being determined by the diagonalization of the matrix in Eq. (182).

In this context another question arises. Can we use this approach for $\sigma\omega_c > 1$? If we examine Eq. (183) for the factor C_μ , we can see that the largest contribution to the integral should occur when the maximum of the impedance and the maximum of $(p\sigma_0) \exp(-p^2\sigma_0^2)$ coincide. If the maxima of the real and imaginary parts of G are near ω_c , then the overlap occurs when

$$\mu = 2 (\sigma\omega_c)^2 \quad (191)$$

Thus, the dimension of the matrix to be diagonalized rises roughly as the square of $(\sigma\omega_c)$. This means that this approach may not be as useful for long bunches.

Rather than continue the analysis of the head-tail problem, we now return to the integral equation to find an approach which eliminates the head-tail approximation. Due to our experience with the analysis of the long bunch problem, we expect to find qualitatively different results.

The Coupling of Modes, A More General Solution

In the previous section we made the restrictive assumption that $\lambda \sim k$. For small currents we expect this to be well satisfied; however, for large currents this will break down and therefore we need to modify our approach.

What do we expect from the solution of the more general problem? We know from the long bunch results that there were two regimes: (1) slow blow up, where head-tail modes dominate thus giving a very strong dependence on the chromaticity, ξ . (2) fast blow up, a coasting-beam-like instability in which essentially all the head-tail modes are excited and coupled, which has a somewhat weaker coasting-beam-like dependence upon the chromaticity.

So far, for the long wavelength case, we have studied the chromaticity dependent head-tail regime, and we find results similar to those for short wavelength, with one important difference. For short wavelength the average value of the impedance is important while for long wavelength, the local value is more relevant.

In the present section we concentrate on the coupling of modes in the long wavelength regime. Qualitatively we expect a new regime of instability which is parallel to the coasting beam like effect for short wavelength.

However, we do not expect to see the large number of modes excited as in the short wavelength case. Once again an expansion in the parameter, $pp'\sigma_o^2$, will be useful. But we do not want an expansion in terms of harmonics of the synchrotron frequency, rather we will expand the factor $B'(pp', \lambda)$ directly. From equation (174) we have

$$B'(pp', \lambda) = \frac{-i}{1-e^{2\pi i\lambda}} \int_0^{2\pi} e^{i\lambda t} e^{pp'\sigma_o^2 \cos t} dt \quad (192)$$

$$= \sum_{\ell=0}^{\infty} \frac{(pp'\sigma_o^2)^\ell}{\ell!} f_\ell(\lambda) \quad (193)$$

where

$$f_\ell(\lambda) = \frac{-i}{1-e^{2\pi i\lambda}} \int_0^{2\pi} e^{i\lambda t} \cos^\ell t dt \quad (194)$$

so,

$$f_0 = 1/\lambda, f_1 = \frac{\lambda}{\lambda^2-1}, f_2 = \frac{(\lambda^2-2)}{(\lambda^2-4)}, \text{ etc.} \quad (195)$$

substituting into equation (173) we can rewrite it in a fashion completely analagous to the matrix equation for the head tail modes. We ultimately find an equation formally similar to equation (179) given by

$$U_\ell = \sum_{\ell'=0}^{\infty} N_{\ell\ell'}^\lambda U_{\ell'}, \quad (196)$$

where

$$U_\ell = \int_{-\infty}^{\infty} dp G(p\omega_o - \omega_\xi) e^{-p^2\sigma_o^2/2} \frac{(p\sigma_o)^\ell}{\sqrt{\ell!}} \bar{D}(p) \quad (197)$$

and

$$N_{\ell\ell'}^{\lambda} = N \int_{-\infty}^{\infty} dp \frac{G(p\omega_0 - \omega_{\ell'})}{4\pi\omega_0\omega_s} e^{-p^2\sigma_0^2} (p\sigma_0)^{\ell+\ell'} \frac{f_{\ell'}(\lambda)}{\sqrt{\ell!\ell'!}} \quad (198)$$

$$= C_{\ell+\ell'} \frac{f_{\ell}(\lambda)}{(\ell!\ell'!)^{1/2}} \quad (199)$$

Notice that the coherent frequency appears as a parameter in a matrix equation with eigenvalue equal to unity. This is formally similar to the integral equation in this respect. It is important to realize that an equation of the form

$$\hat{c}_k V = MV \quad (200)$$

exists only in an approximate sense. In fact it is impossible to write the general equation in the form above, rather one must use the form

$$U = N(\lambda)U \quad (201)$$

In order to understand the new instability which occurs due to the solution of this problem, we will set $\omega_{\ell'} = 0$ in what follows. This eliminates the complication of the head tail effect and simplifies the problem somewhat. The problem is very similar to head tail effect solution just discussed, so we look for solutions to the truncated problem.

From the form of the matrix it is evident that if $C_{\ell+\ell'}$ does not increase with increasing $\ell+\ell'$ then this approach is legitimate. There is however an important difference between the two problems. In the previous section we saw that the matrix for the head-tail

effect was Hermitian when $\omega_\xi = 0$. In the present case this is not true since the matrix contains "odd moments". From Eq. (183) we have

$$C_\mu = \frac{N}{4\pi\omega_y\omega_s} \int_{-\infty}^{\infty} (p\sigma_o)^\mu G(p\omega_o) e^{-p^2\sigma_o^2} dp \quad (\omega_\xi = 0) \quad (202)$$

Then since $G(-p\omega_o) = G^*(p\omega_o)$, we find

$$\mu \text{ even, } C_\mu = \frac{2N}{4\pi\omega_y\omega_s} \int_0^{\infty} (p\sigma_o)^\mu \text{Real}[G(p\omega_o)] e^{-p^2\sigma_o^2} dp \quad (203)$$

$$\mu \text{ odd, } C_\mu = \frac{2iN}{4\pi\omega_y\omega_s} \int_0^{\infty} (p\sigma_o)^\mu \text{Im}[G(p\omega_o)] e^{-p^2\sigma_o^2} dp \quad (204)$$

It is evident that even though we could symmetrize the matrix, it would not be Hermitian since C_μ for μ odd is pure imaginary. So we expect the solution to lead to an instability.

To illustrate the method assume that we can truncate to obtain a 2×2 matrix; this yields

$$\text{Det} \begin{pmatrix} \frac{C_0}{\lambda} - 1 & \frac{C_1 \lambda}{\lambda^2 - 1} \\ C_1 / \lambda & \frac{C_2 \lambda}{(\lambda^2 - 1)} - 1 \end{pmatrix} = 0 \quad (205)$$

which yields the following cubic equation for λ :

$$\lambda^3 - (C_2 + C_0) \lambda^2 + (C_0 C_2 - C_1^2) \lambda + C_0 = 0 \quad (206)$$

Since C_1 is purely imaginary and C_0, C_2 are purely real, the coefficients of this equation are purely real. All of the C 's are just proportional to N , the number of particles in the bunch. If we let $N \approx 0$, then the solutions are

$$\lambda = 0, \pm 1 \quad (207)$$

so what we have here is the coupling between the three lowest modes. As we increase N , the solutions will move and remain real until two of them coincide. At this point, with increasing N , we find one real solution and two others which are complex conjugates. Thus, at a threshold corresponding to the degeneracy of two of the roots, we begin to have instability.

This chromaticity independent instability is the parallel of the coasting-beam-like instability for a long bunch.

In order to estimate the threshold for instability consider just the diagonal terms in the 2×2 matrix equation:

$$\lambda \approx C_0, \quad \frac{\lambda^2 - 1}{\lambda} \approx C_2 \quad (208)$$

Since the solutions must move $\sim$ one unit to be degenerate, we estimate this point to be

$$\begin{aligned} 1 &\approx |C_0| \\ \text{or} \quad 1 &\approx |C_2| \end{aligned} \quad (209)$$

In terms of conventional quantities this threshold is:

$$1 \approx \frac{\left| i e I^{\text{th}} \int_{-\infty}^{\infty} Z(p\omega_0) e^{-p^2\sigma_0^2} dp \right|}{4\pi\nu\omega_s m\gamma c} \quad (210)$$

or

$$1 \approx \frac{\left| i e I^{\text{th}} \int_{-\infty}^{\infty} Z(p\omega_0) (p\sigma_0)^2 e^{-p^2\sigma_0^2} dp \right|}{4\pi\nu\omega_s m\gamma c} \quad (211)$$

We must emphasize, though, that the equations given above are just qualitative estimates; we must solve the cubic equation to find the actual threshold for the truncated problem.

In spite of this the thresholds above do give a good qualitative description of the instability at PETRA⁶ in terms of the scaling with respect to energy, synchrotron frequency, and bunchlength.

Discussion

In the previous sections we have demonstrated another method for solving the head-tail problem, and we have shown the existence of a PETRA-like instability which is only weakly dependent

upon the chromaticity. Both methods are primarily useful for $\sigma\omega_c$ small because, as we have seen, the number of dimensions we must keep in the matrix is approximately $2(\sigma\omega_c)^2$.

We should emphasize three points here. The first is that for the head-tail problem diagonalizing the matrix yields qualitatively different results from the 1×1 problem. The modes of oscillation are not simple "Hermite" modes. Second, for the mode coupling solution which is valid for larger currents, we must keep all the large terms in the matrix. Third, the approach used is not particularly illuminating for longer bunch lengths because the dimension of the matrix increases dramatically, and we know that for the "phase transition" we must couple essentially all of the head tail modes.

VI. CONCLUSION AND SUMMARY

In this paper we have presented a formulation of the problem of transverse coherent instabilities for bunched particle beams. The main effort has been to display the effect of the synchrotron frequency in setting the time scale for the instability without making restrictive assumptions.

For the long bunch with a broad band impedance there are three main effects.

- 1) The region of slow blow up or the head tail effect where $\text{Im}(\omega) < \omega_s$. In the case of a long bunch the instability is sensitive to the local cancellation of the impedance at positive and negative frequencies. This leads to the instability and excitation of all head tail modes $k \ll (\sigma\omega_c)^2$ at low or high frequency depending upon the sign of the chromaticity.
- 2) The "phase transition". This is a sharp coasting beam like threshold for instability. Again it is the local value of the impedance that is important, however in this case the cancellation disappears.
- 3) The Fast blow up region. In this regime, since there are many wavelengths within the bunch length,

the instability is described by the coasting beam like dispersion relation. However, a wave packet rather than a plane wave is excited.

To understand qualitatively the physics for slow blow up consider the coasting beam dispersion integral. From Eq. (71) we have

$$1 = \frac{G(k\omega_0)}{2\omega_y} \int_{-\infty}^{\infty} \frac{\rho_0(\tau) d\tau}{\omega(k) + \omega_y + k\omega_0 \tau} , \quad (212)$$

where we have simplified the expression somewhat while retaining the essentials. It is easy to see that if $\omega(k)$ is a solution of the equation above, then $\omega^*(-k) = \omega(k)$. This means that if $\omega(k)$ corresponds to an unstable solution [$\text{Im}(\omega) > 0$] then $\omega(-k)$ corresponds to a stable, damped solution.

On the other hand for a bunched beam we have seen that k and $-k$ are intrinsically coupled, first due to the bunch factor and second due to the circulation of particles within the bunch. Therefore, it is not surprising that a cancellation occurs which tends to stabilize the bunch. This effect has been known for some time.

What is striking is that for a long bunch there is a region ($\text{Im} \lambda > 1$) where a decoupling of $-k$ and k occurs which leads to a coasting beam like threshold. This "phase transition" can again

be understood by appealing to equation (212). For this equation to be satisfied the integral must have an imaginary part since $G(k\omega_0)$ is complex. If ρ_0 has a finite extent in τ , then the dispersion integral can yield an imaginary part for ω real so long as the pole in the integration lies within the extent of ρ_0 . This well known effect is just Landau damping. If the real part of ω is so large that the pole, in the integrand is outside the range of ρ_0 , then ω must have an imaginary part to satisfy Eq. (212). This argument leads to the qualitative statement that the instability threshold occurs when the shift in frequency is equal to the spread in frequency.

The "Dispersion Integral" $B(pp', \lambda)$ plays the same role in the context of bunched beams, except that it also contains information for the slow blow up region as well. In fact it is through this correspondence between fast and slow blow up that we now know that the coasting beam like behavior results from the excitation of many synchrotron modes collectively. Why do we need so many head-tail modes to excite a coasting beam like wave? This easily seen by examining the plane wave,

$$e^{i(k\omega_0 \tau - \omega t)} = e^{i(k\omega_0 \tau \cos \theta_s - \omega t)} = \sum_{\ell} i^{\ell} J_{\ell}(k\omega_0 \tau) e^{i(\ell \theta_s - \omega t)} \quad (213)$$

Thus in terms of the "circular modes" which yield harmonics of the synchrotron frequency ($\ell \dot{\theta}_s = \ell \omega_s$), we need an infinite

number to describe a plane wave.

In order to stabilize the fast blowup region we can simply increase the spread in frequency due to $\Delta p/p$ just as for a coasting beam or rather increase terms in the denominator of Eq. (121) for the threshold. However, this does not work for the slow blowup region. In this case it is necessary to introduce an additional frequency spread, say with a nonlinear betatron oscillation. But the necessary spread should be only the size of the synchrotron frequency because the growth rate for the slow blowup region is less than ω_s .

To conclude the discussion of the long bunch results we must emphasize that although we have not obtained an exact solution, we expect that the behavior will be qualitatively the same for any high frequency impedance so long as the width of the impedance is larger than the width of the bunch factor. In other words so long as the wave field decays within the bunchlength.

To understand the opposite region where the bunch factor is very broad we have treated the case of a short bunch. There are two main effects here:

- 1) For small currents the physics is dominated by the individual head-tail modes. We have seen that it is still necessary to diagonalize a matrix, and we find that the solutions are not simple "Hermite modes", but rather linear combinations of

them. This, as we have seen, is very important for the calculation of growth rates and frequency shifts.

2) For large currents we have a region of instability which is parallel to the fast blowup of a long bunch. In this case the instability again involves mode coupling as in the long bunch case; however, since there are so few modes excited, the result is qualitatively different. The growth rate here at the threshold is just proportional to the number of particles in the bunch and so the transition is not like the "phase transition" for a long bunch. However, this instability does seem to qualitatively describe the instability at PETRA.

For the short bunch - long wavelength case we have seen that it is moments of the impedance and a bunch factor that determines stability. This makes physical sense because, these moments give the behavior of the wake field $g(t)$ for small t , and for a short bunch one would expect the physics to be determined by the leading edge of $g(t)$.

To conclude the discussion of this case we must remind the reader that for longer bunch lengths we must diagonalize a larger matrix; however, we expect the qualitative results to be similar to those obtained: For the head tail mode problem, including the mixing of "Hermite" modes yields new unstable solutions with small frequency shifts, while for the more general mode coupling

problem, the threshold will correspond to the degeneracy of two roots of an algebraic equation of order $(2n-1)$ if we truncate the matrix at dimension n .

We have described many qualitative features of the instability in the two regions examined here. The mathematical formulation of the problem is very compact and should lead to a better understanding of vertical coherent instabilities in general. A similar fast blow up region has been shown to exist for the longitudinal problem⁴, and so we expect that results very similar to those presented here will apply to the longitudinal problem.

There still needs to be more numerical work done to understand more precisely the solutions to the integral equation and the effect of different impedances, and of course there is also the need for more calculations of contributions to impedances due to structures in the beam environment.

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APPENDIX 1
The ISABELLE Impedance

For the purposes of numerical calculations consider the parameters of the upcoming accelerator ISABELLE given in Table 1. In order to estimate the impedance for ISABELLE we will assume it to be a sum of two parts; a broad band resonance at the cutoff frequency of the vacuum chamber, and the space charge term.

Table 1. ISABELLE Parameters for the Injected Bunch

<u>Parameter</u>	<u>Value</u>
R, Average Radius	610 m
ν , Tune	22.6
ω_0 , Revolution Frequency	$5.0 \times 10^5/\text{sec}$
$\gamma = E/mc^2$	31.3
b, Chamber radius	4.4 cm
ω_s , Synchrotron frequency	40/sec
ξ , Chromaticity	2/22.6
η , Frequency slip factor	2×10^{-3}
$\sigma_E = (\Delta p/p)_{\text{rms}}$	2×10^{-4}
$\omega_c \equiv c/2b$, cutoff frequency	$3 \times 10^9/\text{sec}$
I_{peak} , the peak current	1.5 amp
$ Z_{11}(n\omega_0)/n ^{(a)}$	10 Ω
σ_H , rms horizontal beam size	2.3 mm
σ_V , rms vertical beam size	1.5 mm
x_0/b , beam position	0.6

a) This corresponds to a threshold impedance for longitudinal instabilities.¹³

The broad band part of the impedance can be taken from Eq. (53). If we let $\Gamma = \omega_c/2$, we find

$$Z_{\perp}^{\text{bb}}(\omega) = \frac{-Z_{\perp}(0)\omega_c^2}{i\omega\omega_c + (\omega^2 - \omega_c^2)} \quad (1.1)$$

where $Z_{\perp}(0) = Z_{\perp}(\omega = 0) = Z_{\perp}(\omega_c)/i \quad (1.2)$

In order to estimate $Z_{\perp}(0)$ we take the following approach: we assume a value for the longitudinal impedance, $Z_{\parallel}$, and then use a scaling factor to estimate the corresponding transverse impedance.

The scale factor which is normally used in this type of analysis is given by

$$Z_{\perp}(n\omega_0) = \frac{2c}{b^2} \frac{Z_{\parallel}(n\omega_0)}{n\omega_0} \quad (1.3)$$

where c is the speed of light and b is the chamber radius. In the case of ISABELLE the injected bunch is to be off center in the chamber, and therefore the transverse impedance will be enhanced due to the proximity of the beam environment. To include this effect we make the replacement in Eq. (1.3)

$$b^2 \rightarrow b^2 [1 - (x_0/b)^2]^2 \quad (1.4)$$

where x_0 is the distance of the injected bunch from the center of the chamber. (This enhancement is exact for the resistive part of the smooth wall impedance¹⁴).

To include the effect of space charge we consider a bigaussian beam. Neglecting the small wall contribution we have,¹⁵

$$Z_{\perp}^{sc} = i \frac{RZ_0}{\beta^2 \gamma^2} \frac{1}{\sigma_v(\sigma_v + \sigma_H)} \quad (1.5)$$

where $Z_0 = 377 \Omega$ is the impedance of free space, R is the machine radius, and σ_v , σ_H are the vertical and horizontal rms beam sizes.

If we use Eq. (1.3) to set the magnitude of $Z_{\perp}(0)$, the total impedance becomes

$$Z_{\perp}^{tot} = \frac{2c}{b^2 \omega_0} \left| \frac{Z_{\parallel}(n\omega_0)}{n} \right| \frac{1 \omega_c^2}{i \omega \omega_c + (\omega^2 - \omega_c^2)} + \frac{i R Z_0}{\beta^2 \gamma^2} \frac{1}{\sigma_v(\sigma_v + \sigma_H)} \quad (1.6)$$

The impedance above together with the values given in Table 1 is used in the text to calculate numerical results for ISABELLE.