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Two-Dimensional Analysis of Trapped-Ion Eigenmodes

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We present the first fully two-dimensional eigenmode analysis of the trapped-ion instability in an axisymmetric toroidal geometry. Although attention is focused on this particular instability, the calculations also takes into account the basic dynamics associated with other low frequency modes such as the trapped-electron instability and the ion-temperature-gradient instability. The poloidal structure of the mode is taken into account by Fourier expanding the perturbed electrostatic potential, ϕ , in θ . Assuming that the perturbation varies mildly over a typical ion-banana-width, ρ_{bi} , each poloidal harmonic is then expressed as a truncated Taylor series in the minor radius to account for the radial structure. The resulting set of coupled ordinary second-order differential equations is solved by the method of finite elements after prescribing the appropriate boundary conditions. Our governing equations involve summations over ion bounce and transit harmonics so that they are valid for mode frequencies less than and of the order of the ion bounce and transit frequencies

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($|\omega| \lesssim \bar{\omega}_{bi} \sim \bar{\omega}_{ti}$). Our formalism is also applicable to the familiar radially local and one-dimensional radial approximations. Results obtained in these limits are presented and found to be in reasonable agreement with previous calculations. In low shear plasmas, the full two-dimensional calculation is in qualitative agreement with the one-dimensional radial approximation. However, for higher shear (which gives rise to multiple rational surfaces in the region where the mode is localized) it yields a somewhat different picture. Specifically, the radial structure of the trapped-ion mode is found to differ considerably from that previously obtained in the one-dimensional approximation. Here, the analysis presented is limited to long wavelength ($k_r \rho_{bi} < 1$, k_r being a typical radial wave number) and electrostatic perturbations.

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1. INTRODUCTION

The relevance of microinstabilities to tokamaks is associated with the well-known fact that, even when the plasma is stable against the most dangerous MHD modes, microturbulence in the range of frequency $|\omega| \sim |\omega_{*e}|$ (ω_{*e} being the electron diamagnetic drift frequency defined later) can cause anomalous transport. A thorough review of the present status of microinstability theory in tokamaks has recently been given.¹ In this paper we concentrate on instabilities which occur in the range of frequency $|\omega| \sim |\omega_{*e}| \lesssim \bar{\omega}_{bi} \sim \bar{\omega}_{ti}$, $\bar{\omega}_{bi}$ and $\bar{\omega}_{ti}$ being the typical ion bounce and transit frequencies. In addition, we will require that the plasma be sufficiently collisionless to fall in the so-called "banana regime" (where $\nu_{eff,j} < \bar{\omega}_{bj}$, with $\nu_{eff,j}$ and $\bar{\omega}_{bj}$ being the effective collision frequency for trapped particles and the typical bounce frequency for species j).

Specifically, we present an eigenmode analysis of the trapped-ion instability which can be used to calculate the full two-dimensional structure of low frequency microinstabilities in an axisymmetric toroidal system. In addition to the usual effects associated with the trapped-ion mode, our analysis also includes important dynamics commonly associated with specific instabilities such as the trapped-electron mode, and the ion-temperature-gradient mode. The resultant calculations are, therefore, valid over a wider range in parameter space than previous analyses dealing with each of these instabilities on an individual basis. Moreover, our governing equations include summations over ion bounce and transit harmonics so that they are valid for mode frequencies of the order of or smaller than the typical ion bounce and transit frequencies.

The trapped-ion mode (TIM) was discovered by Kadomtsev and Pogutse.² In this first simple calculation, the authors showed that when both the electrons and the ions were in the banana regime (a situation expected for thermonuclear plasmas) new low frequency instabilities could exist in the plasma. Considering a low β plasma (β being the ratio between the plasma pressure and the magnetic pressure) in a large aspect-ratio tokamak and ignoring temperature gradients, Kadomtsev and Pogutse used a simple Krook collision operator to generate an eigenmode equation in the electrostatic limit. In the radially local limit, they solved the resultant equation and found two types of unstable modes. In the collisionless limit ($v_e, v_i \rightarrow 0$), they obtained a purely growing "trapped-particle interchange mode" and in the dissipative limit ($v_{ii}/c \ll |\omega| \ll v_{ei}/\epsilon$, ϵ being the local inverse aspect ratio), they found a drift-type mode propagating in the electron diamagnetic direction ($\omega_r \propto \omega_{*e}$). For the dissipative mode, the electron and ion collision frequencies were found to be destabilizing and stabilizing respectively. This discovery of a potentially dangerous unstable mode in a parameter range of relevance to thermonuclear plasmas stimulated much subsequent research directed at better understanding the stability criteria and possible effects of such modes on plasma confinement.

With regard to analyzing the spatial structure of these eigenmodes, several contributions have been made by numerous authors in the past several years. For example, Rosenbluth et al.³ used a variational method to study the poloidal structure in the radially local limit. The radial structure was subsequently treated by Gladd and Ross in a one-dimensional model. In this approximation the perturbed potential was assumed to be nearly constant along the field lines.

The analysis presented in this paper is the first to account for the full two-dimensional (poloidal and radial) structure of the trapped-ion instability in toroidal geometry.

The outline of this paper is as follows. In Sec. 2 we derive the basic eigenmode equation, point out the various approximations made as well as the corresponding limitations, and emphasize the new features. Section 3 deals with the radially local limit. Specifically, we concentrate on the dissipative limit in which an analytic dispersion relation can be obtained. Comparison of the analytic results obtained with previous calculations provides us with a good test of our physical model. In Sec. 4 we focus on the one-dimensional approximation of Gladd and Ross⁴ in which the instability is assumed to be nearly flutelike. In Sec. 5 we present our full two-dimensional calculation. The results are compared with those obtained in the one-dimensional approximation. Also, we present results obtained using idealized TFTR parameters. A brief summary of the results together with concluding comments are given in Sec. 6.

2. DERIVATION OF THE EIGENMODE EQUATION

In this section we derive a linear eigenmode equation which describes the trapped-ion instability in a large aspect ratio toroidal system. The analysis is appropriate for a low β plasma, and it is assumed that the radial wavelengths are long compared with to a typical ion banana width. The general procedure consists of solving the linearized drift-kinetic equation and using the resultant perturbed distribution function to calculate the perturbed density

response for each plasma species. In the familiar limit of sufficiently long wavelengths ($k\lambda_D \ll 1$) and sufficiently low frequencies ($|\omega| \ll \omega_{pi}$), the quasineutrality condition can be applied to generate a set of coupled second-order ordinary differential equations which governs the perturbed electrostatic potential. Together with the appropriate boundary conditions, these equations determine the spatial structure and eigenfrequency of the instability. Details of these calculations will now be presented.

2.1 The Drift-Kinetic Equation

The form of the linearized drift-kinetic equation governing the perturbed distribution, f_1 , is given by

$$\frac{\partial f_1}{\partial t} + (\vec{v}_{||} + \vec{v}_D) \cdot \nabla f_1 = -\vec{v}_E^{(1)} \cdot \nabla F_M + e(\vec{v}_{||} + \vec{v}_D) \cdot \nabla \phi_1 \frac{\partial F_M}{\partial E} - v(E, \Lambda) \left\{ f_1 + \frac{e\phi_1}{T} F_M \right\}, \quad (1)$$

where

$$\vec{v}_{||} = \alpha_{||} \hat{n} \left(\frac{2E}{M} \right)^{1/2} [1 - \Lambda/h(\theta)]^{1/2}, \quad (2)$$

$$\vec{v}_D = \frac{\hat{n}}{\Omega} \times [\mu \nabla \vec{B} + v_{||}^2 \hat{n} \cdot \nabla \hat{n}] , \quad (3)$$

$$\vec{v}_E^{(1)} = c \frac{\vec{B} \times \nabla \phi_1}{B^2} , \quad (4)$$

$$v_e(E, \Lambda) = \frac{v_{ei}}{Y^{3/2}} \frac{1 + H(\sqrt{Y})}{|1 - \epsilon - \Lambda|} , \quad (5)$$

$$v_i(E, \Lambda) = \frac{v_{ii}}{Y^{3/2}} \frac{H(\sqrt{Y})}{|1 - \epsilon - \Lambda|} , \quad (6)$$

$$h(\theta) = 1 + \epsilon \cos \theta , \quad (7)$$

$$v_{\alpha i} = \frac{4\pi n_e e^4 \ln \lambda_{\alpha i}}{(2T_\alpha)^{3/2} M_\alpha^{1/2}} , \quad (8)$$

(continued)

$$Y = E/T \quad , \quad (9)$$

$$H(X) = \frac{\exp(-X^2)}{\sqrt{\pi}X} + \left(1 - \frac{1}{2X^2}\right) \text{erf}(X) \quad . \quad (10)$$

In these equations the subscript 1 is used to label perturbed quantities. F_M is a local Maxwellian in which the density and temperature only depend on the minor radius, r ; $\hat{n}$ is a unit vector parallel to $\vec{B}$; Ω is the gyrofrequency; ϵ is the ratio of the minor radius to the major radius, $\epsilon = r/R$; $\ln \lambda_{ei}$ is the usual Coulomb logarithm⁵; and erf is the standard error function. Here, the spatial coordinates used are the usual toroidal variables, r, θ, ζ , (see Fig. 1), and the velocity space variables are $\sigma_{||}, E$, and Λ , with $\sigma_{||}$ representing the sign of the parallel velocity along the field line, E being the total kinetic energy of the particle, and Λ being the pitch-angle variable defined by

$$\Lambda = \frac{\mu B_0}{E} \quad , \quad (11)$$

where μ is the magnetic moment and B_0 is the magnitude of the magnetic field on axis. In Eq. (1), note that collisions are modeled by a simple energy and pitch-angle dependent Krook collision operator. An interesting fact about this operator is that it automatically accounts for the enhancement in the effective collision frequency of trapped and barely circulating particles. As will be discussed later, it is inadequate in the small collision frequency limit ($v_{||}/\epsilon \ll |\omega|$) but becomes quite accurate in the large collisionality limit. Also, it should be noted that for a low β plasma in a large aspect ratio tokamak, it is appropriate to consider circular concentric flux surfaces and a safety factor of order unity; i.e.,

$$q = \frac{rB_c}{R_0 B_\theta} \sim 1. \quad (12)$$

Also, consistent with the large aspect ratio assumption, we shall keep all quantities of orders 1 and $\sqrt{\epsilon}$ and some selected quantities of order ϵ , when the lowest-order terms vanish. We will generally ignore corrections of order ϵ to quantities of order 1. Finally, note that in the equations above, we assume that the plasma consists of a single ion species of charge unity.

Equation (1) can be solved by the method of characteristics. After multiplying by the integration factor $e^{\nu t}$ it can be rewritten as:

$$\frac{d}{dt}(f_1 e^{\nu t}) = e \frac{\partial F_M}{\partial E} \left(\frac{d}{dt}(\phi_1 e^{\nu t}) - \frac{\partial \phi_1}{\partial t} e^{\nu t} \right) - \vec{v}_E^{(1)} \cdot \nabla F_M e^{\nu t}, \quad (13)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}_\parallel + \vec{v}_D) \cdot \nabla \quad (14)$$

represents the total time derivative along the unperturbed guiding-center trajectory. Integrating each side of Eq. (13) from time $t' = -\infty$ to $t' = t$, and assuming that the perturbation vanishes at $t' = -\infty$ we find

$$f_1(t) = -\frac{e\phi_1}{T} F_M + \int_{-\infty}^t dt' \left(\frac{eF_M}{T} \frac{\partial \phi_1}{\partial t} - \vec{v}_E^{(1)} \cdot \nabla F_M \right) \exp[\nu(t' - t)]. \quad (15)$$

Note that in taking $dF_M/dt = 0$, it is assumed that typical radial excursions along the unperturbed trajectory are small compared with the equilibrium scale-length.

Proceeding with the analysis, the integrand in Eq. (15) is written as a function of time so that the integral can be performed explicitly. Since our problem is linear and stationary in time, and since we assume an axisymmetric geometry, the time and azimuthal dependences of all perturbed quantities can be assumed to be of the form $\exp[i(\ell\zeta - \omega t)]$. Since the r and θ dependences of the perturbed potential are not known, we are left with a two-dimensional problem in configuration space. In order to account for the poloidal structure of the mode, we expand ϕ_1 as a Fourier series in θ ; i.e.,

$$\phi(r, \theta, \zeta, t) = \sum_m \phi_m(r) \exp[i(\ell\zeta - m\theta - \omega t)] \quad (15)$$

Substituting this in Eq. (15) gives

$$\begin{aligned} \tilde{f}_1 = & -\frac{e\tilde{\phi}_1}{T} F_M + \frac{eF_M}{T} \int_m \exp(-im\theta) \int_{-\infty}^t dt' (-i\omega + i\omega_{*T}^{(m)}) \phi_m(r') \\ & \times \exp\{i[\ell(\zeta' - \zeta) - m(\theta' - \theta) - (\omega + i\nu)(t' - t)]\} \quad (17) \end{aligned}$$

where

$$\begin{pmatrix} \tilde{f} \\ \tilde{\phi}_1 \end{pmatrix} = \begin{pmatrix} f \\ \phi \end{pmatrix} \exp[-i(\ell\zeta - \omega t)] \quad (18)$$

$$\omega_{*T}^{(m)} = \omega_*^{(m)} [1 + \eta(Y - \frac{1}{2})] \quad (19)$$

$$\omega_*^{(m)} = -\frac{m}{r} \frac{cT}{eB_0} \frac{1}{n} \frac{dn}{dr} \quad (20)$$

$$\eta = \frac{d(\ln T)/dr}{d(\ln n)/dr} \quad (21)$$

This yields an integral equation for the Fourier coefficients $\phi_m(r)$. In the following we shall make the simplifying assumption that each

harmonic varies only weakly over a typical orbit radial excursion. The form of the solution given in Eq. (17) will, then, lead to a set of coupled differential equations which proves to be tractable to numerical and analytical procedures.

If the condition $k_r \rho_{bi} \ll 1$ is well satisfied, where k_r and ρ_{bi} are the typical radial wave number and ion banana width respectively, the potential can be approximately written in the form,

$$\phi_m(r') \approx \phi_m(r) + (r' - r) \phi'_m(r) + \frac{1}{2} (r' - r)^2 \phi''_m(r) , \quad (22)$$

so that Eq. (17) becomes:

$$\begin{aligned} \ddot{f}_1 = & - \frac{e\dot{\phi}_1}{T} F_M + \frac{eF_M}{T} \sum_m \exp(-im\theta) \int_{-\infty}^t dt' \{ -im + i\omega_{*T}^{(m)} \} [\phi_m(r) + (r' - r) \phi'_m(r) \\ & + \frac{1}{2} (r' - r)^2 \phi''_m(r)] \exp\{ i [\ell(\zeta' - \zeta) - m(\theta' - \theta) - (\omega + iv)(t' - t)] \} . \end{aligned} \quad (23)$$

Since the unknown functions, ϕ_m , can now be factored out, the integral in Eq. (23) can, in principle, be evaluated explicitly because it only involves known equilibrium quantities. In practice, this is done by judiciously expanding some of the periodic equilibrium quantities as Fourier series in time. Specifically, for the radial excursions of trapped particles, we take:

$$r - r^{(0)} = \sum_n r^{(n)} \exp(in\omega_b t) , \quad (24)$$

where $r^{(0)}$ is the orbit averaged value of r :

$$r^{(0)} = \frac{1}{\tau_b} \oint dt r , \quad (25)$$

ω_b and τ_b being the bounce frequency and period, respectively. They are defined by:

$$\tau_b = 4 \frac{R_C q}{\sqrt{\gamma} v_{th}} \int_0^{\theta_0} d\theta [1 - \Lambda/h(\theta)]^{-1/2} \quad (26)$$

and

$$\omega_b = 2\pi/\tau_b \quad (27)$$

Here, θ_0 is the angle of the trapped-particle turning points, $\theta_0 = \arccos[(\Lambda-1)/\Lambda]$, and v_{th} is the thermal velocity defined by:

$$v_{th} = \left(\frac{2T}{M}\right)^{1/2} \quad (28)$$

The corresponding quantities for circulating (passing or untrapped) particles are the transit period,

$$\tau_t = 2 \frac{R_C q}{\sqrt{\gamma} v_{th}} \int_0^\pi d\theta [1 - \Lambda/h(\theta)]^{-1/2} \quad (29)$$

and the transit frequency,

$$\omega_t = 2\pi/\tau_t \quad (30)$$

Here, the convention is that ω_b and ω_t are always positive. In the following analysis, we will restrict our attention to radial excursions of trapped ions only. Those of circulating ions and trapped electrons are smaller by factors of order $\sqrt{\epsilon}$ and $(m_e/M_i)^{1/2}$, respectively, and will be neglected.

The exponent appearing in Eq. (23) can be written in the form:

$$\ell(\zeta' - \zeta) - m(\theta' - \theta) = \int_t^{t'} dt'' \left(\ell \frac{d\zeta}{dt''} - m \frac{d\theta}{dt''} \right) \quad (31)$$

With the definitions

$$\omega_D = \ell \left[\frac{dr}{dt} - q(r^{(0)}) \frac{d\theta}{dt} \right] , \quad (32)$$

$$\omega_D^{(0)} = \frac{1}{\tau_{b,t}} \int dt \omega_D , \quad (33)$$

$$\Delta \omega_D = \omega_D - \omega_D^{(0)} , \quad (34)$$

$$S^{(m)}(r) = \ell q(r) - m , \quad (35)$$

$$\Delta S = \ell [q(r) - q(r^{(0)})] , \quad (36)$$

$$W(t) = -\Delta S(\theta' - \theta) + \int_0^t dt' \Delta \omega_D , \quad (37)$$

We can rewrite Eq. (31) as:

$$\begin{aligned} \ell(r' - r) - m(\theta' - \theta) &= \{ \omega_D^{(0)} + H\sigma_{||} S\omega_t \} (t' - t) + W' - W + S(\theta' - \theta) \\ &\quad - H\sigma_{||} S\omega_t (t' - t) , \end{aligned} \quad (38)$$

where $H = H(1 - \epsilon - h)$ vanishes for trapped particles and is equal to one for circulating particles.

In Eq. (38) we omitted the arguments, r , and the superscript, m . We shall continue to do so unless they are needed to avoid ambiguity. Also in Eq. (38), W' and W stand for $W(t')$ and $W(t)$, respectively. From Eq. (38), the exponential appearing in Eq. (23) can be written as:

$$\begin{aligned} e^{i[]}&= \exp[-i(\omega - \omega_D^{(0)} - H\sigma_{||} S\omega_t + i\nu)(t' - t)] \exp[-i[S(\theta' - \theta) - H\sigma_{||} S\omega_t(t' - t) \\ &\quad + W' - W]] . \end{aligned} \quad (39)$$

The second exponential is a periodic function of time and can be expressed as:

$$e^{i[]} = \sum_n a_n \exp[in\omega_{b,t}(t'-t)] \quad (40)$$

When Eqs. (24), (39), and (40) are substituted in Eq. (23), the time integration can be explicitly performed to yield the following solution for $\tilde{f}_1$:

$$\begin{aligned} \tilde{f}_1 = & -\frac{e\tilde{\epsilon}_1}{T} F_M + \frac{eF_M}{T} \sum_m \exp(-im\theta) (\omega - \omega_{*T}) \sum_n a_n \left[\phi_m G(n) + \phi_m' \sum_{n_1} r^{(n_1)} \exp(in_1\omega_b \hat{t}) \right. \\ & \times [G(n+n_1) - G(n)] + 1/2 \phi_m'' \sum_{n_1 n_2} r^{(n_1)} r^{(n_2)} \exp[i(n_1+n_2)\omega_b \hat{t}] \\ & \left. \times [G(n+n_1+n_2) - 2G(n+n_1) + G(n)] \right] , \end{aligned} \quad (41)$$

where we define

$$G(n) = [\omega - \omega_D^{(0)} - (n + H\sigma_{||} S) \omega_{b,t} + iv]^{-1} \quad (42)$$

Explicit expressions for the coefficients $r^{(n)}$ and a_n can be derived straightforwardly.⁶ The results are

$$r^{(n)} = \begin{cases} \rho_\theta \frac{\sqrt{Y} \sqrt{\epsilon} I_n}{\pi L_b} & , \quad n \text{ odd} \\ 0 & , \quad n \text{ even} \end{cases} \quad (43)$$

and

$$a_n|_{\text{trapped}} = \exp[-i(S\theta - n\omega_b \hat{t} + W)] \frac{\sqrt{\epsilon}}{\pi L_b} J_{nm} \quad (44)$$

$$\begin{aligned} a_n|_{\text{circulating}} = & \exp\{-i[S\theta - (n + \sigma_{||} S) \omega_b \hat{t} + W]\} \frac{\sqrt{\epsilon}}{\pi L_t} \int_0^\pi d\theta' [1 - A/h(\theta')]^{-1/2} \\ & \times \cos[S\theta' - (n + \sigma_{||} S) \omega_b \hat{t}' + W'] \quad (45) \end{aligned}$$

with the definitions:

$$\rho_0 = v_{th} \frac{cT}{eB_0} , \quad (46)$$

$$I_n = \int_0^{\theta_0} d\theta \cos(n\omega_b \hat{t}) , \quad (47)$$

$$J_{nm} = \int_0^{\theta_0} d\theta [1 - \Lambda/h(\theta)]^{-1/2} \left[\cos^2\left(\frac{n\pi}{2}\right) \cos(S^{(m)}\theta) \cos(n\omega_b \hat{t} + W) \right. \\ \left. + \sin^2\left(\frac{n\pi}{2}\right) \sin(S^{(m)}\theta) \sin(n\omega_b \hat{t} + W) \right] , \quad (48)$$

$$L_b = \frac{\sqrt{\epsilon}}{2\pi} \int_{-\theta_0}^{\theta_0} d\theta [1 - \Lambda/h(\theta)]^{-1/2} , \quad (49)$$

$$L_t = \frac{\sqrt{\epsilon}}{2\pi} \int_{-\pi}^{\pi} d\theta [1 - \Lambda/h(\theta)]^{-1/2} . \quad (50)$$

Here, $\hat{t}$ is a function of θ and $\sigma_{||}$ and has a one-to-one correspondence with each point of any single orbit. It is defined by

$$\hat{t}(\theta, \sigma_{||}) = \begin{cases} \frac{R_{OQ}}{\sqrt{Y} v_{th}} \int_0^{\theta} d\theta' [1 - \Lambda/h(\theta')]^{-1/2} \Big|_{\sigma_{||}=+1} \\ \tau_b/2 - \hat{t}(\theta, +1) \Big|_{\sigma_{||}=-1} \end{cases} \quad (51)$$

for trapped particles and by:

$$\hat{t}(\theta, \sigma_{||}) = \frac{R_{OQ}}{\sqrt{Y} v_{th}} \sigma_{||} \int_0^{\theta} d\theta' [1 - \Lambda/h(\theta')]^{-1/2} \quad (52)$$

for circulating particles. Note that the expression given in Eq. (41) provides a complete solution for the perturbed distribution function, $\tilde{f}_1$, in terms of an arbitrary electrostatic potential $\tilde{\phi}_1$.

2.2 The Perturbed Density and the Quasineutrality Condition

The next step consists of calculating the perturbed density response, $\tilde{n}_1$, by integrating over velocity space. Making use of the fact that

$$\int d^3v = 2\pi \int_{-\infty}^{\infty} dv_{\parallel} \int_0^{\infty} v_{\perp} dv_{\perp} = \frac{\pi}{2} \left(\frac{2T}{M} \right)^{3/2} \sum_{\sigma_{\parallel}} \int_0^{\infty} dv \sqrt{v} \int_0^{h(\theta)} d\Lambda (h(\theta) [1 - \Lambda/h(\theta)]^{1/2})^{-1} \quad (53)$$

we find that $\tilde{n}_1$ has the form:

$$\tilde{n}_1(r, \theta) = \sum_m \exp(-im\theta) [\hat{P}_m(r, \theta) \phi_m''(r) + \hat{Q}_m(r, \theta) \phi_m'(r) + \hat{R}_m(r, \theta) \phi_m(r)] \quad (54)$$

This can now be substituted in the quasineutrality condition to yield a system of equations of the form:

$$\sum_m \exp(-im\theta) (P_m \phi_m'' + Q_m \phi_m' + R_m \phi_m) = 0 \quad (55)$$

These equations, however, have the disadvantage that they involve the two variables r and θ . It is convenient to remove the explicit dependence on θ by taking the Fourier transform of Eq. (55) with respect to this variable. Hence, operating on Eq. (55) with $\int_{-\pi}^{\pi} d\theta \exp(ip\theta)/2\pi$, reduces it to a set of coupled ordinary differential equations of the form

$$\sum_m (A_{pm} \phi_m'' + B_{pm} \phi_m' + C_{pm} \phi_m) = 0 \quad (56)$$

which is independent of θ . Equation (56) determines the Fourier coefficients of the perturbed electrostatic potential. Using the definitions in Eqs. (47), (48) and defining

$$K_{n,p,m} \equiv \int_0^{\pi} d\theta [1 - \Lambda/h(\theta)]^{-1/2} \cos[S^{(p)}\theta - (n + S^{(m)})\omega_e t] \quad (57)$$

we find, after performing the $q_{||}$ summation, that the matrix elements can be expressed as:

$$A_{pm} = \frac{\tau_e^{3/2} \rho_e}{2\pi^{3/2}} \int_{1-\epsilon}^{1+\epsilon} \frac{d\Lambda}{L_b^3} \int_0^\infty dY Y^{3/2} e^{-Y} \sum_n \frac{\omega - \omega_{*Ti}}{\omega - \omega_{Di}^{(0)} - n\omega_{bi} + i\nu_i} \frac{1}{n_1 n_2} I_{n_1} I_{n_2} \\ \times (J_{n+n_1+n_2, p} J_{n, m} - 2J_{n+n_1, p} J_{n-n_2, m} + J_{n, p} J_{n-n_1-n_2, m}), \quad (58)$$

$$B_{pm} = \frac{\tau_e \rho_e}{\pi^{3/2}} \int_{1-\epsilon}^{1+\epsilon} \frac{d\Lambda}{L_b^2} \int_0^\infty dY Y e^{-Y} \sum_n \frac{\omega - \omega_{*Ti}}{\omega - \omega_{Di}^{(0)} - n\omega_{bi} + i\nu_i} \sum_{n_1 \text{ odd}} I_{n_1} \\ \times (J_{n, p} J_{n-n_1, m} - J_{n+n_1, p} J_{n, m}), \quad (59)$$

and

$$C_{pm} = \frac{\sqrt{\epsilon}}{\pi^{3/2}} \int_{1-\epsilon}^{1+\epsilon} \frac{d\Lambda}{L_b} \int_0^\infty dY Y e^{-Y} \frac{\omega - \omega_{*Te}}{\omega - \omega_{De}^{(0)} + i\nu_e} J_{0, p} J_{0, m} \\ + \frac{1/\sqrt{\epsilon}}{\pi^{3/2}} \int_{1-\epsilon}^{1+\epsilon} \frac{d\Lambda}{L_b} \int_0^\infty dY \sqrt{Y} e^{-Y} \sum_n \frac{\omega - \omega_{*Ti}}{\omega - \omega_{Di}^{(0)} - n\omega_{bi} + i\nu_i} J_{n, p} J_{n, m} \\ + \frac{\tau \sqrt{\epsilon}}{\pi^{3/2}} \int_0^{1-\epsilon} \frac{d\Lambda}{L_t} \int_0^\infty dY \sqrt{Y} e^{-Y} \sum_n \frac{(\omega - \omega_{*Ti})(\omega + i\nu_i)}{(\omega + i\nu_i)^2 - [(n+S^{(m)})\omega_t]^2} K_{n, p, m} K_{n, m, m} \\ - (1 + \tau) \delta_{pm}, \quad (60)$$

where τ is the electron-to-ion temperature ratio defined by $\tau = T_e/T_i$. The only contribution to the A and B matrices comes from the trapped ions, because radial excursions of particles other than trapped ions have been neglected. Here, we ignore the average drift frequency, $\omega_D^{(0)}$, for circulating particles because it is smaller than its trapped particle counterpart and typically much smaller than $|\omega|$ (see

Appendix A of Ref. 7). Also, it should be noted that the C matrix has no contribution from the nonadiabatic circulating electron response and the trapped electron non-time-averaged response. The reason is that, in most of the plasma cross section, these contributions would be smaller than their circulating ion counterparts by a factor of order M_e/M_i and can, therefore, be ignored.

In the following calculations, we will ignore the phase factor, W , which appears in the definition of $J_{n,m}$ given in Eq. (48). This approximation is appropriate because inclusion of this factor only leads to corrections of order $\rho_{bf}/\Delta r$ or $\omega_D^{(0)}/\omega_{b1}$, with Δr being the distance between adjacent rational surfaces. For the parameters of interest, these corrections are always small. As a practical point, by ignoring this factor, the numerical evaluation of the matrix elements can be considerably simplified; i.e., without the W term, the angle integrals, $J_{n,m}$, are now independent of energy.

2.3 Boundary Conditions

To obtain a well-posed eigenmode problem, we need to supplement Eqs. (56) with appropriate boundary conditions. To do so we assume that the plasma is in contact with an axisymmetric conducting limiter at a minor radius, a , at which the tangential electric field must vanish. We also require that, on the axis, the electrostatic energy be integrable and that the perturbed electric field does not cause any accumulation of charge there. These conditions yield:

$$\phi_m(a) = 0 \quad \text{for all } m, \quad (61)$$

$$\phi_m(0) = 0, \quad m \neq 0, \quad (62)$$

$$\left. \frac{d\phi_0}{dr} \right|_{r=0} = 0. \quad (63)$$

The system of equations (56) along with the boundary conditions given in Eqs. (61) to (63) determine the basic form of the trapped-ion eigenmode problem which will be studied.

2.4 General Remarks

The system of eigenmode equations just derived contains some interesting features, and it is appropriate, at this point, to make the following comments.

A. First Derivative Terms

An interesting fact about Eq. (56) is the presence of first derivatives of the perturbed potential (associated with the B matrix). This might seem surprising, since such terms were absent in previous eigenmode calculations.⁴ However, since the presence of these terms is generated by ballooning effects (neglected in the earlier studies), our basic equation is in fact consistent with those appearing in previous studies. Specifically, when the phase factor, W , is neglected in the definition of $J_{n,m}$ given in Eq. (48), it can be shown that the diagonal elements of the B matrix vanish identically. Thus, when the coupling between the various poloidal harmonics is ignored, it follows that no first-order derivative terms appear in the eigenmode equations. This is indeed the case in the earlier calculations, where the mode is assumed to be flutelike; i.e., approximately

constant along the field line.

B. Truncation

In order to solve our eigenmode problem, in practice, it is clearly necessary to truncate the number of unknown functions and equations to a finite number; i.e., to introduce lower and upper bounds, $M_{\min}$ and $M_{\max}$, for the indices p and m in Eq. (56). In doing so we must make certain assumptions concerning the structure of the instability. In what follows we choose to limit our attention to modes which are not very strongly ballooning and take

$$M_{\min} \leq \ell q_{\min} \quad , \quad (64)$$

$$M_{\max} \geq \ell q_{\max} \quad . \quad (65)$$

The influence of this truncation on the final result can be investigated by varying $M_{\min}$ and $M_{\max}$. In the results which will be presented, this truncation error is always less than 10%.

C. Reduced Intervals

Finally, in practical calculations, it proves convenient to solve the eigenmode problem in the subinterval $[x_1, x_2]$, instead of $[0, 1]$ ($0 \leq x_1, x_2 \leq 1$), where x is the normalized radial coordinate, $x = r/a$. This is necessary to avoid certain divergences at $x = 0$ and, as a practical point, it can also be exploited to make the numerical solution of the problem more efficient. The next task is to choose the appropriate boundary conditions to apply on this reduced interval. In what follows we will always use the boundary conditions given in Eqs. (61) to (63) and will treat the left and right boundaries of the interval as if they were located at $x = 0$

and $x=1$. This proves to be adequate, provided the solution is sufficiently well localized within $[x_1, x_2]$. Specifically, the points, x_1 and x_2 , must fall sufficiently far into the evanescent regions of the solution.

3. RADIALLY LOCAL LIMIT

In the limit when the ion banana width is much smaller than the equilibrium scale length, derivatives with respect to r can be ignored, and the system of equations (56) reduces to a simple set of algebraic equations of the form,

$$\sum_m C_{pm} \phi_m = 0 \quad . \quad (66)$$

This radially local eigenmode equation has the virtue of being considerably simpler than its full two-dimensional counterpart. It is readily amenable to numerical solution by standard methods and, in some limiting cases, to a completely analytic solution.

In this section we consider the radially local limit in some detail. First, we find an analytic dispersion relation valid in the dissipative limit. Then, we compare our results with those of Rosenbluth et al.,³ who developed a variational principle that enabled them to study analytically the mode structure along the field line and to treat collisions with the more accurate Lorentz model. This comparison serves as a good test of our simple Krook collision operator.

The approximations involved are that (i) the plasma is in the banana regime,

$$\frac{v}{c} \Big|_{i,e} \ll \bar{\omega}_b \Big|_{i,e} ; \quad (67)$$

(ii) the aspect ratio is large,

$$\frac{a}{R_0} \ll 1 ; \quad (68)$$

and (iii) the familiar dissipative limit (within the banana regime) is appropriate,

$$\frac{v_{ii}}{c} \ll |\omega| \ll \frac{v_{ei}}{c} . \quad (69)$$

In addition, the following simplifying assumptions are made:

$$|\omega| \ll \bar{\omega}_{bi} , \quad (70)$$

$$\ell q = m_0 + 1/2 , \quad (71)$$

and

$$\phi(\theta, \zeta) = \phi_{m_0} \exp[i(\ell \zeta - m_0 \theta)] + \phi_{m_0+1} \exp\{i[\ell \zeta - (m_0+1)\theta]\} . \quad (72)$$

Consistent with Eq. (70) we shall treat terms in $\bar{\omega}/\omega_{bi}$ only to lowest (nonvanishing) order. Note that Eq. (71) corresponds to a mode which is localized halfway between rational surfaces. This assumption (which facilitates the evaluation of several integrals) is suggested by the strong circulating-ion Landau damping that should occur close to rational surfaces. Finally, Eq. (72) is suggested by the results of Ref. 3. It corresponds to a weakly-ballooning mode for which the Fourier series in θ can be truncated to only two poloidal harmonics.

It should also be noted that, consistent with Eqs. (68) and (69), we will neglect $\omega_D^{(0)}$ compared with $|\omega|$ and treat the circulating ions as being collisionless.

Let us now evaluate the elements of the C matrix and solve Eq. (66) analytically. In doing so, it is convenient to distinguish between the various contributions to C and to evaluate each one separately. In practice, given the above assumptions, this is done by rewriting Eq. (60) as:

$$\begin{aligned}
 C_{0m} \approx & \frac{\sqrt{\epsilon}}{\pi^{5/2}} \int_{1-\epsilon}^{1+\epsilon} \frac{d\Lambda}{L_b} \int_0^\infty dY \sqrt{Y} e^{-Y} \frac{\omega - \omega_* T_i}{i v_e} J_{0p} J_{0m} + \frac{\tau \sqrt{\epsilon}}{\pi^{5/2}} \int_{1-\epsilon}^{1+\epsilon} \frac{d\Lambda}{L_b} \int_0^\infty dY \sqrt{Y} e^{-Y} \\
 & \times \left[\frac{\omega - \omega_* T_i}{\omega} - i \frac{v_i}{\omega} \frac{\omega - \omega_* T_i}{\omega + i v_i} \right] J_{0p} J_{0m} + \frac{2\tau \sqrt{\epsilon}}{\pi^{5/2}} \sum_{n>0} \int_{1-\epsilon}^{1+\epsilon} \frac{d\Lambda}{L_b} \int_0^\infty dY \sqrt{Y} e^{-Y} \\
 & \times \frac{\omega (\omega - \omega_* T_i)}{\omega^2 - n^2 \omega_{bi}^2} J_{np} J_{nm} + \frac{\tau \sqrt{\epsilon}}{\pi^{5/2}} \int_0^{1-\epsilon} \frac{d\Lambda}{L_t} \int_0^\infty dY \sqrt{Y} e^{-Y} \sum_n \frac{\omega (\omega - \omega_* T_i)}{\omega^2 - [(n+S^{(m)})_{ti}]^2} K_{npm} K_{nmm} \\
 & - (1 + \tau) \delta_{pm} . \quad (73)
 \end{aligned}$$

The various terms are the trapped-electron, the collisionless and collisional time-averaged trapped-ion, the $(n \neq 0)$ non-time-averaged trapped-ion, the circulating-ion, and the adiabatic contributions. After some algebra,⁶ it follows that the C matrix is of the form:

$$C = \begin{pmatrix} \delta + \alpha + \sigma_1 & \alpha + \sigma_2 \\ \alpha + \sigma_2 & \delta + \alpha + \sigma_1 \end{pmatrix} , \quad (74)$$

where δ , α , and $\sigma_{1,2}$ represent the adiabatic, the time-averaged trapped particle and the non-time-averaged ($n \neq 0$) trapped and circulating-ion resonances, respectively. Equation (66) can now be solved perturbatively. As usual, for the trapped-ion instability, the adiabatic and the time-averaged collisionless trapped-ion terms are taken to be of order unity. The other contributions are all assumed to be small perturbations of the same order.

To lowest order, we have:

$$\delta^{(0)} + 2\alpha^{(0)} = 0, \quad (75)$$

which yields the lowest order mode frequency:

$$\omega^{(0)} = 0.577 \frac{2\sqrt{\epsilon} \omega_{*e}}{1 + \tau}. \quad (76)$$

Here, the superscript indicates the order of the various terms in the small expansion parameter. It is clear from Eqs. (71), (72), (74), and (75) that the lowest order eigenfunction is of the form

$$\phi(0, \zeta) \propto \cos(\theta/2) \exp[i l (\zeta - q\theta)] \quad ; \quad (77)$$

i.e., the mode is ballooning on the outer side of the torus.

To first order, we have

$$2\alpha^{(1)} + \sigma_1^{(1)} + \sigma_2^{(1)} = 0. \quad (78)$$

Here, it is convenient to define

$$\omega^{(1)} = i(\gamma_e + \gamma_i + \gamma_{TL} + \gamma_{CL}) \quad , \quad (79)$$

where γ_e , γ_i , γ_{TL} , γ_{CL} represent the collisional trapped-electron and trapped-ion growth rates and the trapped and circulating-ion

Landau growth rates, respectively. Then, Eq. (78) yields the first-order corrections:

$$\frac{\gamma_e}{\omega_o} = 0.798 \frac{2}{1+\tau} (1 + 1.4 \ln \eta_e) \frac{\epsilon^{3/2} \omega_{*e}}{v_{ei}} , \quad (80)$$

$$\frac{\gamma_i}{\omega_o} = -4.61 \frac{1+\tau}{2} (1 - 1.065 \eta_i) \frac{v_{ii}}{\epsilon^{3/2} \omega_{*e}} \left[\ln \left(\ln \frac{\epsilon^{3/2} |\omega_{*e}|}{(1+\tau) v_{ii}} \right) - 1.06 \right] , \quad (81)$$

$$\frac{\gamma_{TL}}{\omega_o} = -0.03 \left(\frac{2}{1+\tau} \right)^3 (1 - \frac{3}{2} \eta_i) \left(\frac{\omega_{*e}}{\bar{\omega}_t} \right)^3 , \quad (82)$$

and

$$\frac{\gamma_{CL}}{\omega_o} = -3.20 \left(\frac{2}{1+\tau} \right)^3 (1 - \frac{3}{2} \eta_i) \left(\frac{\omega_{*e}}{\bar{\omega}_t} \right)^3 , \quad (83)$$

where $\bar{\omega}_t$ is defined by

$$\bar{\omega}_t = v_{th}/R_O q . \quad (84)$$

A very similar dispersion relation was obtained by Rosenbluth et al.³ Assuming the same ordering as in Eqs. (67) to (71), they found

$$\omega_o = 0.58 \frac{2\sqrt{\epsilon}}{1+\tau} \omega_{*e} , \quad (85)$$

$$\frac{\gamma_e}{\omega_o} = 0.84 (1 + 1.4 \ln \eta_e) \frac{2}{1+\tau} \frac{\epsilon^{3/2} \omega_{*e}}{v_{ei}} , \quad (86)$$

$$\frac{\gamma_i}{\omega_o} = -0.31 (1 - 0.57 \eta_i) \left(\frac{v_{ii}}{\epsilon^{3/2} \omega_{*e}} \frac{1+\tau}{2} \right)^{1/2} \left\{ \ln \left[17 \left(\frac{\epsilon^{3/2} \omega_{*e}}{v_{ii}} \frac{2}{1+\tau} \right)^{1/2} \right] \right\}^{-3/2} \quad (87)$$

$$\frac{\gamma_{CL}}{\omega_o} = -1.49 (1 - \frac{3}{2} \eta_i) \left(\frac{2}{1+\tau} \right)^3 \left(\frac{\omega_{*e}}{\bar{\omega}_t} \right)^3 , \quad (88)$$

with the corresponding lowest order mode structure⁸

$$\phi(\theta, \zeta) \propto (\cos \theta/2 + 0.14 \cos 3\theta/2 - 0.02 \cos 5\theta/2) \exp[i(\ell\zeta - q\theta)] \quad . \quad (89)$$

We recall that this result was derived using a variational principle and that collisional effects were calculated using the Lorentz operator. Comparing these results with the ones found with our Krook operator, we see that, except for the ion collisional growth rates, the two are in basic agreement.

The most important point in this comparison is that it provides a direct test of our simple Krook collision operator in two collisionality regimes. Indeed, since the electrons are assumed to be strongly collisional ($v_{ei}/\epsilon \gg |\omega|$), and the ions, nearly collisionless ($v_{ii}/\epsilon \ll |\omega|$), we see that a comparison of the electron and ion collisional growth rates obtained in both cases will show how accurate the simple Krook model is in both of these limits.

Thus, by inspection of the above dispersion relation, it appears that the Krook collision operator is quite accurate in the large collision frequency limit, but that it fails to predict the correct asymptotic behavior of γ_i in the opposite limit. This lack of agreement in the small collision frequency limit is a consequence of the fact that collisions take place predominantly in a narrow layer at the boundary between trapped and circulating particles and cannot be reproduced accurately with a simple Krook collision operator. Note, however, that in the code, we do include ion-ion collisions using the simple Krook model. The reason is that, in practice, we will consider cases for which the ions are not far into the collisionless regime. In such cases, the asymptotic limit corresponding

to Eqs. (81) and (87) would not be strictly applicable.

It should be noted, however, that for the case of unstable modes far into the collisionless regime, the use of an exact form of the ion collision operator would be inconsequential since both Eqs. (81) and (87) would give a vanishingly small contribution to the growth rate in this instance. Finally, we observe that it would be possible to obtain better agreement in the scaling of γ_i with ν_{ij} and n_i given in Eq. (87), if we choose $[\Lambda - 1 + \epsilon]^{-1}$ instead of $[\Lambda - 1 + \epsilon]$ as the Λ dependence of ν in Eq. (6). This modified Λ dependence of the collision frequency yields an ion collisional growth rate scaling predominantly as $(\nu_{ii})^{1/2}$. Additionally, the n_i dependence here is precisely the same as that given in Eq. (87). However, this modified collision operator fails to give the proper results in the large collision frequency limit. Specifically, the scaling of the electron collisional growth rate with ϵ is incorrect. This is the reason for our use of the more standard collision operator, given in Eqs. (5) and (6), for both electrons and ions.

4. ONE-DIMENSIONAL RADIAL ANALYSIS

In this section, we deal with the one-dimensional radial analysis. Following the approximation of Gladd and Ross,⁴ the instability is assumed to be nearly flutelike. Then, the governing eigenmode equation has the form:

$$\rho_{bi}^2 \frac{d^2 \phi}{dr^2} + Q(r) \phi = 0 \quad , \quad (90)$$

where $\rho_{bi} = \sqrt{\epsilon} \rho_{\theta i}$, with the boundary conditions

$$\phi(0) = 0 \quad , \quad (91)$$

$$\phi(a) = 0 \quad . \quad (92)$$

The radial potential, Q , is given by

$$Q = \rho_{bi}^2 \frac{C_{mm}}{A_{mm}} \quad , \quad (93)$$

for arbitrary m . Here, A_{mm} and C_{mm} have the same form as given in Eqs. (58) and (60) except that: (i) $S = lq - m$ is artificially set equal to zero everywhere in the trapped-particle responses. (ii) In the circulating-ion response, we take $S = lq - m_0$, with m_0 being the closest integer to lq . Note that with this prescription, the factor S only vanishes exactly at the rational surfaces. (iii) The factor m appearing in the definition of ω_* in Eq. (20) is replaced by lq . We note that the particular form of the eigenmode Eq. (90) suggests that the radial potential, Q , can provide a rough measure of $k_r \rho_{bi}$, since

$$(k_r \rho_{bi})^2 \sim \left| \frac{1}{\Phi} \frac{d^2 \Phi}{dr^2} \right| = |Q| \quad . \quad (94)$$

Thus, our assumption, $k_r \rho_{bi} < 1$, requires that $|Q|$ be smaller than one, at least in the region where the mode has most of its amplitude.

The parametrization of the equilibrium and the values of the parameters used are given in the Appendix. Except for the density, they correspond almost exactly to those of Gladd and Ross.⁴ The reason for using a greatly reduced density [$n_e(0) = 5 \times 10^{12}$ instead of $5 \times 10^{13} \text{ cm}^{-3}$] is that without the destabilizing effect of ballooning introduced artificially in Ref. 4 through the η_e and η_i correction factors multiplying the collision frequencies, the modes considered here would be stable.

The results are presented in Figs. 2 to 6. In the calculations it proves convenient to turn on and off the contribution to Q from the nonadiabatic circulating-ion response. This is implemented in the computer code by multiplying this contribution by an artificial weight factor, w_{ci} , which can, then, be varied gradually between 0 and 1. The results presented below are divided into three parts: (i) the influence of circulating ions on the radially local spectrum; (ii) the influence of circulating ions on the radially non-local spectrum; and (iii) the radial structure of the instability. The radially nonlocal spectrum mentioned above is obtained by solving the radial eigenmode Eq. (90) subject to the boundary conditions in Eqs. (91) and (92). The result yields a discrete (quantized) frequency spectrum. As discussed in Sec. 3, to obtain the radially local spectrum, Eq. (90) is solved in the appropriate limit; the radially local equation,

$$Q(r) = 0 \quad , \quad (95)$$

is solved. Since solutions to this equation can be obtained at arbitrary points in the plasma, and since the mode frequency depends continuously on the minor radius at which it is evaluated, the resulting spectrum is continuous. The primary goal in the present section is, of course, to gain an understanding of the radially nonlocal spectrum and the associated eigenmodes, $\phi(r)$. Nevertheless, the radially local spectrum is still of interest since it provides a rough initial estimate of the eigenvalues for the non-local calculations. Before presenting the results in more detail, it is worth mentioning that the following calculations were also carried out using the form of the radial eigenmode equation derived

in Ref. 4. This led to results which were in qualitative agreement with the ones presented (see Ref. 6 for more details). Only the results corresponding to Eq. (93) will now be considered.

A. Influence of the Circulating Ions on the Radially Local Spectrum

The results of interest are displayed in Figs. 2 and 3. Figure 2 shows the radially local spectrum found in the calculations when the nonadiabatic circulating-ion response is ignored ($w_{ci} = 0$). Figure 3 shows the corresponding spectrum when the full circulating-ion dynamics is taken into account. The influence of circulating ions can be summarized as follows. (i) Only one mode corresponding to the conventional trapped-ion mode (TIM) is found to be unstable when the nonadiabatic circulating-ion response is ignored. (ii) When the full circulating-ion dynamics is taken into account, the radially local spectrum has two unstable branches. One corresponds to the conventional TIM, and the other to a new low frequency ($|\omega_{*e}| \lesssim \bar{\omega}_{ti}$) drift-type trapped-electron mode (TEM). (iii) Circulating ions have a strong stabilizing effect on the TIM close to rational surfaces. Accordingly, when the full circulating-ion dynamics is considered, this instability tends to be localized between rational surfaces. In a radially local sense, this mode will grow away from the rational surfaces and will be damped close to them. (iv) In contrast, the TEM tends to be localized at the rational surfaces. For this case, the radially local eigenvalues indicate growth at the rational surfaces and damping far away from them.

The preceding observations, (iii) and (iv), can be understood by recalling the fact that, when the mode frequency is smaller than

a typical ion transit frequency ($|\omega| < \bar{\omega}_{ti}$), the nonadiabatic circulating ion contribution is only important in narrow layers at rational surfaces where $|\omega/(S\bar{\omega}_{ti})| \gtrsim 1$. Away from these layers, the nonadiabatic circulating-ion response is small (of order $|\omega_c|^2/\omega_{ti}^2$) and has little influence on the TIM. However, close to rational surfaces, this contribution becomes important and causes the strong Landau damping of the conventional TIM. Also, it is the fact that the circulating-ion response is nearly hydromagnetic close to rational surfaces, which allows for the new TEM with frequency $\omega_r \sim \omega_{*e}$.

B. Influence of the Circulating Ions on the Radially Nonlocal Spectrum

As the full circulating-ion dynamics is gradually turned on (w_{ci} is increased from 0 to 1), we observe that the various radial eigenmodes behave in two different characteristic ways. This suggests the classification of these modes as (i) radial trapped-ion modes (TIM), and (ii) radial trapped-electron modes (TEM). These are associated respectively with the radially local trapped-ion and trapped-electron branches.

Figure 4 shows that the nonadiabatic circulating-ion response has a stronger influence on the TEM than on the TIM. Indeed, as w_{ci} is artificially increased from 0 to 1, the real part of the frequency for mode #1 (TEM) rapidly increases from $|\omega| < |\omega_{*e}|$, to $|\omega| \sim |\omega_{*e}|$. The frequency of mode #2 (TIM), however, is observed to be fairly insensitive to w_{ci} . This characteristic behavior of ω as a function of w_{ci} can be best understood by examining the radial structure of the instabilities.

C. The Radial Structure

The radial structure of the trapped-electron and trapped-ion eigenmodes along with the corresponding radial potential, Q , are shown in Figs. 5 and 6. In particular, the radial TEM (mode #1) is mainly localized at a particular rational surface, while the radial TIM (mode #2) is mainly localized between rational surfaces.

It is of interest to note that there is a fairly sharp structure in the radial potential, Q , located at rational surfaces ($r/a \sim 0.74$ and 0.94). This comes from the nonadiabatic circulating-ion response. As stated, the width of this structure is roughly given by $\sim 1/(Rq'\bar{\omega}_{ti})$. Specifically, for the (lower frequency) TIM, it is quite narrow and its primary influence on the eigenfunction is to cause a rapid variation of the slope at rational surfaces. On the other hand, for the (higher frequency) TEM, the structure in Q associated with the nonadiabatic circulating-ion response is somewhat broader. Its width is comparable to that of the mode itself, and in this case, it acts as a potential well in which the TEM is analogous to a bound state. Hence, it follows that at low frequencies ($|\omega_{*e}| \lesssim \bar{\omega}_{ti}$), the TEM is localized at a particular rational surface because of the potential well produced by the circulating ions. We note that the circulating-ion response is roughly hydromagnetic close to rational surfaces.

5. TWO-DIMENSIONAL RESULTS

In this section we present results obtained when the full two-dimensional structure of the instability is taken into account. In short the problem is that of solving a set of coupled ordinary differential equations with prescribed boundary conditions. This calculation is carried out numerically by the method of finite elements.^{6,9,10}

In analysing the results below, it proves convenient to provide a quantitative measure of how well the basic assumption, $k_r \rho_{bi} < 1$, is satisfied. This is done by plotting $k_r \rho_{bi}$ as a function of the minor radius, for each poloidal harmonic, and by computing the average value of $k_r \rho_{bi}$ according to:

$$\langle k_r \rho_{bi} \rangle = \frac{\sum_m \int_{x_1}^{x_2} dx \phi_m^* k_r \rho_{bi} \phi_m}{\left(\sum_m \int_{x_1}^{x_2} dx \phi_m^* \phi_m \right)^{-1}}. \quad (96)$$

The radial wave number used is defined for each poloidal harmonic by:

$$k_r(r) = \left\{ \frac{1}{\phi_m} \frac{d\phi_m}{dr} \right\}. \quad (97)$$

Roughly speaking, our calculation will be consistent provided that $\langle k_r \rho_{bi} \rangle$ is smaller than one, and also, provided that $k_r \rho_{bi}$ is smaller than one in the regions where the corresponding poloidal harmonic has significant amplitude.

5.1 Comparison with the One-Dimensional Approximation

First, we consider the same parameters as in Sec. 4 and compare our results with the ones obtained there. The results are displayed in Figs. 7 to 15 and their presentation parallels that of Sec. 4.

The influence of the nonadiabatic circulating-ion response on the radially (non) local spectra is shown in Figs. 7 to 9. The spatial structure of the eigenmodes, along with typical plots of Φ_m and the corresponding value for $k_r \rho_{bi}$, are displayed in Figs. 10 to 15. The perturbed potential shown in Figs. 10 to 13 is the potential along the field line and is defined by

$$\hat{\phi}(r, \theta) = \sum_m \Phi_m(r) \exp[i(\ell q - m)\theta] \quad . \quad (98)$$

A comparison of these results with the corresponding ones obtained in Sec. 4 shows that, for the parameters considered, the basic characteristics found in the one-dimensional radial calculation still persist in the two-dimensional calculation. Specifically, in the radially local limit, the nonadiabatic circulating-ion response contributes strongly to the destabilization of a low frequency ($|\omega_{*e}| \lesssim \bar{\omega}_{ti}$) type of TEM close to rational surfaces. Additionally, it accounts for the strong Landau damping of the conventional TIM near such surfaces. Away from rational surfaces, the nonadiabatic circulating-ion response is small and has no significant influence on the TIM. In carrying out the nonlocal calculation, we still find two characteristic types of eigenmodes. One is localized at a particular rational surface and corresponds to a nonlocal TEM. The other is primarily localized between rational surfaces and corresponds to a two-dimensional TIM. These radial characteristics are consistent with the fact that the nonadiabatic circulating-ion response (which, we recall, is important only close to rational surfaces) has a stronger influence on the nonlocal TEM than on the TIM (see Fig. 9).

Despite the qualitative agreement mentioned above, it is important to emphasize that the two-dimensional calculation does yield new information about the instability, which cannot be obtained from the one-dimensional radial calculation. In particular, it allows us to test the validity of the basic assumption made in the one-dimensional analysis concerning the nearly flutelike poloidal structure of the instability. This feature is exhibited in Figs. 10 and 13. The TEM exhibits a weaker ballooning than the TIM and is, therefore, closer to the flutelike structure assumed in the one-dimensional radial calculation. However, the TIM is seen to balloon strongly to the outside. This suggests that, in the present calculation, the one-dimensional approximation will lead to results that are more reliable in the case of the TEM than in the case of the TIM. For the real part of the eigenfrequency, this behavior tends to be supported by the following comparison of eigenfrequencies obtained in both calculations. In units of 10^4sec^{-1} , these frequencies are [with the notation (ω_r, γ) for $\omega_r + i\gamma$]:

	1-Dimensional	2-Dimensional
TEM	(- 4.71 , 0.65)	(- 4.35 , 2.28)
TIM	(- 0.50 , 0.41)	(- 1.56 , 2.04)

However, it should be noted that ballooning, which is taken into account only in the two-dimensional calculation, has a sizable destabilizing influence on both the TIM and TEM. Finally, we observe that the TIM has some finite structure close to rational surfaces where the mode is nearly flutelike. The main localization of the

mode, however, occurs between rational surfaces where, as stated earlier, the mode is ballooning to the outside.

When the expression given in Eq. (96) is calculated, we find that $\langle k_r \rho_{bi} \rangle = 0.42$ and 0.37 for the TEM and the TIM, respectively. This tends to indicate that our basic assumption, $k_r \rho_{bi} < 1$, is justified. This is further supported by Figs. 12 and 15 which show a typical poloidal harmonic and the corresponding $k_r(r) \rho_{bi}$ for both modes. The important point to note is that $k_r \rho_{bi}$ is small in the regions where ϕ_m has some significant amplitude. We note, also, that $k_r \rho_{bi}$ becomes quite large at the boundaries. This is a consequence of the boundary conditions $\{\phi_m(x_1) = \phi_m(x_2) = 0\}$ imposed here, and of the definition of k_r given in Eq. (97). Mathematically, $k_r \rho_{bi}$ should go to infinity at the boundaries. Numerically, it is large but finite.

5.2 TFTR Parameters

Let us consider the TFTR parameters listed in Table I. These correspond roughly to a high current discharge¹¹ with somewhat idealized profiles. For simplicity, we assume that the plasma has only one ion species. Besides being more relevant to a real machine than the Gladd-Ross parameters just considered, these parameters have the advantage of corresponding to a somewhat smaller value of ρ_{bi}/a . Hence, our basic assumption, $k_r \rho_{bi} < 1$, is more likely to be satisfied here.

We start by considering a case with artificially low shear so that for a chosen toroidal mode number, $l = 5$, the plasma contains

only two rational surfaces. Furthermore, we choose the q profile, such that the rational surfaces are located at $r/a = 0.5$ and 0.8 (where $q = 12$ and 13). These radial positions are away from the region where the mode is localized. The reason is that we first want to consider a case where the presence of rational surfaces (and the associated strong circulating-ion Landau damping) does not have a strong influence on the solution. The influence of increasing shear will be considered later.

The procedure followed here is very similar to that in the previous subsection. In carrying out the calculation, we first find the radially local spectrum neglecting the nonadiabatic circulating ion contribution. These eigenvalues are, then, used as the initial guesses for the eigenvalues in the corresponding two-dimensional calculation. Next, focusing only on the most unstable eigenmode, we gradually turn on the full circulating-ion response.

The spatial characteristics of the mode are displayed in Figs. 16 and 17. In particular, we observe that the instability is strongly ballooning to the outside and that it is well localized radially. Specifically, the mode is well localized between rational surfaces. Because the rational surfaces are located away from where the mode is localized, they have only a weak influence on the mode frequency and spatial structure.

In Fig. 18 we plot various relevant frequencies as a function of the minor radius. Comparing the mode frequency with these, it is interesting to note that the electrons are collisional ($|\omega| \ll v_{ei}/c$) but that the ions are not really collisionless ($v_{ii}/c \lesssim |\omega|$). Hence, in contrast to the dissipative limit, effects associated with finite-ion collision frequency might prove to be important here. This is

noted because, as discussed in Sec. 3, the model collision operator used in our calculations may not be appropriate in this collisionality regime, $v_{ii}/\epsilon \lesssim |\omega|$.

We next examine the influence of increasing shear on the eigenmode just discussed. This is done by varying q on axis and at the limiter according to:

$$q(0) = 2.2718 - 0.3718 w_s, \quad (99)$$

$$q(a) = 2.7846 + 0.1154 w_s, \quad (100)$$

where $q(r) = q(0) + [q(a) - q(0)](r/a)^2$, and where w_s is an artificial weight factor which is varied from 0 to 3.5. Of course, w_s is limited by the requirement that $q(0) \geq 1$ for MHD stability. As w_s is increased, we follow the evolution of the mode frequency as well as the spatial structure of the instability and the value $\langle k_r \rho_{bi} \rangle$. The results are shown in Figs. 19 to 22.

From Fig. 19 we see that the real part of the frequency ω_r , increases monotonically with shear while the growth rate, γ , is fairly constant over a wide range of w_s . Part (b) of this figure shows $\langle k_r \rho_{bi} \rangle$ as a function of w_s . The tendency is that the value of $\langle k_r \rho_{bi} \rangle$ generally increases with shear. This is expected since stronger shear implies more rational surfaces in the region where the mode is localized, and hence, more rapid radial variations. We note that $\langle k_r \rho_{bi} \rangle$ seems to have a maximum at $w_s \approx 3$. However, the numerical evaluation of $\langle k_r \rho_{bi} \rangle$ at this maximum is somewhat inaccurate since the estimates obtained for $N=300$ and $N=350$ (N being the number of grid points used in the finite element method) differ by as much as 15%. This is in contrast to the corresponding

evaluation of the mode frequency which differs by less than 1% from these two values of N . It is generally true that the evaluation of the mode frequency is more accurate than that of $\langle k_r \rho_{bi} \rangle$. The reason is that the latter involves integrations of absolute values of derivatives, which are usually difficult to accurately evaluate numerically.

Let us now look at the eigenfunction in more detail for a selected value of w_s ($w_s > 0$). Specifically, let us consider $w_s = 1.5$ which corresponds to five rational surfaces in the region of interest ($0.4 \leq r/a \leq 1.0$). The spatial structure of the mode is shown in Figs. 20 and 21. Comparing these with Figs. 16 and 17, we see that stronger shear does not influence the global shape of the mode. It merely causes some rapid variations of the eigenfunction close to rational surfaces. This is due to the strong contribution from the nonadiabatic circulating-ion response there.

The value of $\langle k_r \rho_{bi} \rangle$ computed in this case is $\langle k_r \rho_{bi} \rangle = 0.25$. This is consistent with the long wavelength approximation, $k_r \rho_{bi} \ll 1$.

Finally, Fig. 22 shows a typical plot of diagonal elements of the C and A matrices as a function of r . This figure is of interest because it shows the sharp structure in the C matrix localized at rational surfaces. This structure comes from the nonadiabatic circulating ion contribution and it is analogous to that in the radial potential Q introduced in Sec. 4. In part (b) we note that the diagonal elements of A are localized at the corresponding rational surface (for each poloidal harmonic) and that the characteristic width of each element is of the order of the distance between adjacent rational surfaces. This suggests that, for the case considered, only the coupling between the first few adjacent poloidal harmonics might

be important.

6. CONCLUSION

The new formalism developed in this paper was tested by applying familiar approximations and, then, comparing our results with those of previous calculations. In the radially local dissipative limit, we found that our analytic dispersion relation agreed with that of Ref. 3, except for the ion collisional growth rate. We concluded that our Krook collision model was accurate in the large collision frequency limit ($|\omega| \ll \nu/\epsilon$) of primary interest. However, it cannot be adequately applied on the small collision frequency limit ($\nu/\epsilon \ll |\omega|$), and its validity for intermediate collision frequencies ($\nu/\epsilon \sim |\omega|$) is, as yet, untested.

In the one-dimensional radial approximation, our results were in reasonable agreement with those obtained from the eigenmode equations of Ref. 4. Specifically, we found that two types of modes could be unstable; one being localized at a particular rational surface (a low frequency form of the "trapped-electron mode", or TEM) and another, primarily localized between rational surfaces (the "trapped-ion mode", or TIM). Note that no unstable TEM was reported in Ref. 4. The existence of such unstable modes here is probably a consequence of the artificially low density used in the calculations of Sec. 4.

When applied to the Gladd-Ross parameters of Sec. 4, the two-dimensional calculation produced results which were in qualitative agreement with those obtained in the one-dimensional approximation. Here again we found both a low frequency ($\omega_r \sim \omega_{*e} < \bar{\omega}_{ci}$) TEM

localized at a rational surface, as well as a TIM, mainly localized between rational surfaces. The TEM was observed to exhibit weak ballooning and was, therefore, more consistent with the flutelike assumption made in the one-dimensional radial calculations.

The two-dimensional formalism was also applied to more realistic TFTR parameters and the influence of shear was considered. For the particular eigenmode considered, we observed that the real part of the frequency increased monotonically with shear, but that the growth rate was relatively insensitive to this effect. We also noted that, aside from causing rapid radial variations in their neighborhood, multiple rational surfaces did not change the global structure of the instability. In particular, it is important to emphasize that, in contrast to the picture given in Ref. 4, we found that the mode could have finite amplitude at a rational surface (and still remain unstable) and that the radial structure seemed to be determined primarily by the equilibrium variations rather than by the position of the rational surfaces. In addition to toroidal coupling effects, absent in the one-dimensional radial calculations, this may also be due in part to the fact that the equilibrium profiles used here are generally sharper than those of Ref. 4.

Also, we did attempt to reproduce a mode with the same radial characteristics as found by Gladd and Ross⁴ in the presence of multiple rational surfaces. These results were not included in Sec. 5 but will be briefly described here. In particular, we used the same parameters (including the density) as for the $\ell = 5$ case of Ref. 4. We first found an unstable two-dimensional TIM neglecting the non-adiabatic circulating-ion response. Then, we gradually turned on the full circulating-ion response. The result was a marginally stable mode exhibiting many of the characteristics of a TEM (e.g., large

amplitude at rational surfaces and $\omega_r \sim \omega_{*e}$). Because of the computational difficulty of this exercise, we had to limit our attempt to one mode only. Of course, the negative result found here does not definitively rule out the possible existence of a two-dimensional multiple rational surface TIM with similar radial characteristics to that found in Ref. 4.

It is appropriate to recall the main limitations of our analysis, and indicate some possible future developments. The most serious limitation here is that associated with the long-wavelength assumption, $k_r \rho_{bi} < 1$. Even though this inequality was roughly satisfied in the cases discussed in this paper, we found that the computed value of $\langle k_r \rho_{bi} \rangle$ was generally not much smaller than one. Furthermore, it seems likely that, when considering more realistic parameters corresponding to cases with multiple rational surfaces in the plasma and with sharp equilibrium profiles, this basic approximation would break down. It then follows that a more general, integral formulation of the problem, valid for arbitrary values of $k_r \rho_{bi}$, would be necessary.

As noted earlier, the Krook collision model used in this calculation is inaccurate in the low collision frequency limit and its validity is uncertain for intermediate collision frequencies, $v/\epsilon \sim |\omega|$. This regime of collisionality may be of interest in practice (as for the TFTR parameters of Sec. 5). Because, in this instance, the collisional terms might have a sizeable effect, it would be desirable to improve our collision operator in this regime of collisionality. This could be achieved by solving the gyrokinetic equation using the Lorentz collision operator. Unfortunately,

even with present computers, this multi-dimensional calculation in configuration and velocity space is prohibitively time consuming.

The inclusion of other effects, even though not essential to the conventional analysis of trapped particle instabilities, could nevertheless lead to changes in the numerical results by factors of order unity. For example, noncircular flux surfaces or anisotropy in the equilibrium distribution could considerably affect the threshold conditions and even introduce new branches (beam-driven modes) in the dispersion relation. It would thus be of interest to consider generalizing the present approach to treat more general (numerically generated) equilibria.

Finally, it is worth pointing out that, although it does represent the first fully two-dimensional analysis of low frequency instabilities in the trapped-ion regime, the present formulation of this problem is nevertheless somewhat inefficient numerically. Specifically, when dealing with multiple rational surfaces, several poloidal harmonics need to be included in the calculation. In such cases, the numerical solution of Eq. (56) takes several minutes of computer time and very accurate initial guesses are needed (1% or better) for rapid convergence. This puts a rather stringent limitation on the applicability of our approach to cases of practical interest. This difficulty could be alleviated by developing a ballooning formulation of the problem, similar to the one already applied to MHD modes¹² and, more recently, to the trapped-electron mode.¹³

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APPENDIX

PARAMETRIZATION

We define the parametrization of the equilibrium and list the explicit values of the parameters used in each calculation. The parameters are assumed to determine the equilibrium through the model profiles

$$n(x) = n(0) \exp[-(x/x_n)^{\alpha_n}] \quad , \quad (\text{A1})$$

$$T_e(x) = T_e(0) \exp[-(x/x_{Te})^{\alpha_{Te}}] \quad , \quad (\text{A2})$$

$$T_i(x) = T_i(0) \exp[-(x/x_{Ti})^{\alpha_{Ti}}] \quad , \quad (\text{A3})$$

$$q(x) = q(0) + [q(a) - q(0)]x^2 \quad , \quad (\text{A4})$$

where x is the normalized minor radius, $x = r/a$. The specific values of these parameters, used in each calculation, are listed in Table 1.

TABLE 1

	Gladd & Ross	TFTR
R_0 (cm)	80	248
a (cm)	23	85
B_0 (Tesla)	2.8	5.2
$n(0)$ (10^{13} cm^{-3})	0.5	6.0
x_n	1.0	0.75
γ_n	3.5	1.5
$T_e(0)$ (keV)	2.0	5.0
x_{Te}	1.0	0.75
α_{Te}	3.5	2.5
$T_i(0)$ (keV)	1.0	5.0
x_{Ti}	1.0	0.75
α_{Ti}	3.5	2.5
$q(0)$	1.14	variable
$q(a)$	2.1	variable
M_i (amu)	1	2
Z_i	1	1
h_i	1	1
ℓ	3	5

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Fig. 1. Toroidal coordinates.

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 Fig. 2. Radially local (solid line) and radially non-local (dots) spectra found neglecting the non-adiabatic circulating-ion response.

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Fig. 3. Radially local (solid line) and radially non-local (dots) spectra found with the full circulating-ion response.

Fig. 4. Variation of the mode frequencies with the artificial weight factor, w_{ci} , multiplying the non-adiabatic circulating-ion response.

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792495
 Fig. 5. Radial structure of the eigenfunction, Φ , and of the corresponding radial potential, Q , for mode #1 (TEM). Here, $\lambda = 3$ and $\omega = (-4.71, 0.65) \times 10^4 \text{ sec}^{-1}$.

792497
 Fig. 6. Radial structure of the eigenfunction, ϕ , and of the corresponding radial potential, Q , for mode #2 (TIM). Here, $l = 3$ and $\omega = (-0.50, 0.41) \times 10^4 \text{sec}^{-1}$.

792499
 Fig. 7. Radially local (solid line) and radially non-local (dots) spectra obtained with the Gladd-Ross parameters, when the non-adiabatic circulating-ion response is neglected.

Fig. 8. Radially local (solid line) and radially non-local spectra obtained with the Gladd-Ross parameters, including the full circulating-ion response.

Fig. 9. Variation of the mode frequency with the artificial weight factor, w_{ci} , multiplying the non-adiabatic circulating-ion response for modes #1 (TEM) and #2 (TIM), obtained with the Gladd-Ross parameters.

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Fig. 10. Two-dimensional structure of the trapped-electron mode (mode #1) obtained with the Gladd-Ross parameters. Here, $l = 3$, $\omega = (-4.35, 2.28) \times 10^6 \text{ sec}^{-1}$, and $\langle k_r \rho_{bi} \rangle = 0.42$.

792290
 Fig. 11. Radial structure, along $\theta=0$, of the two-dimensional trapped-electron mode (mode #1) found with the Gladd-Ross parameters.

Fig. 12. A typical poloidal harmonic and the corresponding $k_r \rho_{bi}$ for the trapped-electron mode.

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Fig. 13. Two-dimensional structure of the trapped-ion mode (mode #2) obtained with the Gladd-Ross parameters. Here, $\ell = 3$, $\omega = (-1.56, 2.04) \times 10^4 \text{ sec}^{-1}$, and $\langle k_r \rho_{bi} \rangle = 0.37$.

792291
Fig. 14. Radial structure, along $\theta = 0$, of the two-dimensional trapped-ion mode (mode #2) found with the Gladd-Ross parameters.

Fig. 15. A typical poloidal harmonic and the corresponding $k_r \rho_{bi}$ for the trapped-ion mode. 792370

792240
 Fig. 16. Two-dimensional structure of mode #1 obtained with the TFTR parameters, including the full circulating-ion response. Here, $\ell = 5$, $\omega = (0.55, 2.23) \times 10^3 \text{ sec}^{-1}$ and $\langle k_r \rho_{b1} \rangle = 0.12$.

792346
Fig. 17. Radial structure, along $\theta = 0$, of mode #1 found with TFTR parameters, including the full circulating-ion response.

Fig. 18. Various frequencies corresponding to the TFTR parameters. Here, $w_s = 0$.

792347

Fig. 19. Mode frequency and the averaged value of $k_r \rho_{bi}$ as a function of w_s . 792496

Fig. 20. Two-dimensional structure of the instability for $w_s = 1.5$, $\omega = (1.62, 2.53) \times 10^3 \text{ sec}^{-1}$, and $\langle k_r \rho_{bi} \rangle = 0.25$.

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792366
Fig. 21. Radial structure of the instability, along $\theta = 0$,
for $w_s = 1.5$.

792363
Fig. 22. Typical diagonal matrix elements as a function of the minor radius, for $w_s=1.5$.

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