ON THE DENSITY OF MINIMAL FREE SUBFLOWS OF GENERAL SYMBOLIC FLOWS

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This paper studies symbolic dynamical systems $\{0, 1\}^G$, where G is a countably infinite group, $\{0, 1\}^G$ has the product topology, and G acts on $\{0, 1\}^G$ by shifts. It is proven that for every countably infinite group G the union of the minimal free subflows of $\{0, 1\}^G$ is dense. In fact, a stronger result is obtained which states that if G is a countably infinite group and U is an open subset of $\{0, 1\}^G$, then there is a collection of size continuum consisting of pairwise disjoint minimal free subflows intersecting U.

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CHAPTER 1

BACKGROUND

In topological dynamics, a dynamical system (X, G) consists of a Hausdorff space X together with a group G of homeomorphisms of X. The dynamical system is called free if $g(x) \neq x$ for every $x \in X$ and every non-identity $g \in G$, and it is called minimal if $\overline{\{g(x) \mid g \in G\}} = X$ for every $x \in X$. In 1938, G. A. Hedlund and M. Morse began a movement to better understand a tool known as symbolic dynamics and to furthermore view this tool as an object of study in its own right ([5]). As a tool, symbolic dynamics arises through the following process. Take a dynamical system (X, G) with G countable, and choose a partition of X consisting of finitely many sets $A_0, A_1, \ldots, A_n$. We define $\phi : X \to \{0, 1, \ldots, n\}^G$ by setting $\phi(x)(g) = k$ if k is the unique number for which $g(x) \in A_k$. One can define an action of G on $\{0, 1, \ldots, n\}^G$ which makes $(\{0, 1, \ldots, n\}^G, G)$ into a dynamical system (with the product topology) and which also makes the action of G commute with the function ϕ. With this action $\{0, 1, \ldots, n\}^G$ is called a symbolic flow, and $\phi(X)$ is called a subflow. The purpose of this method is to study $\phi(X)$ in order to reveal properties of the original dynamical system.

Traditionally, emphasis in symbolic dynamics has been placed on subflows of symbolic flows of the form $\{1, 2, \ldots, n\}^{\mathbb{Z}}$. This has left much to be unknown about subflows of more general symbolic flows. In particular, the existence and properties of free subflows and free minimal subflows, which are central notions in dynamics, had not been investigated until only the last few years.

In 2007, E. Glasner and V. Uspenskij investigated which countable groups G had the property that $\{0, 1\}^G$ contained a free subflow ([4]). They concluded that abelian groups, residually finite groups, and a few other groups have this property, but they could not draw any conclusions for more general groups. Similarly, in 2007 A. Dranishnikov and V. Schroeder also did work on this problem. They were only able to conclude that torsion free hyperbolic groups have the property ([1]). Around the same time, a complete solution to this problem was found by S. Gao, S. Jackson, and the author ([2]). They proved that $\{0,1\}^G$ contains continuum-many pairwise disjoint free subflows whenever G is a countably infinite group. With Zorn's Lemma it is true that every free subflow contains a minimal free subflow. Therefore their result reveals that $\{0,1\}^G$ contains continuum-many pairwise disjoint free minimal subflows when G is countably infinite.

The purpose of this paper is to strengthen this last result. The main theorem is that for countably infinite groups G the union of the free minimal subflows of $\{0,1\}^G$ is dense. Actually, a stronger result is obtained which states that if G is a countably infinite group and $U \subseteq \{0,1\}^G$ is open, then there is a collection of size continuum consisting of pairwise disjoint free minimal subflows intersecting U. The methods here are self-contained, however they constitute an abstraction and strengthening of the methods found in [2]. In section 2, notation is developed and combinatorial equivalents for dynamical properties are presented. In section 3, general countable groups are studied and useful properties they posses are found. Finally, in the last section it is shown how to construct free minimal subflows intersecting a given open set. A much more in-depth study of symbolic flows using these methods will be available in [3].

CHAPTER 2

PRELIMINARIES

We first give a more detailed description of general symbolic flows. We will work only with $\{0,1\}^G$, however all of our methods and results can be effortlessly modified to work for $\{0,1,\ldots,n\}^G$. We let 2^G denote $\{0,1\}^G$ and

$$2^{<G} = \bigcup_{H \subseteq G} \{0,1\}^H.$$

Fix a countably infinite group G, and let G be enumerated without repetition as $1_G = g_0, g_1, g_2, \ldots$. Define a metric on 2^G by

$$d(x,y) = \begin{cases} 2^{-n}, & \text{if } x \neq y \text{ and } n \in \mathbb{N} \text{ is the least such that } x(g_n) \neq y(g_n), \\ 0, & \text{if } x = y. \end{cases}$$

The metric d is an ultrametric on 2^G compatible with the compact product topology on 2^G.

The action of G on 2^G is given by

$$(g \cdot x)(h) = x(g^{-1}h).$$

One can easily check that each map $x \mapsto g \cdot x$ is a homeomorphism of 2^G. For each $x \in 2^G$ let $[x]$ denote the orbit of x, i.e., $[x] = \{g \cdot x \mid g \in G\}$.

Remark 2.1. The action defined above is the left shift action of G on 2^G. The action referred to in the previous section was the right shift action of G on 2^G, which is defined by

$$(g \cdot x)(h) = x(hg).$$

Using the left shift action is not a problem because as dynamical systems 2^G with the left shift action is isomorphic to 2^G with the right shift action.

Definition 2.2. Let G be a countable group. A *subflow* A of 2^G is a closed subset of 2^G which is invariant under the action of G, meaning $g \cdot A = A$ for each $g \in G$.

Definition 2.3. Let G be a countable group. A subflow A of 2^G is *free* if $g \cdot x \neq x$ for every $x \in A$ and every non-identity $g \in G$.

Definition 2.4. Let G be a countable group. A subflow A of 2^G is *minimal* if $\overline{[x]} = A$ for every $x \in A$.

The previous three definitions come from the general context of dynamical systems. The useful thing about symbolic dynamics is that it brings combinatorics into dynamical systems. The next three definitions are combinatorial in nature, and we will soon see that they are very important.

Definition 2.5. For a countable group G, a 2-*coloring* on G is a function $c : G \to \{0, 1\}$ such that for any $s \in G$ with $s \neq 1_G$ there is a finite set $T \subseteq G$ such that

$$\forall g \in G \; \exists t \in T \; c(gt) \neq c(gst).$$

Definition 2.6. For a countable group G, $c \in 2^G$ is called *minimal* if for every finite $A \subseteq G$ there exists a finite $T \subseteq G$ such that

$$\forall g \in G \; \exists t \in T \; \forall a \in A \; x(gta) = x(a).$$

Definition 2.7. Let G be a countable group and let $c_0, c_1 \in 2^G$. We say that c_0 and c_1 are *orthogonal* if there is a finite set $T \subseteq G$ such that

$$\forall g_0, g_1 \in G \; \exists t \in T \; c_0(g_0 t) \neq c_1(g_1 t).$$

The following lemma appears in [2], but was also independently discovered by Vladimir Pestov.

Lemma 2.8. *If G is a countable group and $x \in 2^G$, then $\overline{[x]}$ is free if and only if x is a 2-coloring on G.*

Proof. (⇒) Assume $\overline{[x]}$ is free. Denote $C = \overline{[x]}$. Fix any $s \in G$ with $s \neq 1_G$. Then for any $y \in C$, $s^{-1} \cdot y \neq y$, and hence there is $t \in G$ with $(s^{-1} \cdot y)(t) \neq y(t)$. Define a function $\tau : C \to G$ by letting $\tau(y) = g_n$ where n is the least so that $(s^{-1} \cdot y)(g_n) \neq y(g_n)$. Then τ is a continuous function. Since C is compact we get that $\tau(C) \subseteq G$ is finite. Let $T = \tau(C)$. Then for any $g \in G$, there is $t \in T$ with $x(gt) = (g^{-1} \cdot x)(t) \neq (s^{-1}g^{-1} \cdot x)(t) = x(gst)$. This proves that x is a 2-coloring.

($\Leftarrow$) Assume that x is a 2-coloring on G. Suppose $z \in \overline{[x]}$, that is, there are $h_m \in G$ with $h_m \cdot x \to z$ as $m \to \infty$. We must show $g \cdot z \neq z$ for $1_G \neq g \in G$. Towards a contradiction suppose $s \cdot z = z$ for $s \neq 1_G$. Then by the continuity of the action we have that $s^{-1}h_m \cdot x \to s^{-1} \cdot z = z$. Let $T \subseteq G$ be a finite set such that for any $g \in G$ there is $t \in T$ with $x(gt) \neq x(gst)$. Let n be large enough so that $T \subseteq \{g_0, \ldots, g_n\}$ and let $m \geq n$ be such that $d(h_m \cdot x, z), d(s^{-1}h_m \cdot x, z) < 2^{-n}$. Now fix $t \in T$ with $(h_m \cdot x)(t) = x(h_m^{-1}t) \neq x(h_m^{-1}st) = (s^{-1}h_m \cdot x)(t)$. Then $z(t) = (h_m \cdot x)(t) \neq (s^{-1}h_m \cdot x)(t) = z(t)$, a contradiction. $\square$

This next lemma is a simple generalization of a well known fact for $\mathbb{Z}$.

LEMMA 2.9. *If G is a countable group and $c \in 2^G$, then $\overline{[c]}$ is minimal if and only if c is minimal.*

PROOF. ($\Rightarrow$) Assume $\overline{[c]}$ is minimal. Let $A \subseteq G$ be finite, and let $k \in \mathbb{N}$ be such that $A \subseteq \{g_0, g_1, \ldots, g_k\}$. Since $\overline{[c]}$ is minimal, for every $y \in \overline{[c]}$ there exists $h \in G$ with $d(h \cdot y, c) < 2^{-k}$. Define $\phi(y)$ to be the least $m \in \mathbb{N}$ such that $d(g_m \cdot y, c) < 2^{-k}$. Then ϕ is continuous and $\overline{[c]}$ is compact so $\phi(\overline{[c]}) \subseteq \mathbb{N}$ is finite. Let $M \in \mathbb{N}$ bound $\phi(\overline{[c]})$ and set $T = \{g_0, g_1, \ldots, g_M\}$. It follows that for any $g \in G$ there is $t \in T$ with $d(t \cdot g^{-1} \cdot c, c) < 2^{-k}$. Therefore, for all $a \in A$ we have $c(gt^{-1}a) = t \cdot g^{-1} \cdot c(a) = c(a)$. Thus, T^{-1} is the desired finite subset of G.

($\Leftarrow$) Now assume c is minimal. Fix $y \in \overline{[c]}$ and let $\epsilon > 0$ be arbitrary. Let $k \in \mathbb{N}$ be such that $2^{-k} < \epsilon$, and set $A = \{g_0, g_1, \ldots, g_k\}$. By our assumption, we may let $T \subseteq G$ be finite such that for all $g \in G$ there is $t \in T$ with $c(gta) = c(a)$ for all $a \in A$. Let h_m be a sequence in G with $h_m \cdot c \to y$ as $m \to \infty$. Let $r \in \mathbb{N}$ be such that $TA \subseteq \{g_0, g_1, \ldots, g_r\}$, and fix $m \in \mathbb{N}$ with $d(h_m \cdot c, y) < 2^{-r}$. Then we have for some $t \in T$, $c(h_m^{-1}ta) = c(a)$ for all $a \in A$. But since $d(h_m \cdot c, y) < 2^{-r}$, $y(ta) = h_m \cdot c(ta) = c(h_m^{-1}ta) = c(a)$ for all $a \in A$. It follows $d(t^{-1} \cdot y, c) < 2^{-k}$ and therefore $d(\overline{[y]}, c) < 2^{-k} < \epsilon$. But ϵ was arbitrary and $\overline{[y]}$ is closed so

$c \in \overline{[y]}$. Additionally, $\overline{[y]}$ is G-invariant so $[c] \subseteq \overline{[y]}$ and therefore $\overline{[c]} \subseteq \overline{[y]}$. Similarly, $\overline{[c]}$ is G-invariant and $y \in \overline{[c]}$ so $\overline{[y]} \subseteq \overline{[c]}$. We conclude $\overline{[c]}$ is minimal. □

The following lemma appears in [2].

LEMMA 2.10. *Let G be a countable group and let $c_0, c_1 \in 2^G$. Then $\overline{[c_0]}$ and $\overline{[c_1]}$ are disjoint if and only if c_0 and c_1 are orthogonal.*

PROOF. ($\Rightarrow$) Conversely, suppose $\overline{[c_0]} \cap \overline{[c_1]} = \emptyset$. Since they are both compact it follows that there is some $\delta > 0$ such that for any $y_0 \in \overline{[c_0]}$ and $y_1 \in \overline{[c_1]}$, $d(y_0, y_1) \geq \delta$. Let n be large enough such that $\delta \geq 2^{-n}$. Then in particular for any $x_0 \in [c_0]$ and $x_1 \in [c_1]$, $d(x_0, x_1) \geq 2^{-n}$. This implies that there is $t \in \{g_0, \ldots, g_n\}$ such that $x_0(t) \neq x_1(t)$.

($\Leftarrow$) Let n be large enough such that $T \subseteq \{g_0, \ldots, g_n\}$. Then for any $x_0 \in [c_0]$ and $x_1 \in [c_1]$, there is $t \in T$ such that $x_0(t) \neq x_1(t)$, and thus $d(x_0, x_1) \geq 2^{-n}$. It follows that $d(y_0, y_1) \geq 2^{-n}$ for any $y_0 \in \overline{[c_0]}$ and $y_1 \in \overline{[c_1]}$, and therefore $\overline{[c_0]} \cap \overline{[c_1]} = \emptyset$. □

CHAPTER 3

FINDING STRUCTURE WITHIN COUNTABLE GROUPS

The sole purpose of this section is to study countable groups in their full generality and develop the tools which we will use in the next section to prove the main theorem. Thus, no mention of 2^G will be made in this section. Our first definition will be central to our studies for the rest of the paper.

Definition 3.1. Let G be a group and let $A, B, \Delta \subseteq G$. We say that the Δ-translates of A are *maximally disjoint within* B if the following properties hold:

(i) for all $\gamma, \psi \in \Delta$, if $\gamma \neq \psi$ then $\gamma A \cap \psi A = \varnothing$;

(ii) for every $g \in G$, if $gA \subseteq B$ then there exists $\gamma \in \Delta$ with $gA \cap \gamma A \neq \varnothing$.

When property (i) holds we say that the Δ-translates of A are *disjoint*. Furthermore, we say that the Δ-translates of A are *contained and maximally disjoint within* B if the Δ-translates of A are maximally disjoint within B and $\Delta A \subseteq B$.

Notice that in the definition above we were referring to the left translates of A by Δ but never explicitly used the term left translates. Throughout this paper when we use the word translate(s) it will be understood that we are referring to left translate(s). Additionally, note that in the definition above there is no restriction on Δ being nonempty. So at times it may be that the $\varnothing$-translates of A are contained and maximally disjoint within B.

Let G be a group and let $A, B \subseteq G$ be finite with $1_G \in A$. Define

$$\rho(B; A) = \min\{|D| \ \mid D \subseteq B \text{ and } \forall g \in B \ (gA \subseteq B \Rightarrow gA \cap DA \neq \varnothing)\}.$$

This is well defined since B is finite. The definition of ρ was tailored so that the following two statements hold:

(i) If $\Delta \subseteq B$ and the Δ-translates of A are maximally disjoint within B, then $|\Delta| \geq \rho(B; A)$;

(ii) If $A' \subseteq A$ then $\rho(B; A') \geq \rho(B; A)$.

The reader should verify the truth of these two statements.

LEMMA 3.2. *Let G be an infinite group and let $A, B \subseteq G$ be finite with $1_G \in A$. For any $\epsilon > 0$ there exists a finite $C \subseteq G$ containing B such that $\rho(C; A) > \frac{|C|}{|A|}(1-\epsilon)$.*

PROOF. Let $\Delta \subseteq G$ be countably infinite and such that the Δ-translates of AA^{-1} are disjoint and $\Delta AA^{-1}A \cap B = \varnothing$. Let $\lambda_1, \lambda_2, \ldots$ be an enumeration of Δ. For each $n \geq 1$, define

$$B_n = B \cup \left(\bigcup_{1 \leq k \leq n} \lambda_k A \right).$$

Fix $n \geq 1$ and let $D \subseteq B_n$ be such that $gA \cap DA \neq \varnothing$ whenever $g \in B_n$ with $gA \subseteq B_n$. It follows that for each $1 \leq i \leq n$ there is $d_i \in D$ with $d_i A \cap \lambda_i A \neq \varnothing$. Then

$$d_i \in \lambda_i AA^{-1}.$$

Since the Δ-translates of AA^{-1} are disjoint, the d_i's are all distinct. Additionally, $d_i A \cap B \subseteq \Delta AA^{-1}A \cap B = \varnothing$ so that $\rho(B_n; A) - n \geq \rho(B; A)$. Therefore we have

$$\rho(B_n; A)\frac{|A|}{|B_n|} \geq \frac{n|A| + \rho(B; A)|A|}{n|A| + |B|}.$$

Clearly as n goes to infinity the fraction on the right goes to 1. So there is $n \geq 1$ with $\rho(B_n; A)\frac{|A|}{|B_n|} > 1 - \epsilon$ and $\rho(B_n; A) > \frac{|B_n|}{|A|}(1-\epsilon)$. Setting $C = B_n$ completes the proof. □

DEFINITION 3.3. A function $f : \mathbb{N} \to \mathbb{N}$ is said to have *subexponential growth* if for every $u > 1$ there is $N \in \mathbb{N}$ so that $f(n) < u^n$ for all $n \geq N$.

LEMMA 3.4. *Let G be an infinite group and let $A, B \subseteq G$ be finite with $1_G \in A$. If $f : \mathbb{N} \to \mathbb{N}$ has subexponential growth then there exists a finite $C \subseteq G$ containing B such that $2^{\rho(C;A)} > f(|C|)$.*

PROOF. Let $N \in \mathbb{N}$ be such that $2^{\frac{n}{2|A|}} > f(n)$ for all $n \geq N$. Let $B' \subseteq G$ be a finite set containing B with $|B'| \geq N$. By Lemma 3.2 there exists a finite $C \subseteq G$ containing B' with $\rho(C; A) > \frac{1}{2}\frac{|C|}{|A|}$. Then $C \supseteq B$ and as $|C|$ is at least N,

$$2^{\rho(C;A)} > 2^{\frac{|C|}{2|A|}} > f(|C|).$$

□

The following proposition is the key result of this section. Unfortunately, at this time it is difficult both to express the importance of this proposition and to explain where it fits in the overall proof of the main theorem.

PROPOSITION 3.5. *Let G be a countably infinite group and let $(H_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of G with $1_G \in H_0$, $\bigcup_{n\in\mathbb{N}} H_n = G$, and*

$$H_{n-1}(H_0^{-1}H_0)(H_1^{-1}H_1)\cdots(H_{n-1}^{-1}H_{n-1}) \subseteq H_n$$

for $n \geq 1$. Then there exists an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of subsets of G and a decreasing sequence $(\Delta_n)_{n\in\mathbb{N}}$ of subsets of G such that

(i) *$F_0 = H_0$;*

(ii) *$1_G \in F_n \subseteq H_n$ for all $n \geq 1$;*

(iii) *$1_G \in \Delta_n$ for all $n \in \mathbb{N}$;*

(iv) *for all $n \in \mathbb{N}$ the Δ_n-translates of F_n are maximally disjoint within G;*

(v) *for all $n \geq 1$ the $\Delta_{n-1} \cap F_n$-translates of F_{n-1} are contained and maximally disjoint within H_n;*

(vi) *for all $n > 0$ and $0 \leq k \leq n$ the $\Delta_k \cap F_n$-translates of F_k are maximally disjoint within H_{n-1};*

(vii) *for every $k \in \mathbb{N}$ and $\gamma \in \Delta_k$, there is $n \geq k$ with $\gamma F_k \subseteq F_n$;*

(viii) *$\gamma F_n \cap \Delta_k f = \gamma(F_n \cap \Delta_k)f$ for all $n \geq k$, $\gamma \in \Delta_n$, and $f \in F_k$.*

PROOF. Set $F_0 = H_0$ so (i) is satisfied. We will construct $(F_n)_{n\in\mathbb{N}}$. Choose $\delta_0^1 \subseteq H_1$ so that $1_G \in \delta_0^1$ and the δ_0^1-translates of F_0 are contained and maximally disjoint within H_1. We then define $F_1 = \bigcup_{\gamma\in\delta_0^1} \gamma F_0$. Note $F_1 \subseteq H_1$.

We will continue the construction inductively. Assume F_0 through F_{n-1} have been defined with $F_m \subseteq H_m$ for $m < n$. Again we choose $\delta_{n-1}^n \subseteq H_n$ so that $1_G \in \delta_{n-1}^n$ and the δ_{n-1}^n-translates of F_{n-1} are contained and maximally disjoint within H_n. Once δ_{n-1}^n through δ_{n-k+1}^n have been defined with $1 < k \leq n$, choose δ_{n-k}^n so that the δ_{n-k}^n-translates of F_{n-k}

are contained and maximally disjoint within

$$B^n_{n-k} - \bigcup_{1\le m<k} \bigcup_{\gamma\in\delta^n_{n-m}} \gamma F_{n-m} = B^n_{n-k} - \bigcup_{1\le m<k} \delta^n_{n-m} F_{n-m}$$

where for $r, s \in \mathbb{N}$ with $r < s$

$$B^s_r = \{g \in G \mid \{g\}(F^{-1}_{r+1}F_{r+1})(F^{-1}_{r+2}F_{r+2})\cdots(F^{-1}_{s-1}F_{s-1}) \subseteq H_s\}.$$

Note $H_{s-1} \subseteq B^s_r$ so $B^s_r \neq \varnothing$.

Finally, we define

$$F_n = \bigcup_{0\le m<n} \bigcup_{\gamma\in\delta^n_m} \gamma F_m = \bigcup_{0\le m<n} \delta^n_m F_m.$$

Note $F_n \subseteq H_n$ since $B^n_k \subseteq H_n$ for all $0 \le k < n-1$. The construction of $(F_n)_{n\in\mathbb{N}}$ is now complete and satisfies (i) and (ii).

The use of the B^n_k's plays a vital role in this proof. Their main function is to achieve conclusion (iv). Let us reveal the important property of the B^n_k's. Fix $n, k \in \mathbb{N}$ with $n > k$. Suppose $g \in G$ satisfies $gF_k \cap F_n \neq \varnothing$. Then $gF_k \cap \delta^n_m F_m \neq \varnothing$ for some $k \le m < n$ (this may be true for several values of m, some of which may be less than k). Let us show why this is true. Suppose $gF_k \cap \delta^n_m F_m = \varnothing$ for all $k < m < n$. It will suffice to show $gF_k \cap \delta^n_k F_k \neq \varnothing$. As $F_n = \bigcup_{0\le t<n} \delta^n_t F_t$, there is $0 \le t \le k$ with $gF_k \cap \delta^n_t F_t \neq \varnothing$. If $t = k$, then we are done. So suppose $t < k$. We have

$$gF_k \subseteq \delta^n_t F_t F^{-1}_k F_k \subseteq \delta^n_t F_t (F^{-1}_{t+1}F_{t+1})(F^{-1}_{t+2}F_{t+2})\cdots(F^{-1}_k F_k)$$

and hence

$$gF_k(F^{-1}_{k+1}F_{k+1})\cdots(F^{-1}_{n-1}F_{n-1}) \subseteq \delta^n_t F_t (F^{-1}_{t+1}F_{t+1})\cdots(F^{-1}_{n-1}F_{n-1}).$$

However, by definition $\delta^n_t F_t \subseteq B^n_t$. So the right hand side of the expression above is contained within H_n, and therefore $gF_k \subseteq B^n_k$. Thus

$$gF_k \subseteq B^n_k - \bigcup_{k<m<n} \delta^n_m F_m.$$

It now follows from the definition of δ^n_k that $gF_k \cap \delta^n_k F_k \neq \varnothing$. This substantiates our claim.

The collection $(\delta_k^n)_{k<n}$ was useful in constructing $(F_n)_{n\in\mathbb{N}}$ but is inadequate for our further needs. For $k \le n$ we wish to recognize exactly how translates of F_k were both explicitly and implicitly used in constructing F_n. For example, for $k < m < n$ we have $\delta_k^m F_k \subseteq F_m$ and $\delta_m^n F_m \subseteq F_n$ so $\delta_m^n \delta_k^m F_k \subseteq F_n$. Thus informally we would say the $\delta_m^n\delta_k^m$-translates of F_k were implicitly used in constructing F_n. However if for $g \in F_n$ we only have $gF_k \subseteq F_n$ we would not necessarily want to say the g-translate of F_k was used in constructing F_n. Hopefully we have made the point that we only wish to consider translates which, in some sense, were either explicitly or implicitly used. Informally, we wish to define D_k^n to be the set of all γ's in F_n such that the γ-translate of F_k was used in constructing F_n. We now give the formal definition for this. For $k \in \mathbb{N}$ define $D_k^k = \{1_G\}$, $D_k^{k+1} = \delta_k^{k+1}$, and in general for $n > k$

$$D_k^n = \delta_{n-1}^n D_k^{n-1} \cup \delta_{n-2}^n D_k^{n-2} \cup \cdots \cup \delta_{k+1}^n D_k^{k+1} \cup \delta_k^n = \bigcup_{k \le m < n} \delta_m^n D_k^m.$$

The D_k^n's are a discrete version of the Δ_n's which we will soon construct. First we must spend the next few paragraphs proving that the D_k^n's possess the following properties for all $k, m, n \in \mathbb{N}$ with $k \le m \le n$:

(1) $D_k^n F_k \subseteq F_n$;

(2) $D_m^n D_k^m \subseteq D_k^n$;

(3) the D_k^n-translates of F_k are disjoint;

(4) the D_k^n-translates of F_k are maximally disjoint with B_k^n.

(Proof of 1) Clearly $D_k^k F_k = F_k$. If we assume $D_k^i F_k \subseteq F_i$ for all $k \le i < n$, then

$$D_k^n F_k = \bigcup_{k \le i < n} \delta_i^n D_k^i F_k \subseteq \bigcup_{k \le i < n} \delta_i^n F_i \subseteq F_n.$$

The claim now immediately follows from induction.

(Proof of 2) Clearly when $n = m$ we have $D_m^n D_k^m = D_n^n D_k^n = D_k^n$. If we assume $D_m^i D_k^m \subseteq D_k^i$ for all $m \le i < n$, then

$$D_m^n D_k^m = \bigcup_{m \le i < n} \delta_i^n D_m^i D_k^m \subseteq \bigcup_{m \le i < n} \delta_i^n D_k^i \subseteq \bigcup_{k \le i < n} \delta_i^n D_k^i = D_k^n.$$

The claim now immediately follows from induction.

(Proof of 3) The D_k^n-translates of F_k are disjoint when $n = k$ and when $n = k+1$. Assume the D_k^i translates of F_k are disjoint for all $k \leq i < n$. Recall $D_k^n = \bigcup_{k \leq i < n} \delta_i^n D_k^i$. If $k \leq r < s < n$, then by the definition of δ_r^n we have $\delta_r^n F_r \cap \delta_s^n F_s = \varnothing$. It then follows from (1) that $\delta_r^n D_k^r F_k \cap \delta_s^n D_k^s F_k = \varnothing$. Additionally, if $k \leq i < n$ and $\gamma, \psi \in \delta_i^n$ are distinct, then $\gamma F_i \cap \psi F_i = \varnothing$ because the δ_i^n-translates of F_i are disjoint by definition. Again by (1) we have $\gamma D_k^i F_k \cap \psi D_k^i F_k = \varnothing$. Finally, by assumption the D_k^i-translates of F_k are disjoint for every $k \leq i < n$. It follows that the D_k^n-translates of F_k must be disjoint. The claim now follows from induction.

(Proof of 4) Here one will see exactly why the B_k^n's were defined. When $n = k$ and $n = k+1$ claim is guaranteed by definition (we take $B_n^n = B_{n-1}^n = H_n$). So fix $k \in \mathbb{N}$ and towards a contradiction suppose $n > k+1$ is such that the D_k^n-translates of F_k are not maximally disjoint within B_k^n. Fix $g \in B_k^n$ such that $gF_k \subseteq B_k^n$ and $gF_k \cap D_k^n F_k = \varnothing$. Our argument will rely on inductively creating a finite sequence of natural numbers. We first detail how the starting number v_0 is determined. Recall that in the construction of F_n we defined δ_{n-1}^n through δ_{k+1}^n first and then chose δ_k^n so that its translates of F_k would be maximally disjoint within $B_k^n - \bigcup_{k<m<n} \delta_m^n F_m$. However, $\delta_k^n = \delta_k^n D_k^k \subseteq D_k^n$ so $gF_k \cap \delta_k^n F_k = \varnothing$. So we cannot have $gF_k \subseteq B_k^n - \bigcup_{k<m<n} \delta_m^n F_m$ as this would violate the definition of δ_k^n. Since $gF_k \subseteq B_k^n$, we must have $gF_k \cap (\bigcup_{k<m<n} \delta_m^n F_m) \neq \varnothing$. Therefore there exists $v_0 \in \mathbb{N}$ with $k < v_0 < n$ and $\gamma_0 \in \delta_{v_0}^n$ such that $gF_k \cap \gamma_0 F_{v_0} \neq \varnothing$. Note that $\gamma_0 D_k^{v_0} \subseteq \delta_{v_0}^n D_k^{v_0} \subseteq D_k^n$ so $\gamma_0^{-1} gF_k \cap D_k^{v_0} F_k = \gamma_0^{-1}(gF_k \cap \gamma_0 D_k^{v_0} F_k) = \varnothing$. For notational convenience, we will set $v_{-1} = n$. Now assume v_0 through v_{i-1} have been defined and $\gamma_j \in \delta_{v_j}^{v_{j-1}}$ has been fixed for each $0 \leq j \leq i-1$ such that

(a) $n > v_0 > v_1 > \cdots > v_{i-1} > k$,

(b) $(\gamma_0 \gamma_1 \cdots \gamma_{i-1})^{-1} gF_k \cap F_{v_{i-1}} \neq \varnothing$, and

(c) $(\gamma_0 \gamma_1 \cdots \gamma_{i-1})^{-1} gF_k \cap D_k^{v_{i-1}} F_k = \varnothing$.

We will find a new number v_i and from here the sequence may either terminate or continue further. By (b) and our earlier comment on the B_r^s's, we have that there is $k \leq v_i < v_{i-1}$

and $\gamma_i \in \delta_{v_i}^{v_{i-1}}$ with

$$(\gamma_0\gamma_1 \cdots \gamma_{i-1})^{-1} gF_k \cap \gamma_i F_{v_i} \neq \varnothing.$$

If $v_i > k$ then $\gamma_i D_k^{v_i} F_k \subseteq \delta_{v_i}^{v_{i-1}} D_k^{v_i} F_k \subseteq D_k^{v_{i-1}} F_k$ which, together with the induction hypothesis (c), shows that (c) is again satisfied. Therefore if $v_i > k$ then (a), (b), and (c) are satisfied and we can continue this construction further. As we are constructing a strictly decreasing sequence with initial term $k < v_0 < n$, we will eventually have $v_i = k$. But if $v_i = k$ then we would have $\gamma_i F_{v_i} \subseteq \delta_k^{v_{i-1}} F_k = \delta_k^{v_{i-1}} D_k^k F_k \subseteq D_k^{v_{i-1}} F_k$ which contradicts (c) of the induction hypothesis. This completes the proof of (4).

Considering (4), in particular we have for all $n > k$ the D_k^n-translates of F_k are maximally disjoint within H_{n-1} since $H_{n-1} \subseteq B_k^n$. We remark that $D_k^n \subseteq D_k^{n+1}$ since $\delta_n^{n+1} D_k^n \subseteq D_k^{n+1}$ and $1_G \in \delta_n^{n+1}$. For $k \in \mathbb{N}$ we define $\Delta_k = \bigcup_{n \geq k} D_k^n$. As $\bigcup_{n \in \mathbb{N}} H_n = G$, we have for each $k \in \mathbb{N}$ the Δ_k-translates of F_k are maximally disjoint within G. Properties (iii) and (iv) are immediately satisfied.

To finish the proof we will show that $\Delta_k f \cap \gamma F_n = \gamma D_k^n f$ for $n \geq k$, $\gamma \in \Delta_n$, and $f \in F_k$. Since $1_G \in \Delta_n$ and $1_G \in F_k$, this will give $F_n \cap \Delta_k = 1_G F_n \cap \Delta_k 1_G = D_k^n$. Conclusion (viii) will then be clear. Conclusion (v) will follow from the definition of $\delta_{n-1}^n = D_{n-1}^n$, and conclusion (vi) will follow from (4) together with the fact that $H_{n-1} \subseteq B_k^n$. For (vii) just note that if $\gamma \in \Delta_k$ then $\gamma \in D_k^n$ for some $n \geq k$ and hence $\gamma F_k \subseteq F_n$ by (1).

Fix $k \leq n$, $f \in F_k$, and $\gamma \in \Delta_n$. Then $\gamma \in D_n^s$ for some $s \geq n$. So

$$\gamma D_k^n f \subseteq D_n^s D_k^n f \subseteq D_k^s f \subseteq \Delta_k f$$

and $\gamma D_k^n f \subseteq \gamma D_k^n F_k \subseteq \gamma F_n$. Therefore $\gamma D_k^n f \subseteq \Delta_k f \cap \gamma F_n$.

For the opposite inclusion, let $\psi \in \Delta_k$ with $\psi f \in \gamma F_n$. Let $s \geq n$ be large enough so that $\psi \in D_k^s$ and $\gamma \in D_n^s$. We will prove $\psi f \in \gamma D_k^n f$ by induction on s. Clearly, if $s = n$ then $\gamma = 1_G$ and $\psi f \in D_k^n f = \gamma D_k^n f$. Now suppose the claim is true for all $n \leq r < s$. By the definition of D_k^s and D_n^s, there are $k \leq i < s$ and $n \leq t < s$ with $\psi \in \delta_i^s D_k^i$ and $\gamma \in \delta_t^s D_n^t$. However, if $i \neq t$ then by the definition of δ_i^s and δ_t^s we have

$$\{\psi f\} \cap \gamma F_n \subseteq \delta_i^s D_k^i f \cap \delta_t^s D_n^t F_n \subseteq \delta_i^s F_i \cap \delta_t^s F_t = \varnothing.$$

So it must be that $i = t$. Let $\lambda, \sigma \in \delta_t^s$ be such that $\psi \in \lambda D_k^t$ and $\gamma \in \sigma D_n^t$. If $\lambda \neq \sigma$ then we would have

$$\{\psi f\} \cap \gamma F_n \subseteq \lambda D_k^t f \cap \sigma D_n^t F_n \subseteq \lambda F_t \cap \sigma F_t = \varnothing.$$

So we must have $\lambda = \sigma$. Then $\lambda^{-1}\psi \in D_k^t \subseteq \Delta_k$, $\lambda^{-1}\gamma \in D_n^t \subseteq \Delta_n$, and $\lambda^{-1}\psi f \in \lambda^{-1}\gamma F_n$. By the induction hypothesis we conclude $\lambda^{-1}\psi f \in \lambda^{-1}\gamma D_k^n f$ and hence $\psi f \in \gamma D_k^n f$. This completes the proof. $\square$

Definition 3.6. Let G be a countably infinite group and $(H_n)_{n\in\mathbb{N}}$ an increasing sequence of finite sets with $1_G \in H_0$, $\bigcup_{n\in\mathbb{N}} H_n = G$, and

$$H_{n-1}(H_0^{-1}H_0)(H_1^{-1}H_1)\cdots(H_{n-1}^{-1}H_{n-1}) \subseteq H_n$$

for $n \geq 1$. Let $(F_n)_{n\in\mathbb{N}}$ and $(\Delta_n)_{n\in\mathbb{N}}$ be respectively increasing and decreasing sequences of subsets of G satisfying the conclusions of Proposition 3.5. Set $\alpha_0 = \beta_0 = 1_G$. If for each $n \geq 1$ there are distinct non-identity $\alpha_n, \beta_n \in \Delta_{n-1} \cap F_n$, then we call $(H_n, F_n, \Delta_n, \alpha_n, \beta_n)_{n\in\mathbb{N}}$ a *blueprint*.

Remark 3.7. Whenever we have a blueprint $(H_n, F_n, \Delta_n, \alpha_n, \beta_n)_{n\in\mathbb{N}}$, the following symbols will have a fixed meaning:

$$\Lambda_n = (\Delta_{n-1} \cap F_n) - \{1_G, \alpha_n, \beta_n\} \text{ (for } n \geq 1);$$

$$a_n = \alpha_n\alpha_{n-1}\cdots\alpha_0 \text{ (for } n \in \mathbb{N});$$

$$b_n = \beta_n\beta_{n-1}\cdots\beta_0 \text{ (for } n \in \mathbb{N}).$$

The following lemma consists of a collection of facts which are easily derived from Proposition 3.5. These facts are frequently needed in the next section. This lemma serves the purpose of collecting these statements together for easy reference for those times when one's intuition or memory is faltering.

Lemma 3.8. *Let G be a countably infinite group and let $(H_n, F_n, \Delta_n, \alpha_n, \beta_n)_{n\in\mathbb{N}}$ be a blueprint. Then*

(i) $\lambda F_{n-1} \subseteq F_n$ *for all* $n \geq 1$ *and all* $\lambda \in \Delta_{n-1} \cap F_n$*;*

(ii) $\lambda_1 F_{n-1} \cap \lambda_2 F_{n-1} = \varnothing$ *for all* $n \geq 1$ *and distinct* $\lambda_1, \lambda_2 \in \Delta_{n-1} \cap F_n$*;*

(iii) $\Delta_n \lambda \subseteq \Delta_{n-1}$ *for all* $n \geq 1$ *and all* $\lambda \in \Delta_{n-1} \cap F_n$*;*

(iv) *both* $\Delta_n a_n$ *and* $\Delta_n b_n$ *are decreasing sequences;*

(v) $a_n, b_n \in F_n$ *for all* $n \in \mathbb{N}$*;*

(vi) $a_n \neq b_n$ *for all* $n \geq 1$*;*

(vii) $\Delta_n a_n \cap \Delta_k b_k = \varnothing$ *for all* $n, k > 0$*;*

(viii) *for* $n > k \in \mathbb{N}$

$$\Delta_n(\Delta_{n-1} \cap F_n)a_{n-1} \cap \Delta_k F_k \subseteq \Delta_k a_k$$

and

$$\Delta_n(\Delta_{n-1} \cap F_n)b_{n-1} \cap \Delta_k F_k \subseteq \Delta_k b_k;$$

(ix) *for* $n > k \in \mathbb{N}$

$$\Delta_n(\Delta_{n-1} \cap F_n)a_{n-1} \cap \Delta_k(\Delta_{k-1} \cap F_k)a_{k-1}$$

$$\subseteq \Delta_k \alpha_k a_{k-1} = \Delta_k a_k$$

and

$$\Delta_n(\Delta_{n-1} \cap F_n)b_{n-1} \cap \Delta_k(\Delta_{k-1} \cap F_k)b_{k-1}$$

$$\subseteq \Delta_k \beta_k b_{k-1} = \Delta_k b_k;$$

(x) $\bigcap_{n\in\mathbb{N}} \Delta_n a_n = \bigcap_{n\in\mathbb{N}} \Delta_n b_n = \varnothing$.

PROOF. (i). By conclusion (viii) of Proposition 3.5

$$\lambda F_{n-1} \subseteq (\Delta_{n-1} \cap F_n)F_{n-1} = \Delta_{n-1}F_{n-1} \cap F_n \subseteq F_n.$$

(ii). λ_1 and λ_2 are in Δ_{n-1}, and the Δ_{n-1}-translates of F_{n-1} are disjoint.

(iii). By conclusion (viii) of Proposition 3.5,

$$\Delta_n \lambda \subseteq \Delta_n(\Delta_{n-1} \cap F_n) = \Delta_{n-1} \cap \Delta_n F_n \subseteq \Delta_{n-1}.$$

(iv). By (iii), $\Delta_n a_n = \Delta_n \alpha_n a_{n-1} \subseteq \Delta_{n-1} a_{n-1}$. The same argument applies to $\Delta_n b_n$.

(v). Clearly $a_0 \in F_0$. If we assume $a_{n-1} \in F_{n-1}$, then by (i)

$$a_n = \alpha_n a_{n-1} \in \alpha_n F_{n-1} \subseteq F_n.$$

By induction, and by a similar argument, we have $a_n, b_n \in F_n$ for all $n \in \mathbb{N}$.

(vi). From (v) we have that $a_n = \alpha_n a_{n-1} \in \alpha_n F_{n-1}$. Similarly, $b_n \in \beta_n F_{n-1}$. Since $1_G \in F_{n-1}$, the claim follows from (ii).

(vii). Suppose $0 < k \leq n$. Since $a_k \neq b_k \in F_k$, and since the Δ_k-translates of F_k are disjoint, from (iv) and (vi) we have

$$\Delta_n a_n \cap \Delta_k b_k \subseteq \Delta_k a_k \cap \Delta_k b_k = \varnothing.$$

The case $0 < n \leq k$ is identical.

(viii). Let $n, k \in \mathbb{N}$ with $n > k$. Suppose $\gamma \in \Delta_n$, $\psi \in \Delta_k$, $\lambda \in \Delta_{n-1} \cap F_n$, and $f \in F_k$ satisfy

$$\gamma \lambda a_{n-1} = \psi f.$$

Then by repeated application of (iii),

$$g = \gamma \lambda \alpha_{n-1} \alpha_{n-2} \ldots \alpha_{k+1} \in \Delta_k.$$

Since $g a_k = \gamma \lambda a_{n-1} = \psi f$, we have $g F_k \cap \psi F_k \neq \varnothing$. As the Δ_k-translates of F_k are disjoint, we must have $g = \psi$. Thus

$$\psi a_k = g a_k = \gamma \lambda a_{n-1} = \psi f \Longrightarrow a_k = f.$$

With b_n in place of a_n the argument is essentially identical.

(ix). This follows immediately from (v), (i), (viii), and the definition of a_k and b_k.

(x). The Δ_n-translates of F_n are disjoint and $F_{n-1} \subseteq F_n$, so by (v)

$$\Delta_n a_n \cap F_{n-1} \subseteq \Delta_n a_n \cap F_n = \{a_n\}.$$

From (ii) and (v) we have $a_n = \alpha_n a_{n-1} \notin F_{n-1}$ since $1_G \in \Delta_{n-1} \cap F_n$. Therefore $\Delta_n a_n \cap F_{n-1} = \varnothing$. Now choose any $g \in \Delta_0 a_0$. Then $g \in \Delta_0$ since $a_0 = 1_G$. By conclusion (vii) of Proposition 3.5, there is $n \in \mathbb{N}$ with $g F_0 \subseteq F_n$. In particular, $g \in F_n$ since $1_G \in F_0$. As

$\Delta_{n+1}a_{n+1} \cap F_n = \varnothing$, we must have $g \notin \Delta_{n+1}a_{n+1}$. It follows $\bigcap_{n\in\mathbb{N}} \Delta_n a_n = \varnothing$. By an identical argument $\bigcap_{n\in\mathbb{N}} \Delta_n b_n = \varnothing$ as well. $\square$

CHAPTER 4

CONSTRUCTION OF SUBFLOWS

This section primarily consists of two long proofs. The first of these is Proposition 4.5 below. This proposition presents a rather general method for constructing elements of 2^G with desirable properties. Definitions 4.1 and 4.4 below play a significant role in deriving the main theorem from Proposition 4.5.

DEFINITION 4.1. Let G be a group, let $1_G \in A \subseteq G$ with A finite, and let $R : A \to \{0,1\}$. We call R *locally recognizable* if for every $x \in 2^G$ with $x|_A = R$

$$\forall a \in A\ (\forall b \in A\ x(ab) = x(b) \Longrightarrow a = 1_G).$$

R is called *trivial* if $|\{a \in A \mid R(a) = R(1_G)\}| = 1$.

DEFINITION 4.2. Let G be a countably infinite group and let $(H_n, F_n, \Delta_n, \alpha_n, \beta_n)_{n\in\mathbb{N}}$ be a blueprint. If $R : A \to \{0,1\}$ is locally recognizable, then we say $(H_n, F_n, \Delta_n, \alpha_n, \beta_n)_{n\in\mathbb{N}}$ is *compatible* with R if $H_0 = A$. If $(p_n)_{n\geq 1}$ is a sequence of functions of subexponential growth, then we say $(H_n, F_n, \Delta_n, \alpha_n, \beta_n)_{n\in\mathbb{N}}$ is *compatible* with $(p_n)_{n\geq 1}$ if $\rho(H_n; H_{n-1}) > 3 + \log_2\ p_n(|H_n|)$ for each $n \geq 1$.

LEMMA 4.3. *If G is a countably infinite group, $R : A \to \{0,1\}$ is locally recognizable, and $(p_n)_{n\geq 1}$ is a sequence of functions of subexponential growth, then there exists a blueprint compatible with R and $(p_n)_{n\geq 1}$.*

PROOF. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of G with $A_0 = A$ and $G = \bigcup_{n\in\mathbb{N}} A_n$. Set $H_0 = A_0$. Once H_0 through H_{n-1} have been defined, apply Lemma 3.4 to find a finite $H_n \subseteq G$ satisfying

$$H_n \supseteq H_{n-1}(H_0^{-1}H_0)(H_1^{-1}H_1)\cdots(H_{n-1}^{-1}H_{n-1})$$

and $\rho(H_n; H_{n-1}) > \log_2\ (8p_n(|H_n|))$. The sequence $(H_n)_{n\in\mathbb{N}}$ will then be compatible with R and $(p_n)_{n\geq 1}$. □

Definition 4.4. Let G be a group, $c \in 2^{<G} \cup 2^G$, and $\Delta \subseteq G$. We say c *admits an extension invariant* Δ *membership test* if there is a finite $V \subseteq G$ with $\Delta V \subseteq \mathrm{dom}(c)$ such that for all $x \in 2^G$ with $x \supseteq c$ and all $\gamma \in \Delta$

$$\{g \in G \mid \forall v \in V \ x(gv) = x(\gamma v)\} = \Delta.$$

Such a set V is called a *test region*.

Proposition 4.5. *Let G be a countably infinite group, let $R : A \to \{0,1\}$ be a non-trivial locally recognizable function, let $(p_n)_{n\geq 1}$ be a collection of functions of subexponential growth, and let $(H_n, F_n, \Delta_n, \alpha_n, \beta_n)_{n\in\mathbb{N}}$ be a blueprint compatible with R and $(p_n)_{n\geq 1}$. Then there exists $c \in 2^{<G}$ with the following properties:*

(i) $(\gamma^{-1} \cdot c)|_A = R$ *for all* $\gamma \in \Delta_1$*;*

(ii) c *admits an extension invariant* Δ_n *membership test with test region a subset of* $F_{n-1} \cap \mathrm{dom}(c)$ *for each* $n \in \mathbb{N}$*;*

(iii) $G - \mathrm{dom}(c)$ *is the disjoint union* $\bigcup_{n\geq 1} \Delta_n \Lambda_n b_{n-1}$*;*

(iv) $|\Lambda_n| > \log_2 \ p_n(|H_n|)$ *for all* $n \geq 1$*;*

(v) $c(g) = 1 - R(1_G)$ *for all* $g \in G - \Delta_1 F_1$*;*

(vi) $c(f) = c(\gamma f)$ *for all* $n \geq 1$*,* $\gamma \in \Delta_n$*, and*

$$f \in F_n - \{a_n, b_n\} - \bigcup_{1\leq k\leq n} \Delta_k \Lambda_k b_{k-1} = (F_n - \{a_n, b_n\}) \cap \mathrm{dom}(c);$$

Proof. By (v) of Proposition 3.5, the $\Delta_{n-1} \cap F_n$-translates of F_{n-1} are contained and maximally disjoint within H_n. So

$$|\Lambda_n| + 3 = |\Delta_{n-1} \cap F_n| \geq \rho(H_n; F_{n-1}) \geq \rho(H_n; H_{n-1}) > \log_2 \ p_n(|H_n|) + 3$$

as $F_{n-1} \subseteq H_{n-1}$. Thus property (iv) is satisfied.

We wish to construct a sequence of functions $(c_n)_{n\geq 1}$ satisfying for each $n \geq 1$:

(1) $\mathrm{dom}(c_n) = G - \Delta_n a_n - \Delta_n b_n - \bigcup_{1\leq k\leq n} \Delta_k \Lambda_k b_{k-1}$

(2) $c_{n+1} \supseteq c_n$;

(3) c_n admits an extension invariant Δ_n membership test with test region a subset of $F_{n-1} \cap \mathrm{dom}(c_n)$.

Let us first dwell for a moment on (1). Condition (1) is consistent with condition (2) because $\Delta_n a_n$ and $\Delta_n b_n$ are decreasing sequences and $\Delta_{n+1}\Lambda_{n+1}b_n \subseteq \Delta_n b_n$ by conclusion (iii) of Lemma 3.8. After considering conclusions (vii) and (viii) of Lemma 3.8, we see that for $n > 1$ we desire $\mathrm{dom}(c_n)$ to be

$$\mathrm{dom}(c_{n-1}) \cup (\Delta_{n-1}a_{n-1} - \Delta_n a_n) \cup (\Delta_{n-1}b_{n-1} - \Delta_n[\Lambda_n \cup \{\beta_n\}]b_{n-1}).$$

It is important to note that these unions are disjoint. This tells us that given c_{n-1}, we can define $c_n \supseteq c_{n-1}$ to have whichever values on $\Delta_{n-1}a_{n-1} - \Delta_n a_n$ and $\Delta_{n-1}b_{n-1} - \Delta_n[\Lambda_n \cup \{\beta_n\}]b_{n-1}$ without worry of a contradiction between the two or with c_{n-1}.

Define

$$c_1 : (G - \Delta_1 a_1 - \Delta_1 b_1 - \Delta_1 \Lambda_1) \to \{0, 1\}$$

by

$$c_1(g) = \begin{cases} R(f) & \text{if } g = \gamma f \text{ where } \gamma \in \Delta_1 \text{ and } f \in A \\ 1 - R(1_G) & \text{otherwise} \end{cases}$$

for $g \in \mathrm{dom}(c_1)$. Note that c_1 satisfies (1) since $b_0 = 1_G$.

We claim c_1 satisfies (3) with test region A. Since Δ_1, $\Delta_1 a_1$, $\Delta_1 b_1$, and $\Delta_1\Lambda_1$ are pairwise disjoint subsets of Δ_0, we have that $\Delta_1 A$ is disjoint from $\Delta_1 a_1 \cup \Delta_1 b_1 \cup \Delta_1 \Lambda_1$ (since $1_G \in A = F_0$). Thus $\Delta_1 A \subseteq \mathrm{dom}(c_1)$ and $A \subseteq F_0 \cap \mathrm{dom}(c_1)$ as required.

To finish verifying condition (3), we let $c \in 2^G$ be an arbitrary extension of c_1. Since $1_G \in \Delta_1$, it is enough to show $g \in \Delta_1$ if and only if $c(ga) = c(a)$ for all $a \in A$. If $\gamma \in \Delta_1$, then $\gamma A \subseteq \mathrm{dom}(c_1)$ (see previous paragraph) and hence $c(\gamma a) = R(a) = c(a)$ for all $a \in A$. Now suppose $g \in G$ satisfies $c(ga) = c(a)$ for all $a \in A$. This implies $(g^{-1} \cdot c)|_A = R$. Note that $c(h) = c_1(h) = 1 - R(1_G)$ for all $h \in \mathrm{dom}(c_1) - \Delta_1 A$. As $c(g) = c(1_G) = c_1(1_G) = R(1_G)$, either $g \in \Delta_1 A$ or $g \notin \mathrm{dom}(c_1)$. But g cannot be in $G - \mathrm{dom}(c_1) \subseteq \Delta_0$, for then since the

Δ_0-translates of F_0 are disjoint we would have

$$gA - \{g\} = gF_0 - \{g\} \subseteq \mathrm{dom}(c_1) - \Delta_1 F_0 = \mathrm{dom}(c_1) - \Delta_1 A.$$

Then $c(ga) = c_1(ga) = 1 - R(1_G)$ for all $1_G \neq a \in A$ and $R = (g^{-1} \cdot c)|_A$ would be trivial, a contradiction. So $g \in \Delta_1 A$. Let $\gamma \in \Delta_1$ and $a \in A$ be such that $g = \gamma a$. By construction, $(\gamma^{-1} \cdot c)|_A = (\gamma^{-1} \cdot c_1)|_A = R$, and we have that for all $b \in A$

$$(\gamma^{-1} \cdot c)(ab) = c(\gamma ab) = c(gb) = c(b) = R(b).$$

Therefore, it follows from the definition of a locally recognizable function that $a = 1_G$ and $g \in \Delta_1$.

Now suppose that $c_1, c_2, \ldots c_{k-1}$ have been constructed and satisfy (1) through (3). We pointed out earlier that we desire c_k to have domain

$$\mathrm{dom}(c_{k-1}) \cup (\Delta_{k-1} a_{k-1} - \Delta_k a_k) \cup (\Delta_{k-1} b_{k-1} - \Delta_k [\Lambda_k \cup \{\beta_k\}] b_{k-1}).$$

We define c_k to satisfy $c_k \supseteq c_{k-1}$ and:

$$c_k(\Delta_{k-1} a_{k-1} - \Delta_k \{1_G, \alpha_k\} a_{k-1}) = \{0\};$$

$$c_k(\Delta_k a_{k-1}) = \{1\};$$

$$c_k(\Delta_k b_{k-1}) = \{1\};$$

$$c_k(\Delta_k \alpha_k b_{k-1}) = \{0\};$$

$$c_k(\Delta_{k-1} b_{k-1} - \Delta_k [\Delta_{k-1} \cap F_k] b_{k-1}) = \{0\}.$$

From our earlier remarks on (1), we know c_k is well defined. It is easily checked that c_k satisfies (1) and (2) (recall that $\Delta_k a_k = \Delta_k \alpha_k a_{k-1}$).

Let $V \subseteq F_{k-2} \cap \mathrm{dom}(c_{k-1})$ be the test region referred to in (3) for $n = k - 1$. We claim that c_k satisfies (3) with test region $W = V \cup \{a_{k-1}, b_{k-1}\}$. Clearly $W \subseteq F_{k-1} \cap \mathrm{dom}(c_k)$. Also, $\Delta_k V \subseteq \Delta_{k-1} V \subseteq \mathrm{dom}(c_{k-1}) \subseteq \mathrm{dom}(c_k)$, and clearly $\Delta_k \{a_{k-1}, b_{k-1}\} \subseteq \mathrm{dom}(c_k)$. Thus $\Delta_k W \subseteq \mathrm{dom}(c_k)$ as required.

Let $c \in 2^G$ be an arbitrary extension of c_k. Since $1_G \in \Delta_k$, it suffices to show that for all $g \in G$,

$$g \in \Delta_k \iff \forall w \in W \;\; c(gw) = c(w).$$

Suppose $g \in G$ satisfies the condition on the right. Then for all $w \in V$ we have $c(gw) = c(w)$, and since $1_G \in \Delta_{k-1}$ we must have $g \in \Delta_{k-1}$. Now $c(ga_{k-1}) = c(a_{k-1}) = c_k(a_{k-1}) = 1$, and $ga_{k-1} \in \Delta_{k-1}a_{k-1} \subseteq G - \mathrm{dom}(c_{k-1})$. From how we defined c_k, we have for $h \in G$

$$h \in \Delta_{k-1} \text{ and } c(ha_{k-1}) = 1 \Longrightarrow h \in \Delta_k \text{ or } h \in \Delta_k \alpha_k.$$

However, $g \notin \Delta_k \alpha_k$, for then $c(gb_{k-1}) = c_k(gb_{k-1}) = 0 \neq 1 = c(b_{k-1})$. We conclude $g \in \Delta_k$. The converse, that each $\gamma \in \Delta_k$ satisfies $c(\gamma w) = c(w)$ for all $w \in W$, is easy to check (recall $\Delta_k \subseteq \Delta_{k-1}$). Thus c_k satisfies (3).

Finally, define $c = \bigcup_{n \geq 1} c_n$. Properties (i) and (v) clearly hold due to how c_1 was defined. Property (iv) was verified near the beginning of the proof. Property (ii) holds since $c \supseteq c_n$ for each $n \geq 1$, and property (iii) follows from (1) and conclusions (ix) and (x) of Lemma 3.8. We proceed to verify property (vi).

The second equality in (vi) is easy to verify since $F_n - \{b_n\}$ is disjoint from $\Delta_k \Lambda_k b_{k-1}$ for $k > n$ (use conclusion (viii) of Lemma 3.8). Fix $n \geq 1$, $\gamma \in \Delta_n$, and $f \in (F_n - \{a_n, b_n\}) \cap \mathrm{dom}(c)$. Then $f \notin \{a_n, b_n\}$ and hence $f, \gamma f \notin \Delta_n\{a_n, b_n\}$ since the Δ_n-translates of F_n are disjoint. However, $\mathrm{dom}(c_m) - \mathrm{dom}(c_n) \subseteq \Delta_n\{a_n, b_n\}$ for $m > n$, and since $f, \gamma f \in \mathrm{dom}(c)$ there must be a $m \leq n$ with $f, \gamma f \in \mathrm{dom}(c_m)$. Let $k \leq n$ be minimal with either f or γf in $\mathrm{dom}(c_k)$. We proceed by cases.

<u>Case 1</u>: $k = 1$. Let B be a subset of F_1. From conclusion (viii) of Proposition 3.5, we have

$$f \in \Delta_1 B \iff f \in \Delta_1 B \cap F_n \iff \gamma f \in \gamma(\Delta_1 B \cap F_n)$$

$$\iff \gamma f \in \Delta_1 B \cap \gamma F_n \iff \gamma f \in \Delta_1 B_1.$$

By looking back at the definition of c_1 and using the above result for different choices of $B \subseteq F_0$, we concluded that both f and γf are in $\mathrm{dom}(c_1)$. More importantly, $c(\gamma f) = c_1(\gamma f) = c_1(f) = c(f)$.

Case 2: $1 < k \leq n$. By the same computation as above, for each $i \leq n$ and each $B \subseteq F_i$ we have that $f \in \Delta_i B$ if and only if $\gamma f \in \Delta_i B$. It follows that both $f, \gamma f \in \mathrm{dom}(c_k)$. After considering the five equations defining c_k and using the above note, we conclude $c(\gamma f) = c_k(\gamma f) = c_k(f) = c(f)$. □

We are now ready to prove the main theorem. Just as the proof of Proposition 4.5 relied heavily on Proposition 3.5, the proof of this theorem will rely heavily on Proposition 4.5. Conclusion (i) is used for density. Conclusions (ii), (iii), and (iv) are used to get 2-colorings. While conclusions (v) and (vi) aid in achieving minimality, minimality is actually quite tedious to get, and it is the requirement of minimality which makes this proof so long.

THEOREM 4.6. *Let G be a countably infinite group, $x \in 2^G$, and $\epsilon > 0$. Then there is a perfect set of pairwise orthogonal minimal 2-colorings in the ϵ-ball about x.*

PROOF. Let $r \in \mathbb{N}$ be such that $2^{-r} < \epsilon$ and let $B_1 = \{g_0, g_1, \ldots, g_r\}$ where $g_0, g_1, \ldots$ is the fixed enumeration of G used in defining the metric d on 2^G. Choose any $a \neq b \in G - B_1$ and set $B_2 = B_1 \cup \{a, b\}$. Next chose any $c \in G - (B_2 B_2 \cup B_2 B_2^{-1})$ and set $B_3 = B_2 \cup \{c\} = B_1 \cup \{a, b, c\}$. Let $A = B_3 B_3$ and define $R : A \to \{0, 1\}$ by

$$R(g) = \begin{cases} x(g) & \text{if } g \in B_1 \\ x(1_G) & \text{if } g \in \{a, b, c\} \\ 1 - x(1_G) & \text{if } g \in A - B_3 \end{cases}$$

We claim R is a locally recognizable function (it is clearly non-trivial). Towards a contradiction suppose there is $y \in 2^G$ extending R and $1_G \neq g \in A$ with $y(gh) = y(h)$ for all $h \in A$. In particular, $y(g) = y(1_G) = R(1_G)$ so $g \in B_3$. We first point out that at least one of a, b, or c is not an element of gB_3. We prove this by cases. Case 1: $g \in B_2$. Then $c \notin gB_2 \subseteq B_2 B_2$ and $c \neq gc$ since $g \neq 1_G$. Thus $c \notin gB_3$. Case 2: $g \in B_3 - B_2 = \{c\}$. Then $g = c$. Since $c \notin B_2 B_2^{-1}$, it must be that $a, b \notin cB_2$. If $a, b \in cB_3$ then we must have $a = c^2 = b$, contradicting $a \neq b$. We conclude $\{a, b\} \not\subseteq cB_3 = gB_3$.

The key point now is that $\{a,b,c\} \subseteq \{h \in A \mid y(h) = y(1_G)\} \subseteq B_3$ but $\{a,b,c\} \not\subseteq gB_3 \subseteq A$. Therefore

$$|\{h \in B_3 \mid y(gh) = y(1_G)\}| < |\{h \in A \mid y(h) = y(1_G)\}|$$

$$= |\{h \in B_3 \mid y(h) = y(1_G)\}| = |\{h \in B_3 \mid y(gh) = y(1_G)\}|.$$

This is clearly a contradiction.

For $n \geq 1$ and $k \in \mathbb{N}$ define $p_n(k) = 4k^5$. Then $(p_n)_{n\geq 1}$ is a sequence of functions of subexponential growth. Apply Lemma 4.3 to get a blueprint $(H_n, F_n, \Delta_n, \alpha_n, \beta_n)_{n\in\mathbb{N}}$ compatible with R and $(p_n)_{n\geq 1}$, and let $c \in 2^{<G}$ be as in the conclusion of Proposition 4.5.

For each $n \geq 1$ let Γ_n be the graph with vertex set Δ_n and edge relation given by

$$(\gamma, \psi) \in E(\Gamma_n) \Longleftrightarrow \gamma^{-1}\psi \in H_n H_n^{-1} H_n^2 H_n^{-1} \text{ or } \psi^{-1}\gamma \in H_n H_n^{-1} H_n^2 H_n^{-1}.$$

This graph is not to have loops, so $(\gamma, \gamma) \notin E(\Gamma_n)$.

We proceed to reveal an important property the graphs $(\Gamma_n)_{n\geq 1}$. Recalling that the sequence $(H_n)_{n\in\mathbb{N}}$ satisfied certain conditions, we calculate

$$H_n H_n^{-1} H_n^2 H_n^{-1} = H_n H_n^{-1} H_n H_n H_n^{-1} \subseteq H_{n+1} H_n H_n^{-1}$$

$$\subseteq H_{n+1} H_{n+1} \subseteq H_{n+2} \subseteq H_{n+3},$$

and

$$(H_n H_n^{-1} H_n^2 H_n^{-1})^{-1} = H_n H_n^{-1} H_n^{-1} H_n H_n^{-1} \subseteq H_{n+1} H_n^{-1} H_{n+1}^{-1} H_n H_{n+1}^{-1}$$

$$\subseteq H_{n+2} H_n H_{n+1}^{-1} \subseteq H_{n+3}.$$

Therefore if $(\gamma, \psi) \in E(\Gamma_n)$, then either $\gamma \in \psi H_{n+3}$ or $\psi \in \gamma H_{n+3}$. Since $H_{n+3} \cup H_{n+3}^{-1} \subseteq H_{n+4}$, we have $(\gamma, \psi) \in E(\Gamma_n)$ implies $\gamma \in \psi H_{n+4}$ and $\psi \in \gamma H_{n+4}$.

Let $i \in \mathbb{N}$, $\sigma \in \Delta_{n+7+i}$, $\gamma \in \sigma F_{n+4+i}$, and suppose $(\gamma, \psi) \in E(\Gamma_n)$. Then

$$\psi \in \Delta_n \cap \gamma H_{n+4} \subseteq \Delta_n \cap \sigma H_{n+4+i} H_{n+4} \subseteq \Delta_n \cap \sigma H_{n+5+i}$$

and

$$\psi F_n \subseteq \sigma H_{n+5+i} F_n \subseteq \sigma H_{n+5+i} H_n \subseteq \sigma H_{n+6+i}.$$

By conclusions (vi) and (viii) of Proposition 3.5 the $(\Delta_n \cap \sigma F_{n+7+i})$-translates of F_n are maximally disjoint within σH_{n+6+i}. So $\psi F_n \cap (\Delta_n \cap \sigma F_{n+7+i}) F_n \neq \varnothing$. However, $\psi \in \Delta_n$ and the Δ_n-translates of F_n are disjoint. So we must have $\psi \in \sigma F_{n+7+i}$. We have demonstrated the following fact:

$$\forall i \in \mathbb{N}\ \forall \sigma \in \Delta_{n+7+i}\ [(\gamma, \psi) \in E(\Gamma_n) \text{ and } \gamma \in \sigma F_{n+4+i} \Longrightarrow \psi \in \sigma F_{n+7+i}].$$

In particular, one important aspect of this conclusion is that if $\lambda \neq \sigma \in \Delta_{n+7+i}$, $\gamma \in \lambda F_{n+4+i}$, and $\psi \in \sigma F_{n+4+i}$ then $(\gamma, \psi) \notin E(\Gamma_n)$.

Fix $n \geq 1$. Define $m_n(k) = n + 4 + 3k$ for $k \in \mathbb{N}$. We will construct a sequence of functions $(\mu_i^n)_{i \geq 1}$ mapping into $\{0, 1, \ldots, 2|H_n|^5\}$ satisfying for each $i \geq 1$:

(1) $\mu_{i+1}^n \supseteq \mu_i^n$;

(2) $\mathrm{dom}(\mu_i^n) = \Delta_n \cap (\bigcup_{0 \leq t < i} \Delta_{m(t+1)} F_{m(t)})$;

(3) $(\gamma, \psi) \in E(\Gamma_n) \Rightarrow \mu_i^n(\gamma) \neq \mu_i^n(\psi)$ whenever $\gamma, \psi \in \mathrm{dom}(\mu_i^n)$;

(4) $\mu_{i+1}^n(\gamma) = \mu_{i+1}^n(\sigma\gamma)$ for all $\sigma \in \Delta_{m(i+1)}$ and all $\gamma \in F_{m(i)} \cap \Delta_n$.

We begin by constructing μ_1^n. Since every vertex of Γ_n has degree at most $2|H_n|^5$, we can find a labeling of $F_{n+4} \cap \Delta_n$ using only the labels $\{0, 1, \ldots, 2|H_n|^5\}$ such that two members are labeled differently if they are $E(\Gamma_n)$-adjacent. Note that it is a simple consequence of conclusion (viii) of Proposition 3.5 that for $\gamma, \psi \in F_{n+4} \cap \Delta_n$ and $\sigma \in \Delta_{n+7}$

$$(\gamma, \psi) \in E(\Gamma_n) \Longleftrightarrow (\sigma\gamma, \sigma\psi) \in E(\Gamma_n).$$

We can therefore copy this labeling to every Δ_{n+7}-translate of $\Delta_n \cap F_{n+4}$ to get the function μ_1^n. Clearly properties (2) and (4) are satisfied. Property (3) also holds due to our earlier comment.

Now suppose μ_i^n has been constructed. Again we note that for $\sigma \in \Delta_{m(i+1)}$ and $\gamma, \psi \in \Delta_n \cap F_{m(i)}$, conclusion (viii) of Proposition 3.5 gives $\sigma\gamma, \sigma\psi \in \Delta_n$. Therefore for every $\sigma \in \Delta_{m(i+1)}$

$$(\gamma, \psi) \in E(\Gamma_n) \Longleftrightarrow (\sigma\gamma, \sigma\psi) \in E(\Gamma_n).$$

Let $\gamma \in \Delta_n \cap F_{m(i)}$ and let $\sigma \in \Delta_{m(i+1)}$. Then for every $0 \leq t < i$ we have

$$\gamma \in \Delta_{m(t+1)} F_{m(t)} \Longleftrightarrow \gamma \in \Delta_{m(t+1)} F_{m(t)} \cap F_{m(i)} \Longleftrightarrow \sigma\gamma \in \sigma(\Delta_{m(t+1)} F_{m(t)} \cap F_{m(i)})$$

$$\Longleftrightarrow \sigma\gamma \in \Delta_{m(t+1)} F_{m(t)} \cap \sigma F_{m(i)} \Longleftrightarrow \sigma\gamma \in \Delta_{m(t+1)} F_{m(t)}.$$

By (2) we conclude that $\gamma \in \mathrm{dom}(\mu_i^n)$ if and only if $\sigma\gamma \in \mathrm{dom}(\mu_i^n)$. Suppose it is the case that $\gamma \in \mathrm{dom}(\mu_i^n)$. Let $t < i$ and $\lambda \in \Delta_{m(t+1)}$ be such that $\gamma \in \lambda F_{m(t)}$. By conclusion (viii) of Proposition 3.5, there is $\psi \in \Delta_n \cap F_{m(t)}$ with $\gamma = \lambda\psi$. Since $\mu_i^n \supseteq \mu_{t+1}^n$, we have $\mu_i^n(\gamma) = \mu_i^n(\psi)$ by (4). By conclusion (viii) of Proposition 3.5, $\sigma\lambda \in \Delta_{m(t+1)}$. Therefore by (4) we have

$$\mu_i^n(\sigma\gamma) = \mu_i^n(\sigma\lambda\psi) = \mu_i^n(\psi) = \mu_i^n(\lambda\psi) = \mu_i^n(\gamma).$$

We have verified the three following facts for $\gamma, \psi \in F_{m(i)} \cap \Delta_n$:

$$(\gamma, \psi) \in E(\Gamma_n) \Longleftrightarrow \forall \sigma \in \Delta_{m(i+1)}\ (\sigma\gamma, \sigma\psi) \in E(\Gamma_n);$$

$$\gamma \in \mathrm{dom}(\mu_i^n) \Longleftrightarrow \forall \sigma \in \Delta_{m(i+1)}\ \sigma\gamma \in \mathrm{dom}(\mu_i^n);$$

$$\gamma \in \mathrm{dom}(\mu_i^n) \Longrightarrow \forall \sigma \in \Delta_{m(i+1)}\ \mu_i^n(\sigma\gamma) = \mu_i^n(\gamma).$$

By (4) we can find a $\{0, 1, \ldots, 2|H_n|^5\}$-labeling of $\Delta_n \cap F_{m(i)}$ which extends μ_i^n on $\mathrm{dom}(\mu_i^n) \cap F_{m(i)}$ with the property that if $(\gamma, \psi) \in E(\Gamma_n)$ then γ and ψ are labeled differently. We then copy this labeling to all $\Delta_{m(i+1)}$-translates of $F_{m(i)}$ and then union with μ_i^n to get μ_{i+1}^n. Properties (1) through (4) are then satisfied.

For $n \geq 1$ define $\mu^n = \bigcup_{i \geq 1} \mu_i^n$ and let $\{\lambda_1^n, \lambda_2^n, \ldots, \lambda_{s(n)}^n\}$ be an enumeration for Λ_n. Note that by (2) and conclusion (vii) of Proposition 3.5 we have $\mathrm{dom}(\mu^n) = \Delta_n$. For $i \geq 1$ define $B_i : \mathbb{N} \to \{0, 1\}$ to be such that $B_i(k)$ is the i^{th} digit from least to most significant in the binary representation of k when $k \geq 2^{i-1}$ and $B_i(k) = 0$ when $k < 2^{i-1}$. Now for $\tau \in 2^\omega$ (Cantor space) we let $c_\tau \in 2^G$ be such that $c_\tau \supseteq c$ and satisfies

$$c_\tau(\gamma\lambda_i^n b_{n-1}) = B_i(\mu^n(\gamma)) \text{ and}$$

$$c_\tau(\gamma\lambda_{s(n)}^n b_{n-1}) = \tau(n-1)$$

for $n \geq 1$, $\gamma \in \Delta_n$, and $1 \leq i < s(n)$. As each $c_\tau \supseteq c$, it follows from conclusion (i) of Proposition 4.5 and our choice of R that $d(c_\tau, x) < 2^{-r} < \epsilon$. Also, since the map $\tau \mapsto c_\tau$ is continuous and one-to-one $\{c_\tau \mid \tau \in 2^\omega\}$ is a perfect subset of 2^G.

Fix $\tau \in 2^\omega$. We will first show c_τ is a 2-coloring. Note that since $|\Lambda_n| > \log_2 \, p_n(|H_n|)$ we have

$$s(n) > \log_2 \, (4|H_n|^5).$$

So all numbers 0 through $2|H_n|^5$ can be represented in binary using $s(n)-1$ digits. Therefore if $\mu^n(\gamma) \neq \mu^n(\psi)$ then there is $1 \leq i < s(n)$ with $c_\tau(\gamma\lambda_i^n b_{n-1}) \neq c_\tau(\psi\lambda_i^n b_{n-1})$.

Let $1_G \neq s \in G$. Since $\bigcup_{n \geq 1} H_n = G$, we may let $n \geq 1$ be least such that $s \in H_n$. Set $T = F_n F_n^{-1} F_n$, and let $g \in G$ be arbitrary. Since the Δ_n-translates of F_n are maximally disjoint within G, there is $\gamma \in \Delta_n$ with $\gamma F_n \cap g F_n \neq \varnothing$. So there is $f \in F_n F_n^{-1}$ with $gf = \gamma \in \Delta_n$. We proceed by cases.

<u>Case 1</u>: $gsf \notin \Delta_n$. Let $V \subseteq F_{n-1} \cap \mathrm{dom}(c)$ be the test region for the extension invariant Δ_n membership test admitted by c. Since $c_\tau \supseteq c$, $gf \in \Delta_n$, and $gsf \notin \Delta_n$, there is $v \in V$ such that $c_\tau(gfv) \neq c_\tau(gsfv)$. This completes this case since $fv \in T$.

<u>Case 2</u>: $gsf \in \Delta_n$. Then

$$(gf)^{-1}(gsf) = f^{-1}sf \in F_n F_n^{-1} H_n F_n F_n^{-1} \subseteq H_n H_n^{-1} H_n^2 H_n^{-1}$$

since $F_n \subseteq H_n$. Thus $(gf, gsf) \in E(\Gamma_n)$ so $\mu^n(gf) \neq \mu^n(gsf)$. Consequently, there is $1 \leq i < s(n)$ with $c_\tau(gf\lambda_i^n b_{n-1}) \neq c_\tau(gsf\lambda_i^n b_{n-1})$. This completes this case since $f\lambda_i^n b_{n-1} \in T$. We conclude c_τ is a 2-coloring.

Now suppose $\tau \neq \sigma \in 2^G$, and let $n \geq 1$ satisfy $\tau(n-1) \neq \sigma(n-1)$. We will show c_τ and c_σ are orthogonal. Let $T = F_n F_n^{-1} F_n$ and let $g_1, g_2 \in G$ be arbitrary. Then there is $f \in F_n F_n^{-1}$ with $g_1 f \in \Delta_n$. We proceed by cases.

<u>Case 1</u>: $g_2 f \notin \Delta_n$. Let $V \subseteq F_{n-1}$ be the test region for the extension invariant Δ_n membership test admitted by c. Since $g_1 f \in \Delta_n$ and $g_2 f \notin \Delta_n$, there is $v \in V$ with $c_\tau(g_1 fv) \neq c_\sigma(g_2 fv)$.

Case 2: $g_2 f \in \Delta_n$. Then $c_\tau(g_1 f \lambda^n_{s(n)} b_{n-1}) = \tau(n-1) \neq \sigma(n-1) = c_\sigma(g_2 f \lambda^n_{s(n)} b_{n-1})$. We conclude c_τ and c_σ are orthogonal.

Fix $\tau \in 2^\omega$. All that is left is to show that c_τ is minimal. Fix $n \geq 1$. We will show that $c_\tau(\gamma h) = c(h)$ for all $\gamma \in \Delta_{n+13}$ and all $h \in H_n$. Let $1 \leq k \leq n$ and let $\psi \in \Delta_k \cap F_{n+3}$. Then there is $m \in \mathbb{N}$ with $n+7 \leq k+4+3m < n+10$, and we know that $\mu^k(\psi) = \mu^k(\sigma\psi)$ for all $\sigma \in \Delta_{k+4+3m+3} \supseteq \Delta_{n+13}$. Therefore we have

$$g \in (G - \mathrm{dom}(c)) \cap F_{n+3} \text{ and } \sigma \in \Delta_{n+13} \Longrightarrow c_\tau(g) = c_\tau(\sigma g).$$

When we combine this with conclusion (vi) of Proposition 4.5 we find that $c_\tau(g) = c_\tau(\sigma g)$ for all $g \in F_{n+3} - \{a_{n+3}, b_{n+3}\}$ and all $\sigma \in \Delta_{n+13}$. Since $F_{n+2} \subseteq F_{n+3} - \{a_{n+3}, b_{n+3}\}$, we have $c_\tau(g) = c_\tau(\sigma g)$ for all $g \in F_{n+2}$ and all $\sigma \in \Delta_{n+13}$.

Let $h \in H_n - F_{n+2}$. It is enough to show that $\Delta_{n+13} h \cap \Delta_1 F_1 = \varnothing$. It will follow from conclusion (v) of Proposition 4.5 that $c(h) = c(\sigma h)$ for all $\sigma \in \Delta_{n+13}$. Towards a contradiction suppose $\sigma h \in \Delta_1 F_1$ for some $\sigma \in \Delta_{n+13}$. Let $\psi \in \Delta_1$ be such that $\sigma h \in \psi F_1$. Note $\psi F_1 \subseteq \sigma h F_1^{-1} F_1 \subseteq \sigma H_n H_2 \subseteq \sigma H_{n+1}$. By conclusions (vi) and (viii) of Proposition 3.5, the $\Delta_1 \cap \sigma F_{n+2}$-translates of F_1 are maximally disjoint within σH_{n+1}. So $\psi F_1 \cap (\Delta_1 \cap \sigma F_{n+2}) F_1 \neq \varnothing$. Since $\psi \in \Delta_1$ and the Δ_1-translates of F_1 are disjoint, we must have $\psi \in \Delta_1 \cap \sigma F_{n+2}$. Consequently,

$$\sigma h \in \psi F_1 \subseteq (\Delta_1 \cap \sigma F_{n+2}) F_1 = \Delta_1 F_1 \cap \sigma F_{n+2} \subseteq \sigma F_{n+2}.$$

This implies $h \in F_{n+2}$, a contradiction. Thus $\Delta_{n+13} h \cap \Delta_1 F_1 = \varnothing$. We conclude $c_\tau(\sigma h) = c_\tau(h)$ for all $h \in H_n$ and all $\sigma \in \Delta_{n+13}$.

Now let $B \subseteq G$ be finite. Let $n \geq 1$ be such that $B \subseteq H_n$. Set $T = F_{n+13} F_{n+13}^{-1}$, and let $g \in G$ be arbitrary. Clearly there is $t \in T$ with $gt \in \Delta_{n+13}$ and hence $c_\tau(gtb) = c_\tau(b)$ for all $b \in B$. We conclude c_τ is minimal. $\square$

The interested reader should consult [3] for an extensive generalization of these methods and for further study of 2^G. In particular, each of the methods used in this last proof are isolated and presented in a more abstract setting in [3].

BIBLIOGRAPHY

[1] A. Dranishnikov and V. Schroeder, *Aperiodic colorings and tilings of Coxeter groups*, Groups Geom. Dyn. 1 (2007), 311-328.

[2] Su Gao, Steve Jackson, and Brandon Seward, *A coloring property for countable groups*, Math. Proc. Cambridge Philos. Soc., to appear.

[3] ________, *Group colorings and Bernoulli subflows*, tentative title, in preparation, http://www.math.unt.edu/~sgao.

[4] Eli Glasner and Vladimir Uspenskij, *Effective minimal subflows of Bernoulli flows*, Proc. Amer. Math. Soc., to appear.

[5] Gustav A. Hedlund and Marston Morse, *Symbolic dynamics*, Amer. J. Math. 60 (1938), no. 42, 815-866.